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Introduction

As to the so-called elastic plane stress problems, many papers have already been proposed and in special cases the solutions are fully discussed or obtained in the functional forms, the applied forces on the boundaries and the forms of the domains being simple. But in the latter solutions, such as the Muschelisvili's paper etc., it is yet difficult to find the stress functions so as to satisfy the given conditions in the practical calculations.

In the present paper, it is intended to treat the general case referring to a thin plate with holes acted on the boundaries by external forces in the middle plane, the applied forces on the respective boundary and its twisting moments being not necessarily in equilibrium and the form of the plate, such as the forms of the outer and inner boundaries of the plate, being quite arbitrary.

Fundamental Principle of the Theory

In such a plate as stated at the introduction, if the stress state is approximately regarded as a mean stress state of plane stress, the stress distribution may be expressed by the following formulae

$$(\widehat{xx},\ \widehat{yy},\ \widehat{xy}) = \left(\frac{\partial^2}{\partial y^2},\ \frac{\partial^2}{\partial x^2},\ -\frac{\partial^2}{\partial x \partial y}\right)\chi,\ \widehat{zx} = \widehat{yz} = \widehat{zz} = 0 \qquad (1)$$

where the $x-$ and $y-$ co-ordinate axes lie on the middle plane of the plate with a right angle and the $z-$axis is normal to the plane and the χ is the function which satisfies the following conditions;

(A) At any point in the domain:

$$\nabla_1^4\chi = 0, \qquad \nabla_1^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \tag{2}$$

(B) On the boundaries:

In the case of the plate with n holes, on the i—th boundary

$$\left.\begin{aligned} X_\nu &= \widehat{xx}\cos(x\nu) + \widehat{xy}\cos(y\nu) \\ Y_\nu &= \widehat{xy}\cos(x\nu) + \widehat{yy}\cos(y\nu) \end{aligned}\right\} \tag{3_1}$$

in which the ν is the outward-drawn normal of an elemental arc ds of the i—th boundary and (X_ν, Y_ν) are the x— and y—direction components of an external force on the boundary.

Here, using the Eqs. (1), the above equations can be rewritten into the form

$$\left.\begin{aligned} \left(\frac{\partial\chi}{\partial x}\right)_{P_i} &= -\int_{A_i}^{P_i} Y_\nu\, ds + C_{1i} \\ \left(\frac{\partial\chi}{\partial y}\right)_{P_i} &= \int_{A_i}^{P_i} X_\nu\, ds + C_{2i} \end{aligned}\right\} \tag{3_2}$$

and, moreover, these equations are reduced to

$$\left.\begin{aligned} \left(\frac{\partial\chi}{\partial\nu}\right)_{P_i} &= -\cos(x\nu)_{P_i}\int_{A_i}^{P_i} Y_\nu\, ds + \cos(y\nu)_{P_i}\int_{A_i}^{P_i} X_\nu\, ds \\ &\qquad + C_{1i}\cos(x\nu)_{P_i} + C_{2i}\cos(y\nu)_{P_i} \\ (\chi)_{P_i} &= -\int_{A_i}^{P_i} dx\int_{A_i}^{Q_i} Y_\nu ds + \int_{A_i}^{P_i} dy\int_{A_i}^{Q_i} X_\nu ds + C_{1i}\Big|x\Big|_{A_i}^{P_i} + C_{2i}\Big|y\Big|_{A_i}^{P_i} + C_{3i} \end{aligned}\right\} \tag{3_3}$$

where A_i denotes the starting point of the integration along the i—th boundary, P_i a given point and Q_i any point within the range of the arc length $\widehat{A_iP_i}$.

(C) Along an arbitrary closed curve.

From the conditions that the displacement and its inclinations for the x— and y—axes at a given point in the domain are, in the present problem, one-valued functions of its position, the equations

$$\left.\begin{aligned}
&\oint \frac{\partial}{\partial \nu}(\nabla_1^2\chi)ds = 0\\
&\oint\left(y\frac{\partial}{\partial \nu}-x\frac{\partial}{\partial s}\right)\nabla_1^2\chi ds-(1+\sigma)\left|\frac{\partial \chi}{\partial x}\right|_{A_i}^{A_i}=0\\
&\oint\left(x\frac{\partial}{\partial \nu}+y\frac{\partial}{\partial s}\right)\nabla_1^2\chi ds+(1+\sigma)\left|\frac{\partial \chi}{\partial y}\right|_{A_i}^{A_i}=0
\end{aligned}\right\} \tag{4}$$

must be satisfied when the integration is taken along an arbitrary closed curve in the domain.

