On the Decay of Vibration of an Elastic Body due to Energy Flow into External Systems: I The Case of the Torsional Vibration of a Wire-and-Rotator System

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https://doi.org/10.15017/7159649

出版情報：九州帝國大學彈性工學研究所報告. 4 (3), pp.27-45, 1947-09. 九州帝國大學彈性工學研究所
バージョン：

九州帝國大學彈性工學研究所報告

第4卷 第3號　昭和22年9月

REPORTS

OF THE

RESEARCH INSTITUTE FOR ELASTICITY ENGINEERING

KYUSHU IMPERIAL UNIVERSITY, FUKUOKA, JAPAN.

Vol. IV, No. 3. September, 1947.

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On the Decay of Vibration of an Elastic Body due to Energy Flow into External Systems.

PUBLISHED BY THE UNIVERSITY

These publications are issued at irregular intervals.

All communications relating to these Reports should be addressed to the Director of the Research Institute for Elasticity Engineering of the Kyûshû Imperial University.

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On the Decay of Vibration of an Elastic Body due to Energy Flow into External Systems.

I (The Case of the Torsional Vibration of a Wire-and-Rotator System).

(Delivered at the Monthly Meeting of the Research Institute, May 21, 1946)

By

Kanae SENDA.

Contents.

I. Introduction.

It is a well-known fact that free vibrations of every so-called elastic body will gradually decay, on account of, for example, resistance of air, internal friction, plastic properties and so on. In actual conditions, on the other hand, the vibrating system is usually connected to other elastic systems (generally very large compared with the system in question) by some clamping or supporting method, and consequently the energy of vibration will be transmitted to them through the junctions, which also gives rise to another source of decay. This sort of decay is the very one that the present author wishes to discuss in the following. On this standpoint of view, therefore, it is quite natural to assume that the vibrating body is perfectly elastic and that the decay of vibration arises only from the source just described. The actual rate of decay will be approximately obtained by summing up those characteristic to several different sources.

We shall be far from success in carrying out the theoretical calculation of the decay even along the line of the above-mentioned simplification, unless the mechanical and geometrical boundary conditions as well as the geometrical form of the vibrating system are also quite simple. Thus the possibility of success lies only in several particular cases which are, however, in themselves, of fundamental importance and of practical interest.

The present writer deals with, as the first example, the case of torsional vibration, in which a discal rotator is suspended by a wire, the latter, in turn, being clamped to another large elastic system, and he discusses the rate of its decay from the above standpoint. This case has been frequently utilized as a method to measure the transverse coefficient of internal friction of the material composing the wire, on the assumption that the amount of energy flow (in the form of diverging elastic waves) from the upper clamped end is negligibly small. Therefore, if this assumption should fail to hold, the above method of measurement would be a nonsense. To verify the legitimacy of the assumption on a theoretical ground is one of the chief objects the present author aims at.

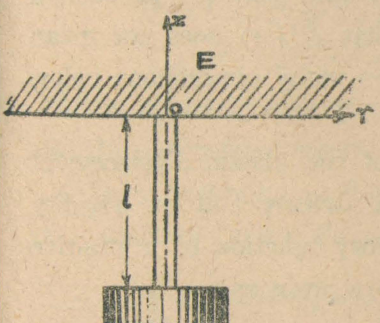
It remains here to add a notice that he dare not assert, judging from the conclusions to be drawn from our investigation, some opinions relative to the appropriateness of the usual idea as well as the common notion of the existence of the so-called internal viscosity in solid materials.

II. Fundamental Principles of the Theory.

When the rotator of our system installed in a vacuum chamber be rotated through an angle from its equilibrium position and then released, it will execute, in general, a rotational vibration of quite approximately simple harmonic nature. The main part of this slight discrepancy between these two motions (i.e. the actual and the simple harmonic) may plausibly arise, as we have just pointed out above, from the very fact that the external clamping system at the upper end of the wire is not rigid in the strict sense but elastic, and therefore the clamping portion undergoes a periodically variable elastic deformation in accordance with the torsional vibration of the wire, which, in turn, plays the part of a source of diverging elastic waves, thus causing the decay of the torsional vibration of our system. The essential part of our investigation consists, therefore, in the calculation of the rate of energy flow in question. The present chapter is devoted to the general principles which enable us to reach this object.

Now, in treating such a problem, it is inevitably necessary to assume the boundary conditions at the junction between the wire and the external clamping system as well as the geometrical form of the latter to be extremely simple, in order that we may put the calculation within our reach. Thus we take the simplest case, in which the upper end of the wire is ideally welded to the free surface of an semi-infinitely large elastic body, and a rigid discal rotator is suspended at the lower as in the adjacent figure.

Since there is the energy flow from the upper junction, it is impossible to take the vibration as exactly periodic; but considering the slightness of the rate of energy flow in unit time, we can assume it approximately to be so. Thus we are led to the problem to calculate the energy flow when the vibration is exactly periodic. Such a manner of treatment seems, at first sight, to be apparently paradoxical, because the original problem to be solved is related to the fact that any free periodic vibration is impossible owing to the energy flow into the external system. A further review, however, convinces us of its legitimacy.[1]



Let us now take cylindrical coordinates r, θ, z, z-axis coinciding with the axis of the wire as in the above figure, and assume all quantities to be independent of θ. Let S_r, S_θ, S_z be the components of elastic displacement in the wire respectively, then we can assume $S_r=0=S_z$ and as the simplest form of S_θ an expression which contains a factor e^{ipt}, where p is a positive constant and t denotes the time variable. Thus the possible general expression for S_θ can be written as[2]

$$S_\theta=\sum_{n=1}^{\infty}(C_n e^{i\gamma_n z}+D_n e^{-i\gamma_n z})\,J_1(\lambda_n r)\,e^{ipt},\quad \gamma_n=\sqrt{\frac{\rho_1}{\mu_1}p^2-\lambda_n^2},\quad \cdots\cdots\cdots (1)$$

1) We can also start by assuming an damped oscillation and then proceed to calculate its damping coefficient.

2) Love, The Mathematical Theory of Elasticity, p. 288.

where C_n and D_n are arbitrary constants; ρ_1 and μ_1 denote the density and and the rigidity of the wire respectively; $J_1(\lambda_n r)$ stands for Bessel function of order unity and of the first kind; and if a is the radius of the wire, $a\lambda_n$'s are determined as not-negative numbers satisfying the following equation;

$$\frac{d}{da}\{J_1(a\lambda_n)/a\}=0, \quad \text{or } J_2(a\lambda_n)=0, \qquad (2)$$

which represents the boundary condition that the side surface of the wire is free from traction; in other words, $a\lambda_n$ are not-negative zeros of Bessel function of order 2: $J_2(a\lambda_n)$.

