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ISIBASI, Tadasi
Research Institute for Elasticity Engineering, Kyushu Imperial University

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Tadasi ISIBASI

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Form-factors of Solid of Revolutions in Relation to Those of Plane Plates.

By

Tadasi ISIBASI, Kogaku-hakushi.

(Professor of Elasticity and Strength of Materials in the Faculty of Engineering, Member of the Research Institute for Elasticity Engineering.)

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Contents

1. Introductory.

The factors of stress concentration (i.e. the form-factors) of a solid of revolution under axial tension or compression are scarcely known so far as the author is aware, and only the case of a bar with a round indentation is mathematically investigated [1, 2]*. On the other hand the geometrical resemblance of the stress-state in a meridian section of a solid of revolution to that of a two-dimensional plate of the same form leads to an idea that there will exist some simple relation between the form-factors of the former with those of the latter if they are under the same loading condition.

In this connection it is recognized by some investigators [3, 4, 5, 6] that there is the relation

$$\alpha_u \leq \alpha_e \qquad (1)$$

where

α_u; form-factor of solid of revolution under axial load,

α_e; form-factor of a plane plate of which the form of the boundary is

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*Numbers in the square bracket refer to the Bibliography at the end of the paper.

the same as that of a meridian section of the solid of revolution under consideration.

The relation (1) will also be affirmed by the results given in the Neuber's famous work [1].

On the basis of Neuber's mathematical work on deep hyperbolic notches already referred to, Kuntze [7] and Krisch [8] proposed a formula

$$\alpha_u = 0.75\,\alpha_e + 0.25 \tag{2}$$

This relation will be valid for deep notches of symmetrical type (Fig. 1), but it is doubtful whether this gives correct values for notches of little depth especially in case of notches of fillet-type (Fig. 1).

Neuber's conclusion that $\alpha_u \sim \alpha_e$ when the depth of the notch is little is the only mathematical investigation on this subject.

It must be remarked here that Weber's idea of obtaining the stress-state of a solid of revolution from that of a two-dimensional stress-state and vice versa, by using special operation is worth mensioning [9] but it seems too difficult to find the Airy's function for the wanted case even if the form of the subsequent operator is fortunately found.

In this paper the author propose a formula of α_u for notches of little depth. By this formula we can obtain the values of α_u from the corresponding values of α_e which are obtainable by some practical methods, for example, by photo-elastic measurements on transparent models.

In the appendix some graphs of α_u calculated from the said formula are given for practical use.

2. Stress Function.

We take the axis of a solid of revolution as x-axis of a cylindrical coordinates x and r (r-axis being perpendicular to x-axis). Since the deformation is symmetrical with respect to x-axis it follows that the stress components are independent of the coordinates θ. In the plate rectangular coordinates x and r are used. In both cases the r-axis passes through the point at which the stress becomes maximum.

The stress components in the solid of revolution are denoted by σ_x, σ_r, σ_t and τ_z, while in the plate they are σ_x^*, σ_r^* and τ_z^* as the stress-state in the latter being two-dimensional. (σ denotes normal stress, τ shearing stress, suffix t means tangential direction).

The stress-components of the two-dimensional plate can be derived, as is well known, from Airy's function F_e as follows;

$$\left.\begin{aligned} &\sigma_x{}^* = \frac{\partial^2 F_e}{\partial r^2}, \quad \sigma_r{}^* = \frac{\partial^2 F_e}{\partial x^2}, \quad \tau_z{}^* = -\frac{\partial^2 F_e}{\partial r \partial x} \\ &\Delta_1 \Delta_1 F_e = 0 \quad \Delta_1 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} \end{aligned}\right\} \quad (3)$$

