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STRENGTH OF A SOLID CIRCULAR CYLINDER STRAINED BY LATERAL PRESSURE

By

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1. Introduction

Among various items of problems relating to high pressure technique, it was pointed out by P.W. Bridgman in the International Congress for Applied Mechanics at Zürich, 1926, and is also described in his work on "The Physics of High Pressure", that a solid cylinder with free ends and acted on by the lateral fluid pressure is liable to rupture, usually in the middle part, when the pressure rises to a limit approximately equal in magnitude to the tensile strength of the material of cylinder; this takes place, it is said, as if a cylinder is pulled apart by the axial forces acting at ends actually free from any traction; the appearance of the fractured cylinder seems to be like an ordinary tensile specimen with or without local contraction according as the material is ductile or brittle. This phenomenon, called the "pinching-off" effect, drew attention of the attendance at Zürich. But looking over the literature, it reminds us that the above effect may probably be brought in the same category of problems as the tensile rupture of materials surrounded by a pressure medium, as had been investigated by the school of W. Voigt in Göttingen towards the end of the foregoing century and also by some others since then. If this be inadmissible, the reason ought to come from the stress distribution being not identical. So taking the homogeneous stress-state $\sigma_1=0$, $\sigma_2=\sigma_3=-p$ ($p=a$ positive constant), as a standard, any deviation due to a particular surface condition should be made clear.

* Delivered at the Monthly Meeting of the Research Institute, April 30, 1948.

In the present note the elastic calculation of stresses in a solid circular cylinder with infinite length strained by uniform pressure acting in a finite zone of the lateral surface, Figs. 1 and 1a, will be performed to determine

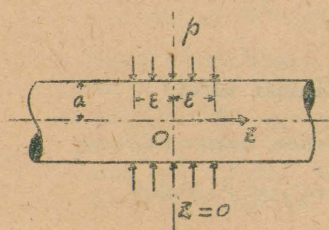

Fig. 1

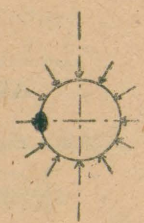

Fig. 1a

the strength of a cylinder with particular attention to the tensile stress in the axial direction. With regard to the present problem, it is to be remarked that an approximate method of the calculation based on energy principle is given in the book, "Drang und Zwang" by *A.* and *L.* Föppl, 2nd vol., 1928, *p.* 141, but the result of the calculation seems to be inapplicable for the present, owing to a great deviation from the exact calculation. Remembering that a method of the exact calcuation was formerly developed by L.N.G. Filon, Phil. Trans. Roy. Soc., Ser. A, vol. 198, 1902, *p.* 147, we find also in the above book an analysis of stresses made by integrating the differential equations of elastic displacements for a case of the pressure distributed practically in accordance with the inverse square law of the distance from the section, where the pressure attains its peak; this pressure distribution is improper to the present case; moreover, the calculation shown there is limited to the stress near the boundary surface.

2. General Formulae.

The deduction and intergration of differential equations of the elastic displacements in a cylinder strained symmetrically about its axis are fully explained in the above works by Filon and Föppl. Also in the author's book (in Jap.) on Mechanics of Materials (Zairyo Rikigaku) the basic formulae of this problem are ready for use; so it will suffice for the present to note that the displacement components, u and w, in the radial and axial directions, respectively, or more definitely their derivatives, $\partial w/\partial r$ and

$\partial u/\partial z$, taken along the radial and axial directions, are bound to the equations of the same form

$$\left(D^2+\frac{\partial^2}{\partial z^2}\right)^2\left(\frac{\partial w}{\partial r},\ \frac{\partial u}{\partial z}\right)=0,$$

where $D^2=\frac{\partial^2}{\partial r^2}+\frac{\partial}{\partial r}\frac{1}{r}$, and that the result of the integration can be represented in circular and Bessel functions; so it is also with the stress-components. Thus, writing $J_0(ikr)=I_0(kr)$, $i^{-1}J_1(ikr)=I_1(kr)$, and leaving out unnecessary terms,

$$\left.\begin{aligned}
\sigma_z &= A\cos kz\left\{4\left(1-\frac{1}{m}\right)I_0(kr)-(1+\alpha)\left[2\left(2-\frac{1}{m}\right)I_0(kr)+kr\,I_1(kr)\right]\right\},\\
\sigma_r &= A\cos kz\left\{4\left(1-\frac{1}{m}\right)\left(\frac{I_1(kr)}{kr}-I_0(kr)\right)+(1+\alpha)\left[\left(1-\frac{2}{m}\right)I_0(kr)+kr\,I_1(kr)\right]\right\},\\
\sigma_t &= A\cos kz\left\{(1+\alpha)\left(1-\frac{2}{m}\right)I_0(kr)-4\left(1-\frac{1}{m}\right)\frac{I_1(kr)}{kr}\right\},\\
\tau &= A\sin kz\left\{2\left(1-\frac{1}{m}\right)(1-\alpha)I_1(kr)-(1+\alpha)kr\,I_0(kr)\right\},
\end{aligned}\right\}$$

and (1)

$$\Theta=\sigma_r+\sigma_t+\sigma_z=-A\,2\left(1+\frac{1}{m}\right)(1+\alpha)\cos kz\,I_0(kr).$$

where σ_z, σ_r, σ_t are the normal stresses in the axial, radial and tangential directions, and τ is the shear stress causing the obliquity of the angle between the linear elements dr and dz; further A and k are arbitrary constants.[1] The normal stresses shown above are even functions of both z and r, while the shear stress is an odd function of the same. Here, the normal stresses being all symmetrical about the plane $z=0$, the expressions (1) suit the stress-state caused by the pressure distributed symmetrically about the said plane. Lastly, in order to make the shear stress vanish on the cylindrical surface, $r=$ the radius a, the constant α in (1) should be taken as

$$\alpha=\frac{2I_1(ka)}{\frac{m}{2(m-1)}ka\,I_0(ka)+I_1(ka)}-1, \qquad (2)$$

ka being simply denoted by y in the following.

---

[1] 'A' is written for $\frac{G}{2}\frac{m}{m-1}A_1$ of the author's book referred to.

Now it is to be observed that the stress-components have to satisfy the equations of equilibrium

$$\left.\begin{aligned} \frac{\partial\sigma_z}{\partial z}+\frac{1}{r}\frac{\partial(r\tau)}{\partial r} &= 0\,, \\ \frac{\partial(r\sigma_r)}{\partial r}-\sigma_t+r\frac{\partial\tau}{\partial z} &= 0\,, \end{aligned}\right\} \tag{3}$$

as may be verified by substituting (1), and making use of the formulae,

$$\frac{\partial I_0(x)}{\partial x}=I_1(x)$$

and

$$\frac{\partial I_1(x)}{\partial x}=I_0(x)-\frac{I_1(x)}{x}\,.$$

As the shear stress disappears on the circular boundary, there is no surface traction in the axial direction; then the stress force on any cross-section must vanish, as it is actually the case, for

$$\int_0^a \sigma_z r dr = A\cos kz\int_0^a\left\{4\left(1-\frac{1}{m}\right)rI_0(kr)-(1+\alpha)\left[2\left(2-\frac{1}{m}\right)rI_0(kr)+kr^2I_1(kr)\right]\right\}dr\,,$$

or with $ka=y$,

$$\int_0^a \sigma_z r dr = A\cos kz\,\frac{2a^2}{y}\left(1-\frac{1}{m}\right)\left\{2I_1(y)-(1+\alpha)\left(\frac{m}{2(m-1)}yI_0(y)+I_1(y)\right)\right\}\,,$$

in virtue of the relations

$$\int xI_0(x)dx = xI_1(x)\,,$$

$$\int x^2I_1(x)dx = x^2I_0(x)-2xI_1(x)\,,$$

and the expression in { }, becomes nought because of the expression (2) for α. Thus, no axial stress-force being present on any cross-section, there is neither pull nor thrust acting on the cylinder. Accordingly, the stress components (1), more particularly σ_z, may be considered to be solely due to the normal traction on the circular boundary, provided that the cylinder is so long that no end effect exists (Saint-Venant's principle) at the part coming into consideration.

3. Preliminaries, Cosine-Distribution of Pressure.

Before going to investigate the stress-state in a long cylinder, as was specified in Art. 1, a survey may be made in a simple case taken for reference, i.e. a cylinder with the length $2l$, Fig. 2, acted on by the pressure varying in such a manner that the stress σ_r at $r=a$ is

$$\sigma_{ra} = -p_1 \cos kz . \tag{4}$$

As the length is finite, the end effect should be made possibly small; this condition can be partially satisfied by taking

$$kl = \frac{\pi}{2} . \tag{5}$$

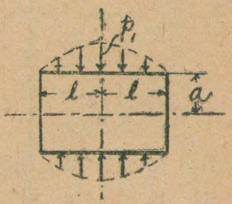

Fig. 2

Then the normal stress σ_z is zero at $z=\pm l$, though the shear stress remains there. Next, to comply with the condition (4), we have from the first expression of (1)

$$A = p_1 \frac{\frac{m}{2(m-1)} y I_0(y) + I_1(y)}{2y\left(I_0^2(y) - I_1^2(y)\right) - 4\left(1-\frac{1}{m}\right)\frac{I_1^2(y)}{y}}, \quad y = ka . \tag{6}$$

With these constants the stresses at $r=0$ and $r=a$ are

$$\left.\begin{aligned} \sigma_{z0} &= \frac{p_1}{\Delta} \cos kz \left(y\frac{I_0(y)}{I_1(y)} - 2\right), \\ \sigma_{r0} &= \sigma_{t0} = -\frac{p_1}{\Delta} \cos kz \left(\frac{y}{2}\frac{I_0(y)}{I_1(y)} + \frac{1}{m}\right), \end{aligned}\right\} \tag{7}$$

and

$$\left.\begin{aligned} \sigma_{za} &= -\frac{p_1}{\Delta} \cos kz\, I_0(y)\left[2 - y\left(\frac{I_0(y)}{I_1(y)} - \frac{I_1(y)}{I_0(y)}\right)\right], \\ \sigma_{ta} &= -\frac{p_1}{\Delta} \cos kz\, 2\left[\frac{I_0(y)}{m} + \left(1-\frac{1}{m}\right)\frac{I_1(y)}{y}\right], \end{aligned}\right\} \tag{8}$$

where

$$\Delta = y I_0(y)\left(\frac{I_0(y)}{I_1(y)} - \frac{I_1(y)}{I_0(y)}\right) - 2\left(1-\frac{1}{m}\right)\frac{I_1(y)}{y} . \tag{9}$$

These stresses are the principal stresses, as $\tau = 0$ for $r=0$ and a.