Since $\lambda_1=0$, $J_1(\lambda_1 r)$ becomes identically zero, but in this case we must take r instead of $J_1(\lambda_1 r)$, and therefore $J_1(\lambda_1 r)$ must be replaced by r whenever it appears in (1) and other expressions derivable from it. In such a state of affairs, we make a conventional rule that $J_1(\lambda_1 r)$ does not mean zero but the simple function r, which rule renders many expressions below compact.

Next we have to express the possible form of the elastic displacement S'_θ in the external body E. Let ρ_2 and μ_2 be the density and the rigidity of E respectively, and let $f(\lambda)$ represent an arbitrory function of a positive parameter λ. Then the expression for S'_θ can be written as

$$S'_\theta=\int_0^\infty f(\lambda)\,e^{i\gamma z}J_1(\lambda r)\,d\lambda e^{ipt}, \quad \gamma=\left(\frac{\rho_2}{\mu_2}p^2-\lambda^2\right)^{1/2}, \qquad (3)$$

where the branch of the two-valued function γ should be taken in accordance with the rule:

$$\left.\begin{array}{ll} 1), & \gamma<0, \quad \text{if } \lambda^2<\rho_2 p^2/\mu_2, \\ 2), & \frac{\gamma}{i}>0, \quad \text{if } \lambda^2>\rho_2 p^2/\mu_2. \end{array}\right\} \qquad (4)$$

The legitimacy of this selection rule will immediately follow, if we consider the physical circumstance of affairs at infinity.

We have assumed the discal rotator to be rigid, and so its rotatory vibration can be expressed by

$$S''_\theta=Ure^{ipt}, \quad (U: \text{const.}) \qquad (5)$$

Such a simplification has not a substantial influence on the results to be obtained, but it reduces the tediousness of calculation considerably.

The boundary condisions on the plane $z=0$ are such that, at the junction between the wire and E, the elastic displacements and the shearing stresses on both sides are equal, and the other portion of the surface is free from traction. Thus we have

$$\int_0^\infty f(\lambda) J_1(\lambda r)\, d\lambda = \sum_{n=1}^{\infty}(C_n + D_n) J_1(\lambda_n r), \quad r<a, \quad (6)$$

$$\mu_2 \int_0^\infty f(\lambda)\, \gamma J_1(\lambda r) d\lambda = \mu_1 \sum_{n=1}^{\infty}(C_n - D_n)\gamma_n J_1(\lambda_n r), \quad r<a, \quad (7)$$

$$\int_0^\infty f(\lambda)\, \gamma J_1(\lambda r)\, d\lambda = 0, \quad r>0. \quad (8)$$

Moreover, the condition that the lower end of the wire is clamped to the rotator is given by the following two equations:

$$\left.\begin{aligned} &C_1 e^{-i\gamma_1 l} + D_1 e^{i\gamma_1 l} = U, \\ &C_n e^{-i\gamma_n l} + D_n e^{i\gamma_n l} = 0, \quad n \geq 2, \end{aligned}\right\} \quad (9)$$

All the boundary conditions we have to condsider with regard to the wire are thus exhausted by the above equations (2), (6), (7), (8) and (9),[1]

Now, according to Fourier-Bessel's integral theorem,[2] if $\phi(r)$, a function of r, satisfies Dirichlet's conditions in a closed interval $0 \leq r \leq a$, we have

$$\begin{aligned} \int_0^\infty d\lambda \int_0^a \lambda\xi \phi(\xi) J_1(\lambda\xi) J_1(\lambda r) d\xi &= \frac{1}{2}\{\phi(r+0) + \phi(r-0)\}, \quad 0<r<a, \\ \text{〃} \qquad &= \frac{1}{2}\phi(+0), \quad r=0, \\ \text{〃} \qquad &= \frac{1}{2}\phi(a-0), \quad r=a, \\ \text{〃} \qquad &= 0, \qquad\quad r>a. \end{aligned}$$

Hence if we assume, from physical standpoint of view, that the right-hand side of (7) is continuous in $0 \leq r < a$ and finite when $r \to a$, both the conditions (7) and (8) are satisfied, provided that the form of the arbitrary function $f(\lambda)$ is given by

$$f(\lambda)\gamma = \frac{\mu_1}{\mu_2}\int_0^a \lambda\xi \left[\sum_{n=1}^{\infty}(C_n - D_n)\gamma_n J_1(\lambda_n \xi)\right] J_1(\lambda\xi) d\xi, \quad (10)$$

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1) When we consider the vibration of the rotator, its equation of motion should be taken as another condition.

2) Gray, Mathews, McRobert, Bessel Functions, pp. 96-97.

On the other hand, the theorem of Dini's expansions[1] teaches us

$$\left.\begin{aligned}&\int_0^\infty f(\lambda)J_1(\lambda r)d\lambda=\sum_{m=1}^\infty A_m J_1(\lambda_m r),\\ \text{where}\quad &A_m=2\int_0^a rJ_1(\lambda_m r)dr\int_0^\infty f(\lambda)J_1(\lambda r)d\lambda/a^2\{J_1(\lambda_m a)\}^2,\end{aligned}\right\}\quad\dots\dots\dots (11)$$

It is to be noted here that, as we have already remarked, in the terms corresponding to $m=1$, $J_1(\lambda_1 r)$ and $J_1(\lambda_1 a)$ do mean r and a respectively, and hence the series (11) commences with term $A_1 r$.

Now we apply (10) and (11) to (6) and get

$$C_m+D_m=\frac{2\mu_1}{a^2\mu_2\{J_1(\lambda_m a)\}^2}\sum_{n=1}^\infty (C_n-D_n)\gamma_n\int_0^a rJ_1(\lambda_m r)$$
$$\times\int_0^\infty\frac{\lambda}{\gamma}J_1(\lambda r)\int_0^a \xi J_1(\lambda_n\xi)J_1(\lambda\xi)d\xi\, d\lambda\, dr,\quad\dots\dots\dots (12)$$

(9) and (12) constitute a system of linear algebraic equations for infinite number of unknowns, C_n's and D_n's. Therefore if these constants can be determined from this system, our dynamical problem will be completely solved. Unfortunately, it is hopelessly difficult, but we can, nevertheless, proceed further in the calculation of energy flow without explicitely determining whole the unknown constants, as we shall see in Chap. III.

