所謂平面弾性應力の一解法

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On a Method of Solving the So-called Elastic Plane Stress Problem

By

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Summary

A general method of solving the so-called elastic plane stress problems for a multiply connected domain is, in the present paper, investigated theoretically with full discussion, the forms of the outer and inner boundaries of the domain being assumed to be arbitrary, and the chief items of the paper are as follows:

(1) We know from the general expression of the plane stress, that the present problems are solved, if we determine properly the stress function (so-called Airy function) of a given domain, which satisfies the equation

$$\nabla^4{}_1\chi = 0 \quad \cdots\cdots (1)$$

at any point on the domain, and the equation

$$(\chi)_{P_i} = -\int_{A_i}^{Q_i} dx \int_{A_i}^{Q_i} Y_\nu ds + \int_{A_i}^{P_i} dy \int_{A_i}^{Q_i} X_\nu ds + C_{1i}\left|x\right|_{A_i}^{P_i} + C_{2i}\left|y\right|_{A_i}^{P_i} + C_{3i} \quad \cdots\cdots (2)$$

along any one of the boundary curves, in which X_ν, Y_ν are the x- and y-components of the given force acting on the boundary, and $(c_1, c_2, c_3)_i$ integrating constants.

Then, from the observation that the displacement and its inclination at a point are, in the present problems, one-valued functions of its position, the following equations are obtained as the conditions for the determination of the constants in Eqs. (2):

$$\left.\begin{aligned}&\oint \frac{\partial}{\partial \nu}(V_1^2\chi)\,ds\\&\oint\left\{y\frac{\partial}{\partial \nu}(V_1^2\chi-x)\frac{\partial}{\partial s}(V_1^2\chi)\right\}ds-(1+\sigma)\left|\frac{\partial\chi}{\partial x}\right|_{A_i}^{A_i}=0\\&\oint\left\{x\frac{\partial}{\partial \nu}(V_1^2\chi+y)\frac{\partial}{\partial s}V(_1^2\chi)\right\}ds-(1+\sigma)\left|\frac{\partial\chi}{\partial y}\right|_{A_i}^{A_i}=0\end{aligned}\right\}\quad\cdots\cdots\quad(3)$$

In these equations, the integrations are performed around arbitrarily closed curves in the domain, ν being the normal of an elementary arc ds of the closed curves. Eqs. (1) and (2) together with Eqs. (3) are pointed out to be the necessary and sufficient conditions for the solutions of the present problems.

(2) The values of the constants in Eqs. (2) at the respective boundaries (and accordingly the stress function) are determined as follows. The function χ is expressed by

$$\chi=\chi_0+\sum_{i=1}^{n}(c_{1i}x+c_{2i}y+c_{3i})\,\xi_{2i}\quad\cdots\cdots\quad(4)$$

in which n is the number of the holes in the domain, and χ_0 and ξ_{2i} satisfy the equations

$$V_1^4\chi_0=0\,,\qquad V_1^2\xi_{2i}=0\quad\cdots\cdots\quad(5)$$

at any point of the domain, as well as the equations

$$\left.\begin{aligned}&\chi_0=-\int_{A_i}^{P_i}dx\int_{A_i}^{Q_i}Y_\nu ds+\int_{A_i}^{P_i}dy\int_{A_i}^{Q_i}X_\nu ds\ \text{(along each of boundaries)}\\&\xi_{2i}=\ 1\qquad\text{(along the } i\text{-th boundary)}\\&\xi_{2i}=\ 0\qquad\text{(along all the boundaries except the } i\text{-th)}\end{aligned}\right\}\quad(6)$$

The χ expressed by Eq. (4) satisfies clearly conditions (1) and (2), and $(c_1, c_2, c_3)_i$ denote the constants in Eqs. (2), along the i-th boundary, and we see that arbitrary 3 constants referring to any one boundary, say $(c_1, c_2, c)_0$ of the constants $(c_1, c_2, c_3)_i$ can be neglected as effectless upon the stress distribution. Then, if the χ_0 & ξ_{2i} functions can be determined, in the expression of χ given by Eq. (4), the unknown quantities become the $3n$ constants.

Now, the χ has to satisfy the condition in Eqs. (3) at each of the so-called independent closed curves, which encloses a so-called independent boundary, and we have n such curves in a domain with n holes. From this condition, we obtain $3n$ simutlaneous equations of the first degree of the c's, the solutions of which give of the values the c's.

(3) We have ascertained that the ξ_{2i} defined by Eqs. (5) & (6) are always found by means of the method of approximate solution of a plane harmonic fnnction and χ_0 can also be determined by using the following numerical method:

The plane biharmonic equation $(V_1^4\chi=0)$ can be approximately re-written into the form

$$\begin{aligned} b_{00}\chi_{00}+b_{10}\chi_{10}+b_{20}\chi_{20}+b_{11}\chi_{11}+b_{02}\chi_{02} \\ +b_{30}\chi_{30}+b_{21}\chi_{21}+b_{12}\chi_{12}+b_{03}\chi_{03} \\ +b_{40}\chi_{40}+b_{22}\chi_{22}+b_{04}\chi_{04}=0 \end{aligned} \quad \cdots\cdots (7)$$

by using the two- dimensional interpolation formulæ, in which we have $\chi_{is}=\chi(x_i, y_s)\ (i, s=0, 1, 2, 3, \ldots\ldots)$

$$b_{00}=\frac{3}{p_{01}p_{02}p_{03}p_{04}}+\frac{1}{p_{01}p_{02}q_{01}q_{02}}+\frac{3}{q_{01}q_{02}q_{03}q_{04}},\quad b_{10}=\frac{3}{p_{10}p_{12}p_{13}p_{14}}+\frac{1}{p_{10}p_{12}q_{01}q_{02}}$$

$$b_{01}=\frac{1}{p_{01}p_{02}q_{10}q_{12}}+\frac{3}{q_{10}q_{12}q_{13}q_{14}},\quad b_{20}=\frac{3}{p_{20}p_{21}p_{23}p_{24}}+\frac{1}{p_{20}p_{21}q_{01}q_{02}},\quad b_{11}=\frac{1}{p_{10}p_{12}q_{10}q_{12}}$$

$$b_{02}=\frac{1}{p_{01}p_{02}q_{20}q_{21}}+\frac{3}{q_{20}q_{21}q_{23}q_{24}},\quad b_{30}=\frac{3}{p_{30}p_{31}p_{32}p_{34}},\quad b_{21}=\frac{1}{p_{10}p_{12}q_{20}q_{21}}$$

$$b_{03}=\frac{3}{q_{30}q_{31}q_{32}q_{34}},\quad b_{40}=\frac{3}{p_{40}p_{41}p_{42}p_{43}},\quad b_{22}=\frac{1}{p_{20}p_{21}q_{20}q_{20}},\quad b_{04}=\frac{3}{q_{40}q_{41}q_{42}q_{43}}$$

$$p_{ij}=x_i-x_j,\quad q_{ij}=y_i-y_j.$$

Now, a finite number of rectangular net lines being drawn upon a given figure, the Eqs. (7) considered at all the points of intersection of the net lines become simultaneous equations for $\chi_{i,s}$, and these equations can be solved numerically by means of the so-called "Iteration Method".

Eventually, we reach the conclusion that the required function χ and accordingly the present problems for quite arbitrary domains can always be solved practically by using the method mentioned above.