Now putting $k = y/a$ in (5), the length-diameter ratio is

$$\frac{l}{a} = \frac{\pi}{2y} . \tag{10}$$

For some assigned values of y and the corresponding ratios l/a, as are written in the first two columns of Table 1, the numerical values of the stresses and also of the stress-differences at the middle section $z=0$ are shown in the same table.

Table 1. Stresses at the middle section $z=0$ due to the pressure $p_1 \cos kz$.

		$r=0$			$r=a$		
y	l/a	σ_{r0}/p_1	σ_{z0}/p_1	$(\sigma_{z0}-\sigma_{r0})/p_1$	σ_{ta}/p_1	σ_{za}/p_1	$(\sigma_{za}-\sigma_{ta})/p_1$
0	∞	−1	0	1	−1	0	1
0.25	2π	−0.998	0.012	1.010	−1.004	−0.012	0.992
0.5	π	−0.992	0.046	1.038	−1.014	−0.047	0.967
0.75	$2\pi/3$	−0.980	0.098	1.078	−1.029	−0.103	0.926
1	$\pi/2$	−0.960	0.162	1.122	−1.048	−0.177	0.871
2	$\pi/4$	−0.774	0.387	1.161	−1.108	−0.539	0.569
3	$\pi/6$	−0.492	0.390	0.882	−1.091	−0.810	0.281
4	$\pi/8$	−0.264	0.266	0.530	−1.030	−0.938	0.092

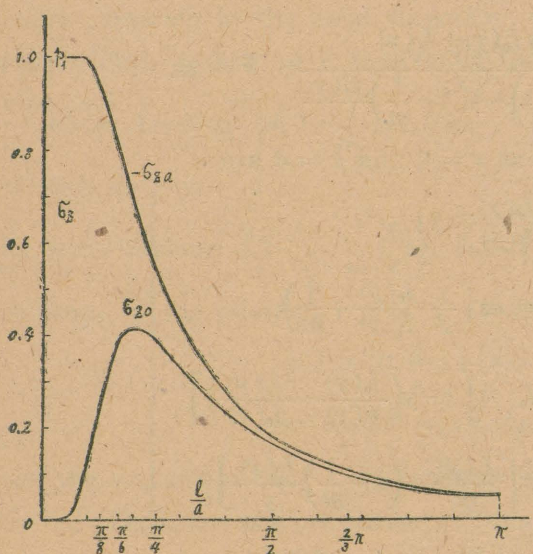

Fig. 3

We see in the table that σ_{z0} is always positive and has a maximum at a point between $\pi/4$ and $\pi/6$ of l/a (cp. Fig. 3), while σ_{za} is negative, and of the stress-differences, $\sigma_{z0}-\sigma_{r0}$ and $\sigma_{za}-\sigma_{ta}$, which are equal to twice the greatest shear stresses at $r=0$ and a, respectively, the former is greater than the latter for all the values of l/a; these circumstances may be referred to as a guide in the following analysis.

4. Uniform Pressure.

Now we have to consider the surface condition as signified by Fig. 1; then the radial stress σ_{ra} at the surface should be made numerically equal to the uniform pressure intensity p in the part between $z=+\varepsilon$ and $-\varepsilon$, while it is zero in the other parts, both positive and negative, extending to infinity, i.e.

$$\sigma_{ra} = -p\frac{2}{\pi}\int_0^\infty \frac{\sin\mu y \cos ny}{y}\,dy\,, \tag{11}$$

where $y = ka$, $\mu = \varepsilon/a$ and $n = z/a$.

The stress-state conforming to this condition can be built up by taking the coefficient A in (1) as a function of k, putting dA in the place of A, such that

$$dA = A'\,a\,dk = A'\,dy\,,$$

and integrating the elementary stress-components with respect to y from zero to infinity. Here, A' is

$$A' = p\,\frac{2}{\pi}\,\frac{\sin\mu y}{y}\;\frac{\dfrac{m}{2(m-1)}\,yI_0(y)+I_1(y)}{2y\Big(I_0^2(y)-I_1^2(y)\Big)-4\Big(1-\dfrac{1}{m}\Big)\dfrac{I_1^2(y)}{y}}\,. \tag{12}$$

With this expression the stress-components at any point (r, z) can be found; particularly, for $r = 0$

$$\left.\begin{aligned}
\sigma_{z0} &= \int_0^\infty A'\cos ny\Big\{4\Big(1-\frac{1}{m}\Big)-2(1+\alpha)\Big(2-\frac{1}{m}\Big)\Big\}dy\,,\\
\sigma_{r0} = \sigma_{t0} &= \int_0^\infty A'\cos ny\Big\{(1+\alpha)\Big(1-\frac{2}{m}\Big)-2\Big(1-\frac{1}{m}\Big)\Big\}dy\,,\\
\Theta_0 = 2\sigma_{r0}+\sigma_{z0} &= -2\Big(1+\frac{1}{m}\Big)\int_0^\infty A'\cos ny(1+\alpha)\,dy\,;
\end{aligned}\right\} \tag{13}$$

and for $r = a$

$$\left.\begin{aligned}
\sigma_{za} &= \int_0^\infty A'\cos ny\Big\{4\Big(1-\frac{1}{m}\Big)I_0(y)-(1+\alpha)\Big[2\Big(2-\frac{1}{m}\Big)I_0(y)+yI_1(y)\Big]\Big\}dy\,,\\
\sigma_{ta} &= \int_0^\infty A'\cos ny\Big\{(1+\alpha)\Big(1-\frac{2}{m}\Big)I_0(y)-4\Big(1-\frac{1}{m}\Big)\frac{I_1(y)}{y}\Big\}dy\,,\\
\Theta_a = \sigma_{r0}+\sigma_{ta}+\sigma_{za} &= -2\Big(1+\frac{1}{m}\Big)\int_0^\infty A'\cos ny(1+\alpha)I_0(y)\,dy\,;
\end{aligned}\right\} \tag{14}$$

τ is zero both for $r = 0$ and $r = a$.

Substituting the expressions (2) and (12) for α and A', respectively, we have from (13) and (14)

$$\left.\begin{aligned}
\sigma_{z0} &= p\,\frac{2}{\pi}\int_0^\infty \mathfrak{A}\,\frac{\sin\mu y\cos ny}{y}\,dy\,,\\
\sigma_{r0} = \sigma_{t0} &= -p\frac{2}{\pi}\int_0^\infty \mathfrak{B}\,\frac{\sin\mu y\cos ny}{y}\,dy\,,\\
\Theta_0 &= -p\frac{2}{\pi}\,2\left(1+\frac{1}{m}\right)\int_0^\infty \frac{1}{\varDelta}\,\frac{\sin\mu y\cos ny}{y}\,dy\,,
\end{aligned}\right\} \tag{15}$$

and

$$\left.\begin{aligned}
\sigma_{za} &= -p\,\frac{2}{\pi}\int_0^\infty \mathfrak{C}\,\frac{\sin\mu y\cos ny}{y}\,dy\,,\\
\sigma_{ta} &= -p\frac{2}{\pi}\int_0^\infty \mathfrak{D}\,\frac{\sin\mu y\cos ny}{y}\,dy\,,\\
\Theta_a &= -p\frac{2}{\pi}\,2\left(1+\frac{1}{m}\right)\int_0^\infty \frac{I_0(y)}{\varDelta}\,\frac{\sin\mu y\cos ny}{y}\,dy\,,
\end{aligned}\right\} \tag{16}$$

where

$$\left.\begin{aligned}
&\mathfrak{A} = \frac{1}{\varDelta}\left(y\frac{I_0(y)}{I_1(y)}-2\right), \qquad \mathfrak{B} = \frac{1}{\varDelta}\left(\frac{y}{2}\,\frac{I_0(y)}{I_1(y)}+\frac{1}{m}\right),\\
&\mathfrak{C} = \frac{I_0(y)}{\varDelta}\left[2-y\left(\frac{I_0(y)}{I_1(y)}-\frac{I_1(y)}{I_0(y)}\right)\right], \qquad \mathfrak{D} = \frac{2I_0(y)}{\varDelta}\left[\frac{1}{m}+\left(1-\frac{1}{m}\right)\frac{I_1(y)}{yI_0(y)}\right]
\end{aligned}\right\} \tag{17}$$

and

$$\varDelta = yI_0(y)\left(\frac{I_0(y)}{I_1(y)}-\frac{I_1(y)}{I_0(y)}\right)-2\left(1-\frac{1}{m}\right)\frac{I_1(y)}{y}\,. \tag{18}$$

Take $z=\varepsilon$ for the present; then the stresses at each end of the pressure zone are

$$\left.\begin{aligned}
\sigma_{z0} &= \frac{p}{\pi}\int_0^\infty \mathfrak{A}\,\frac{\sin 2\mu y}{y}\,dy\,,\\
\sigma_{r0} = \sigma_{t0} &= -\frac{p}{\pi}\int_0^\infty \mathfrak{B}\,\frac{\sin 2\mu y}{y}\,dy\,,\\
\Theta_0 &= -\frac{p}{\pi}\,2\left(1+\frac{1}{m}\right)\int_0^\infty \frac{1}{\varDelta}\,\frac{\sin 2\mu y}{y}\,dy\,,
\end{aligned}\right\} \tag{19}$$

and

$$\left.\begin{aligned}\sigma_{za} &= -\frac{p}{\pi}\int_0^\infty \mathfrak{C}\,\frac{\sin 2\mu y}{y}\,dy\,,\\ \sigma_{ta} &= -\frac{p}{\pi}\int_0^\infty \mathfrak{D}\,\frac{\sin 2\mu y}{y}\,dy\,,\\ \theta_a &= -\frac{p}{\pi}\,2\left(1+\frac{1}{m}\right)\int_0^\infty \frac{I_0(y)}{\varDelta}\,\frac{\sin 2\mu y}{y}\,dy\,,\end{aligned}\right\} \qquad (20)$$

showing that these are one half the respective stresses to be caused at the centre ($z=0$) of the pressure zone with a double range, i.e. 2ε instead of ε; this is quite natural, as the latter stresses may be obtained by combining two equal systems with the semi-zone ε placed side by side. More generally, the stresses belonging to two systems with equal or unequal lengths of the pressure zone may be combined to find the resultant stress.