Since the triple integral in (12) is rather complicated we had better reduce it to a treatable torm. Changing the order of integration and then treating it in some usual way we have,

$$a^2J_1(\lambda_m a)J_1(\lambda_n a)\int_0^\infty\frac{\lambda^3\{J_2(\lambda a)\}^2}{\gamma(\lambda^2-\lambda_m^2)(\lambda^2-\lambda_n^2)}d\lambda=I_{mn},\ \text{say},$$

or

$$I_{mn}=a^2J_1(\lambda_m a)J_1(\lambda_n a)\left[\frac{\lambda^2_m}{\lambda^2_m-\lambda^2_n}\int_0^\infty\frac{\lambda\{J_2(\lambda a)\}^2}{\gamma(\lambda^2-\lambda^2_m)}d\lambda\right.$$
$$\left.+\frac{\lambda^2_n}{\lambda^2_n-\lambda^2_m}\int_0^\infty\frac{\lambda\{J_2(\lambda a)\}^2}{\gamma(\lambda^2-\lambda^2_n)}d\lambda\right],\quad m\neq n,\quad\dots\dots\dots (13)$$

$$I_{nn}=a^2\{J_1(\lambda_n a)\}^2\int_0^\infty\frac{\lambda^3\{J_2(\lambda a)\}^2}{\gamma(\lambda^2-\lambda^2_n)^2}d\lambda,\quad\dots\dots\dots (14)$$

and in particular

$$I_{n1}=a^3J_1(\lambda_n a)\int_0^\infty\frac{\lambda\{J_2(\lambda a)\}^2}{\gamma(\lambda^2-\lambda^2_n)}d\lambda,\quad\dots\dots\dots (15)$$

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1) Watson, loc. cit., pp. 596–602.

$$I_{11}=a^4\int_0^\infty \frac{\{J_2(\lambda a)\}^2}{\gamma\lambda}d\lambda, \quad \text{(16)}$$

It is only I_{11} that is of essential importance in the calculation of energy flow, as we shall see also in Chap. III below.

Since $\gamma=(\omega^2-\lambda^2)^{1/2}$, where $\omega^2\equiv\rho_2 p^2/\mu_2$, I_{mn}'s are continuous functions of ω and moreover, as a slight examination shows, they are expansible in power series of ω (not of ω^2) in the neighbourhood of $\omega=0$, in other words they are analytic functions of ω in the vicinity of the origin. By these series we can evaluate I_{mn}'s when ω is fairly small. More powerful methods which enable us to evaluate I_{mn}'s are given in the appendix below, whereas the series forms are sufficient to arrive at our present object.

Now that the external clamping body E can be assumed as a common metallic one having large value of rigidity, $\omega\equiv\sqrt{\frac{\rho_2}{\mu_2}}\,p$ becomes a small quantity when the frequency p is not too large; for instance, if we take $\rho_2=10$, $\mu_2=10^{12}$ (in C.G.S. units) and $p=10^3$, we get $\omega=10^{-2.5}$. Thus we can evaluate I_{mn} approximately by expanding them in powers of ω and then taking their first two or three terms. The series expressions for I_{mn} save I_{11} are

$$\left.\begin{aligned} I_{mn}=ia^2J_1(\lambda_m a)J_1(\lambda_n a)\Big[&\frac{\lambda^2_m}{\lambda^2_n-\lambda^2_m}\Big(J_{m0}+\frac{1}{2}J_{m2}\omega^2+\frac{3}{4}J_{m4}\omega^4\Big)\\ &+\frac{\lambda^2_n}{\lambda^2_m-\lambda^2_n}\Big(J_{n0}+\frac{1}{2}J_{n2}\omega^2+\frac{3}{4}J_{n4}\omega^4\Big)\Big]+O(\omega^5),\\ &m\neq n,\ m,\ n\neq 1, \end{aligned}\right\} \quad \text{(17)}$$

$$I_{nn}=-ia^2\{J_1(\lambda_n a)\}^2\Big[K_{n0}+\frac{1}{2}K_{n2}\omega^2+\frac{3}{4}K_{n4}\omega^4\Big]+O(\omega^5),\ n\neq 1, \quad \text{(18)}$$

$$I_{n1}=-ia^3J_1(\lambda_n a)\Big[J_{n0}+\frac{1}{2}J_{n2}\omega^2+\frac{3}{4}J_{n4}\omega^4\Big]+O(\omega^5),\ n\neq 1, \quad \text{(19)}$$

where J_{mn} and K_{mn} represent the following expression respectively

$$J_{mn}=a^{n+1}\int_0^\infty \frac{\{J_2(\xi)\}^2}{\xi^n(\xi^2-\xi^2_m)}d\xi, \quad \text{(20)}$$

$$K_{mn}=a^{n-1}\int_0^\infty \frac{\{J_2(\xi)\}^2}{\xi^{n-2}(\xi^2-\xi^2_m)}d\xi, \quad \text{(21)}$$

with $J_2(\xi_n)=0,\ \xi_n>0,\ i.e,\ \xi_n=a\lambda_n$.

Thus it will suffice to evaluate the integrals:

$$I_\beta^\nu(\xi_n)\equiv\int_0^\infty\frac{\xi^\nu\{J_2(\xi)\}^2}{(\xi^2-\xi_n^2)^\beta}d\xi=\lim_{\tau\to\xi_n}\int_0^\infty\frac{\xi^\nu\{J_2(\xi)\}^2}{(\xi^2-\tau^2)^\beta}d\xi$$

$$\equiv\lim_{\tau\to\xi} I_\beta^\nu(\tau),\ \text{say};\quad \Im(\tau)\neq 0,\ \cdots\cdots \quad (22)$$

where ν denotes even integers such that $2\geqq\nu\geqq-4$, and β is either 1 or 2.

Evaluation of I_{11} will be executed in another way.

Let us now express $\{J_2(\xi)\}^2$ in $I_\beta^\nu(\tau)$ by a Neumann's integral and then change the order of integration, then we get after a slight calculation

$$I_\beta^\nu(\tau)=\frac{1}{\pi}\int_0^\pi d\theta\int_0^\infty\frac{\xi^\nu J_4(2\xi\sin\theta)}{(\xi^2-\tau^2)^\beta}d\xi$$

$$=\frac{1}{\pi}(\mp i\tau)^{\nu+5-2\beta}\frac{\Gamma\left(2+\frac{\nu+1}{2}\right)\Gamma\left(\beta-2-\frac{\nu+1}{2}\right)}{2\cdot 4!\,\Gamma(\beta)}$$

$$\times\int_0^\pi {}_1F_2\left(\frac{\nu+5}{2};\frac{\nu+5}{2}+1-\beta,\ 5;\ -\tau^2\sin^2\theta\right)\sin^4\theta\, d\theta$$

$$+\frac{1}{\pi}\frac{\Gamma\left(2-\beta+\frac{\nu+1}{2}\right)}{2\Gamma\left(\beta+3-\frac{\nu+1}{2}\right)}\int_0^\pi {}_1F_2\left(\beta;\ \beta+\frac{5-\nu}{2},\beta-\frac{3+\nu}{2};\ -\tau^2\sin\theta^2\right)$$

$$\times(\sin\theta)^{2\beta-\nu-1}d\theta,\cdots\cdots \quad (23)^{1)}$$

where the $\mp$ signs in the last equation should be taken according as the imaginary part of τ, say $\Im(\tau)$, is positive or negave; and ${}_1F_2(\ )$ stands for the function defined in the following way:

$${}_1F_2(p_1;\ p_2;\ p_3;\ z)\equiv\sum_{n=0}^\infty\frac{(p_1)_n}{n!\,(p_2)_n(p_3)_n}z^n,\quad (p)_n\equiv\frac{\Gamma(p+n)}{\Gamma(p)}.$$