While the stresses in a solid of revolution will be given by Weber's function F_u as follows;

$$\left.\begin{aligned} &\sigma_x = \frac{\partial^2 F_u}{\partial r^2} + \frac{1}{r}\frac{\partial F_u}{\partial r} \\ &\sigma_r = -\frac{\partial^2}{\partial r^2}(F_u + 2\phi_2) - \frac{2}{m}\frac{1}{r}\frac{\partial \phi_2}{\partial r} \\ &\sigma_t = -\frac{\partial}{\partial r}(F_u + 2\phi_2) - \frac{2}{m}\frac{\partial^2 \phi_2}{\partial r^2} \\ &\tau_z = -\frac{\partial^2 F_u}{\partial r \partial x} \end{aligned}\right\} \quad (4)$$

where

$$\left.\begin{aligned} &F_u = \phi_1 + x\frac{\partial \phi_2}{\partial r} \\ &\Delta\phi_1 = \Delta\phi_2 = 0 \\ &\Delta = \Delta_1 + \frac{1}{r}\frac{\partial}{\partial r} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \end{aligned}\right\} \quad (5)$$

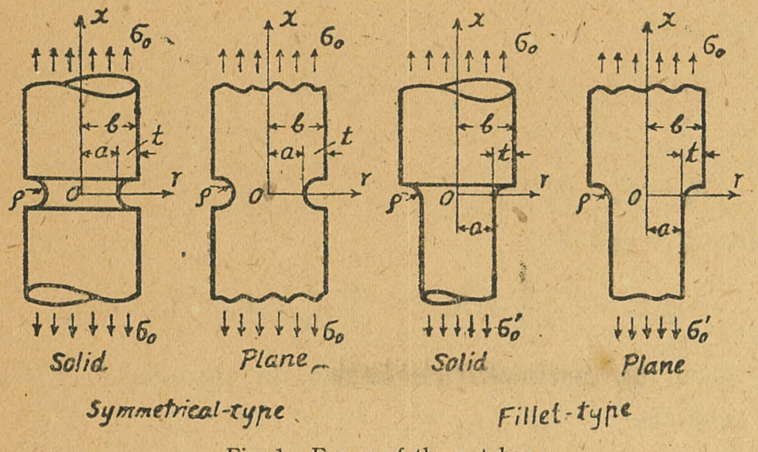

Fig. 1. Forms of the notches.

therefore we have

$$\left.\begin{aligned} \varDelta F_u &= 2\frac{\partial^2\phi_2}{\partial x^2} \\ \varDelta\varDelta F_u &= 0 \end{aligned}\right\} \qquad (6)$$

The relation between the operators $\varDelta\varDelta$ and $\varDelta_1\varDelta_1$ is

$$\varDelta\varDelta = \varDelta_1\varDelta_1 + \frac{2}{r}\frac{\partial\varDelta_1}{\partial r} + \frac{1}{r^3}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial r^2}$$

In the case of a notch of little depth the region of non-uniform stress distribution is confined within a narrow part near the notch and the other part of the body is stressed uniformly, therefore the stress-state of a solid of revolution with a notch of little depth differs from that of a plane plate, of which the form of the boundary is the same as that of a meridian section of the former, only within a limitted region near the notch, and so it is desirable to have the solution of the problem applicable for this limitted part; that is to say a solution is needed which is valid only for the part within which the value of the coordinates r is large compared with other dimensions of the body. In the following discussions, therefore, we will neglect those terms which are of higher order than $\frac{1}{r}$.

Thus we have

$$\varDelta\varDelta = \varDelta_1\varDelta_1 + \frac{2}{r}\frac{\partial\varDelta_1}{\partial r}$$

therefore we have

$$\varDelta\varDelta F_u = \varDelta_1\varDelta_1 F_u + \frac{2}{r}\frac{\partial}{\partial r}\varDelta_1 F_u. \qquad (7)$$

Putting

$$F_u = F_e + F_0 \qquad (8)$$

we have from (7) and (8)

$$\varDelta_1\varDelta_1 F_0 + \frac{2}{r}\frac{\partial}{\partial r}\varDelta_1 F_0 = -\frac{2}{r}\frac{\partial}{\partial r}\varDelta_1 F_e \qquad (9)$$

Thus when the function F_e is known the function F_0 will be found from this equation.

A solution of (9) is

$$\Delta_1 F_0 = \frac{1}{r} h(x, r) - \Delta_1 F_e \tag{10}$$

as will easily be verified by substituting the expression (10) in (9), $h(x, r)$ being a plane harmonic function satisfying Laplace equation $\Delta_1 h = 0$.