In the above formulae we have, however, to observe the discontinuity existing at $z=\varepsilon$, particulary on the surface $r=a$; this point will be studied later on.

5. Numerical Calculation of Stresses at an Axial Point.

First, to find the stresses at $r=0$, the evaluation of the integrals was performed by dividing the range of y into two parts, viz. from zero to $y_1 = 10$ and from y_1 to infinity. In the first part, the numerical value of the integrand was calculated for several values of y (0, 0.5, 1, 1.5, 2, 3, ..., 10) by the aid of the tables of Bessel and circular functions, and the quadrature was done by means of Simpson's rule. Table 2 shows the values of $1/\varDelta$, $\mathfrak{A}$ and $\mathfrak{B}$.

Table 2. Values of $\frac{1}{\varDelta}$, $\mathfrak{A}$ and $\mathfrak{B}$.

y	$\frac{1}{\varDelta}$	$\mathfrak{A}$	$\mathfrak{B}$
0	0.7692	0	1
0.5	0.7452	0.0461	0.9918
1	0.6758	0.1623	0.9597
1.5	0.5698	0.2941	0.8879
2	0.4466	0.3869	0.7741
3	0.2286	0.3896	0.4919

4	0.1010	0.2658	0.2642
5	0.0418	0.1503	0.1295
6	0.0167	0.0766	0.0601
7	0.0066	0.0367	0.0269
8	0.0026	0.0169	0.0118
9	0.0010	0.0075	0.0051
10	0.0004	0.0033	0.0022

In the second part beyond $y_1 = 10$, the asymptotic expansions of Bessel functions were used, viz. (cp. Jahnke und Emde, Funktionentafeln, p. 203)

$$I_0(y) = \frac{e^y}{\sqrt{2\pi y}} S_0 , \qquad I_1(y) = \frac{e^y}{\sqrt{2\pi y}} S_1 ,$$

where

$$S_0 = 1 + \frac{c_1}{y} + \frac{c_2}{y^2} + \cdots ,$$

and

$$S_1 = 1 - \frac{d_1}{y} - \frac{d_2}{y^2} - \cdots ,$$

with $c_1 = 0.125$, $c_2 = 0.0703125, \ldots$, $d_1 = 0.375$, $d_2 = 0.1171875, \ldots$. Then, we have approximately

$$\Delta = \frac{e^y}{\sqrt{2\pi y}}\left(1 - \frac{0.775}{y}\right) ,$$

and with $\frac{1}{m} = 0.3$

$$\mathfrak{A} = e^{-y} y \sqrt{2\pi y}\left(1 - \frac{0.725}{y}\right) ,$$

$$\mathfrak{B} = e^{-y} y \sqrt{\frac{\pi}{2} y}\left(1 + \frac{1.875}{y}\right) .$$

These expressions give for $y_1 = 10$, $1/\Delta = 0.00039$, $\mathfrak{A} = 0.00332$ and $\mathfrak{B} = 0.00214$, showing in comparison with the values in Table 2 that the accuracy of the approximate calculation is quite sufficient for the present purpose. Thus, putting

$$F(\mu, n ; y) = \frac{\sin \mu y \cos n y}{y} , \qquad \left(\mu = \frac{\varepsilon}{a} \text{ and } n = \frac{z}{a}\right),$$

we have from (15)

$$\sigma_{z0} = p \frac{2}{\pi}\left\{\int_0^{y_1} \mathfrak{A} F(\mu , n ; y) dy + \int_{y_1}^{\infty} e^{-y} y \sqrt{2\pi y}\left(1 - \frac{0.725}{y}\right) F(\mu , n ; y) dy ,\right\} ,$$

$$\left.\begin{aligned}\sigma_{r0} = \sigma_{t0} &= -p\frac{2}{\pi}\left\{\int_0^{y_1} \mathfrak{B} F(\mu, n; y) dy \right.\\ &\left. + \int_{y_1}^{\infty} e^{-y} y \sqrt{\frac{\pi}{2} y}\left(1 + \frac{1.875}{y}\right) F(\mu, n; y) dy\right\},\\ \Theta_0 &= -p\frac{2}{\pi} 2\left(1+\frac{1}{m}\right)\left\{\int_0^{y_1} \frac{F(\mu, n; y)}{\Delta} dy \right.\\ &\left. + \int_{y_1}^{\infty} e^{-y}\sqrt{2\pi y}\left(1 + \frac{0.775}{y}\right) F(\mu, n; y)\, dy\right\}.\end{aligned}\right\} \quad (21)$$

Leaving out the factor $F(\mu, n; y)$, which is less than μ, for the moment, we have to calculate the integrals

$$\int_{y_1}^{\infty} e^{-y} y^{\frac{3}{2}} dy \quad \text{and} \quad \int_{y_1}^{\infty} e^{-y} y^{\frac{1}{2}} dy$$

in the first two expressions. The first integral may be reduced to the second, for by the partial integration

$$\int e^{-y} y^{\frac{3}{2}} dy = -e^{-y} y^{\frac{3}{2}} + \frac{3}{2}\int e^{-y} y^{\frac{1}{2}} dy .$$

Again, by the partial integration

$$\int e^{-y} y^{\frac{1}{2}} dy = -e^{-y} y^{\frac{1}{2}} + \frac{1}{2}\int e^{-y} y^{-\frac{1}{2}} dy .$$

Substituting in the last integral $y = t^2$,

$$\int e^{-y} y^{-\frac{1}{2}} dy = 2\int e^{-t^2} dt .$$

Accordingly,

$$\int_{y_1}^{\infty} e^{-y} y^{\frac{1}{2}} dy = e^{-y_1}\sqrt{y_1} + \int_{\sqrt{y_1}}^{\infty} e^{-t^2} dt = e^{-y_1}\sqrt{y_1} + \frac{\sqrt{\pi}}{2} - \int_0^{\sqrt{y_1}} e^{-t^2} dt .$$

For $y_1 = 10$, we have

$$\int_0^{\sqrt{10}} e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$$

very nearly, the difference being of the order 10^{-5}; hence, with a sufficient degree of accuracy

$$\int_{10}^{\infty} e^{-y} y^{\frac{1}{2}} dy \cong e^{-10}\sqrt{10} = 0.00014,$$

and also

$$\int_{10}^{\infty} e^{-y} y^{\frac{3}{2}} dy \cong e^{-10} 10\sqrt{10} + \frac{3}{2} \times 0.00014 = 0.00165.$$

These values might modify the last figure (third decimal place) of the coefficients of p/π in Table 3; however, neglecting this order of magnitude, we omit the second integrals beyond $y_1 = 10$ in the expressions of (21); then the stresses at each end of the pressure zone, $2\mu a$ in length, are calculated by putting $\mu = n$, as shown in the table.

Here the values of $\sigma_{r_0} = \sigma_{t_0}$ were found as $\sigma_{r0} = \frac{1}{2}(\Theta_0 - \sigma_{z0})$. The table shows that σ_{z0} is positive; it is greatest when μ is equal to some 0.3.

Table 3. Stresses at $r = 0$ and $z = \varepsilon$ due to uniform pressure p acting in a zone with the length $2\mu a$.

$\mu = \frac{\varepsilon}{a}$	Θ_0	σ_{z0}	$\sigma_{r0} = \sigma_{t0}$	σ_{z0}	$\sigma_{r0} = \sigma_{t0}$
0.1	$-0.962\,p/\pi$	$0.274\,p/\pi$	$-0.618\,p/\pi$	$0.087\,p$	$-0.197\,p$
0.3	-2.365	0.433	-1.399	0.138	-0.445
0.5	-2.963	0.250	-1.606	0.080	-0.511
0.7	-3.137	0.089	-1.613	0.028	-0.513
0.9	-3.179	0.024	-1.602	0.008	-0.510
1.0	-3.200	0.021	-1.611	0.007	-0.513

Now taking two continuous pressure zones, as signified by ε and ε', respectively, each with the same value of p, the resultant stress at the point of continuation is found by adding two stress systems; thus with reference to (19)

$$\sin 2k\varepsilon + \sin 2k\varepsilon' = 2\sin k(\varepsilon+\varepsilon')\cos k(\varepsilon-\varepsilon').$$

In this way we obtain the stress at the point $z = \varepsilon - \varepsilon'$ measured from the centre of the combined zone with the total length $\lambda = 2(\varepsilon + \varepsilon')$; particularly, when $\varepsilon = \varepsilon'$, the point just refererred to coincides with the centre. Table 4 shows the values of the stresses σ_{z0} and σ_{r0} and also of their difference, at some points of the zone with $\lambda = 2a(\mu + \mu') = 2a$, as can be readily found from the component stresses in Table 3; in the most tables following the figure in the third decimal is omitted.

Table 4. Stresses at some points of the zone with $\lambda = 2a$.

Distance from centre	σ_{z0}	σ_{r0}	$\sigma_{z0} - \sigma_{r0}$
0	0.16 p	-1.02 p	1.18 p
0.4 a	0.17	-0.96	1.13
0.8 a	0.10	-0.71	0.81
1.0 a	0.01	-0.51	0.52

Again, Table 5 gives the values of the stresses at the centre of a zone with the total length $\lambda = 2a(\mu + \mu') = 4\mu a$, as were obtained from Table 3.