The first term in the right-hand side of (23) seems to have two different limiting values when τ tends ξ_n. But, according to (22), this is impossible, unless these two limiting values coincide to zero. In fact, the term in question has $\{J_2(\tau)\}^2$ or $J_2(\tau)$ as its factor according as $\beta=1$ or 2, and so it tends to zero with $\tau\to\xi_n$. Thus we get

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1) Watson, loc. cit., p. 434.

$$I_\beta^\nu(\xi_n)=(-)^{\beta+\frac{\nu}{2}}\sum_{k=0}^{\infty}\frac{(-)^k\Gamma(\beta+k)}{k!\,\Gamma\left(\beta+\frac{5-\nu}{2}+k\right)\Gamma\left(\beta-\frac{3+\nu}{2}+k\right)}$$

$$\times\frac{(2k+2\beta-\nu-2)!!}{(2k+2\beta-\nu-1)!!}\xi_n^{2k}, \quad\text{..............................} \quad (24)$$

where $\qquad (2m)!!\equiv 2m(2m-2)(2m-4)\text{.....................}\ 2,$

and $\qquad (2m+1)!!\equiv(2m+1)(2m-1)(2m-3)\ \text{...........}\ 1.$

Next let us consider I_{11}. By the definition of I_{11}

$$\frac{1}{a^4}I_{11}=-i\int_0^\infty\frac{\{J_2(a\lambda)\}^2}{\lambda(\lambda^2-\omega^2)^{1/2}}d\lambda=-i\lim_{k\to\omega}\int_0^\infty\frac{\{J_2(a\lambda)\}^2d\lambda}{\lambda\{\lambda^2+(ik)^2\}^{1/2}},\ \Im(k)<0.$$

Expressing $\{J_2(a\lambda)\}^2$ by a Neumann's integral as before and changing the order of integration, we get

$$\int_0^\infty\frac{\{J_2(a\lambda)\}^2}{\lambda\{\lambda^2+(ik)^2\}^{1/2}}d\lambda=\frac{1}{\pi}\int_0^\pi d\theta\int_0^\infty\frac{J_4(2a\lambda\sin\theta)}{\lambda\{\lambda^2+(ik)^2\}^{1/2}}d\lambda$$

$$=-i\frac{a^4k^3}{2}\sum_{r=0}^{\infty}\frac{(-)^r(ak)^{2r}}{(r+2)r!(r+4)!}-a\sum_{r=0}^{\infty}\frac{(-)^r(ak)^{2r}}{(2r+1)\Gamma\left(r+\frac{7}{2}\right)\Gamma\left(r-\frac{1}{2}\right)},\ ^{1)}$$

$$=a\left(\frac{4}{15\pi}+\frac{16}{315\pi}a^2k^2-\frac{i}{96}a^3k^3+\ldots\ldots\right)$$

or

$$I_{11}=-ia^5\left(\frac{4}{15\pi}+\frac{16}{315\pi}a^2\omega^2-\frac{i}{96}a^3\omega^3+\cdots\cdots\right),\quad\text{......................}\quad (25)$$

Expressions necessary for the evaluation of I_{mn}'s when ω is small have thus been obtained.

III. Calculation of the Rate of Dissipation of Energy.

Let us divide the expression for S_θ in (1) into its real and imaginary parts and denote them as

$$\left.\begin{aligned}S_\theta&=(V_1+iV_2)(\cos pt+i\sin pt),\\ S_1&\equiv\Re(S_\theta)=V_1\cos pt-V_2\sin pt,\\ S_2&\equiv\Im(S_\theta)=V_2\cos pt+V_1\sin pt.\end{aligned}\right\}\quad\text{..............................}\quad(26)$$

Next we consider a torsional vibration of the wire represented by S_1 or S_2. The energy flow into E during one period T is equal to the amount of

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1) Watson, loc. cit., p. 434.

work done by the wire against E during the same interval of time. Hence if we denote the latter quantity by $\mathfrak{G}_1$ or $\mathfrak{G}_2$, according as the mode of vibration is S_1 or S_2, then we have

$$\mathfrak{G}_1 = -2\pi\mu_1\int_0^T\int_0^a\left(\frac{\partial S'_1}{\partial z}\right)_{z=0}\cdot\left(\frac{\partial S_1}{\partial t}\right)_{z=0} r\,dr\,dt$$

$$= 2\pi^2\mu_1\int_0^a\left(V_2\frac{\partial V_1}{\partial z} - V_1\frac{\partial V_2}{\partial z}\right)_{z=0} r\,dr, \qquad (27)$$

In the similar way

$$\mathfrak{G}_2 = \mathfrak{G}_1 \equiv \mathfrak{G}, \text{ say.}$$

Let $\overline{C_n}$, $\overline{D_n}$ and $\overline{\gamma_n}$ denote the conjugate complex quantities of C_n, D_n and γ_n respectively, then

$$\left(V_2\frac{\partial V_1}{\partial z} - V_1\frac{\partial V_2}{\partial z}\right)_{z=0} = \Im\left\{(V_1 + iV_2)_{z=0}\left(\frac{\partial V_1}{\partial z} - i\frac{\partial V_1}{\partial z}\right)_{z=0}\right\}$$

$$= -\Re\left[\{\sum_{n=1}^{\infty}(C_n + D_n)J_1(\lambda_n r)\}\{\sum_{m=1}^{\infty}\bar{\gamma}_m(\overline{C_m} - \overline{D_m})J_1(\lambda_m r)\}\right] \quad \ldots (28)$$

On the other hand, there exists an orthogonal relation:

$$\int_0^a rJ_1(\lambda_m r)J_1(\lambda_n r)\,dr = 0, \quad (m \neq n),$$

Applying these equations to (27) we get

$$\mathfrak{G} = -\pi^2 a^2\mu_1\Re\sum_{n=1}^{\infty}\bar{\gamma}_n(C_n + D_n)(\bar{C}_n - \bar{D}_n)\{J_1(\lambda_n a)\},$$

