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ON THE WAVES OF SHIPS

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§ 1—INTRODUCTION

The theoretical and experimental investigations of ship waves have been performed by many authors with respect to the wave profiles and the wave-making resistance of ships; the calculation which follows is a tentative extension of the theory already established. In practical naval engineering we are interested naturally in the wave-making resistance only, but this is a *surface integral* of pressure and moreover cannot be measured directly in the experiment, while on the other hand surface elevation can be expressed by a *line integral* of elementary waves and direct observation in water-tanks is possible. Therefore in order to test the accuracy of a new theory of ship waves which will be developed in later sections comparison with an experiment must be preferably made about the surface elevation instead of the wave-making resistance of ships.

The wave profiles of ships calculated theoretically by well-known methods can be regarded as showing a good agreement on the whole with the observations, but if we examine more closely, some differences will be still enumerated.

1. The theoretical amplitudes of waves are larger than those of observed.

2. The wave systems are observed much weakened at the stern.

3. At low Froude numbers the positions of the theoretical crests and

troughs of wave systems are not in sufficient coincidence with the observations.[1]

The first and the second discrepancies above-mentioned were explained by Havelock[2] and Wigley[3] as due to neglection of viscosity of water in the theory, for these theoretical surface elevations were calculated under the assumption of a perfect fluid. It is a very difficult problem, however, to formulate the expressions of wave profiles taking account of viscosity of water, because in a viscous fluid a velocity potential does not usually exist. Havelock (*loc. cit.*) proposed an approximate method in which he assumed that "the frictional effect upon the waves can be described as a diminution in the effective relative velocity of the model and the surrounding water as we pass from bow to stern. This is not very satisfactory from a theoretical point of view; but, on the other hand, it leads to a comparatively simple modification of expressions for the waves produced by the model. From a formal point of view, we may regard the modification as an empirical introduction of a reducing factor to allow for decrease in efficiency of the elements of the ship's surface as we pass from bow to stern."[4] He assumed that this reducing factor diminishes uniformly from the bow to the midship, from unity to $\beta(<1)$, preserving this constant value (β) over the rear portion from the midship to the stern.

Wigley's method (*loc. cit.*) is also essentially similar to Havelock's. However, in addition to the correcting factor β, he introduces another parameter, α, which stands for the reduction in the interference effects due to the damping of the bow waves and he puts $\alpha=\beta$ more or less arbitrarily. Thus in his calculation it has been assumed that the height of all the component wave systems of the afterbody is to be reduced in the ratio β, including the components of the symmetrical disturbance of the surface. Further, the forebody components have been similarly reduced in this ratio

(1) This is the case remarkably with Shigekawa's experiment, cf. On the Waves of Ships in Japanese, *Reports of the Shipping Laboratory,* No. 5 (1942) in which a model with a long parallel middle body was used.

(2) Havelock, Ship Waves: The Relative Efficiency of Bow and Stern. *Proc. Roy. Soc. London* A, 149 (1935).

(3) Wigley, Effects of Viscosity on the Wave-Making of Ships. *Trans. Inst. of Engineers and Shipbuilders in Scotland,* vol. 81, Part 3 (1938).

(4) Quoted with a slight modification from his paper, p. 417, *ibid.*

for all points aft of a station 1 ft. forward of the after perpendicular (this model being 16 ft. in length). Assuming an appropriate value for β he verified the weakenings of stern waves (Figure 7, *ibid.*), but his interest is not concentrated in the wave profiles but in the wave resistance and so far as the surface elevations are concerned, it cannot be said, in our opinion, that the calculated result has been satisfactorily improved in the general aspect.

These two methods above-mentioned are very natural in their ideas and successful in taking into consideration the effect of viscosity of water upon the waves in the simplest way as possible. But the parameter β introduced in their papers is, strictly speaking, a kind of an empirical factor and there can be contained in it all other kinds of corrections arising from miscellaneous and ambiguous causes, among which we must mention above all the inaccuracy accompanying the approximation of ship forms by means of the corresponding source distributions. That is to say, in the ordinary theory of waves the intensity of the equivalent source is determined by the well-known equation

$$M(X)=\frac{U_0}{2\pi}\frac{\partial Y}{\partial X}, \tag{1.1}$$

where M is the intensity, U_0 the general flow and $Y(X,Z)$ is the surface of the ship. When we proceed to improve the theory taking account of viscosity, we must at the same time take one step forward in the approximation of the source distribution. (This problem will be discussed in §3 and the effect of viscosity will be estimated in §4.)

§2—EQUATION OF THE SURFACE ELEVATION

The X-axis is taken in the direction of motion of the undisturbed stream, Y on the free surface, Z vertically upwards with the ZX-plane corresponding to the central vertical plane of the ship. ζ, the surface elevation on the flank of a ship when the doublet extending from the free surface to $Z=-\infty$, whose intensity is constant with respect to Z, is distributed on the ZX plane, was given by Havelock already,[1] and if his ex-

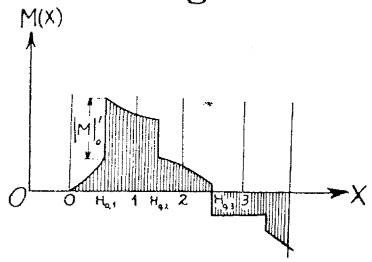
[1] Havelock, Ship Waves: The Calculation of Wave Profiles, *Proc. Roy. Soc. London*, A, vol. 135 (1932).

pression is rewritten in terms of the source distribution $M(X)$ instead of the doublet, after slight modification we get the alternative form

$$\left.\begin{aligned}\zeta &= \zeta_l+\zeta_w\,,\\ \zeta_l &= \frac{4}{\pi U_0}\sum_{r=0}^{n}\Big[\,|M(H)|_r^{r+1}\Big\{-\frac{\pi}{2x_0}Q_0(x_0\overline{H_{r,r+1}-X})\\ &\quad+\int_r^{r+1}\frac{\partial M}{\partial H}\Big\{-\frac{\pi}{2x_0}Q_0(x_0\overline{H-X})\,dH\Big\}\Big],\\ \zeta_w &= \frac{4}{\pi U_0}\sum_{r=0}^{n}\Big[\,|M(H)|_r^{r+1}\Big\{\frac{2\pi}{x_0}P_0(x_0\overline{H_{r,r+1}-X})\\ &\quad+\int_r^{r+1}\frac{\partial M}{\partial H}\Big\{\frac{2\pi}{x_0}P_0(x_0\overline{H-X})\,dH\Big\}\Big],\end{aligned}\right\} \tag{2.1}$$

where $x_0 = gU_0^{-2}$ and H is a variable of integration which is the same as X in its nature. In this formulation in order to admit some possible discontinuities of M the whole length of the ship was divided into n segments, in each of which, $(r,\ r+1)$ *say*, at most one point of discontinuity of M exists, whose abscissa has been denoted by $H_{r,\,r+1}$ and the magnitude of this discontinuity by $|M(H)|_r^{r+1}$, *see* Figure 1. $P_0(X)$ and $Q_0(X)$ are the functions given by Havelock, viz.

Fig. 1



$$P_0(X) = -\frac{\pi}{2}\int_0^X Y_0(t)dt\,,$$

$$Q_0(X) = \frac{\pi}{2}\int_0^X \{H_0(t)-Y_0(t)\}dt\,,$$

where $H_0(t)$ and $Y_0(t)$ are the Bessel functions. These functions are already tabulated for various values of X. ζ_l and ζ_w are called the local disturbance and the regular wave disturbance respectively. In the equations of (2.1) the first terms of ζ_l and ζ_w represent the waves generated by the discontinuity of the source distribution, while the second terms stand for the waves due to the gradual change of M within each of these intervals. $P_0(X)$ is defined to be zero when $X<0$, which corresponds to the physical fact that no regular waves can travel upstream from the point where it has been originated.

In the last place it must be noticed in the equations (2.1) that this value of ζ is in reality the surface elevation at $Y=0$ as is usually the case with the theory of ship waves. For under the assumption of slenderness the surface of the ship can be well approximated by the equation $Y=0$.

The subsequent calculation of the wave profiles is based upon the expression above-mentioned but the practical procedure of the computation will be explained later in § 5.