The preceding statements tell us that the present problem is fully solved, if we find a proper method, by which the χ is determined so as to satisfy the conditions (2), (3) and (4).

Now, let us firstly examine the mathematical character of the χ function of the present problem over again. The twisting moment about the origin caused by the external forces acting on the i-th boundary is expressed by the equation

$$M_{P_i}=\int_{A_i}^{P_i}(xX_\nu,-yX_\nu)ds=-\left[x\frac{\partial \chi}{\partial x}+y\frac{\partial \chi}{\partial y}\right]_{A_i}^{P_i}+\left[\chi\right]_{A_i}^{P_i}$$

in which M_{P_i} denotes the twisting moment at the point P_i and, as the integration may be taken through the whole closed curve, we have

$$\begin{aligned}
\left|\chi\right|_{A_i}^{A_i} &= M_{A_i}+x_{A_i}\left|\frac{\partial \chi}{\partial x}\right|_{A_i}^{A_i}+y_{A_i}\left|\frac{\partial \chi}{\partial y}\right|_{A_i}^{A_i},\\
&= M_i-x_{A_i}(F_y)_i+y_{A_i}(F_x)_i
\end{aligned} \tag{5_1}$$

where M_i is the total twisting moment caused by all of the external forces acting on the i-th boundary and $(F_x)_i$, $(F_y)_i$ are the x- and y-components of the total force.

Moreover, from the Eq. (5_1) and the 2nd. equation of the Eqs. (3_3), we have

$$-\int_{A_i}^{A_i}dx\int_{A_i}^{Q_i}Y_\nu ds+\int_{A_i}^{A_i}dy\int_{A_i}^{Q_i}X_\nu ds=M_i-x_{A_i}(F_y)_i+y_{A_i}(F_x)_i\,. \tag{5_2}$$

We see from the above equations that the boundary value of the χ increases the amount of the righthand side value of the Eq. (5_1) for one turn of the point P_i through along the i-th boundary, provided that the total force and its twisting moment are not in equilibrium on the i-th boundary alone, and accordingly the χ is, in the present problems, a many-valued plane biharmonic function.

Now, in considering of the conditions (5_1), let us put the χ in the form

$$\chi=\frac{-1}{2\pi}\sum_{k=1}^{n}\{M_k-x(F_y)_k+y(F_x)_k\}\operatorname{arctg}\frac{y-y_k}{x-x_k}+\chi^* . \tag{6}$$

That is, the first term is a known many-valued biharmonic function, the χ^* an unknown one-valued biharmonic one, and (x_k, y_k) denote the co-ordinates at any point in the inside domain of the k-th boundary, $(F_x)_k$ $(F_y)_k$ the x- and y-components of the total force acting on the boundary and M_k is the twisting moment of the force.

Here, as to the χ^* in order that the χ satisfies the conditions (3_3) and (4), we have the following conditions:

Substituting the Eq. (6) into the Eqs. (3) and (4), we have the equations

$$\left.\begin{aligned}
\chi^*_{P_i}&=\frac{1}{2\pi}\sum_{k=1}^{n}\{M_k-x_{P_i}(F_y)_k+y_{P_i}(F_x)_k\}\operatorname{arctg}\frac{y_{P_i}-y_k}{x_{P_i}-x_k}\\
&\quad-\int_{A_i}^{P_i}dx\int_{A_i}^{Q_i}Y_\nu\,ds+\int_{A_i}^{P_i}dy\int_{A_i}^{Q_i}X_\nu\,ds+C_{1i}\Big|x\Big|_{A_i}^{P_i}+C_{2i}\Big|y\Big|_{A_i}^{P_i}+C_{3i}\\
\left(\frac{\partial\chi^*}{\partial\nu}\right)_{P_i}&=\frac{1}{2\pi}\sum_{k=1}^{n}\Big[\{(F_x)_k\cos(y\nu)_{P_i}-(F_y)_k\cos(x\nu)_{P_i}\}\operatorname{arctg}\frac{y_{P_i}-y_k}{x_{P_i}-x_k}\\
&\quad+\{M_k-x_{P_i}(F_y)_k+y_{P_i}(F_x)_k\}\frac{(x_{P_i}-x_k)\cos(y\nu)_{P_i}-(y_{P_i}-y_k)\cos(x\nu)_{P_i}}{(x_{P_i}-x_k)^2+(y_{P_i}-x_k)^2}\Big]\\
&\quad-\operatorname{Cos}(x\nu)_{P_i}\int_{A_i}^{P_i}Y_\nu\,ds+\cos(y\nu)_{P_i}\int_{A_i}^{P_i}X_\nu\,ds+C_{1i}\cos(x\nu)_{P_i}+C'_{2i}\cos(y\nu)_{P_i}
\end{aligned}\right\}\tag{7}$$

along the i-th boundary from the Eqs. (3) and the equations

$$\left.\begin{aligned}
&\oint\frac{\partial}{\partial\nu}(\nabla_1^2\chi^*)=0\\
&\oint\left(y\frac{\partial}{\partial\nu}-x\frac{\partial}{\partial s}\right)\nabla_1^2\chi^*\,ds+\frac{1}{2\pi}(1-\sigma)\left|\sum_{k=1}^{n}(F_y)_k\operatorname{arctg}\frac{y-y_k}{x-x_k}\right|_{A_i}^{A_i}=0\\
&\oint\left(x\frac{\partial}{\partial\nu}+y\frac{\partial}{\partial s}\right)\nabla_1^2\chi^*\,ds+\frac{1}{2\pi}(1-\sigma)\left|\sum_{k=1}^{n}(F_x)_k\operatorname{arctg}\frac{y-y_k}{x-x_k}\right|_{A_i}^{A_i}=0
\end{aligned}\right\}\tag{8}$$

along an arbitrary closed curve in the domain from the Eqs. (4).

In the Eqs. (7), the point P_i turning around along the i-th boundary, the value of the arctg $(y_{P_i}-y_k)/(x_{P_i}-x_k)$ is 2π $(i=0)$ or -2π $(i\neq 0)$ when the point (x_k, y_k) is in the inside domain of the i-th boundary and is zero when the point is in the outside domain of the boundary, so that the functions $\chi^*_{P_i}$ and $(\partial\chi^*/\partial\nu)_{P_i}$ expressed by the Eqs. (7) are obviously one-valued functions in considering of the Eq. (5_2) as well as the fact that the total force acting along all the boundaries of the domain and its twisting moment are always in equilibrium. Hence, we see that conditions (7) satisfy the assumption that χ^* is a one-valued biharmonic function.

Above all, the results obtained above tell us that for solving the present problem, we have to find the function χ^* which is a one-valued biharmonic function and satisfies the conitions (7) on the i-th boundary and the conditions (8) along an arbitrary closed curve in the give domain.

However, we have yet unknown constants $(C_1, C_2, C_3)_i$ in the Eqs. (7) at the respective boundaries and accordingly the boundary values of the χ^* can not be determined.

Hence, let us put the χ^* again as the form

$$\chi^* = \chi_0 + \sum_{i=0}^{n}(C_1\chi_1 + C_2\chi_2 + C_3\chi_3)_i \tag{9}$$

in which we have at any point in the domain

$$\nabla_1^4(\chi_0, \chi_{1i}, \chi_{2i}, \chi_{3i}) = 0 \tag{10}$$

and on the i-th boundary, from the Eqs. (7) and (9),

$$\left.\begin{aligned}
(\chi_0)_{P_i} &= \frac{1}{2\pi}\sum_{k=1}^{n}\{M_k - x_{P_i}(F_y)_k + y_{P_i}(F_x)_k\}\operatorname{arctg}\frac{y_{P_i}-y_k}{x_{P_i}-x_k} \\
&\quad -\int_{A_i}^{P_i}dx\int_{A_i}^{Q_i}Y_\nu ds + \int_{A_i}^{P_i}dy\int_{A_i}^{Q_i}X_\nu ds \\
(\chi_1, \chi_2, \chi_3) &= \left(\Big|x\Big|_{A_i}^{P_i},\ \Big|y\Big|_{A_i}^{P_i},\ 1\right)C_i(s)
\end{aligned}\right\} \tag{11_1}$$