3. Boundary Conditions.

Let the stress acting at a point on the boundary of a solid of revolution be p of which the direction cosines are λ and ν. The outward normal drawn at this point has the direction cosines α and β (Fig. 2), then we have the relations

$$p\lambda = \sigma_r \alpha + \tau_z \beta$$
$$p\mu = \sigma_x \beta + \tau_z \alpha$$
$$\alpha = \frac{dx}{ds}, \quad \beta = -\frac{dr}{ds}$$

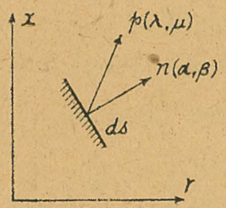

Fig. 2. A part of boundary of solid of revolution.

Substituting (4) and the relation $\Delta F_u = 2\frac{\partial^2 \phi_2}{\partial x^2}$ in the above equations we have

$$\left.\begin{aligned} p\lambda &= \frac{d}{ds}\left(\frac{\partial F_u}{\partial x}\right) + \frac{1}{r}\left(\frac{\partial F_u}{\partial r} + m'\frac{\partial \phi_2}{\partial r}\right)\frac{dx}{ds} \\ p\mu &= -\frac{d}{ds}\left(\frac{\partial F_u}{\partial r}\right) - \frac{1}{r}\frac{\partial F_u}{\partial r}\frac{dr}{ds} \end{aligned}\right\} \tag{11}$$

where $m' = 2\left(1 - \frac{1}{m}\right)$, m being the Poisson's number.

If the boundary is free from any stresses we have

$$\left.\begin{aligned} &\frac{d}{ds}\left(\frac{\partial F_u}{\partial x}\right) + \frac{1}{r}\left(\frac{\partial F_u}{\partial r} + m'\frac{\partial \phi_2}{\partial r}\right)\frac{dx}{ds} = 0 \\ &\frac{d}{ds}\left(\frac{\partial F_u}{\partial r}\right) + \frac{1}{r}\frac{\partial F_u}{\partial r}\frac{dr}{ds} = 0 \end{aligned}\right\} \tag{12}$$

A solution of the second equation of (12) is

$$\left(\frac{\partial F_u}{\partial r}\right)_0 = \text{const.} = b. \tag{13}$$

The suffix 0 means that the term in the round blacket takes the value on the bondary. (13) is an important equation which the function F_u must satisfy on free surface of a solid of revolution.

Substituting (13) in the first equation of (12) we have

$$\frac{d}{ds}\left(\frac{\partial F_u}{\partial x}\right)+\left(\frac{b_0}{r^2}+\frac{m'}{r}\frac{\partial \phi_2}{\partial r}\right)\frac{dx}{ds}=0$$

Integrating this condition along the boundary s of the body;

$$\left(\frac{\partial F_u}{\partial x}\right)_s=\left(\frac{\partial F_u}{\partial x}\right)_{s=0}-\int_0^x\left(\frac{b_0}{r^2}+\frac{m'}{r}\frac{\partial \phi_2}{\partial r}\right)ds+k_2$$

s being the length of the boundary curve in the meridian section measured from the point $x=0$ and k_2 is a constant.

On the other hand we are able to put

$$\left(\frac{\partial F_u}{\partial x}\right)_{s=0}+k_2=0$$

therefore we have

$$\left(\frac{\partial F_u}{\partial x}\right)_s=-\int_0^x\left(\frac{b_0}{r^2}+\frac{m'}{r}\frac{\partial \phi_2}{\partial r}\right)ds \tag{14}$$

The integral in (14) is to be taken along the boundary of a meridian section of a solid of revolution.

(13) and (14) are the boundary conditions which Weber's function F_u must satisfy on a free boundary of a solid of revolution.