Now equation (12) can be rewritten as

$$(C_n + D_n)(\bar{C}_n - \bar{D}_n)\bar{\gamma}_n a^2\{J_1(\lambda_n a)\}^2 = \frac{2\mu_1}{\mu_2}\bar{\gamma}_n(\bar{C}_n - \bar{D}_n)\sum_{m=1}^{\infty}(C_m - D_m)\gamma_m I_{mn},$$

This leads the last expression of $\mathfrak{G}$ to the form

$$\mathfrak{G} = -\frac{2\pi^2\mu_1}{\mu_2}\Re\sum_{m,\,n=1}^{\infty}\sum^{\infty}\gamma_m\bar{\gamma}_n(C_m - D_m)(\bar{C}_n - \bar{D}_n)I_{mn}.$$

Utilizing the relation $I_{mn} = I_{nm}$ and denoting by $\sum_{m,\,n=1}^{\infty}{}'\sum^{\infty}{}'$ the summation of all terms save those for which $m = n$, we finally get

$$\mathfrak{G} = -\frac{2\pi^2\mu_1^2}{\mu_2}\sum_{m,\,n=1}^{\infty}{}'\sum^{\infty}{}'\,\Re\{\gamma_m\bar{\gamma}_n(C_m - D_m)(\bar{C}_n - \bar{D}_n)\} \times \Re(I_{mn})$$

$$-\frac{2\pi^2\mu_1^2}{\mu_2}\sum_{n=1}^{\infty}|\gamma_n^2(C_n - D_n)^2| \times \Re(I_{nn}). \qquad (30)$$

As we have pointed out in the preceding chapter, ω is, in general, a small quantity. Hence if we neglect small quantities of higher order than ω^3, it is easily seen, from the formulae (17)~(25), that I_{mn}'s save I_{11} become purely imaginary quantities, and consequently they have no contribution to the right-hand side of (30). That is to say, in this degree of approximation, the only substantial term in (30) is the very one having the factor $-\frac{1}{96}a^8\omega^3$ which appears in the third term of the formula for I_{11} (see (25)). It must be noted that in the above discussion we have tacitly assumed that the magnitudes of $(C_n-D_n)\gamma_n$ are, at the highest, of the order of $(C_1-D_1)\gamma_1$, which can be permitted from a physical standpoint of view. We thus get

$$\mathfrak{G} \doteqdot \frac{\pi^2}{48}\frac{\mu_1^2}{\mu_2}|\gamma_1^2(C_1-D_1)^2|a^8\omega^3 \quad \text{...} \quad (31)$$

This formula enables us, if only two constants C_1 and D_1 are known, to evaluate the amount of the rate of energy flow in one period of the vibration, without having any further knowledge concerning all the unknown constants C_n and D_n. The exact determination of C_1 and D_1 depends, of course, upon the solution of the system of equations (2) and (12), which is practically impossible. But luckily, we have a short cut leading to our object, if we satisfy ourselves with some approximations, by considering provisionally as if the clamping body E were perfectly rigid, not elastic. In this way we have approximately

$$C_1-D_1 \doteqdot 2C_1 = \frac{iU}{\sin\gamma_1 l}.$$

Hence (31) gives

$$\mathfrak{G} \doteqdot \frac{\pi^2}{48}\frac{U^2a^8\rho_1\mu_1}{\mu_2\sin^2\gamma_1 l}\left(\frac{\rho_2}{\mu_2}\right)^{3/2}p^5, \qquad \gamma_1 = \sqrt{\frac{\rho_1}{\mu_1}}\,p, \quad \text{.............................} \quad (32)$$

If $\gamma_1 l$ is fairly small (this is the case, for instance, when the material of the wire is a common metal and l is of the order 1 *m.*, provided p is not too larger than 10^2), we can take $\gamma_1 l$ in place of $\sin\gamma_1 l$ and hence obtain

$$\mathfrak{G} \doteqdot \frac{\pi^2}{48}\frac{\mu_1^2}{\mu_2}\frac{U^2a^8}{l^2}\left(\frac{\rho_2}{\mu_2}\right)^{3/2}p^3. \quad \text{...} \quad (33)$$

IV. Damping Coefficient.

At this stage of our investigation we can reach the object stated in Chap. I and II. We have considered in the foregoing discussions provisionally as if the free vibration of our dynamical system were exactly periodic, although it can not be so in the strict sense. This assumption implies that our system performs, in fact, a certain forced vibration produced by a suitable periodically varying moment of force applied to the discal rotator, thus supplying just the same amount of energy that will compensate the energy flow from the upper junction. Therefore, if we let our system alone, there will be a natural damping of its 'free vibration' (let its frequency be p) originated by the energy flow which is approximately equal to that corresponding to the above mentioned periodic 'forced vibration' with the same period p, provided that the damping is not too great. In the free vibration the frequency p can take only definite values instead of arbitrary ones. We determine them by considering the equation of motion of the rotator:

$$\frac{\pi a^4}{2I}\gamma_1\mu_1 = p^2 \tan \gamma_1 l, \qquad (34)$$

where I is the moment of inertia of the rotator about its axis of rotation. If $\gamma_1 l$ is fairly small, this reduces to

$$p^2 \doteq \frac{\pi a^2 \mu_1}{2Il}. \qquad (35)$$

From (32) and (33) combined with (34) and (35) respectively we can calculate approximately the rate of energy dissipation when our dynamical system is performing its free vibration.

Now let us consider a fairly long (but not too long) interval of time and the vibration of the rotator during this interval. Let the angle of rotation of the rotator at an instant t measured from its equilibrium position be θ, then we can express it, in general, with a good degree of approximation, by the form

$$\theta = Ue^{-\lambda t}\cos pt,$$

where U and λ are constants independent of t. From this we con easily calculate the kinetic energy of the rotator and see clearly that its maximum values occur regularly at constant time interval of $2\pi/p$, taking monotonously decreasing values. If we construct the difference between two successive maxima and denote the logarithmic decrement of the vibration by D, then

$$\frac{I}{2}\left[\left(\frac{d\Theta}{dt}\right)^2_{t=\frac{\pi}{2p}}-\left(\frac{d\Theta}{dt}\right)^2_{t=\frac{5\pi}{2p}}\right]\doteq 2\pi I U^2 p\lambda = 2I U^2 p^2 D.$$