Table 5. Stresses at the centre of a zone with the total length $4\mu a$.

Length of zone $\lambda = 4\mu a$	σ_{z0}	$\sigma_{r0} = \sigma_{t0}$	$\sigma_{z0} - \sigma_{r0}$
0.4 a	0.17 p	-0.39 p	0.56 p
1.2	0.28	-0.89	1.17
2.0	0.16	-1.02	1.18
2.8	0.06	-1.03	1.09
3.6	0.02	-1.02	1.04
4.0	0.01	-1.03	1.04

It will be seen in the above table that the radial stress at the centre is approximately equal to the pressure applied at the surface, except when the length of the pressure zone is much smaller than the diameter of the cylinder; as to the axial stress, which is positive, there is a peak value at a certain

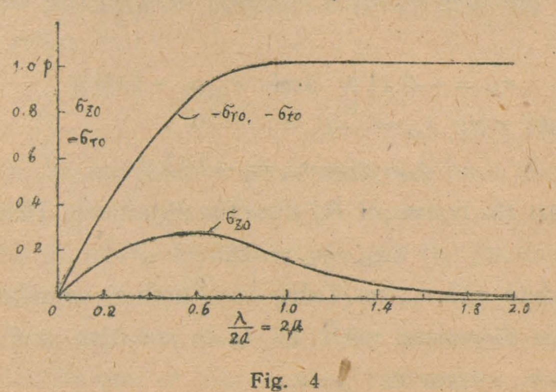

Fig. 4

point of the length-diameter ratio, about $\lambda/2a = 0.6$ say; cp. Fig. 4. So far, it appears that the amount of tensile stress induced in the cylinder is given by σ_{z0}, and further that the stress-difference found above is an important item in estimating the shear strength. However, we have still to calculate the stresses in the surface layer to compare with the above ones.

6. Stresses in the Surface Layer.

Next, to calculate the stresses (16) in a similar manner to the preceding article, i.e. by numerical integration up to $y_1 = 10$ say, and then adding the remainder for higher values of y, we have approximately

$$\mathfrak{C} = 1 + \frac{0.4}{y}, \qquad \mathfrak{D} = 0.6 + \frac{1.94}{y}$$

and

$$\frac{I_0(y)}{\Delta} = 1 + \frac{0.9}{y}, \quad \text{for } y \geqq y_1 = 10 \text{ say.}$$

If we neglect the second term in each of these expressions, the stresses (16) are

$$\left.\begin{aligned} \sigma_{za} &= -\frac{2p}{\pi}\left\{\int_0^{y_1}(\mathfrak{C}-1)F(\mu, n; y)\,dy + \int_0^\infty F(\mu, n; y)\,dy\right\}, \\ \sigma_{ta} &= -\frac{2p}{\pi}\left\{\int_0^{y_1}(\mathfrak{D}-0.6)F(\mu, n; y)\,dy + 0.6\int_0^\infty F(\mu, n; y)\,dy\right\}, \\ \Theta_a &= -\frac{2p}{\pi}2\left(1+\frac{1}{m}\right)\left\{\int_0^{y_1}\left(\frac{I_0(y)}{\Delta}-1\right)F(\mu, n; y)\,dy + \int_0^\infty F(\mu, n; y)\,dy\right\}. \end{aligned}\right\} \tag{22}$$

Taking e.g. $\varepsilon = a$ and $z = 0$, i.e $\mu = 1$ and $n = 0$, the values found from (22) are

$$\sigma_{za} = -0.11\,p \quad \text{and} \quad \sigma_{ta} = -1.06\,p,$$

which give together with $\sigma_{ra} = -p$

$$\sigma_{za} + \sigma_{ta} + \sigma_{ra} = -2.17\,p$$

in coincidence with the value of Θ_a directly obtained. Hence, also

$$\sigma_{za} - \sigma_{ta} = 0.95\,p,$$

a result smaller than $\sigma_{z0} - \sigma_{r0}$ in Table 5. Further calculation for different values of ε may be dispensed with, as σ_{za} is negative and $|\sigma_{ta}|$ is not likely so large that $\sigma_{za} - \sigma_{ta}$ surpasses $\sigma_{z0} - \sigma_{r0}$.

However, the stress at $z=\varepsilon$ attracts special attention because of the discontinuity of the surface condition at that point. Let the rectangular area with the height $\partial r = a(1-\rho)$ and small width $2\alpha = 2a|\mu - n|$, Fig. 5, represent the projection of a cylindrical element at $z=\varepsilon$. Then a review of the condition of equilibrium in the axial direction with regard to this elementary part is necessary, since $\partial\sigma_z/\partial z$ and $\partial\tau/\partial r$ are found to be infinite in the limit when $\alpha \to 0$ and $r \to a$; so the first equation of (3) is subject to a certain revision. The second equation of the same may be left as it is, every term in the equation being finite.

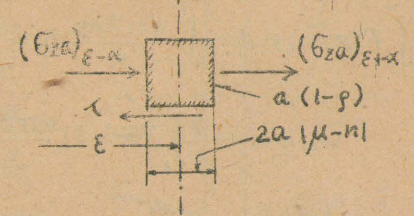

Fig. 5. The stresses in the radial direction are left out for the sake of clearness.

To comply with the condition stated above, we have to retain $(\partial\tau/\partial r)dr$ as well as τ in the first equation of equilibrium; thus we write for the moment

$$\frac{\partial\sigma_z}{\partial z}\left(r+\frac{dr}{2}\right)+\tau+\frac{\partial\tau}{\partial r}(r+dr)=0\,.$$

By making $\rho \to 1$ in $\partial r = a(1-\rho)$, $\tau+(\partial\tau/\partial r)dr$ is the shear stress at the surface, i.e.

$$\tau+\frac{\partial\tau}{\partial r}\,dr=0\,.$$

Accordingly, τ and $(\partial\tau/\partial r)dr$ are of the same order of magnitude; as they disappear from the equation of equilibrium, we have by neglecting dr in comparison with r,

$$\frac{\partial\sigma_z}{\partial z}+\frac{\partial\tau}{\partial r}=0\,. \qquad (3a)$$

This equation may be interpreted as a particular case of the first equation of (3), because the latter reduces to this form, if we omit τ/r, which is small as compared with $|\partial\sigma_z/\partial z|$ and $|\partial\tau/\partial r|$. So the stresses σ_z and τ bound by (3) are correct also at this particular point. Therefore, we pass on to find the values of σ_z and τ at and near the point $z=\varepsilon$. Speaking more specifically, the distribution of the stresses just under the surface layer is to be made clear because of the peculiarity of this point. This item of the subject will be reserved for the next article but one, where the equilibrium of stresses will be considered further in detail.

In the expressions (22) we put first $z=\varepsilon$ or $n=\mu$. Then, since $2\int_0^\infty F(\mu,\mu;y)dy=\int_0^\infty \frac{\sin 2\mu y}{y}dy=\frac{\pi}{2}$, we have

$$\left.\begin{aligned}(\sigma_{za})_\varepsilon &= -\frac{p}{\pi}\int_0^{y_1}(\mathfrak{C}-1)\frac{\sin 2\mu y}{y}dy-\frac{p}{2};\\ (\sigma_{ta})_\varepsilon &= -\frac{p}{\pi}\int_0^{y_1}(\mathfrak{D}-0.6)\frac{\sin 2\mu y}{y}dy-0.6\frac{p}{2},\\ (\Theta_a)_\varepsilon &= -\frac{p}{\pi}2\left(1+\frac{1}{m}\right)\int_0^{y_1}\left(\frac{I_0(y)}{\Delta}-1\right)\frac{\sin 2\mu y}{y}dy-\frac{p}{2}2\left(1+\frac{1}{m}\right).\end{aligned}\right\}\quad(23)$$

Next, we take a point just inside the pressure zone and put $z=\varepsilon-\alpha$ with $\alpha\to 0$; then the first term in each expression of (22) – the integral with the finite limits – will be almost same as in (23), but the second term becomes twice as large as in the above formulae, as

$$2\int_0^\infty F(\mu,n;y)dy=\pi, \quad \text{when} \quad n<\mu.$$

Lastly, as to a point just outside the pressure zone, i.e. $z=\varepsilon+0$, the second term $\int_0^\infty$ disappears.

To illustrate the above statement we take for example $\varepsilon=a$ or $\mu=1$. Then, we find with $y_1=10$ that

$$(\sigma_{za})_\varepsilon=0, \quad (\sigma_{ta})_\varepsilon=-0.50\,p, \quad (\Theta_a)_\varepsilon=-1.00\,p.$$

As $(\sigma_{ra})_\varepsilon=-0.5\,p$ at this point, $(\sigma_{za})_\varepsilon+(\sigma_{ta})_\varepsilon+(\sigma_{ra})_\varepsilon=-1.00\,p$, coinciding with $(\Theta_a)_\varepsilon$ in the present degree of accuracy. The values of σ_{za} on both sides of the very point of the discontinuity are $-p/2$ just within and $+p/2$ just without the zone, Fig. 6, while the values of σ_{ta} in inside and outside differ from $(\sigma_{ta})_\varepsilon$ by $\mp 0.3\,p$, respectirely, It is to be noted that the tensile stress at the point $z=\varepsilon+0$ is much greater than the greatest tensile stress, as may be found in Table 5. But the tensile stress stated above refers to a particular case when $\mu=1$; so the value of the said stress in a more general case comes into question.