§ 3—FURTHER APPROXIMATION OF THE SOURCE DISTRIBUTION

Our next step is to discuss the source distribution $M(X)$ corresponding to a given ship form.

The usual way of finding M is, as was mentioned before, to assume after Havelock that

$$M(X)=\frac{U_0}{2\pi}\frac{dY_1}{dX}, \tag{3.1}$$

where $Y=Y_1(X)$ is the equation of the surface of a ship with an infinite draught. Although this assumption had been used for many years without further examinations, Inui remarked recently that its approximation can be rough to some extent especially when the ship form is full, and he proposed a new method of determining the source distribution.[1] His method is to assume $M(X)$ as a parabola of the 5th order and to determine its coefficients so that the streamline (Verzweigungsstromlinie) passes through 9 points appropriately chosen on the contour of the ship from bow to stern. He tabulated numerical values necessary for computations but his table can be available only for a symmetrical ship form and is inconvenient for our present purpose. So we developed another method which is, so to speak, a graphical method of successive approximation.

We take for convenience' sake the origin at the bow and introduce the non-dimensional lengths x and y defined by

$$X=2Lx-L, \quad \text{and} \quad Y=2Ly, \tag{3.2}$$

[1] Inui, *Proc. Soc. of Naval Architects of Japan* (1949) (unpublished).

where L is the half length of the ship (*see* Figure 2), and we denote anew the original ship form by the equation

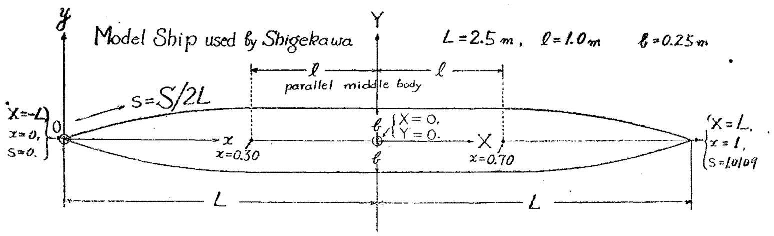
$$y = y_1(x)\,. \tag{3.3}$$

As the first approximation (M', *say*) we take the source distribution defined by (3.1) and draw graphically a ship form corresponding to it;[1] this contour is named y' conveniently. In the next place by applying the same procedure as (3.1) to the figure defined by (y_1-y'), the differentiation being performed graphically, we get the residual (correcting) distribution m', which together with M' above-mentioned constitutes the second approximation $M''(=M'+m')$. This process can be repeated until we get the closest approximation as we please, viz.

$$M(x) = \frac{U_0}{2\pi}\left\{\frac{dy_1}{dx} + \frac{d(y_1-y')}{dx} + \frac{d(y_1-y'')}{dx} + \cdots\right\}.$$

Our method is the same in its idea as Inui's and $M(x)$ thus determined corresponds exactly to the ship form infinitely below the free surface. But strictly speaking in order to find out the *exact* value of $M(x)$ (or rather

Fig. 2



$M(x, z)$) we must take account of the influence of the free surface, thus $M(x)$ together with its image added so as to satisfy the condition on the free surface must generate the ship surface exactly. Therefore $M(x)$ determined by the above procedure is only an approximation of the source distribution $M(x, z)$. But practically it is impossible here to consider the effect of the image-source, and assuming that this correction would be of the higher order, we hope we could have furthered the approximation of the older method to some extent.

(1) The method of constructing the ship form is due to D.W. Taylor, On Ship-Shaped Stream Forms, *T.I.N.A.* vol. 35 (1894). The brief account of the paper will be found in the Supplementary Notes at the end of the present paper.

The ship used in the subsequent calculation of the wave profiles is a cylinder of infinite draught having water lines of the equation

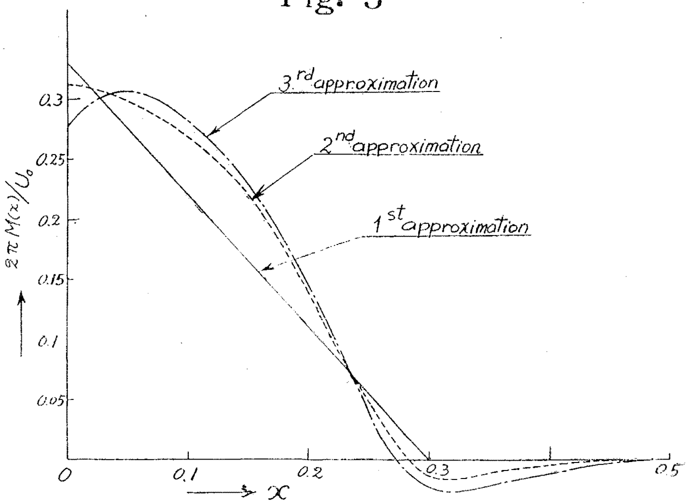
$$\left.\begin{aligned} y_1 &= \frac{1}{3}x - \frac{5}{9}x^2 \qquad \text{from} \quad x = 0 \quad \text{to} \quad x = 0.3\,, \\ &= \frac{1}{20} \qquad\qquad\quad \text{from} \quad x = 0.3 \quad \text{to} \quad x = 0.5\,, \end{aligned}\right\} \tag{3.4}$$

with $L = 2.5$ m., while the afterbody is symmetrical to the forebody with respect of $x = 0.5$ and may be left out of consideration for the time being (cf. Figure 2). This is the same model as Shigekawa (*loc. cit.*) used in computations and observations of the wave profiles for various speeds, and the results of our theory will be compared with his observations in a later section.[1]

The slope of the contour can be reckoned from

$$\left.\begin{aligned} \frac{dy_1}{dx} &= \frac{1}{3}\left(1 - \frac{10}{3}x\right) \qquad \text{from} \quad x = 0 \quad \text{to} \quad x = 0.3\,, \\ &= 0 \qquad\qquad\qquad\quad \text{from} \quad x = 0.3 \quad \text{to} \quad x = 0.5\,, \end{aligned}\right\} \tag{3.5}$$

Fig. 3



(1) The forebody of his model can be expressed by the equation

$$\begin{aligned} Y &= b\left\{1 - \left(\frac{X+l}{L-l}\right)^2\right\} \quad \text{from } X = -L \text{ to } -l\,, \\ &= b \qquad\qquad\qquad\qquad \text{from } X = -l \text{ to } \quad 0, \end{aligned}$$

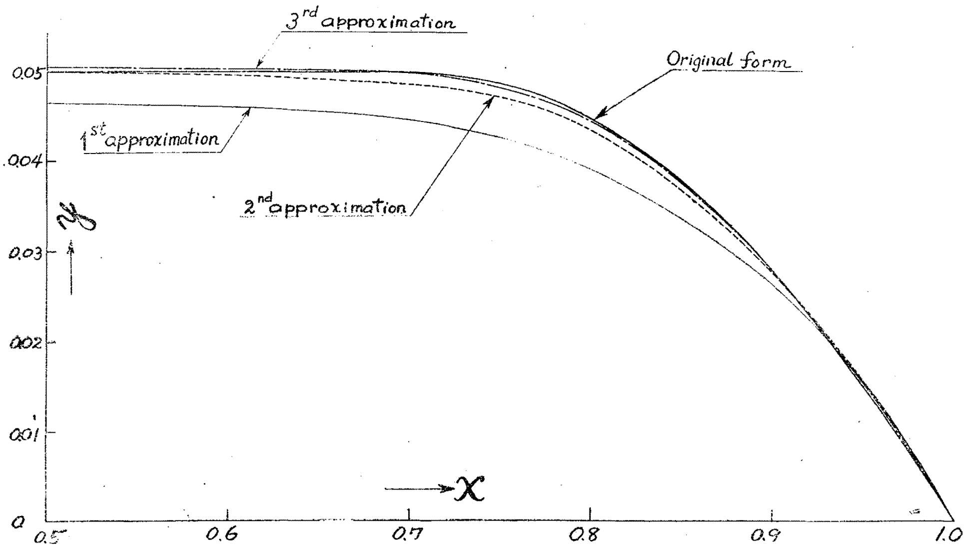
with $L = 2.5$ m., $l = 1.0$ m. and $b = 0.25$ m. (p. 210 of his paper). By the transformation of (3.2), this equation can be identified with (3.4) above. His calculation is based on the Michell's method.