4. General Expression of the Relation between α_u and α_e.

Let the maximum stress in a solid of revolution and in a plate be σ_u and σ_e respectively, then they appear at the bottom of the notch when it is a symmetrical type, but in case of a notch of fillet-type the point at which the maximum stress appears is different according as the body is a solid of revolution or a two-dimensional plate. But the difference of the positions of its appearance will be very small compared with other dimensions of the body as can be inferred from the fact that in case of bending the point at which the maximum stress appears is nearly the same position in a solid of revolution [3] as well as in the corresponding plate [10]. Thus for notches of symmetrical as well as of fillet-type we write

$$\sigma_u = (\sigma_x)_{x=0}, \quad \sigma_e = (\sigma_x^*)_{x=0}$$

Consider now the difference δ of these stresses

$$\delta = \sigma_u - \sigma_e \tag{15}$$

then from Neuber's investigation we are able to conclude that $\delta = 0$ according as r (or a) tends to infinity.

Since the normal stress normal to the boundary surface is zero, when the surface is free from any stresses, we have

$$\sigma_u = (\sigma_r + \sigma_x + \sigma_t)_0 - (\sigma_t)_0$$
$$= \left(\frac{1}{r}\frac{\partial F_u}{\partial r} - 2\frac{\partial^2\phi_2}{\partial r^2} - \frac{2}{m}\frac{1}{r}\frac{\partial\phi_2}{\partial r}\right)_0$$

where the suffix 0 means that the quantities in the round blacket take the values on the surface of the body.

Since

$$2\frac{\partial^2\phi_2}{\partial r^2} = -\Delta_1 F_0 - \Delta_1 F_e - \frac{1}{r}\frac{\partial F_u}{\partial r} - \frac{2}{r}\frac{\partial\phi_2}{\partial r}$$

therefore

$$\sigma_u = \left(\frac{2}{r}\frac{\partial F_u}{\partial r} + \Delta_1 F_0 + \frac{m'}{r}\frac{\partial\phi_2}{\partial r}\right)_0 + (\Delta_1 F_e)_0$$

On the other hand we have from (13)

$$\left(\frac{1}{r}\frac{\partial F_u}{\partial r}\right)_0 = \left(\frac{b_0}{r^2}\right)_0$$

In the plate there is the relation $\sigma_x^* + \sigma_r^* = \Delta_1 F_e$, therefore on the free boundary it must be

$$\sigma_e = (\Delta_1 F_e)_0$$

Thus we have

$$\delta = \sigma_u - \sigma_e = \left(2\frac{b_0}{r^2} + \frac{m'}{r}\frac{\partial\phi_2}{\partial r}\right)_0 + (\Delta_1 F_0)_0 \tag{16}$$

From (16) we are able to find δ if the values $(\Delta_1 F_0)_0$, b_0 and $\left(\frac{\partial\phi_2}{\partial r}\right)_0$ are known at the bottom of the notch.

Substituting (10) in (16) we have

$$\delta = 2\left(\frac{b_0}{r^2}\right)_0 + \left(\frac{1}{r}\right)_0\left[h(x, r) + m'\frac{\partial\phi_2}{\partial r}\right]_0 - (\Delta_1 F_e)_0$$

At the bottom of a symmetrical notch of a plate we have the relation

$$(\Delta_1 F_e)_0 = \left(\frac{\partial^2 F_e}{\partial r^2}\right)_0$$

The function $h(x, r)$ is plane harmonic, as is explained before, so it will be written

$$h(x, r) = h_1(x, r) + c'r = h_1(x, r) + (\Delta_1 F_e)_{\substack{0\\x=0}} r$$

where the function $h_1(x, r)$ is again plane harmonic.

Thus we have

$$\delta = 2\left(\frac{b_0}{r^2}\right)_0 + \left(\frac{1}{r}\right)_0 \left(h_1(x, r) + m' \frac{\partial \phi_2}{\partial r}\right)_0 \tag{17}$$

(17) is correct for notches of symmetrical as well as of fillet-type.