Before going, however, into the calculation for a series of values of μ, as will be carried out in the next article, a remark concerning (22) seems to be wanted for the present. In the expressions given there, $1/y$ is neglected in comparison with unity; so it is desirable to see the effect, if any, of this secondary term left out. Now if we suppose to retain this term, an additional term made free from a constant factor is $2\int_{y_1}^{\infty}\frac{F(\mu,n;y)}{y}dy$, which may be written as

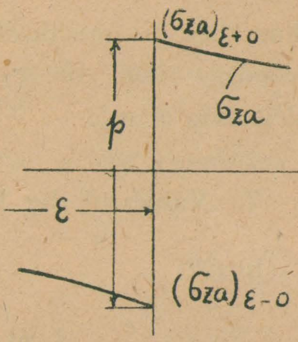

Fig. 6

$$\int_{y_1}^{\infty}\frac{\sin(\mu+n)y}{y^2}dy+\int_{y_1}^{\infty}\frac{\sin(\mu-n)y}{y^2}dy.$$

Putting $t=(\mu+n)y$ and $\eta=(\mu+n)y_1$ in the first integral, it becomes

$$(\mu+n)\int_{\eta}^{\infty}\frac{\sin t}{t^2}dt,$$

or after partial integration

$$(\mu+n)\left[\frac{\sin\eta}{\eta}+\int_{\eta}^{\infty}\frac{\cos t}{t}dt\right].$$

In a similar manner the second integral may be transformed by putting $t=|\mu-n|y$ and $\eta=|\mu-n|y_1$, as

$$\pm\frac{\eta}{y_1}\int_{\eta}^{\infty}\frac{\sin t}{t^2}dt=\pm\frac{\eta}{y_1}\left[\frac{\sin\eta}{\eta}+\int_{\eta}^{\infty}\frac{\cos t}{t}dt\right]$$

according as $\mu\gtrless n$.

In the first case, where it may be assumed that $\eta=(\mu+n)y_1\gg 1$,

$$\int_{\eta}^{\infty}\frac{\cos t}{t}dt\cong-\frac{\sin\eta}{\eta};\text{[2)]}$$

therefore, the expression in the square brackets vanishes. As to the second case, where $\eta=|\mu-n|y_1$ tends to zero,

$$\int_{\eta}^{\infty}\frac{\cos t}{t}dt=-\ln\gamma\eta,\text{[2)]}$$

2) Jahnke-Emde, Funktionentafeln, 2nd ed., p. 78.

$\ln\gamma$ being the Euler's constant. Hence, in the limit

$$\lim_{\eta=0} \eta\int_{\eta}^{\infty} \frac{\cos t}{t}\, dt = -\lim_{\eta=0} \eta \ln\eta = 0 .$$

Thus, there is practically no appreciable error due to the omission of the secondary term; the extent of the actual error will be considered in the calculation of σ_{za}.

7. Values of the Tensile Stress at Ends of the Pressure Zone.

In (22)

$$I_0(y) = 1+\left(\frac{y}{2}\right)^2+\frac{1}{4}\left(\frac{y}{2}\right)^4+\cdots,$$

$$I_1(y) = \frac{y}{2}\left[1+\frac{1}{2}\left(\frac{y}{2}\right)^2+\frac{1}{12}\left(\frac{y}{2}\right)^4+\cdots\right].$$

Accordingly,

$$\frac{y}{2}\left[\frac{I_0(y)}{I_1(y)}-\frac{I_1(y)}{I_0(y)}\right] = 1-\frac{1}{2}\left(\frac{y}{2}\right)^2+\frac{5}{12}\left(\frac{y}{2}\right)^4+\cdots,$$

$$\frac{\varDelta}{I_0(y)} = y\left[\frac{I_0(y)}{I_1(y)}-\frac{I_1(y)}{I_0(y)}\right]-2\left(1-\frac{1}{m}\right)\frac{I_1(y)}{yI_0(y)} = \left(1+\frac{1}{m}\right)\left[1-\frac{1}{2}\left(\frac{y}{2}\right)^2\right],$$

$$\mathfrak{C} = \frac{I_0(y)}{\varDelta}\left[2-y\left(\frac{I_0(y)}{I_1(y)}-\frac{I_1(y)}{I_0(y)}\right)\right] = \frac{m}{1+m}\left(\frac{y}{2}\right)^2\left[1-\frac{1}{3}\left(\frac{y}{2}\right)^2\right].$$

Taking simply the first term only in the last expression, we have in the case when $n=\mu$,

$$2\int_0^{y_1} \mathfrak{C}\, F(\mu, n; y)\, dy = \frac{m}{4(1+m)}\int_0^{y_1} y\sin 2\mu y\, dy .$$

With $\frac{1}{m} = 0.3$,

$$2\int_0^{y_1} \mathfrak{C}\, F(\mu, n; y)\, dy = \frac{1}{5.2(2\mu)^2}(\sin 2\mu y_1 - 2\mu y_1 \cos 2\mu y_1) .$$

If μ is small as compared with unity, this becomes approximately equal to $2\mu y_1^3/15.6$, while it diminishes as μ^{-1} for large values of μ.

Next, the following integral may be also considered in two extreme cases, viz. for small values of μ

$$\int_0^{y_1} 2F(\mu,\mu;y)\,dy = \int_0^{y_1} \frac{\sin 2\mu y}{y}\,dy = 2\mu y_1 - \frac{(2\mu y_1)^3}{3\times 3!},$$

and for large values of μ

$$\int_0^{y_1} 2F(\mu,\mu;y)\,dy = \frac{\pi}{2} - \frac{\cos 2\mu y_1}{2\mu y_1} - \frac{\sin 2\mu y_1}{(2\mu y_1)^2}.$$

Thus, according as μ is very small or large, the second expression of (22) may be written as

$$\sigma_{za} = \frac{p}{\pi} 2\mu y_1\left(1 - \frac{y_1^2}{15.6}\right) - \frac{p}{\pi}\int_0^{\infty} 2F(\mu,n;y)\,dy$$

or

$$\sigma_{za} = \frac{p}{\pi}\left[\frac{\pi}{2} - \frac{1}{(2\mu)^2}\left(\frac{1}{5.2} + \frac{1}{y_1^2}\right)\sin 2\mu y_1 + \frac{1}{2\mu}\left(\frac{1}{5.2} - \frac{1}{y_1^2}\right) y_1 \cos 2\mu y_1\right]$$

$$- \frac{p}{\pi}\int_0^{\infty} 2F(\mu,n;y)\,dy.$$

In either case the first term on the right side is positive, and particularly when $n = \mu + 0$ and $\mu \gg 1$, we have by the latter expression

$$(\sigma_{za})_{\varepsilon+0} \cong \frac{p}{2}, \tag{24}$$

coinciding practically with what was found for the case when $\mu = 1$.

The gap between the above extreme cases may be filled by the stress calculated for some moderate values of μ, as shown in Table 6. The last figure of the stress values is the result slightly modified by the term left out in (22); taking the second term in the approximate expression of $\mathfrak{C}$ for large values of y into account, the term under question is

$$-\frac{p}{\pi}\,0.4\int_{y_1}^{\infty} \frac{\sin 2\mu y}{y^2}\,dy \quad \text{or} \quad -\frac{p}{\pi}\,\frac{0.2}{\mu}\,\frac{\cos 2\mu y_1}{y_1^2} \ \text{approximately.}$$

The correction due to this cause was found to be equal to two units or less in the third decimal place. The table shows that the value of $(\sigma_{za})_{\varepsilon+0}$ is approximately equal to $0.5p$, if μ is not very small as compared with unity.

Table 6. The tensile stress $(\sigma_{za})_{\varepsilon+0}$ acting at just outside $(z=\varepsilon+0)$ of the pressure zone.

$\mu=\frac{\varepsilon}{a}$	$(\sigma_{za})_{\varepsilon+0}$
0.2	0.245 p
0.4	0.402
0.6	0.472
0.8	0.497
1.0	0.505

8. The Stress-Distribution in the Vicinity of the Point of Discontinuity.

In continuation of the condition of equilibrium considered in Art. 6, we have the gradient of σ_z for dz just without the boundary $z=\varepsilon$,

$$\frac{\partial\sigma_z}{\partial z}=\lim_{\alpha=0}\left[\left((\sigma_z)_{\varepsilon+\alpha}-(\sigma_z)_{\varepsilon}\right)/\alpha\right].$$

A similar expression can be written for dz just within the boundary. On the other hand $(\partial\tau/\partial r)\,dr=-\tau$, τ being the shear stress in a layer just under the surface; hence by (3 a)

$$\lim_{\alpha=0,\ \delta r=0}\left[\left((\sigma_z)_{\varepsilon+\alpha}-(\sigma_z)_{\varepsilon-\alpha}\right)\delta r-2\alpha\tau\right]=0. \tag{25}$$

This equation may be directly established by observing the equilibrium of the stress forces acting on an elementary part of the cylinder. Here it is necessary to pay attention to the variation of the stresses even in a very small area, as they can not be considered as uniform; so we have to take the mean values such that

$$\bar{\sigma}_z\,a\,\delta r=\int_{a-\delta r}^{a}\sigma_z\,r\,dr \quad\text{and}\quad 2\alpha\bar{\tau}=\int_{\varepsilon-\alpha}^{\varepsilon+\alpha}\tau\,dz. \tag{26}$$

The expressions for σ_z and τ are by (1), (2) and (12)

$$\sigma_z=-p\frac{2}{\pi}\int_0^\infty F(\mu,n;y)\frac{I_0(\rho y)}{\Delta}\left[2-y\left(\frac{I_0(y)}{I_1(y)}-\rho\frac{I_1(\rho y)}{I_0(\rho y)}\right)\right]dy, \tag{27}$$

$$\tau=p\frac{2}{\pi}\int_0^\infty\frac{\sin\mu y\sin ny}{\Delta}I_0(y)\left[\frac{I_1(\rho y)}{I_1(y)}-\rho\frac{I_0(\rho y)}{I_0(y)}\right]dy, \tag{28}$$

where $\rho y=kr$, ρ being equal to r/a. From (27)

$$\int_{\rho a}^{a} \sigma_z r dr = -p\frac{2a^2}{\pi}\int_0^\infty F(\mu, n; y)\frac{I_0(y)}{\Delta}dy\int_\rho^1 \left[\left(\frac{2}{I_0(y)}-\frac{y}{I_1(y)}\right)I_0(\rho y)+\rho y\frac{I_1(\rho y)}{I_0(y)}\right]\rho d\rho.$$