and S which is the length of the arc measured from the bow, or non-dimensionally $s = S/2L$, can be given by the formula

$$s = \frac{1}{20}\left\{3+(10x-3)\sqrt{1-\frac{2}{3}x+\frac{10}{9}x^2}\right\}+\frac{27}{20\sqrt{10}}\left[\log_e\left\{\sqrt{1-\frac{2}{3}x+\frac{10}{9}x^2}+\frac{1}{3\sqrt{10}}(10x-3)\right\}-\log_e\left(1-\frac{1}{\sqrt{10}}\right)\right], \qquad 0\leqq x\leqq 0.3. \tag{3.6}$$

The results of our approximation, viz. the source distributions and the ship forms generated therefrom, are embodied in Figures 3 and 4, and for ready

Fig. 4



comparison the result of the well-known Havelock's method is denoted in the figures by the "1st approximation." In our case if the process is repeated to the third approximation the error can be reduced to 1 or 2% with respect to the ship form

§4—CALCULATION OF THE BOUNDARY LAYER

In accordance with the general plan of this paper explained in §1 our next step is to study theoretically the boundary layer formed along the flank of the model ship and to discuss the effect upon the calculated wave profiles. The boundary layer or the friction belt is a very thin region of a retarded fluid (laminar or turbulent) in which the large velocity of the general flow falls rapidly to zero on the surface of a body. In consequence of the presence of this retarded layer, the outer field of the potential flow is displaced out from the surface by the amount of

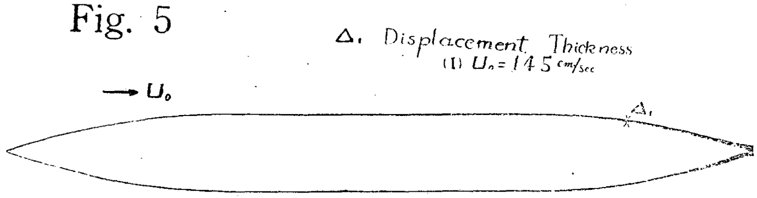
$$\Delta_1 = \int_0^{\Delta} \frac{U_1 - U}{U_1} dN,$$

where Δ is the thickness of that layer, N the outward-drawn normal on the surface and U and U_1 are the velocities of the fluid in and just outside the boundary layer respectively. Effectively, therefore, the ship becomes fuller than the actual shape by Δ_1 when moving through a viscous fluid, which is called the displacement thickness of the boundary layer. But as was mentioned before in the introduction of this paper, this hydrodynamical correction of forms has been neglected in the theoretical treatments of ship waves already published except in half empirical methods by Havelock and by Wigley (*loc. cit.*).

Δ_1 or non-dimensionally δ_1 (i.e. $\Delta_1/2L$) is very thin on the forebody but will become thicker and thicker on the afterbody until near the stern it will be negligible no longer. Thus the tapering of the hinder part of the model being much compensated apparently, the waves generated by the stern will be fairly weakened. One of the remarkable discrepancies between the calculated and the observed wave profiles, i.e. the absence of the marked stern waves may be due to the *no longer negligible thickness* of the boundary layer at the stern.

We are now going to calculate the boundary layer approximately and examine how far we can justify the above conjecture theoretically. In Figure 5 Δ_1 of the model ship of Shigekawa has been reproduced in advance from the data of the subsequent calculations to illustrate its smoothing effect at the stern.

Fig. 5



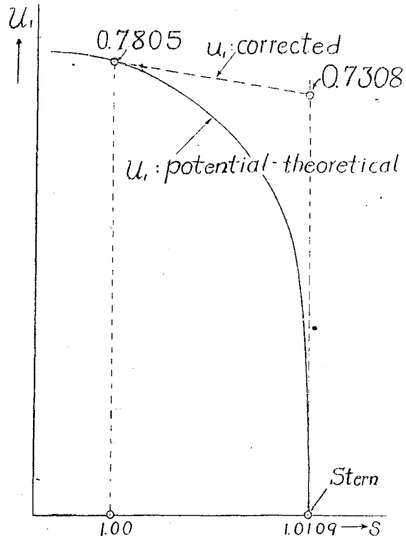
(I) CALCULATION OF THE INVISCID FLOW (GRUNDSTRÖMUNG)

In order to begin the calculation it is necessary to know beforehand about the distribution of the velocity of the inviscid flow on the surface of the ship. It is enabled by the aid of the theory developed by Moriya(1) (*see* (II) of the Supplementary Notes). $u_1(s)$, $(P_1-P_0)/(\frac{1}{2}\rho U_0^2)$ and (du_1/ds) thus obtained are shown in Figures 6*(2) and 7*, where $u_1 \equiv U_1/U_0$ (U_0 being the general flow) is the dimensionless velocity at a point on the ship, $s \equiv S/2L$ the dimensionless arc-length measured from the bow, cf. § 3 and Figure 2. P_0 and P_1 are the pressures at a great distance from the body and on its surface respectively. They are connected by so-called Bernoulli's theorem, viz.

$$P_0+\frac{1}{2}\rho U_0^2 = P_1+\frac{1}{2}\rho U_1^2 .$$

These quantities are all shown in the figures as the functions of s ($s=0$ at the bow and $s=1.0109$ at the stern) on the abscissa.

Fig. 8



In concluding this paragraph we must add a few words about the condition of the stern: according to the potential theory of flow a stagnation must be formed at the rear end just like at the leading edge (cf. Figure 6), but in an actual motion of a fluid around a body with a sharp trailing edge this theory does not hold good: u_1, the velocity of the fluid just outside the boundary layer, has a finite (non-zero) value at the stern and the fluid flows away smoothly. In order to reproduce this state of affairs approximately we take the following ex-

(1) Moriya, A Method of Calculating Aerodynamic Characteristics of an Arbitrary Wing Section (in Japanese), *the Journal of the Society of Aeronautical Science of Nippon*, vol. 5, No. 33 (1938).

(2) Those figures which are asterisked are placed together at the end of this paper for convenience of space.

pedient: by prolonging the curve of u_1 in a straight line for the part of s beyond 1.00 (corresponding to $x \doteqdot 0.99$) we read $u_1 = 0.7308$ at $s = 1.0109$ on the graph of Figure 6. Of course no physical meaning can be found in this value of u_1 thus obtained but it is hardly probable that any serious error may be caused in the final results by the slight uncertainty contained in this approximation. The process of this approximation is shown schematically in Figure 8.

(II) TRANSITION OF LAMINAR MOTION TO TURBULENCE

As is well-known, motion in a boundary layer becomes turbulent when the Reynolds number is too high. But precise criterion, theoretical and experimental, for the boundary layer of a model ship in a water-tank has not yet been established. In this paper the ratio of the (breadth/length) of the model being as small as 1/10, we think we can safely refer to the data obtained for a flat plate placed edgeways to the stream.

The temperature of water when the observations of waves by Shigekawa were made is not given explicitly in his paper, but if we assume tentatively that it was 15°C, then ν (water) $= 0.01141$ cm.^{2}/sec.,[1] and $R_L = U_0 \cdot 2L/\nu$, the Reynolds numbers of the model, are found to be

$$\left.\begin{aligned} R_L &= 6.354 \times 10^6 && (U_0 = 145 \text{ cm./sec.}), \\ &= 7.888 \times 10^6 && (U_0 \doteqdot 180 \text{ cm./sec.}), \\ \text{and} \quad &= 11.394 \times 10^6 && (U_0 = 260 \text{ cm./sec.}). \end{aligned}\right\} \tag{4.1}$$

(These three speeds were selected from the series of his observations as examples of low, moderate and high speeds, the corresponding Froude numbers being 0.207, 0.257 and 0.371; these three cases will be mentioned afterwards for brevity as I, II and III respectively.) If we adopt as the criterion for the transition

$$U_0 \cdot X_t/\nu = 3 \times 10^5, \tag{4.2}$$

where X_t denotes the distance of the transition point from the leading edge, which is applicable for the flat plate (*see* (III) of the Supplementary Notes), then we readily find

$$X_t/2L = 0.047,\ 0.038 \text{ and } 0.026 \tag{4.3}$$

(1) Goldstein, *Modern Developments in Fluid Dynamics*, vol. I (1938), p. 5, Table 1 (*c*).

for the cases of I, II and III respectively. We can conclude from these data that the transition, although whose precise position cannot be mentioned definitely, must have occurred at any rate just near the leading edge of the model.