$$\left.\begin{aligned}
\left(\frac{\partial\chi_0}{\partial\nu}\right)_{P_i} &= \frac{1}{2\pi}\sum_{k=1}^{n}\{(F_x)_k\cos(y\nu)_{P_i} - (F_y)_k\cos(x\nu)_{P_i}\}\operatorname{arctg}\frac{y_{P_i}-y_k}{x_{P_i}-x_k} \\
&\quad + \{M_k - x_{P_i}(F_y)_k + y_{P_i}(F_x)_k\}\frac{(x_{P_i}-x_k)\cos(y\nu)_{P_i} - (y_{P_i}-y_k)\cos(x\nu)_{P_i}}{(x_{P_i}-x_k)^2 + (y_{P_i}-y_k)^2} \\
&\quad -\cos(x\nu)_{P_i}\int_{A_i}^{P_i}Y_\nu ds + \cos(y\nu)_{P_i}\int_{A_i}^{P_i}X_\nu ds \\
\left(\frac{\partial\chi_1}{\partial\nu}, \frac{\partial\chi_2}{\partial\nu}, \frac{\partial\chi_3}{\partial\nu}\right)_i &= \{\cos(x\nu)_{P_i}, \cos(y\nu)_{P_i}, O\}C_i(s)
\end{aligned}\right\} \tag{11_2}$$

the $C_i(s)$ being the characteristic function of the $i-$th boundary which denotes

$$C_i(s) = 1 \qquad \text{(along the } i\text{-th boundary)}$$

$$C_i(s) = 0 \qquad \text{(along all the boundaries except the } i\text{-th).}$$

Then it is obvious that the χ^* expressed by the Eq. (9) is a biharmonic function and satisfies the conditions (7).

Here, the boundary values of the functions (χ_0, χ_{1i}, χ_{2i} and χ_{3i}) and their inclinations of the outward normal direction of the $i-$th boundary become known quantities from the conditions (11), so that these biharmonic functions under the conditions (11) can always be found, as is well known, by using the numerical calculating method.

Therefore, the χ^* can be determined from the Eq. (9), provided that the unknown constants $(C_1, C_2, C_3)_i$ in the Eq. (9) can be obtained. And these constants are determined as follows:

Firstly, substituting the Eq. (9) into the Eqs. (8), we have

$$\left.\begin{aligned}
&\oint \frac{\partial}{\partial \nu}\left[\nabla_1^2\left\{\chi_0 + \sum_{i=1}^{n}(C_1\chi_1 + C_2\chi_2 + C_3\chi_3)_i\right\}\right]ds = 0 \\
&\oint \left[\left(y\frac{\partial}{\partial \nu} - x\frac{\partial}{\partial s}\right)\nabla_1^2\left\{\chi_0 + \sum_{i=1}^{n}(C_1\chi_1 + C_2\chi_2 + C_3\chi_3)_i\right\}\right]ds \\
&\qquad + \frac{1}{2\pi}(1-\sigma)\left|\sum_{k=1}^{n}(F_y)_k \operatorname{arctg}\frac{y-y_k}{x-x_k}\right|_{A_i}^{A_i} = 0 \\
&\oint \left[\left(x\frac{\partial}{\partial \nu} + y\frac{\partial}{\partial s}\right)\nabla_1^2\left\{\chi_0 + \sum_{i=1}^{n}(C_1\chi_1 + C_2\chi_2 + C_3\chi_3)_i\right\}\right]ds \\
&\qquad + \frac{1}{2\pi}(1-\sigma)\left|\sum_{k=1}^{n}(F_x)_k \operatorname{arctg}\frac{y-y_k}{x-x_k}\right|_{A_i}^{A_i} = 0
\end{aligned}\right\} \tag{12}$$