As

$$\int \rho I_0(\rho y) d\rho = \frac{\rho}{y} I_1(\rho y)$$

and

$$\int \rho^2 I_1(\rho y) d\rho = \frac{\rho^2}{y} I_0(\rho y) - \frac{2\rho}{y^2} I_1(\rho y),$$

the integral for ρ becomes

$$\int_\rho^1 \left[\qquad\right]\rho d\rho = \rho\frac{I_1(\rho y)}{I_1(y)} - \rho^2\frac{I_0(\rho y)}{I_0(y)}.$$

Thus, by the first expression of (26)

$$\begin{aligned}\bar{\sigma}_z \delta r &= \frac{1}{a}\int_{\rho a}^{a} \sigma_z r dr \\ &= -p\frac{2a}{\pi}\int_0^\infty F(\mu, n; y)\frac{I_0(y)}{\Delta}\left[\rho\frac{I_1(\rho y)}{I_1(y)} - \rho^2\frac{I_0(\rho y)}{I_0(y)}\right]dy. \qquad (29)\end{aligned}$$

Again, calculating the second expression of (26), we have since $ny = kz$,

$$\begin{aligned}\int_{\varepsilon-a}^{\varepsilon+a} \sin kz\, dz &= -\frac{1}{k}\cos k(\varepsilon+a) + \frac{1}{k}\cos k(\varepsilon-a) \\ &= -\frac{a}{y}\cos\left(\mu+\frac{a}{a}\right)y + \frac{a}{y}\cos\left(\mu-\frac{a}{a}\right)y.\end{aligned}$$

Accordingly,

$$\begin{aligned}2a\bar{\tau} = \int_{\varepsilon-a}^{\varepsilon+a} \tau\, dz &= -p\frac{2a}{\pi}\int_0^\infty \frac{\sin \mu y \cos\left(\mu+\frac{\alpha}{a}\right)y}{y}\frac{I_0(y)}{\Delta}\left[\frac{I_1(\rho y)}{I_1(y)} - \rho\frac{I_0(\rho y)}{I_0(y)}\right]dy \\ &+ p\frac{2a}{\pi}\int_0^\infty \frac{\sin \mu y \cos\left(\mu-\frac{\alpha}{a}\right)y}{y}\frac{I_0(y)}{\Delta}\left[\frac{I_1(\rho y)}{I_1(y)} - \rho\frac{I_0(\rho y)}{I_0(y)}\right]dy. \qquad (30)\end{aligned}$$

The above calculation shows that when ρ tends to unity,

$$(\bar{\sigma}_z)_{\varepsilon+a}\delta r - (\bar{\sigma}_z)_{\varepsilon-a}\delta r - 2a\bar{\tau} = 0$$

for any relative values of α and $(1-\rho)a$, confirming the validity of (25).

Now to find the stresses acting on the surface of the elementary part under consideration, we have first for σ_z the integral

$$\int_0^\infty F(\mu, n; y)\frac{I_0(\rho y)}{\varDelta}\left[2-y\left(\frac{I_0(y)}{I_1(y)}-\rho\frac{I_1(\rho y)}{I_0(\rho y)}\right)\right]dy.$$

Writing this in the form

$$\int_0^{y_1} + \int_{y_1}^\infty ,$$

and expanding $I_0(\rho y)$ and $I_1(\rho y)$ in the first integral such that

$$I_0(\rho y)=I_0(y-\overline{1-\rho}y) = I_0(y)-(1-\rho)yI_1(y),$$

$$I_1(\rho y)=I_1(y-\overline{1-\rho}y) = I_1(y)-(1-\rho)(yI_0'(y)-I_1(y)),$$

we obtain

$$I_0(\rho y)\left[2-y\left(\frac{I_0(y)}{I_1(y)}-\rho\frac{I_1(\rho y)}{I_0(\rho y)}\right)\right] = I_0(y)\left[2-y\left(\frac{I_0(y)}{I_1(y)}-\frac{I_1(y)}{I_0(y)}\right)\right]-2(1-\rho)yI_1'(y)$$

$$+ \text{ terms of higher powers of } (1-\rho).$$

As ρ is very near unity, the product of $(1-\rho)$ and a finite integral is small; so it is with the second term on the right side. Therefore, we have approximately

$$\int_0^{y_1} = \int_0^{y_1} \mathfrak{C}\, F(\mu, n; y)\,dy$$

or

$$\int_0^{y_1} = \int_0^\infty \mathfrak{C}\, F(\mu, n; y)dy-\int_{y_1}^\infty F(\mu, n; y)\,dy.$$

Next, we expand the integrand of the second integral $\int_{y_1}^\infty$ in the form

$$\frac{I_0(\rho y)}{\varDelta}\left[2-y\left(\frac{I_0(y)}{I_1(y)}-\rho\frac{I_1(\rho y)}{I_0(\rho y)}\right)\right] = \frac{e^{-(1-\rho)y}}{\sqrt{\rho}}\Big[1-(1-\rho)y\Big]+ \text{ terms of } \frac{1}{y}.$$

The integral is now approximately

$$\int_{y_1}^\infty F(\mu, n; y)e^{-(1-\rho)y}\Big[1-(1-\rho)y\Big]dy,$$

in which we have

$$\int_{y_1}^\infty F(\mu, n; y)e^{-(1-\rho)y}dy = \int_0^\infty F(\mu, n; y)e^{-(1-\rho)y}dy$$

$$-\int_0^{y_1} F(\mu, n; y)\Big[1-(1-\rho)y\Big]dy$$

$$=\frac{1}{2}\int_0^\infty \frac{\sin(\mu+n)y}{y}\, e^{-(1-\rho)y}\,dy$$

$$+\frac{1}{2}\int_0^\infty \frac{\sin(\mu-n)y}{y}\,e^{-(1-\rho)y}dy-\int_0^{y_1}F(\mu,n;y)dy$$

$$+(1-\rho)\int_0^{y_1}F(\mu,n;y)ydy\,.$$

Assuming that $\mu\neq n$ and leaving out the last term, we see that

$$\int_{y_1}^\infty F(\mu,n;y)e^{-(1-\rho)y}dy$$

$$=\frac{1}{2}\tan^{-1}\left(\frac{\mu+n}{1-\rho}\right)+\frac{1}{2}\tan^{-1}\left(\frac{\mu-n}{1-\rho}\right)-\int_0^{y_1}F(\mu,n;y)dy\,.$$

The other part of the above integral, viz. $\int_{y_1}^\infty F(\mu,n;y)e^{-(1-\rho)y}ydy$ is finite; so the product $(1-\rho)\int_{y_1}^\infty$ may be neglected. Thus, if ρ is near unity, the result of the integration is

$$\int_0^\infty F(\mu,n;y)\frac{I_0(\rho y)}{\varDelta}\left[2-y\left(\frac{I_0(y)}{I_1(y)}-\rho\frac{I_1(\rho y)}{I_0(\rho y)}\right]dy\right.$$

$$=\int_0^\infty \mathfrak{C}\,F(\mu,n;y)dy-\int_0^\infty F(\mu,n;y)dy+\frac{1}{2}\tan^{-1}\left(\frac{\mu+n}{1-\rho}\right)+\frac{1}{2}\tan^{-1}\left(\frac{\mu-n}{1-\rho}\right).$$

Therefore, with $\sigma_{za}=-p\frac{2}{\pi}\int_0^\infty \mathfrak{C}\,F(\mu,n;y)dy$ and $\lim_{\rho\to1}\tan^{-1}\left(\frac{\mu+n}{1-\rho}\right)=\frac{\pi}{2}$,

$$\sigma_z=\sigma_{za}+p\frac{2}{\pi}\int_0^\infty F(\mu,n;y)dy-\frac{p}{2}-\frac{p}{\pi}\tan^{-1}\left(\frac{\mu-n}{1-\rho}\right). \tag{31}$$

This gives, when we put $\sigma_{za}=\pm p/2$ according as $n\lessgtr\mu$,

$$\sigma_z=\frac{p}{\pi}\tan^{-1}\left(\frac{n-\mu}{1-\rho}\right). \tag{32}$$

If $(1-\rho)$ is far smaller than $|n-\mu|$, σ_z tends to $\pm p/2$ according as $n\lessgtr\mu$, as was just assumed.

To find the value of τ for $n=\mu\pm a$ and $\rho\to1$, the integral in (28) is also divided in two parts:

$$\int_0^{y_1}+\int_{y_1}^\infty\,.$$

In the first integral we write

$$\frac{I_1(\rho y)}{I_1(y)}-\rho\frac{I_0(\rho y)}{I_0(y)}=(1-\rho)\left[2-y\left(\frac{I_0(y)}{I_1(y)}-\frac{I_1(y)}{I_0(y)}\right)\right]$$

correctly to the first degree of $(1-\rho)$; so the first integral diminishes with $(1-\rho)$, and it may be neglected.

As to the second integral for a large value of y, we have

$$I_0(y)\left[\frac{I_1(\rho y)}{I_1(y)}-\rho\frac{\dot{I}_0(\rho y)}{I_0(y)}\right]=\frac{e^{\rho y}}{\sqrt{2\pi y}}\frac{1-\rho}{\sqrt{\rho}}\left(1-\frac{0.375}{\rho y}\right).$$

Substituting this expression as well as the approximate expression for Δ, and neglecting the term $1/y$,

$$\tau=p\frac{2}{\pi}\frac{1-\rho}{\sqrt{\rho}}\int_{y_1}^{\infty}\sin\mu y\sin ny\,.\,e^{-(1-\rho)y}\,dy\,. \tag{33}$$

Writing here

$$2\sin\mu y\sin ny=\cos(\mu-n)y-\cos(\mu+n)y\,,$$

$$\int_{y_1}^{\infty}e^{-(1-\rho)y}\Big[\cos(\mu-n)y-\cos(\mu+n)y\Big]dy$$

$$=\frac{e^{-(1-\rho)y_1}[(1-\rho)\cos(\mu-n)y_1-(\mu-n)\sin(\mu-n)y_1]}{(1-\rho)^2+(\mu-n)^2}$$

$-$ a similar term with $(\mu+n)$ instead of $(\mu-n)$.