Even if, however, some error may be contained in the above estimation, considering that we are interested exclusively in the thickness of the friction belt and, on the other hand, that it is very thin near the bow if turbulent or not, we can readily conclude that no marked discrepancy may possibly result from this uncertainty Bearing it in mind therefore, in the subsequent numerical calculations we can make use of the following assumption:—let us assume more simply that the boundary layer is turbulent right from its beginning at the bow.

(III) CALCULATION OF THE BOUNDARY LAYER

As in the preceding paper by one of the present authors[1] the thickness of the turbulent boundary layer can be reckoned approximately from the following formula:[2]

$$\delta^{\frac{5}{4}} = 0.2983\ R_L^{-\frac{1}{4}} u_1^{-\frac{115}{28}} \int_0^s u^{\frac{27}{7}}\, ds\,, \tag{4.4}$$

where δ is the non-dimensional thickness of the viscous layer (i.e. $\Delta/2L$) measured vertically to the surface at the point where the arc-length from the bow equals s. $R_L = U_0 \cdot 2L/\nu$, cf. (4.1), and $u_1(s)$ is the velocity of the inviscid flow on the surface (calculated in (I) of this section).

The above formula was first given by C. B. Millikan for the surface of revolution;[3] the approximation of this method originated from the following idea. In a viscous flow around a body, so slender as to cause no marked gradient of pressure along its surface, we may safely neglect the

(1) Okabe, A Contribution to the Theory of the Frictional Resistance of Ships (in Japanese), *Reports of the Research Institute for Fluid Engineering*, vol. 4, No. 2 (1947). English Abstract, *loc. cit.* vol. 6, No. 2 (1950).

(2) In deriving this formula the effect of the free surface upon the boundary layer was neglected. To take it into consideration seems very difficult at the stage of our knowledge.

(3) Millikan, The Boundary Layer and Skin Friction for a Figure of Revolution, *Trans. A.S.M.E.*, vol. 54, No. 2 (1932).

deformation of the velocity profile from that of the flow along a flat plate, taking partly into account, however, the consequence of the finite breadth of the body by only means of the distribution of $u_1(s)$, variable from point to point. In other words, replacing the actual body by a train of flat plates of infinitesimal lengths placed one after another edgeways to the main stream we regard the velocity just outside the boundary layer as varying continuously from plate to plate, while the velocity profile within the layer is assumed invariable (conveniently expressed by the so-called $\frac{1}{7}$th power law).

Obviously this assumption breaks down where the sharp pressure gradient exists, especially in the neighbourhood of the stern,(1) where even the separation of the boundary layer from the surface may be found. Because of this unsurmountable difficulty, however, very few methods of practical calculation have been proposed that enable us to take into consideration all the situation completely.(2) With respect to the separation of the boundary layer a brief discussion will be found in (IV) of the Supplementary Notes at the end of this paper.

The displacement thickness, or non-dimensionally $\delta_1 = \Delta_1/2L$, can be given by the formula(3)

$$\delta_1 = \frac{1}{8}\,\delta. \tag{4.5}$$

In Figure 9* δ_1 has been shown for three values of U_0: I, II and III. $(d\delta_1/ds)$ which is necessary in the subsequent computations can be reckoned from

(1) This criticism holds good for the bow as well, but since the viscous layer is very thin here no serious error can be caused finally.

(2) Among them a method by Buri (*Zürich Dissertation*; 1931), its alternative by Howarth (*Proc. Roy. Soc. London*, A, 149, 1935) and another method by Gruschwitz (*Ingenieur-Archiv*, 2, 1931) must be mentioned, but no remarkable difference was found in the final results when the formula (4.4) was used and compared with the conclusion drawn out of the solution of the differential equation obtained by Howarth. As to this comparison reference is to be made to Okabe's paper, *loc. cit.* An inconvenience of Howarth's method is that the curve necessary to the computation has not been prepared for the accelerated region. In Gruschwitz's method we must solve a pair of simultaneous equations and it will be quite troublesome in such a problem of ours.

(3) This formula is derived readily from the $\frac{1}{7}$th power law above-mentioned; if we assume that the velocity varies as nth power of the distance from the surface, (4.5) must be replaced by $\delta_1 = n\delta/(n+1)$.

$$\frac{d\delta_1}{ds} = 0.02983\, R_L^{-\frac{1}{4}}\, u^{-\frac{1}{4}}\, \delta^{-\frac{1}{4}} - \frac{23}{56} \frac{\delta}{u_1} \frac{du_1}{ds}\,;$$

the fact is that this expression can be easily deduced from the differential equation which underlies the formula (4.4). $(d\delta_1/ds)$ is shown in Figure 10* (a) and (b).

If we define the outer edge of the displacement thickness by the equation

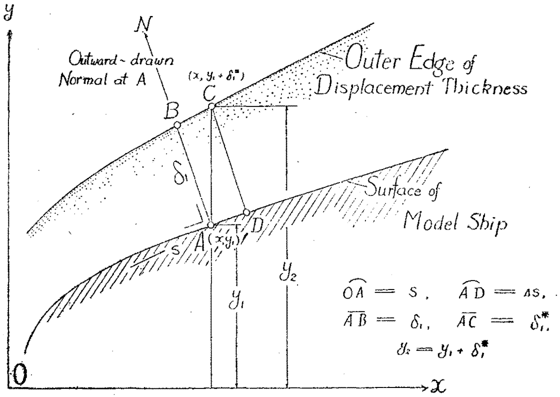
$$y = y_2(x)\,, \tag{4.6}$$

(dy_2/dx) i.e. the corrected value of (dy_1/dx) due to the presence of the friction belt can be calculated from the formula

$$\frac{dy_2}{dx} = \left(\frac{dy_1}{dx} + \frac{d\delta_1}{ds}\right) \div \left(1 - \frac{dy_1}{dx} \frac{d\delta_1}{ds}\right), \tag{4.7}$$

which is reproduced in Figure 11* for the case of I; in the remaining two other cases the curves are found almost the same and are omitted. Strictly speaking, however, $(d\delta_1/ds)$ in (4.7) must be taken at $(s+\Delta s)$ instead of s (*see* Figure 12), because the boundary layer is always measured vertically to the surface and not to the centre-line (the x-axis). But Δs is

Fig. 12



found to be quite small everywhere from bow to stern,[1] so that the correction was neglected altogether.

[1] E.g. in the case of I, at the points of $s=0.05$, 0.10, 0.20, 0.80, 0.90, 0.95 and 0.975, Δs amounts to only 0.00004, 0.00005, 0.00005, -0.0002, -0.0005, -0.0009 and -0.001 respectively. From $s=0.3$ to 0.7, Δs vanishes identically of course.

The new contour of the model ship effective to wave-making is thus defined by the equation (4.6), which can be expressed approximately by

$$y_2 = y_1 + \delta_1^* = y_1 + \delta_1 \left\{1 + \left(\frac{dy_1}{dx}\right)^2\right\}^{1/2} \left(1 - \frac{dy_1}{dx}\frac{d\delta_1}{ds}\right)^{-1}. \qquad (4.8)$$

In Figure 13* y_2 of case II has been embodied for ready comparison with the original form. The curves for other two cases are quite similar and are omitted.[1]

§ 5—CALCULATION OF THE SURFACE ELEVATION

In the equation (2.1) cited before if we substitute an appropriate expression for $M(x)$, the source distribution, we can obtain at least mathematically the expression of the surface elevation. In the ordinary calculations of ship waves the source distribution was assumed without exception according to (1.1) or (3.1), therefore the practical computations have been retained treatable if we approximate the ship form by means of a parabola of the 2nd or the 3rd degree.[2] But as was shown in Figure 3, $M(x)$ obtained as the closer approximation at present has been found to be much complicated, so even if we should succeed to represent its feature in terms of a comparatively simple expression the subsequent mathematics would be found still to be too troublesome for practical purpose. To avoid this difficulty, therefore, we take an alternative approximation.