These equations have to be held along an arbitrary closed curve in the given domain and the unknown quantities are constants $(C_1, C_2, C_3)_i$ only in the equations as the functions χ_0 and $(\chi_1, \chi_2, \chi_3)_i$ are already known functions. Now, adopting as the arbitrary closed curves the so-called independent closed one, each of which encloses at least a new closed boundary, we have n such curves in the domain with n holes and accordingly $3n$ simultaneous equations of the first degree for $3n$ unknown constants. Therefore, the solution of the equations gives us one set of the values of the unknown constants and the present problem may perfectly be solved. That is to say, firstly the bihormonic functions χ_0 and (χ_1, χ_2, χ_3) under the boundary conditions (Eqs. (7)) are found by a numerical calculating method and secondly the constants $(C_1, C_2, C_3)_i$ from the simultaneous equations (Eqs. (12)) and then the χ^* function is determined from the Eq. (9) and next the χ function from the Eq. (6). The χ being thus obtained for the present problem, the required stress distribution in the given domain is found from the Eq. (1).

Conclusion

In the present paper, it has been shown that the Airy's stress function is a many-valued plane biharmonic function when a given domain is multiply-connected and the applied forces of the respective boundary and its twisting moments are not in equilibrium, and that the function can be separated into two parts, the one being a known bihermonic function expressed by the arctangent series, the other an unknown biharmonic one as stated in the Eq. (6) and we have fully discussed that the function can always be found by utilizing the numerical calculating method, however arbitrary the domain may be.

Of course, in such a special case that the applied forces on the respective houndary and its moments are in equilibrium, the present method may be adopted in the similar way, if we put all the terms concerning with the M_i, $(F_x)_i$ and $(F_y)_i$ in the conditions for the functions stated in the present paper to zero.

Hence, we see that the so-called elastic plane stress problems can always be solved by the method proposed in this paper.

However, we do not consider that the present method is satisfactory, because, at the practical calculation of finding the Airy's function the method is fairly tedious, while the suitable numerical calculating instruments for finding the biharmonic functions satisfing the given boundary conditions can not be used freely, and the accuracy falls, in general, considerably in the twice numerical differentiation of the Airy's function which is necessary for finding the required stress distribution in the given plate.

Thus we are still in course of research for the method of finding the stress distribution directly without the twice numerical differentiation metioned above.

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References

1) Kolosoff: Über einige Eigenschaften des ebenen Problems der Elastizitäts=theorie. Ztschr. ü. Math. u. Phys., Bd. 62, 1914.

2) S. Yokota: General Expression for Stress-Components in Two-Dimensional Problems of Elasticity. Jour. of Mech. Eng., Japan, No. 38, Bd. 18, April 1915.

3) I. Malkin: Zusammenfassende Berichte über einige neuere Arbeiten auf dem Gebiete der Elastizitätslehre. Ztshr. f. angew. Math., Bd. 10, Heft 2, April 1930.

4) E. Kohl: Beitrag zur Lösung des ebenen Spannungsproblems. Ztschr. f. angew. Math. u. Mech., Bd. 11, Heft 2, April 1931.

5) L. Föppl: Konforme Abbildung ebener Spannungszustände. Ztschr. f. angew. Math. u. Mech., Bd. 11, Heft 2, April 1931.

6) E. Weinel: Die Integralgleichungen des ebenen Spannungszustandes und der Plattentheorie. Ztschr. f. angew. Math. u. Mech., Bd. 11, Heft 5, Okt. 1931.

7) N. Muschelisvili: Recherches sur les problèmes aux limites retatifs à l'equation biharmonique et aux equation de l'èlasticite à deux dimensions. Math. Ann. Bd. 107, 1932.

8) N. Muschelisvili: Praktische Lösung der fundementalen Randwertaufgaben der Elastizitäts-theorie in der ebene für einige Berandungsformen. Ztschr. f. angew. Math. u. Mech., Bd. 13, Heft 4, Aug. 1933.

9) E. Wiedemann: Der Formänderungszustand einer Quadratischenplatte mit quartischer Öffnung. Ingenisur-Archiv, Bd. 7, 1936.

10) S. Negoro: Notes on the Solutions of the So-called Two-Dinensional Elastic Problems. Jour. of Mech. Eng., Japan, No. 23, Bd. 6, May 1940.

11) H.M. Westergaard: Displacements Determined by Airy's Stress Function. Jour. of Applied Mech., March 1941.