When the mass of the wire is small compared with that of the rotator (this case is rather common than otherwise), it is quite legitimate to assume that the amount of dissipation of our system during one quasi-period be approximately equal to the above value. Furthermore, this value must be equal to $\mathfrak{G}$, because we assumed that the decay of vibration arises from energy flow $\mathfrak{G}$. Thus we get

$$D \doteq \frac{\pi}{24}\frac{a^4\sqrt{\rho_1\mu_1}}{l\sin 2\gamma_1 l}\left(\frac{\rho_2}{\mu_2}\right)^{3/2}p^4,\ \lambda = Dp/\pi, \qquad (36)$$

and if $\gamma_1 l$ is fairly small

$$D \doteq \frac{\pi}{48}\frac{\mu_1}{\mu_2}\frac{a^4}{l}\left(\frac{\rho_2}{\mu_2}\right)^{3/2}p^3,\ \lambda = Dp/\pi, \qquad (37)$$

As can be easily seen from (36) or (37, D and λ are, in general, very small quantities: for instance, their values calculated in the case when $\mu_1=\mu_2=10^{12}$, $\rho_1=\rho_2=10$ (in C.G.S. Units) are given in the table below,

1), $a=10^{-1}$, $l=10^2$, $p=10$; $\lambda\doteq 6.6\times10^{-21}$, $D\doteq 2.1\times10^{-21}$,

2), $a=1$, $l=5\times10$, $p=10^2$; $\lambda\doteq 1.3\times10^{-12}$, $D\doteq 4.1\times10^{-14}$,

3), $a=10$, $l=2\times10$, $p=10^3$; $\lambda\doteq 3.2\times10^{-4}$, $D\doteq 1.0\times10^{-6}$,

We conclude from these results that the decay of torsional vibration of a wire-and-rotator system, considered as having its source in the energy flow from its clamped portion, as compared with those due to other different sources, is very small and can be neglected with safety.

V. Summary.

Problems concerning the decay of vibration of an elastic system fixed at some portions of its boundary are much complicated, partly because of the variety of sources of the decay and partly due to manifold geometrical and mechanical boundary conditions of the system. Therefore satisfactorily theoretical treatments of such problems are hopelessly difficult without prescribing some limitations to these circumstances.

The present writer takes the energy flow into the external system due to the propagation of elastic waves originated at the junction as a source of the decay, and he discusses theoretically the problem in the case of the torsional vibration of a perfectly elastic wire ideally clamped at its upper end to a semi-infinitely large elastic body and clamped to a rigid rotator at the lower. The results obtained show that the decay due to the above cause is exceedingly feeble and can be neglected compared with other sorts of decay. This conclusion has hitherto been believed to be true by every one without any reliable reasoning, but we have got now a theoretical verification. Thus the constant of logarithmic decrement of the torsional vibration (its decay is assumed to be due to the above cause only) can be written as

$$D \doteq \frac{\pi}{24}\frac{a^4\sqrt{\rho_1\mu_1}}{\mu_2 \sin 2\gamma_1 l}\left(\frac{\rho_2}{\mu_2}\right)^{3/2} p^4, \quad \gamma_1 = \sqrt{\frac{\rho_1}{\mu_1}}\, p.$$

where ρ_1, μ_1, a and l denote the density, the rigidity, the radius and the length of the wire respectively; ρ_2 and μ_2 represent the density and the rigidity of the external semi-infinite elastic body respectively; and p is the frequency of the vibration in 2π units of time.

If $\gamma_1 l$ is fairly small, D can be reduced to a still simpler form:

$$D \doteq \frac{\pi}{48}\frac{\mu_1}{\mu_2}\frac{a^4}{l}\left(\frac{\rho_2}{\mu_2}\right)^{3/2} p^3.$$

It is to be repeatedly noted that these results have been obtained theoretically on several artificial assumptions, and therefore in other cases further appropriate mathematical treatments will be necessary, and the conclusions may be somewhat different from ours. The present author wishes to discuss these cases in the near future.

In conclusion the author wishes to express his cordial thanks to Prof. Yamada of the Research Institute for Fluid Engineering for his kind and valuable advices.

IV. Appendix (evaluation of the integral I_{mn})

In Chap. II we have considered only the case in which the parameter ω is small and hence I_{mn}'s are expansible in powers of ω, giving some approximate values of them. The power series method is surely an easy-going and a powerful one, but since it involves two power series having respectively λ_m and λ_m as their radii of convergence, when $|\omega|$ approaches to these values the convergence of the series becomes very slow, and moreover, when $|\omega|$ surpases them the original power series must be replaced by other appropriate expressions (for example series of negative powers). This circumstance of affairs is very troublesome and incovenient in evaluating I_{mn}'s for arbitrary range of $|\omega|$. Here we shall take another way.

To evaluate the integrals I_{mn}'s, it is sufficient to discuss the following integral in the case when α and β are a complex number and a positive integer respectively, and ω and τ are complex parameters such that $\Im(\omega)<0$ and $\Im(\tau)>0$ respectively.

$$I(\alpha, \beta; \omega, \tau)=\int_0^\infty \frac{\lambda^\alpha \{J_2(a\lambda)\}^2}{\gamma(\lambda^2-\tau^2)^\beta} dy, \quad -5<\Re(\alpha)<2\beta+1, \quad \text{.............. (1)}$$

where the branch of $\gamma=\sqrt{\omega^2-\lambda^2}$ should satisfy the condition:

$\lim_{\lambda\to\infty} \arg \sqrt{\omega^2-\lambda^2}=\frac{\pi}{2}$, in accordance with (4) in Chap. II.

The original integrals to be evaluated can be easily obtained by the process $\Im(\omega)\to 0$ in $\lim_{\tau\to\lambda n} I(\alpha, \beta; \omega, \tau)$.

To begin with, let us consider provisionally the case in which $-4<\Re(\alpha)<2\beta-4$. Utilizing the formula[1)]

$$\{J_2(a\lambda)\}^2=\frac{a^4\lambda^4}{2^5\pi i}\int_{-\infty i}^{\infty i} \frac{\Gamma(-s)\Gamma(2s+5)}{\{\Gamma(s+3)\}^2\Gamma(s+5)}\left(\frac{a\lambda}{2}\right)^{2s} ds, \quad \text{.................. (2)}$$

and changing the order of integration, we get

$$I(\alpha, \beta; \omega, \tau)=\frac{a^5}{2^5\pi i}\int_{-\infty i}^{\infty i} \frac{\Gamma(-s)\Gamma(2s+5)}{\{\Gamma(s+3)\}^2\Gamma(s+5)}$$

$$\times\left(\frac{a}{2}\right)^{2s}\int_0^\infty \frac{\lambda^{\alpha+4+2s}}{\gamma(\lambda^2-\tau^2)^\beta} d\lambda\, ds, \quad \text{.................................. (3)}$$

1) Watson, loc. cit., p. 436.