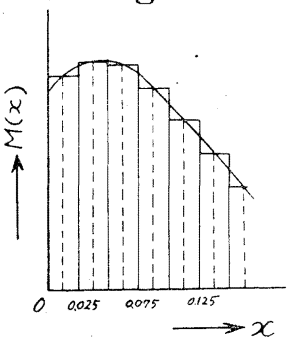
To this end we divided the length of the ship (in reality from the bow ($x = 0$) to the point in the wake 1/10 of the length of the ship behind the stern ($x = 1.1$)) into 44 intervals of equal length, in each of which the small variation of M being neglected M was replaced by a constant which is equal to the true value at the mid point of this interval. The whole curve of $M(x)$ was thus approximated by a sequence of these horizontal segments, cf. Figure 14. It must be noticed here by the way that the division mentioned in § 2 as e.g. $(r, r+1)$ admitting at most one point of discontinuity of the source distribution within it does not correspond ex-

[1] Acknowledgements of the authors are due to Miss Takaki who assisted them in the numerical calculations as well as in the drawings necessary for the discussion of §4.

[2] Generally parabolas of the 2nd degree were used.

actly to this division above explained but to its alternative shown by dotted lines through each mid point. Referring to the latter division (dotted lines) the one constituent of the elementary waves (2.1), i.e. that part of waves due to the discontinuities of M, originates from the mid point of each interval, while the remaining component corresponding to the second terms of these equations vanishes identically. And as was stated before P_0-waves must be put null in the region of its negative argument.

Fig. 14



Next we have to estimate the error contained in this process of approximation. This problem, however, is quite the same in its nature as that treated by Havelock[1] previously and availing ourselves of his conclusion it will be found accordingly that the order of the error varies roughly as gdU_0^{-2}, where d is the interval of the division. His discussion relates, however, to the approximation of the wave profiles when the generation of the elementary waves distributed over a certain range has been replaced by a wave from one concentrated source situated at the end of that range, while in our present calculation this concentrated source is placed at the mid point of each interval, thus some decrease in the error would be naturally expected. Estimated from the graph in Havelock's paper or calculated actually ourselves, it is readily known that except cases of low speed when the error can amount to *ca.* 3% it is almost negligible through out moderate and higher speeds.

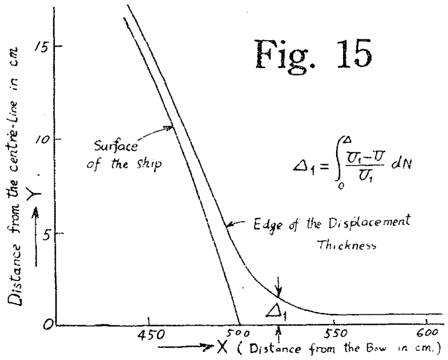
In calculating the stern waves it is necessary to know the displacement thickness of the wake just behind the stern, but the theoretical treatment has not been established, even approximately, for the turbulent wake just behind a body where the permanent type of velocity distribution has not yet been attained. In order to solve this difficulty without much deliberate labour we prolonged the curve of δ_1, calculated in § 4 from bow to stern, smoothly and rather arbitrarily in the wake to the point 1/10 of the length

(1) Havelock, Ship Waves: Their Variation with Certain Systematic Changes of Forms, *Proc. Roy. Soc. London*, A, vol. 136 (1932).

of the ship behind the stern from where downstream the displacement thickness was assumed constant, Figure 15.

The numerical calculations of surface elevations were carried out for three cases of I, II and III (cf. (4.1)) by the following methods:

(A): the ordinary method of Havelock, viz. in which the source distribution proportional to (dy_1/dx) was assumed.[1]



(B): the ordinary method, but taking account of the effect of the boundary layer the source distribution proportional to $dy_2/dx = d(y_1+\delta_1^*)/dx$ was assumed.

(C): the further approximation of the source distribution, but the boundary layer was left out of consideration.

(D): the further approximation of the source has been carried out with respect to the corrected contour defined by $y = y_2$.[2]

The results of these calculations are embodied in Figures 16*–18*, but to avoid unnecessary complications (B) and (C) are reproduced for II only (Figure 17*). For ready comparison the observations by Shigekawa (curves of mean values) were shown by broken lines.

If reference is made to these figures the features listed below will be observed.

---

(1) In Shigekawa's original paper (*loc. cit.*) the surface elevations calculated by (A) are shown for various speeds but the present authors' opinion is that some mistake was made in his computation.

(2) In deriving the source distribution equivalent to y_2, δ_1 of the case II was used throughout for the cases of I and III to lighten the labour neglecting the slight variation of δ_1 with the Reynolds number, cf. Figure 9.

(i): Comparing with the observations, (A) shows the discrepancies enumerated in § 1, viz. too large amplitudes, very marked stern waves and some difference in positions of crests and troughs.

(ii): By the further approximation of the theory by taking account of the effect of the displacement thickness in the ordinary method of determining the source distribution, (B), no better agreement could be attained except some weakening of stern waves.

(iii): By closer approximation of the source distributions, (C), crests and troughs were brought into better agreement in position with the observations but the amplitudes have been much enlarged; no weakening of stern waves can take place of course.

(iv): In (D) the weakening of stern waves and the agreement in crests and troughs were obtained at last but the amplitudes are as large as in (C), much too larger than the observed.

Comparing the above data we shall arrive at some conclusions which will be summarized in the next section.

§ 6 –SUMMARY AND CONCLUSION

The calculated wave profiles by the ordinary theory of ship waves show some discrepancies as explained in § 1 between the observations. In § 3 we proposed a new method of the successive approximation of the source distribution corresponding to the ship form and at the same time in § 4 estimated the thickness of the boundary layer (displacement thickness) formed over the flank of the ship. By comparing the surface elevations calculated in § 5 with the observations by Shigekawa, we can arrive at the following conclusions:

(1) The absence of the marked stern waves in the observations is ascribed to the smoothing effect of the boundary layer no longer negligible at the stern.

(2) The difference in phases of waves between the observed and the calculated is due to the insufficient approximation of the source distribution assumed in the older theory.

But we cannot explain appropriately the amplitudes of waves resulting from our theory, which are too larger than the older method much more larger than the observations. The study of this discrepancy must be postponed to later investigations. (February 12, 1950)

SUPPLEMENTARY NOTES

(I) Graphical Method of Determining Two Dimensional Ship Form (Summary of Taylor's Paper, *loc. cit.*): The stream function due to a point source, whose intensity is m, situated at the origin of the coördinates is readily found to be $m\theta$, where θ is the angle measured from the x-axis. If the source is distributed along the x-axis, then the stream function becomes

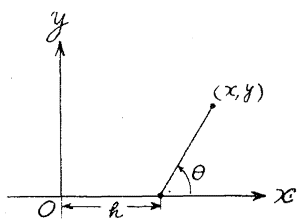
$$\int m(h)\theta(x,y;h)dh,$$

θ being the angle indicated in Figure 19. On the other hand the stream function corresponding to the general flow U_0 being $-U_0 y$, the combined function ψ of these two flows is given obviously by

$$\psi = -U_0 y + \int m(h)\theta(x,y;h)\,dh.$$

The curve $\psi = 0$ gives the boundary of two regions of flow, i.e. in our case the outer form of the ship. When the distribution of m is quite general the integral of the second term of the above formula can be carried out numerically for various pairs of (x, y) by dividing the x-axis into a number of intervals of equal length and assigning to each the corresponding value of m. Taylor tabulated θ as a function of x and y when $h = 0$.(1) Availing ourselves of his table this integral was evaluated as a function of x and y for the source distribution equivalent to the given ship form (cf. §3), and the curves were drawn in reference to y (the ordinate) and ψ (the abscissa) for various values of x as a parameter. If we read in the figure the values of (x,y) of the points of intersections of these curves with a straight line given by $\psi = U_0 y$, they give immediately the ship form. In Figures 20*–22* these stream functions are reproduced, from which the ship form

Fig. 19



(1) His table, in reality, gives $90° - \theta$ in our notation.

previously shown in Figure 4 was obtained. Figure 20* corresponds to Havelock's approximate assumption of source distribution and Figures 21* and 22* to the further approximations.