When $(1-\rho)\ll1$ and $|\mu-n|\ll1$, we obtain by neglecting terms of higher powers of $(1-\rho)$ and $(\mu-n)$,

$$\tau=\frac{p}{\pi}\frac{(1-\rho)^2}{(1-\rho)^2+(\mu-n)^2}\,. \tag{34}$$

τ is thus stationary at $\mu=n$, and it is equal to p/π, so long as $(1-\rho)\neq0$; further it diminishes rapidly with $|\mu-n|$ in both directions. In the surface layer $\rho=1$, τ is really zero, whereas the above expression becomes indeterminate when $n=\mu$, showing that $(1-\rho)$ can be made as small as we please but not equal to zero, i.e., there is a rapid fall of the stress between these two close layers, as was already stated.

So far we have considered the stresses σ_z and τ; as for the other two stresses σ_r and σ_t, which run as

$$\left.\begin{aligned}\sigma_r&=p\frac{2}{\pi}\int_0^{\infty}\frac{F(\mu,n;y)}{\Delta}\left[\left(\frac{I_0(y)}{\rho I_1(y)}+\rho y+\frac{2}{\rho y}\frac{m-1}{m}\right)I_1(\rho y)-\left(y\frac{I_0(y)}{I_1(y)}+1\right)I_0(\rho y)\right]dy\,,\\ \sigma_t&=p\frac{2}{\pi}\int_0^{\infty}\frac{F(\mu,n;y)}{\Delta}\left[\frac{m-2}{m}I_0(\rho y)-\left(y\frac{I_0(y)}{I_1(y)}+2\frac{m-1}{m}\right)\frac{I_1(\rho y)}{\rho y}\right]dy\,,\end{aligned}\right\} \tag{35}$$

a similar procedure to the foregoing analysis leads to the following expressions.

$$\left.\begin{aligned}&\sigma_r=\frac{p}{\pi}\tan^{-1}\left(\frac{n-\mu}{1-\rho}\right)-\frac{p}{2}\\ \text{and}\qquad&\\ &\sigma_t=\sigma_{ta}+0.3\,p\left[\frac{2}{\pi}\tan^{-1}\left(\frac{n-\mu}{1-\rho}\right)\pm1\right].\end{aligned}\right\}\qquad(36)$$

In the latter expression σ_{ta} stands for the value of σ_t at the surface, and the positive or negative sign before unity should be taken according as μ is greater or less than n. If we put $\sigma_{ta}=-0.50\,p\mp0.3\,p$, as was found for $\mu=1$ (Art. 6), then

$$\sigma_t=p\frac{0.6}{\pi}\tan^{-1}\left(\frac{n-\mu}{1-\rho}\right)-0.50\,p\,.$$

Having found all the stresses, we have now to determine the greatest tensile stress and stress-difference. One of the principal stresses being σ_t, the other two can be calculated by combining σ_z, σ_r and τ. However, the greatest tensile stress is found to be given simply by $(\sigma_{za})_{\varepsilon+a}=p/2$.

Lastly, to find the stress-difference $\sqrt{(\sigma_z-\sigma_r)^2+4\tau^2}$, we have by (32) and (36) $\sigma_z-\sigma_r=p/2$, and $\max\tau=p/\pi$ by (34); so the difference is equal to $0.81\,p$. The part played by σ_t seems to be less marked.

9. Concentrated Load.

In the foregoing calculation $2p\varepsilon$ means the force acting in the area $2\varepsilon.1$; so we put

$$P=2p\varepsilon.\quad\text{or}\quad 2p=\frac{P}{\mu a}.$$

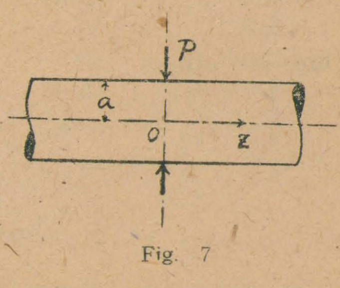

Fig. 7

Then, making ε diminish and tend to zero, P becomes a concentrated load per unit length of the circumference of the cross-section at $z=0$, Fig. 7. If ε or μ be taken so small that $\sin\mu y=\mu y$, then writing $2p=\dfrac{P}{\mu a}$ in (15),

$$\left.\begin{aligned}
\sigma_{z0} &= \frac{P}{\pi a}\int_0^\infty \mathfrak{A}\cos ny\,dy\,,\\
\sigma_{r0} = \sigma_{t0} &= -\frac{P}{\pi a}\int_0^\infty \mathfrak{B}\cos ny\,dy\,,\\
\theta_0 &= -\frac{P}{\pi a}\,2\left(1+\frac{1}{m}\right)\int_0^\infty \frac{\cos ny}{\Delta}\,dy\,,
\end{aligned}\right\} \tag{37}$$

where $\mathfrak{A}$, $\mathfrak{B}$ and Δ are given in (17) and (18).

To compute the integrals numerically, we proceed as before by separating each integral into two parts and applying the approximate expressions for $\mathfrak{A}$, $\mathfrak{B}$ and Δ for large values of y beyond y_1. Particularly, when z or $n=0$,

$$\left.\begin{aligned}
\sigma_{z0} &= \frac{P}{\pi a}\int_0^{y_1}\mathfrak{A}\,dy+\frac{P}{\pi a}\int_{y_1}^\infty e^{-y}y\sqrt{2\pi y}\left(1-\frac{0.725}{y}\right)dy\,,\\
\sigma_{r0} = \sigma_{t0} &\doteq -\frac{P}{\pi a}\int_0^{y_1}\mathfrak{B}\,dy-\frac{P}{\pi a}\int_{y_1}^\infty e^{-y}y\sqrt{\frac{\pi}{2}y}\left(1+\frac{1.875}{y}\right)dy\,,\\
\theta_0 &= -\frac{P}{\pi a}\,2\left(1+\frac{1}{m}\right)\int_0^{y_1}\frac{dy}{\Delta}-\frac{P}{\pi a}\,2\left(1+\frac{1}{m}\right)\int_{y_1}^\infty e^{-y}\sqrt{2\pi y}\left(1+\frac{0.775}{y}\right)dy\,.
\end{aligned}\right\} \tag{38}$$

Taking $y_1=10$, the second term in each expression of (38) was found to contribute a significant figure in the third decimal place, as

$$\sigma_{z0} = \frac{P}{\pi a}(1.494+0.004),$$

$$\sigma_{r0}=\sigma_{t0} = -\frac{P}{\pi a}(3.223+0.002),$$

$$\theta_0 = -\frac{P}{\pi a}(4.952+0.001).$$

Thus, we obtain

$$\left.\begin{aligned}
\sigma_{z0} &= 1.498\,\frac{P}{\pi a} = 0.477\,\frac{P}{a}\,,\\
\sigma_{r0}=\sigma_{t0} &= -3.225\,\frac{P}{\pi a} = -1.027\,\frac{P}{a}\,,\\
\theta_0 &= -4.953\,\frac{P}{\pi a} = -1.577\,\frac{P}{a}\,.
\end{aligned}\right\} \tag{39}$$

The expressions (37) afford the means of calculating the stresses due to uniform pressure. The stress at $z=na$ caused by P acting at $z=0$ is equal to the stress at $z=0$ due to the same load at z; therefore, putting

$$P = p\,dz = p\,a\,dn\,, \quad p \text{ constant,}$$

and integrating the stress for a given pressure zone, from n_1 to n_2, the stresses at $z=0$ are

$$\left.\begin{aligned} \sigma_{z0} &= \frac{p}{\pi}\int_{n_1}^{n_2} dn \int_0^{\infty} \mathfrak{A} \cos ny\, dy\,, \\ \sigma_{r0} = \sigma_{t0} &= -\frac{p}{\pi}\int_{n_1}^{n_2} dn \int_0^{\infty} \mathfrak{B} \cos ny\, dy\,. \end{aligned}\right\} \tag{40}$$

The quadrature of the integrals $\int_0^{10} \mathfrak{A} \cos ny\, dy$ and $\int_0^{10} \mathfrak{B} \cos ny\, dy$ was made by taking several values of y between the limits; the remaining parts of the integrals beyond $y_1=10$ might modify the last figure in Table 7; this degree of accuracy can not be claimed in the present computation. But the figure in the table may be used to know approximately the stress distribution under the concentrated load P at various points of the axis or to calculate the stresses (40) under the distributed load. The stresses given in Tables 3 to 5, Art. 5, can be obtained in this way.

Table 7. Values of $\int_0^{10} \mathfrak{A} \cos ny\, dy$ and $\int_0^{10} \mathfrak{B} \cos ny\, dy$.

n	$\int_0^{10} \mathfrak{A} \cos ny\, dy$	$\int_0^{10} \mathfrak{B} \cos ny\, dy$	n	$\int_0^{10} \mathfrak{A} \cos ny\, dy$	$\int_0^{10} \mathfrak{B} \cos ny\, dy$
0	1.498	−3.225	1.0	−0.509	−0.143
0.1	1.398	−3.121	1.2	−0.407	0.001
0.2	1.133	−2.836	1.4	−0.274	0.037
0.3	0.765	−2.422	1.6	−0.159	0.024
0.4	0.372	−1.951	1.8	−0.062	−0.011
0.5	0.022	−1.485	2.0	0.024	−0.050
0.6	−0.243	−1.073	2.2	0.083	−0.064
0.8	−0.500	−0.473			

10. Deformation of a Cylinder.

The radial contraction of a cylinder particularly at the middle section of the pressure zone may be found to see the effect of the localized action of pressure as compared with the uniform deformation caused by the pressure acting all over the surface. The radial displacement u corresponding to the stress system (1) is generally

$$u = \frac{A}{G}\left(1-\frac{1}{m}\right)\frac{2}{k}\cos kz\left[\frac{m}{4(m-1)}(1+\alpha)krI_0(kr)-I_1(kr)\right], \tag{41}$$

where G is the modulus of rigidity. Replacing A by $dA=A'\,a\,dk=A'\,dy$, and substituting $1+\alpha$ and A' from (2) and (12), respectively, the displacement at $r=a$ is

$$\frac{u_a}{a} = -\frac{p}{G}\frac{2}{\pi}\left(1-\frac{1}{m}\right)\int_0^\infty \frac{\sin \mu y \cos ny}{y\left[y^2\left(\frac{I_0^2(y)}{I_1^2(y)}-1\right)-2\left(1-\frac{1}{m}\right)\right]}dy. \tag{42}$$

For a large value of y it can be shown that

$$y^2\left(\frac{I_0^2(y)}{I_1^2(y)}-1\right) = y+0.25.$$

If we write this expression simply equal to y and further neglect the constant term $2(1-1/m)=1.4$, a part of the integral in (42) becomes for $n=0$

$$\int_{y_1}^\infty \frac{\sin \mu y}{y^2}dy = 0$$

approximately, if $y_1 \gg 1$. Accordingly, the radial contraction at the median section is given by

$$-\frac{u_a}{a} = \frac{p}{E}\frac{4}{\pi}\left(1-\frac{1}{m^2}\right)\int_0^{y_1} \frac{\sin \mu y}{y\left[y^2\left(\frac{I_0^2(y)}{I_1^2(y)}-1\right)-2\left(1-\frac{1}{m}\right)\right]}dy. \tag{43}$$

Putting $1/m=0.3$, the values of the contraction are as shown in Table 8.