Let us denote the second integral in this expression by K and discuss the cases corresponding to different values of β separately.

1) the case: $\beta=1$.

If we differentiate K with respect to ω, we get a linear differential equation for K. This immediately admit of integration and gives

$$K=\frac{-i}{\sqrt{\tau^2-\omega^2}}\left[\frac{\tau}{2}(-i)^{\alpha+2+2s}\Gamma\left(2+\frac{\alpha}{2}+s\right)\Gamma\left(-1-\frac{\alpha}{2}-s\right)\right.$$

$$\left.+\frac{i^{\alpha+2s}}{\sqrt{\pi}}\Gamma\left(\frac{\alpha+5}{2}+s\right)\Gamma\left(-1-\frac{\alpha}{2}-s\right)\int_0^\infty\frac{x^{\alpha+3+2s}}{\sqrt{\tau^2-x^2}}dx\right], \quad \text{.........} (4)$$

In this result we have to choose the branches of two-valued functions so that both arg $\sqrt{\tau^2-\omega^2}$ and arg $\sqrt{\tau^2-x^2}$ tend to zero when we let $\Re(\tau)\to\infty$ keeping $\Im(\omega)$ constant. We substitute (4) in (3) and change the order of integration. Let us now consider in s-plane an integration path C consisting of the part of the imaginary axis between the points $\pm iR$ (R is a large positive number) and a semicircle Γ, of radius R, having its centre at the origin, on the right of the imaginary axis, the origin being made an interior point of C by constructing a small indentation. If the semicircle Γ is described so that its shortest distances from the poles of the integrands (i.e., $s=0, 1, 2, \ldots\ldots$ and $s=r-1-\frac{\alpha}{2}$, where $r=0, 1, 2, \ldots\ldots$) take the possibly largest values, then all the integrals along Γ tend to zero when $R\to\infty$, thus leaving only those taken along the imaginary axis, and consequently, we can evaluate them by summing up the residues of the integrands within C. In this way we have

$$I(\alpha, 1; \omega, \tau)=\frac{i\sqrt{\pi}}{\sin\frac{\alpha\pi}{2}}\frac{\left(\frac{a}{2}\right)^4}{\sqrt{\tau^2-\omega^2}}\left[(-i)^\alpha\sqrt{\pi}\frac{\tau^{\alpha+3}}{2}\left(\frac{2}{a\tau}\right)^4\{J_2(a\tau)\}^2\right.$$

$$+\frac{8\tau}{a^{\alpha+2}}\sum_{r=0}^{\infty}\frac{(-)^r\Gamma\left(r+\frac{3-\alpha}{2}\right)}{\Gamma\left(r-\frac{\alpha}{2}\right)\Gamma\left(r+2-\frac{\alpha}{2}\right)\Gamma\left(r+4-\frac{\alpha}{2}\right)}(a\tau)^{2r}$$

$$-\frac{2^4}{\sqrt{\pi}}i^\alpha\sum_{r=0}^{\infty}\frac{(-)^r\Gamma\left(\frac{\alpha+5}{2}+r\right)\Gamma\left(r+\frac{5}{2}\right)a^{2r}}{r!\,\Gamma\left(r+2+\frac{\alpha}{2}\right)\Gamma(r+3)\,\Gamma(r+5)}\int_0^\omega\frac{x^{\alpha+3+2r}}{\sqrt{\tau^2-x^2}}dx$$

$$-\frac{2^4}{\sqrt{\pi}\,a^{\alpha+2}}\sum_{r=0}^{\infty}\frac{(-)^r\Gamma\left(r+\frac{3}{2}\right)\Gamma\left(r+\frac{3-\alpha}{2}\right)a^{2r}}{r!\,\Gamma\left(r-\frac{\alpha}{2}\right)\Gamma\left(r+2-\frac{\alpha}{2}\right)\Gamma\left(r+4-\frac{\alpha}{2}\right)}$$

$$\times\int_0^{\omega}\frac{x^{2r+1}}{\sqrt{\tau^2-x^2}}dx\Bigg], \qquad (5)$$

At this stage of development we can extend, by the principle of analytic continuation, the temporary range of α to a wider one: $-4<\Re(\alpha)<3$. Of course, the poles at $\alpha=-2, 0, 2$ appearing in the right-hand side of (5) are merely apparent. Let τ in (5) tend to λ_n, then, by the definition of λ_n, the first term becomes zero. Further, let us make use of the following formulae:

$$\int_0^{\omega}\frac{x^{\alpha+3+2r}}{\sqrt{\lambda^2_n-x^2}}dx=\frac{\sqrt{\pi}}{2}\lambda_n^{2r+3+\alpha}\frac{\Gamma\left(r+2+\frac{\alpha}{2}\right)}{\Gamma\left(r+\frac{5+\alpha}{2}\right)}+\int_{\lambda_n}^{\omega}\frac{x^{2r+3+\alpha}}{\sqrt{\lambda^2_n-x^2}}dx,$$

$$\int_0^{\omega}\frac{x^{2r+1}}{\sqrt{\lambda_n^2-x^2}}dx=\frac{\sqrt{\pi}}{2}\lambda_n^{2r+1}\frac{r!}{\Gamma\left(r+\frac{3}{2}\right)}+\int_{\lambda_n}^{\omega}\frac{x^{2r+1}}{\sqrt{\lambda_n^2-x^2}}dx,$$

then the expression in the square bracket of (5) takes the form which is composed of terms containing integrals of the form $\int_{\lambda_n}^{\omega}$ and those independent of ω. Since the former tend to zero as $\sqrt{\lambda^2_n-\omega^2}$ when ω tend to λ_n, if the latter did not vanish identically, the right-hand side of (5) would become infinite when we first make τ tend to λ_n and then ω to λ_n, which is clearly inconsistent with the fact. Thus we get

$$I(\alpha, 1; \omega, \lambda_n)=\frac{-i}{\sin\frac{\alpha\pi}{2}}\frac{a^4}{\sqrt{\lambda_n^2-\omega^2}}\Bigg[i^{\alpha}\sum_{r=0}^{\infty}\frac{(-)^r\Gamma\left(\frac{\alpha+5}{2}+r\right)\Gamma\left(r+\frac{5}{2}\right)}{r!\,\Gamma\left(r+2+\frac{5}{2}\right)\Gamma(r+3)\Gamma(r+5)}$$

$$\times a^{2r}\int_{\lambda_n}^{\omega}\frac{x^{\alpha+3+2r}}{\sqrt{\lambda_n^2-x^2}}dx$$