(II) The Outline of Moriya's Method (*loc. cit.*): Let the (non-dimensional) ship form be expressed in terms of ξ, a parameter,

$$x = \frac{1}{2}(1+\cos\xi),$$

and
$$y = \sum_n a_n \cos n\xi + \sum_n b_n \sin n\xi, \tag{1}$$

then on the surface of the ship the velocity and its derivative of the inviscid flow which is U_0 parallel to the x-axis at an infinite distance can be given approximately by the formulae

$$\left.\begin{aligned} u_1(s) &= \left(\frac{1}{2}\sin\xi + J_2\right)\left(\frac{1}{4}\sin^2\xi + J_1^2\right)^{-1/2}, \\ \text{and}\quad du_1/ds &= -\left(\frac{1}{2}\cos\xi + K_1\right)\left(\frac{1}{4}\sin^2\xi + J_1^2\right)^{-1} \\ &\quad + \left(\frac{1}{2}\sin\xi + J_2\right)\left(\frac{1}{8}\sin 2\xi - J_1K_2\right)\left(\frac{1}{4}\sin^2\xi + J_1^2\right)^{-2}, \end{aligned}\right\} \tag{2}$$

where J_1, J_2, K_1 and K_2 are given by

$$J_1 = \sum nb_n \cos n\xi, \qquad J_2 = \sum nb_n \sin n\xi,$$
$$K_1 = \sum n^2 b_n \cos n\xi, \qquad K_2 = \sum n^2 b_n \sin n\xi.$$

The values of ξ corresponding to various values of x are tabulated in Tables 1 and 2 of the original paper. Because our ship is of course symmetrical with respect to the x-axis all the a_n's are found to to be identically zero, which simplifies the calculation to a great extent; in deriving the formulae (2) this advantage has been taken already.

The expansion expressed by (1) is an ordinary process of the Fourier analysis and if we take the terms up to $n = 7$, we obtain

$$\begin{aligned} y = {} & 0.050932\sin\xi - 0.003532\sin 3\xi \\ & -0.005004\sin 5\xi - 0.000290\sin 7\xi; \end{aligned} \tag{3}$$

b_2, b_4 and b_6 vanish also in virtue of symmetry of the fore- and after part of the ship. Since all the subsequent computations will be carried out on the basis of this approximated contour, it will be necessary here to test the accuracy of this expansion. The result of this comparison is shown in Figure 23*; the approximation is thus found to be fairly good.

(III) Some Remarks about the Transition: The following is a quotation from Goldstein:[1] "…In a moderately steady air stream, when the front of the plate is sharpened to a knife edge, the transition to turbulence takes place when $U_0\Delta/\nu$ is about 3,000, where Δ is the thickness of the boundary layer. This corresponds to a value of about 3×10^5 for U_0X/ν, where X is the distance from the leading edge. The critical Reynolds number depends here also on the disturbances present and on the conditions at the front of the plate. Values varying from 1,650 to 5,750 for $U_0\Delta/\nu$, corresponding roughly to 9×10^4 to 1.1×10^6 for U_0X/ν, have been observed.…"

In Shigekawa's paper cited before observations were made using a model of paraffin and although no details of the condition of the surface were given, we conjecture it must have been rougher than the surface used in the wind tunnel experiment to which the above quotation relates. About the condition of water, on the other hand, it is stated in his paper that the experiments were repeated after intervals of about 20 minutes and that the surface of water when the model was put again in motion was almost quiet except the remaining very long waves. It is very difficult to infer from this brief description the magnitude of turbulence present in the main stream but it will not be so far from the fact to conclude that the stream was *moderately turbulent*. Finally, the leading edge of the model subtends an angle of 40° to the stream instead of being sharpened to a knife edge and this tends to introduce additional disturbances in the boundary layer downstream. If reference is made again to the preceding statement by Goldstein taking into account these various situations, the value of U_0X/ν where turbulence took place in the boundary layer in Shigekawa's experiment will be deduced less than 3×10^5 to some extent.

(IV) Separation of the Boundary Layer: So far no definite criterion has been established about the separation of the turbulent boundary layer; only Γ', a parameter proposed originally by Buri (*loc. cit.*), is of some use

(1) Goldstein, *op. cit.* vol. I, p. 71, with slight changes in notations, viz. U_0, Δ and X for U, δ and x respectively.

for the discussion. According to him the critical value of this parameter corresponding to the separation was found experimentally equal to -0.06, while after Prandtl it lies within the range from -0.05 to -0.09.[1] And judging from the experience of one of the authors,[2] Γ may be reckoned rather roughly by means of the procedure explained in (III) of § 4. For, in spite of the apparent paradox that the separation caused by the adverse pressure gradient will be discussed on the basis of a parameter calculated neglecting the pressure gradient itself, because of the very rapid rise of Γ in the vicinity of $\Gamma=-0.06$ even the considerable error committed in its estimation can have little consequence in determining the position of separation.[3]

Now Γ in question is defined by

$$\Gamma \equiv \frac{\vartheta}{u_1} \frac{du_1}{ds} R_\vartheta^{\frac{1}{4}},$$

where ϑ is the non-dimensional momentum thickness $(\Theta/2L)$ and $R_\vartheta = U_1\Theta/\nu$, which can be transformed for convenience' sake into

$$\Gamma = \left(\frac{7}{72}\right)^{\frac{5}{4}} R_L^{\frac{1}{4}} u_1^{-\frac{3}{4}} \delta^{\frac{5}{4}} \frac{du_1}{ds}$$

in virtue of the simple relation[4]

$$\vartheta = \frac{7}{72}\delta.$$

Γ is reproduced in Figure 24* for various values of s and the position corresponding to $\Gamma=-0.06$ has been marked. The curves of Γ are sensibly coincident for three cases of I, II and III. According to this criterion the separation point is found in the neighbourhood of $s=0.94$ (i.e. $x \doteqdot 0.93$); we think that this is tolerably reasonable. Therefore from the vicinity of $s=0.94$ downstream to the stern the various data obtained in the foregoing paragraph (§ 4) have been revealed to be less reliable. But having no practical means to revise them at present, we are going to use them

(1) Durand, *Aerodynamic Theory*, vol. III, p. 158 (1935).

(2) Okabe, *ibid.*

(3) And from just the same reason the choice of the critical value ($-0.05 \sim -0.09$) is usually a matter of practical indifference.

(4) This formula is derived from the $\frac{1}{7}$th power law as before. If n is taken instead of $\frac{1}{7}$, it must be replaced by $n\delta/(n+1)(2n+1)$.

uncorrectedly for the stern: in other words we prolong smoothly, so to speak, the curves calculated fairly accurately in the region up to the midship to the stern where we are helpless to discuss the situation.

(V) Surface traction: Surface traction, τ_0, calculated from the data already obtained, viz.

$$\frac{\tau_0}{\rho U_0^2} = 0.0232\, R_L^{-\frac{1}{4}}\, u_1^{\frac{7}{4}}\, \delta^{-\frac{1}{4}}\,,$$ (1)

is shown in Figure 25*. In this figure the point characterized by $\Gamma = -0.06$ is indicated by an arrow. Perhaps the actual curves of τ_0 will fall rapidly near this point showing the feature tentatively interpolated by dotted lines; reference can be made about it to Okabe's paper (Figure 4, *loc. cit.*).

After completing this paper, I had a chance to attend Professor Havelock's lecture "Wave Resistance Theory and its Application in Ship Problems" in the Hotel Commander, Cambridge, Mass., U.S.A. on August 28, 1950. I should like to express here our cordial gratitude to this eminent professor for the interest he took in our work and for the encouragement he gave us during the discussion. (J.O.)

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(1) 0.0232 is the revised value from 0.0225, cf. (62) pp. 361~2, vol. II of Goldstein (*op. cit.*), in order that C_f may become equal to 0.074 $R_L^{-1/5}$ instead of 0.072 $R_L^{-1/5}$.