Let now

$$f_1(x, r) = h_1(x, r) + m' \frac{\partial \phi_2}{\partial r} \tag{18}$$

As said before δ tends to 0 according as r tends to infinity so it must be

$$\lim_{r\to\infty} f_1(x, r) = \text{finite}$$

Therefore the idea of asymptotic expansion of a function leads to the following expression of $f_1(x, r)$ for large value of r

$$f_1(x, r) \sim K_0 + O\left(\frac{1}{r}\right) \tag{19}$$

The symbol $O\left(\frac{1}{r}\right)$ does not mean a function of the oder of $\frac{1}{r}$. It may be a function of the order of $\frac{1}{r^2}$ or of more higher order with respect to $\frac{1}{r}$. In the following discussion any terms of the order of higher than $\frac{1}{r}$ being neglected it is not necessary to discuss the order of this function further.

The constant K_0 does not involve r but it will be a function of the dimensions of the notch, i.e. of such quantities as a, ρ and t (Fig. 1). Further the term a in K_0 must appear in the form $\left(\frac{t}{a}\right)$, for if it not so there will be a case when $\lim\limits_{r\to\infty}\frac{f_1}{r}$ = finite; a fact which is not admissible as stated above.

Therefore up to the order of $\left(\frac{1}{r}\right)$ we have

$$K_0=\left[1+K_0'\left(\frac{t}{a}\right)\right]C_0'(\rho,\ t)$$

Now let the nominal stress at the minimum section of the solid of revolution and of the plate be σ_{nu} and σ_{ne} respectively then we have

$$\sigma_{nu}=\left(1+\frac{t}{a}\right)^2\sigma_0,\quad \sigma_{ne}=\left(1+\frac{t}{a}\right)\sigma_0.$$

where the stress σ_0 is the stress acting at the larger part of the body (Fig. 1).

Then we have

$$\sigma_u=\alpha_u\sigma_{nu}=\left(1+\frac{t}{a}\right)^2\alpha_u\sigma_0$$

$$\sigma_e=\alpha_e\sigma_{ne}=\left(1+\frac{t}{a}\right)\alpha_e\sigma_0$$

therefore we have

$$\delta=\left(1+\frac{t}{a}\right)\sigma_0\left[\left(1+\frac{t}{a}\right)\alpha_u-\alpha_e\right]$$

and therefore

$$\left(1+\frac{t}{a}\right)\alpha_u-\alpha_e=\frac{1}{\left(1+\frac{t}{a}\right)\sigma_0}\left[2\frac{b_0}{r}+\frac{f_1(x,\ r)}{r}\right]_0$$

Choosing the coordinates of the point at which the maximum stress occurs to be $x=0$ and $r=a$ in the solid of revolution as well as in the plate and putting, for the sake of simplicity, $1+K_0'\left(\frac{t}{a}\right)=1+\left(\frac{t}{a}\right)$ we have

$$\left(1+\frac{t}{a}\right)\alpha_u-\alpha_e=\frac{C_0'(\rho,\ t)}{a}+\frac{1}{a}O\left(\frac{1}{a}\right)\qquad(20)$$

As a first approximation we retain only the first term in the r.h.s. of (20) then

$$\left[\left(1+\frac{t}{a}\right)\alpha_u-\alpha_e\right]=\frac{C_0'(\rho,t)}{a} \tag{21}$$

To have a dimensionless expression of the r.h.s. of (21) we remind the fact that the depth of the notch t has a predominant effect in the case of notches of little depth so we put

$$\left(\frac{C_0'}{r}\right)_0=\left(\frac{t}{a}\right)C_0''$$

C_0'' contains no term of a; the terms ρ and t will appear in the form $\sqrt{\frac{t}{\rho}}$, as is suggested by Neuber's investigation. We put therefore