$$+a^{-\alpha-2}\sum_{r=0}^{\infty}\frac{(-)^r\Gamma\left(r+\frac{3}{2}\right)\Gamma\left(r+\frac{3-\alpha}{2}\right)a^{2r}}{r!\,\Gamma\left(r-\frac{\alpha}{2}\right)\Gamma\left(r+2-\frac{\alpha}{2}\right)\Gamma\left(r+4-\frac{\alpha}{2}\right)}\int_{\lambda_n}^{\omega}\frac{x^{2r+1}}{\sqrt{\lambda_n^2-x^2}}dx\Bigg], \qquad (6)$$

and in particular

$$I(1,1;\omega,\lambda_n)=\frac{a^4}{\sqrt{\lambda_n^2-\omega^2}}\Bigg[\sum_{r=0}^{\infty}\frac{(-)^r a^{2r}}{r!(r+4)!}\int_{\lambda_n}^{\omega}\frac{x^{2r+4}}{\sqrt{\lambda_n^2-x^2}}dx$$

$$-\frac{i}{a^3}\sum_{r=0}^{\infty}\frac{(-)^r a^{2r}}{\Gamma\left(r-\frac{1}{2}\right)\Gamma\left(r+\frac{7}{2}\right)}\int_{\lambda_n}^{\omega}\frac{x^{2r+1}}{\sqrt{\lambda_n^2-x^2}}dx\Bigg], \dots (7)$$

This last formula is the one that is essential in our investigation.

(6) and (7) hold good in the range $-5<\Re(\alpha)<3$, and the convergence of their series is very rapid. In these formulae we can, of course, also take ω as real; that is to say, they hold whenever $\Im(\omega)\leq 0$. All integrals in (7) are expressible in terms of elementary functions, but here we shall not write down them in detail.

2) the case: $\beta=2$.

Consider first a narrower range of α: $-2<\Re(\alpha)<3$, then

$$I(\alpha,2;\omega,\tau)=\frac{1}{2\tau}\frac{\partial}{\partial\tau}I(\alpha,1;\omega,\tau), \quad \dots\dots\dots (8)$$

Next, we reform the right-hand side of (8) by the following formulae,

$$\int_0^{\omega}\frac{x^p}{\sqrt{\tau^2-x^2}}dx=-\omega^{p-1}\sqrt{\tau^2-\omega^2}+(p-1)\int_0^{\lambda_n}x^{p-2}\sqrt{\tau^2-x^2}\,dx$$

$$+(p-1)\int_{\lambda_n}^{\omega}x^{p-2}\sqrt{\tau^2-x^2}\,dx,\ p>1;$$

$$\int_0^{\omega}\frac{x}{\sqrt{\tau^2-x^2}}dx=\tau-\sqrt{\tau^2-\omega^2};$$

$$\int_0^{\omega}\frac{x^p}{(\tau^2-x^2)^{3/2}}dx=\frac{\omega^{p-1}}{\sqrt{\tau^2-\omega^2}}-(p-1)\int_0^{\lambda_n}\frac{x^{p-2}dx}{\sqrt{\tau^2-x^2}}$$

$$-(p-1)\int_{\lambda_n}^{\omega}\frac{x^{p-2}\,dx}{\sqrt{\tau^2-x^2}},\ p>1;$$

$$\int_0^{\omega}\frac{x}{(\tau^2-x^2)^{3/2}}dx=\frac{1}{\sqrt{\tau^2-\omega^2}}-\frac{1}{\tau}.$$

Since the result from the reformed expression of $I(\alpha,2;\omega,\tau)$ to be obtained when we make τ tend to λ_n is nothing else than the required integral, it must be finite and determinate when ω tends to λ_n. Further, the terms containing the integrals of the form $\int_{\lambda_n}^{\omega}$ are in themselves finite and determinate. Hence the sum of the remaining terms should also be so. But as we can easily see from its form, this last conclusion is impossible, unless it becomes identically zero. Such a manner of reasoning to utilize the functional properties of I renders it possible to avoid troublesome and tedious calculations and gives immediately

$$I(\alpha, 2; \omega, \lambda_n)=\frac{i}{\sin\frac{\alpha\pi}{2}}\frac{a^4}{(\lambda_n^2-\omega^2)^{3/2}}\left[i^\alpha\sum_{r=0}^{\infty}\frac{(-)^r\Gamma\left(\frac{\alpha+5}{2}+r\right)\Gamma\left(r+\frac{5}{2}\right)}{r!\,\Gamma\left(1+\frac{\alpha}{2}+r\right)\Gamma(r+3)\Gamma(r+5)}\right.$$

$$\times a^{2r}\int_{\lambda_n}^{\omega}x^{\alpha+1+2r}\frac{\omega^2-x^2}{\sqrt{\lambda_n^2-x^2}}dx$$

$$-a^{-\alpha}\sum_{r=0}^{\infty}\frac{(-)^r\Gamma\left(\frac{r+5}{2}\right)\Gamma\left(r+\frac{5-\alpha}{2}\right)a^{2r}}{r!\,\Gamma\left(r+1-\frac{\alpha}{2}\right)\Gamma\left(r+3-\frac{\alpha}{2}\right)\Gamma\left(r+5-\frac{\alpha}{2}\right)}$$

$$\left.\times\int_{\lambda_n}^{\omega}x^{2r+1}\frac{\omega^2-x^2}{\sqrt{\lambda_n^2-x^2}}dx\right], \quad\cdots\cdots\cdots\cdots\cdots\cdots (9)$$

This holds good, by dint of the principle of analytic continuation, even for a wider range of α: $-5<\Re(\alpha)<5$. The poles which appear when α is even are, of course, merely apparent.

In particular

$$I(3, 2; \omega, \lambda_n)=-\frac{a^4}{(\lambda_n^2-\omega^2)^{3/2}}\left[\sum_{r=0}^{\infty}\frac{(-)^r(r+3)}{r!\,(r+4)!}a^{2r}\int_{\lambda_n}^{\omega}x^{2r+4}\frac{\omega^2-x^2}{\sqrt{\lambda_n^2-x^2}}dx\right.$$

$$\left.-i\sum_{r=0}^{\infty}(-)^r\frac{\left(r+\frac{3}{2}\right)a^{2r-3}}{\Gamma\left(r-\frac{1}{2}\right)\Gamma\left(r+\frac{7}{2}\right)}\int_{\lambda_n}^{\omega}x^{2r+1}\frac{\omega^2-x^2}{\sqrt{\lambda_n^2-x^2}}dx\right], \quad\cdots (10)$$

This is the expression we aim to obtain and holds for any range of $|\omega|$ such that $\Im(\omega)\leq 0$. The convergence of the above series is very rapid as in the case (1), and all integrals appearing in these formulae are expressible in terms of elementary functions.