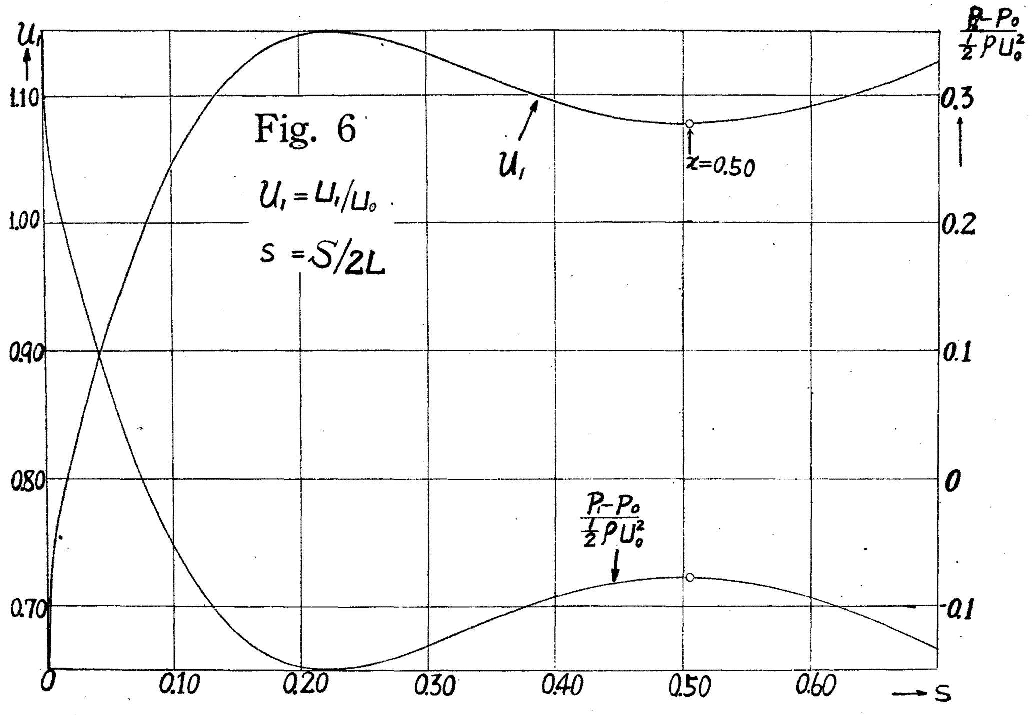


Fig. 6
$U_1 = U_1/U_0$
$s = S/2L$
U_1
$x=0.50$
$\frac{P_1-P_0}{\frac{1}{2}\rho U_0^2}$
1.10
1.00
0.90
0.80
0.70
0.3
0.2
0.1
0
-0.1
0
0.10
0.20
0.30
0.40
0.50
0.60
→ s

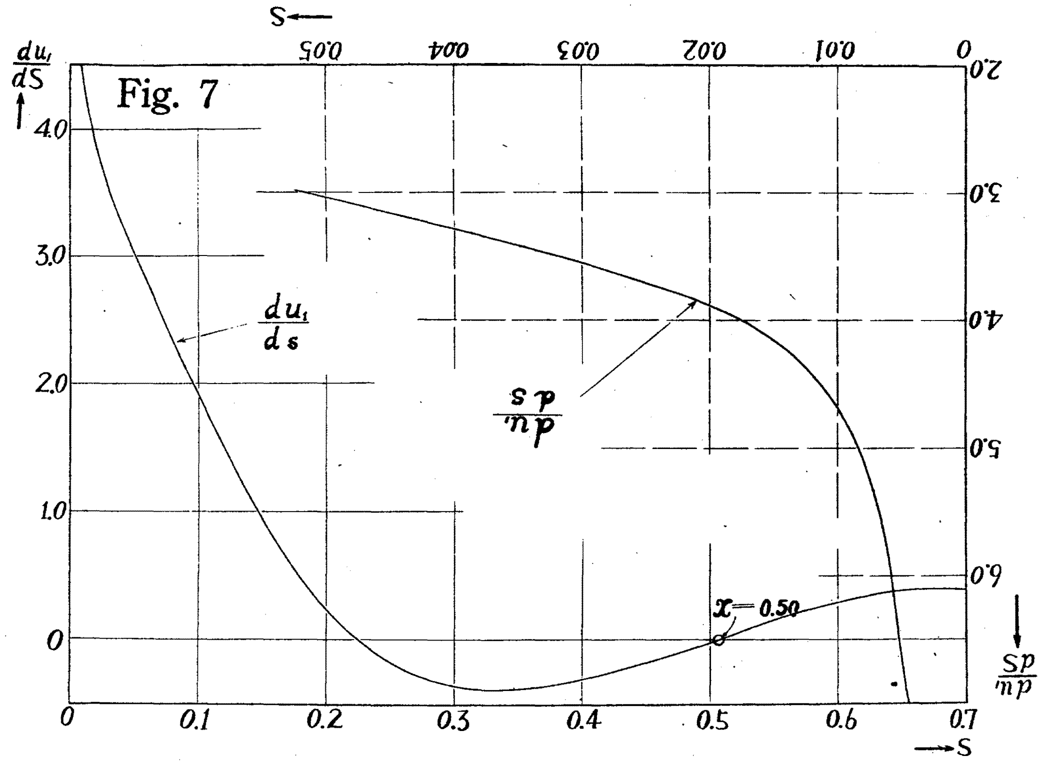


Fig. 7
$\frac{du_1}{dS}$
$\frac{du_1}{ds}$
$x=0.50$
4.0
3.0
2.0
1.0
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
→ s

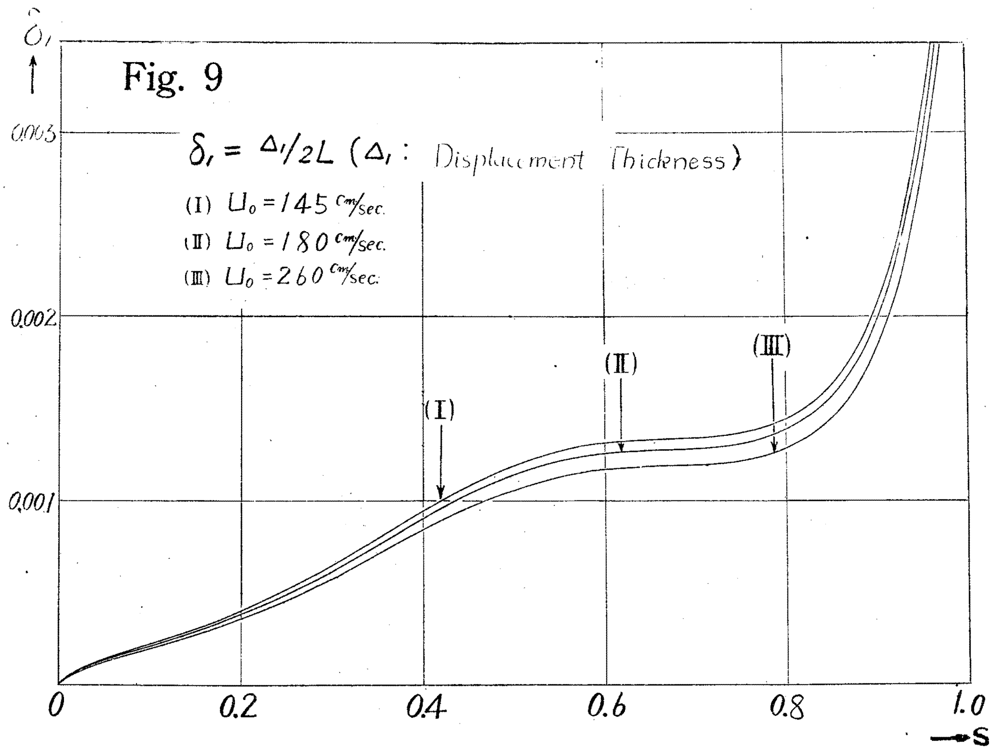


Fig. 9
δ₁ = Δ₁/2L (Δ₁ : Displacement Thickness)
(I) U₀ = 145 cm/sec.
(II) U₀ = 180 cm/sec.
(III) U₀ = 260 cm/sec.
δ₁
0.003
0.002
0.001
(I)
(II)
(III)
0
0.2
0.4
0.6
0.8
1.0
s