$$C_0''=C_0\left(1+C_1\sqrt{\frac{t}{\rho}}\right) \tag{22}$$

Consequently (20) will be transformed into

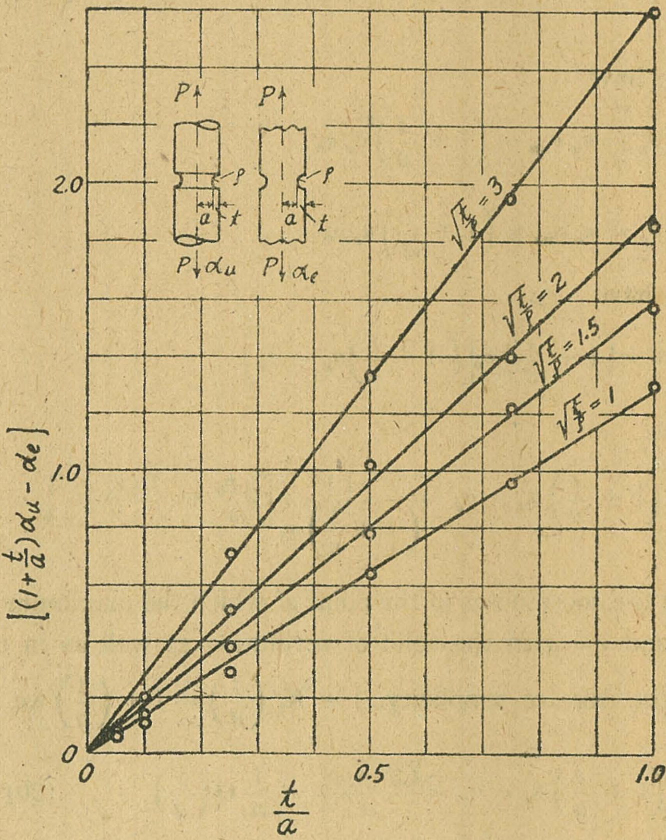

Fig. 3. Straight-line relation between $\left[\left(1+\frac{t}{a}\right)\alpha_u-\alpha_e\right]$ and $\left(\frac{t}{a}\right)$. The points in the figure are taken from Neuber's nomographs.

$$\left[\left(1+\frac{t}{a}\right)\alpha_u-\alpha_e\right]=C_0\left(1+C_1\sqrt{\frac{t}{\rho}}\right)\left(\frac{t}{a}\right) \qquad (23)$$

Both the constants C_0 and C_1 will be independent of the terms a, ρ and t.

5. The Values of the Constants C_0 and C_1.

To determine the numerical values of the constant C_0 and C_1 in (23) the results of the Neuber's nomographs [1] are available. The points in Fig. 3 are taken from them. These points lie on the straight lines for which $\sqrt{\frac{t}{\rho}}=$ constant. This fact shows that the straight-line relation between $\left[\left(1+\frac{t}{a}\right)\alpha_u-\alpha_e\right]$ and $\left(\frac{t}{a}\right)$, which is suggested by (23), is quite right.

The inclinations of these lines in Fig. 3 give the values of the constant C_0'' which, according to (22), must be again a straight line in $C_0''-\sqrt{\frac{t}{\rho}}$ plane. Fig. 4 proves this fact satisfactory.

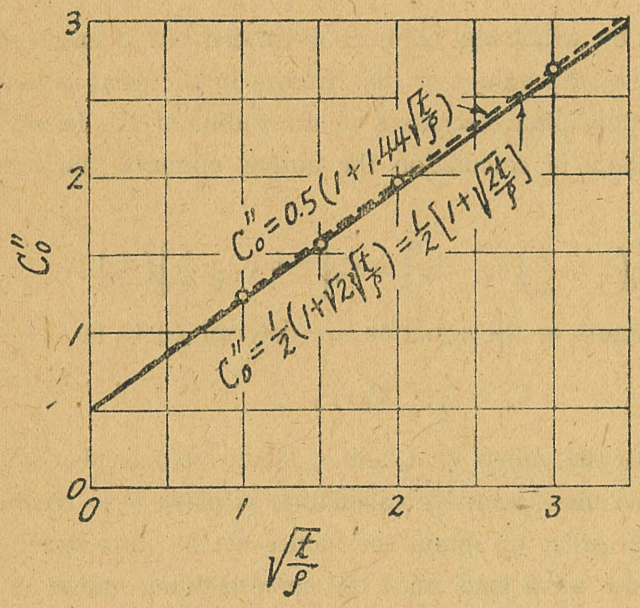

Fig. 4. The relation between C_0'' and $\sqrt{\frac{t}{\rho}}$.