Table 8. Radial contraction at the median section.

$\mu=\frac{\varepsilon}{a}$	0.2	0.6	1.0	1.4	1.8	2.0
$-\frac{u_a}{a}$	$0.463\frac{p}{E}$	0.675	0.727	0.708	0.702	0.707

When $\sigma_z=0$, the radial contraction of a long solid cylinder deformed uniformly by the pressure is equal to $-u_a/a=(1-1/m)p/E=0.7\,p/E$. The contraction given in the table is first much smaller than this value; it rises with μ to a maximum and then decreases a little to take a value, which is nearly same as the uniform contraction.

11. Strength of Cylinder.

The greatest tensile stress σ_{z0} at $r=0$, Table 5, is equal to $0.28p$; this stress is probably near a maximum value, as may be seen in Fig. 4. On the other hand, the greatest stress-difference in the same table is $\sigma_{z0}-\sigma_{r0}=1.18p$. Besides, in the surface layer $r=a$, there acts the greatest tensile stress equal to $0.5p$ at each end of the pressure zone extending to a length at least comparable with the diameter of the cylinder. Thus, if we put aside any secondary effect and take simply the greatest tensile stress or the greatest shear stress as the determining factor of the strength, the critical condition for the start of yielding or breaking according to the case will be reached under a certain assumption concerning the length-diameter ratio, when the fluid pressure is raised so high that one of the following equations is satisfied:

(1) $0.28p$ or $0.50p = K_1$ —— tensile fracture,

(2) $1.18p = 2K_3$ —— yielding or shear-fracture,

where K_1 is the tensile strength and K_3 denotes the breaking or the elastic strength in shear, according as the material is brittle or ductile. The stress $0.5p$ acts at a point; it is not likely to be so dominant.

Comparing these equations, (2) will be first satisfied in iron and steel, for even if we take $0.50p$ in (1) for the moment, as the concurrent stress,

$$\frac{K_3}{K_1} < \frac{1.18}{2\times 0.50} = 1.18,$$

the ratio of the yield point and the tensile strength being less than unity; so ductile materials will yield at $p=2K_3/1.18=1.69K_3$, causing the local contraction of the cross-sectional area; this may accompany the tensile action of the pressure acting on the curved surface of the specimen. Here it is interesting to note that 1.69 times yield point is not far from the tensile strength in steels. In brittle materials with a small amount of plastic deformation, shear-fracture instead of yielding comes naturally into consideration; in this case too the rupture by tensile stress does not precede, if the shear strength is within the limit of the above inequality.

Although the above calculation is based on a rather particular value of the length-diameter ratio, the part played by the shear stress does not alter virtually in any other cases, unless the said ratio is very small, as may be seen from the ralues of $\sigma_{z0}-\sigma_{r0}$ in Table 5.

The stress-difference in the surface layer at the middle of a pressure zone is a little less than that in the centre. For example, in the case when the length-diameter ratio is equal to unity, this difference was found to be $0.95p$. However, the elastic failure of the innermost part causes necessarily the redistribution of stresses over the cross section—probably the decrease of the axial stress in magnitude and the increase of the stress-difference $\sigma_{za}-\sigma_{ta}$.

The distribution of pressure in the part of a cylinder contained in the stuffiing box for tightening pressure fluid is really unknown. But any deviation of the distribution from what was assumed in the present investigation will not be so important as to modify the view concerning the strength in the innermost part of a cylinder; of course the stresses in the surface layer at ends of the pressure zone may thereby suffer a ceatain change.

Now the inference that the rupture of ductile materials will be brought about first by yielding, is consistent with the fact observed by Bridgman in respect to the amount of the pressure and the position of the breaking section, while the rupture of brittle materials, e.g. glass or glass-hard steel, wants explanation quite different from that of ductile materials, since the experiment shows that the rupture takes place on a clean surface coinciding with a cross-sectional plane—a fact which is inexplicable by supposing shear fracture. This kind of rupture comes obviously in the category of tensile fracture.

Generally speaking, brittle materials, which do not yield, breaks by tension or shear; the possibility of either kind of rupture depends on the relative values of p in the equations:

(1a) $xp = K_1$,

(2a) $yp = 2K_3$,

where x and y stand for numerical constants, something like in (1) and (2), respectively. However, the value of K_3 depends on the stress state. In the present case σ_z is numerically much smaller than σ_r and σ_t. So we may practically replace (2*a*) by the condition for compressive strength, K_2 say; thus

(2b) $y'p = K_2$.

Really, the compressive strength is often taken as a standard in brittle materials, though its nature may not be always the same. From (1a) and (2b), we see that the fracture by the first condition precedes, if

$$x > y' \frac{K_1}{K_2}.$$

Putting $y'=1$, as may be approximately taken so in consideration of the above calculation, and $K_2/K_1=9$ to 18 for glass, we find that

$$x > 0.11 \sim 0.056.$$

These values are not only far less than those given in (1) but also less than the values for σ_{z0} given in Table 5 except when the length of the pressure zone is much greater than the diameter. Hence, if we disregard the rupture at ends of the pressure zone because of the pressure distribution, besides the reason already mentioned, deviating from the assumed state, the rupture at the middle part is possible. But the amount of pressure is, according to the theory, much higher than the tensile strength; there must be a cause or causes to be considered to clarify the discrepancy.

12. Concluding Remarks.

As an incidental cause for tensile rupture, it may be supposed that there is uneven distribution of pressure round the cylinder-surface contained in the stuffing box; this may give rise to a bending moment acting on the cylinder. If the bending may really take part in rupture is not likely, however. For such an effect, if any, is hardly constant in different cases.

Next, minute flaws may originally exist in materials, and they may grow by the action of stress. So the penetration of pressure fluid into the flaws may occur; the stress-concentration at the bottom of a flaw will be increased; further growth of flaws induces the axial force of the pressure coming into play. This view is, however, hypothetical for the present, as Bridgman states in his book that he found no effect indicating ordinary liquids other than mercury being forced into steel to any slight extent, though there is an evidence speaking for it (Th. v. Kármán's test on marble and sand stone).

A fundamental question is the effect of the lateral compressive stress or stresses on the tensile strength in the axial direction. More generally, the part played by each principal stress in the tensile rupture should be made clear. This subject has been studied, but so far as the investigation goes up to present, the effect of the lateral compression alone is not so great that the tensile rupture occurs at pressure equal to the tensile strength, if the pressure fluid does not get into flaws.

Lastly, the reference made in Art. 1 reminds us of the difference existing between the stresses found in the above calculation and those given by Föppl's approximate calculation. For example, the stresses at the centre of the middle section of the pressure zone with the length equal to the diameter of a cylinder are, according to Table 5,

$$\sigma_{z0} = 0.16\,p \quad \text{and} \quad \sigma_{r0} = \sigma_{t0} = -1.02\,p\,,$$

whereas the said approximate calculation gives

$$\sigma_{z0} = 1.01\,p \quad \text{and} \quad \sigma_{r0} = \sigma_{t0} = -1.55\,p\,.$$

Such a great discordance seems to originate from the basic assumption made for the approximate expressions of σ_z and σ_r. Anyhow the comparison shows that the approximate formulae mentioned above do not answer the present purpose.

Summing up the results of the above calculation, we may conclude that the strength of a cylinder acted by the fluid pressure on its curved surface is mainly conditioned by the greatest stress-difference, if the material is ductile; the rupture starts probably with yielding, when the stress-difference reaches a certain limit depending on the nature of materials.

The rupture of brittle materials under a similar action of the pressure has been contemplated with reference to the critical conditions of tension and shear—eventually also of compression. Various causes contributing to the tensile rupture being considered, the question remains still open, if the pressure necessary to bring about the rupture may be identified with the tensile strength of materials.

As was stated, the tensile stress due to the abrupt transition of the pressure at ends of its zone can not claim to have much importance in the present problem, because the assumption made with regard to the pressure distribution is rather arbitrary, and the stress concentration is limited to a very single point. But the result given by this calculation may serve as design data suggesting the occurrence of a similar effect in solid bodies in contact, such as bodies built up by shrinkage or force-fit.

After all we see that the tensile stress found in the present analysis can not play a leading part in the rupture of a cylinder surrounded by pressure fluid. Practically, the "pinching-off" effect may be included in the same subject as the tensile rupture of materials in a pressure medium—a problem known since the time of Voigt. The rupture of material in a stress-state, for example, $\sigma_1=0$, $\sigma_2=\sigma_3=-p$, wants further investigation.

In conclusion I have to note my gratitude to Prof. M. Higuchi, who undertook an independent check on the numerical value of σ_{z0} for the case $\mu=0.6$ in his own way of calculation and found that $\sigma_{z0}=0.27p$ instead of $0.28p$—rather close for the present purpose. His task is valuable as the conclusion given here depends on the value of the tensile stress.