From Fig. 4 we have the values of the constants

$$C_0 = \frac{1}{2}, \quad C_1 = 1.44 \simeq \sqrt{2}.$$

Using these values we have the formula

$$\left[\left(1 + \frac{t}{a}\right)\alpha_u - \alpha_e\right] = \frac{1}{2}\left[1 + \sqrt{\frac{2t}{\rho}}\right]\left(\frac{t}{a}\right) \tag{24}$$

This is the relation between the form-factors α_u (solid of revolution) and α_e (corresponding plane plate) for notches of little depth. The author think that this relation will be valid not only for the case of symmetrical notches but also in notches of fillet-type.

6. Summary and Conclusions.

In this paper the author explained the relation between the form-factors of a solid of revolution and that of a plate of which the form of the boundary is the same as a meridian section of the former, both being under the same loading condition (axial load). The depth of the notches is assumed to be little as compared with other dimensions of the solid and the plate.

The idea on which the analysis is carried out is based on the fact that the difference of the values of the stresses in the solid of revolution and in the corresponding plate decreases as the radius of the former increases.

As the result of calculation the author obtained the following relation (Fig. 1)

$$\left[\left(1 + \frac{t}{a}\right)\alpha_u - \alpha_e\right] = C_0\left[1 + C_1\sqrt{\frac{t}{\rho}}\right]\left(\frac{t}{a}\right).$$

The constants in the equation are determined to be

$$C_0 = \frac{1}{2}, \quad C_1 = \sqrt{2}$$

by referring to the values of α_u and α_e taken from Neuber's nomographs [1].

Until some more accurate calculation is made it is recommended to use this relation in order to obtain the values of the form-factors of a solid of revolution under axial load since the corresponding values of α_e can easily be found in any wanted cases.

The author wishes to express his indebtedness to the Imperial Academy for its financial support made for this investigation.

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Appendix.

Using the formula (24) the form-factors of solid of revolution for some cases of loading are calculated and shown in the following figures. The curves subscribed Kuntze, Krisch are the values obtained from (2) and those obtained from (24) have the subscription Isibasi.

Since (24) is valid for small values of $\left(\frac{t}{a}\right)$ while (2) is for large value, the thick lines giving α_u are drawn so that they touch these two kinds of curves in the respective region of $\left(\frac{t}{a}\right)$.

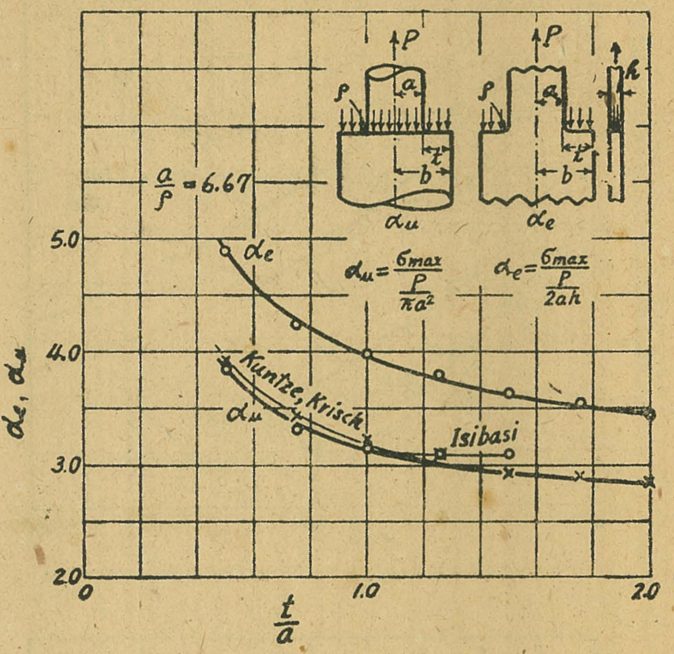

Fig. 9

N.B.: Values of α_e in Figs. 5, 6, 7 and 8 are taken from Bibliography 11) and those in Fig. 9 from 12). Some of them are obtained from the original curves by interpolation.

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