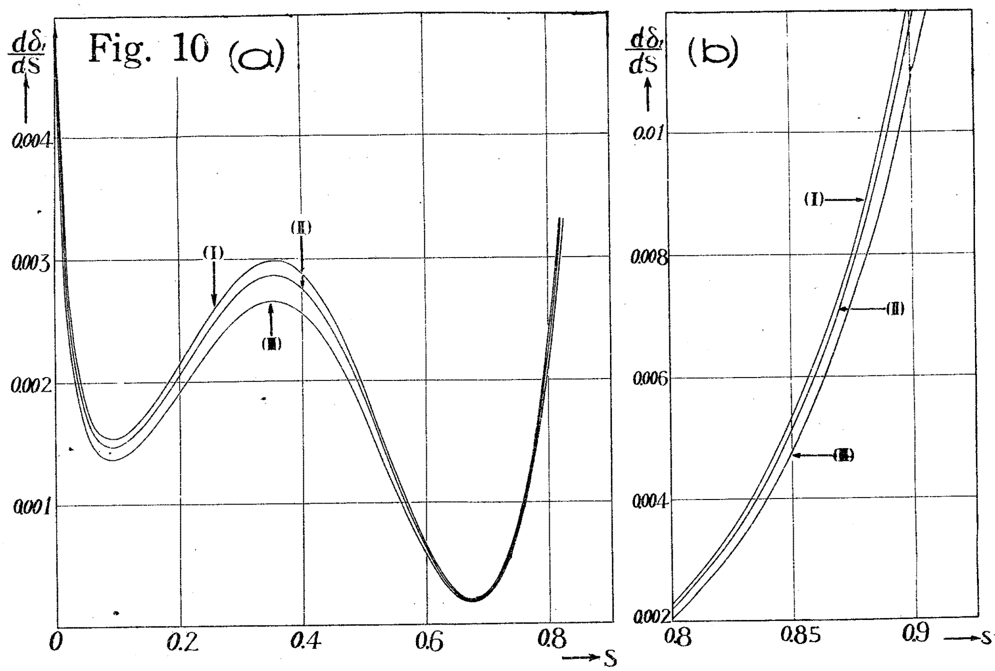
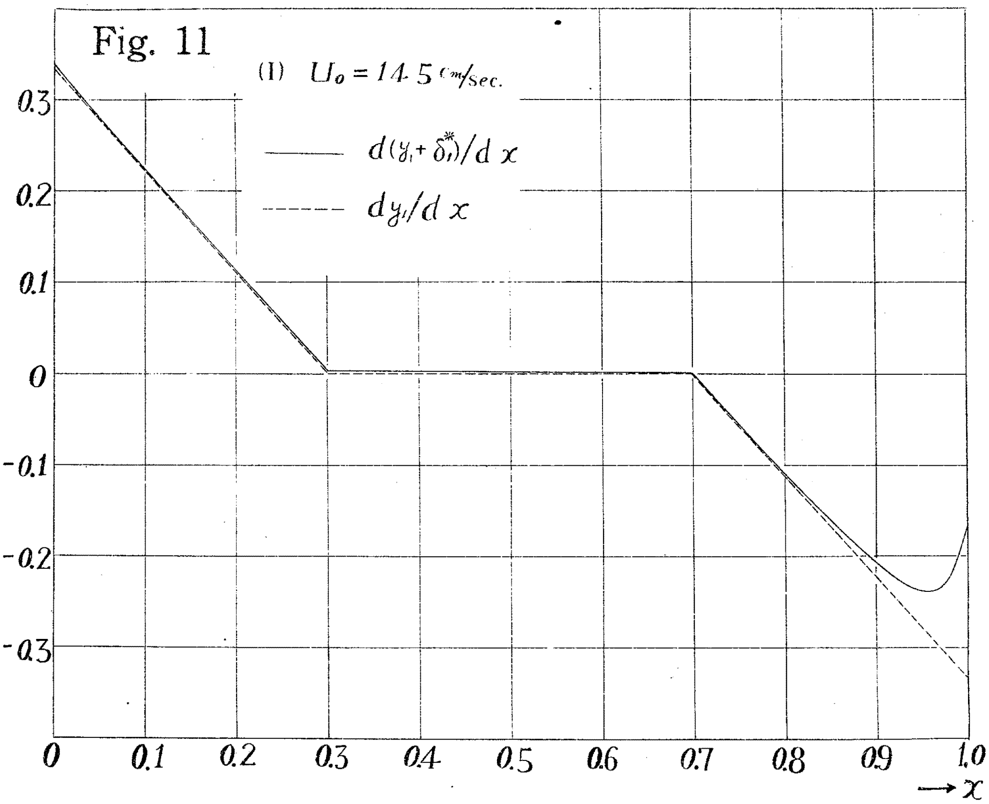


Fig. 10 (a)
dδ₁/ds
0.004
0.003
0.002
0.001
(I)
(II)
(III)
0
0.2
0.4
0.6
0.8
s
(b)
dδ₁/ds
0.01
0.008
0.006
0.004
0.002
(I)
(II)
(III)
0.8
0.85
0.9
s

Fig. 11



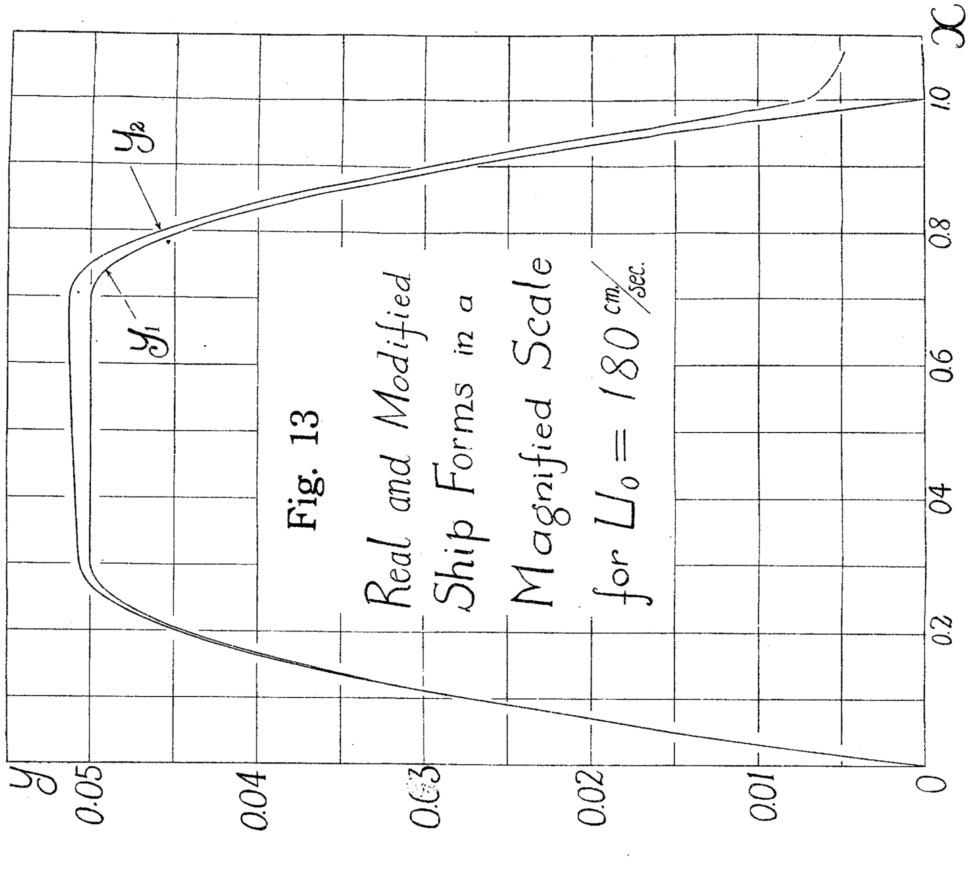
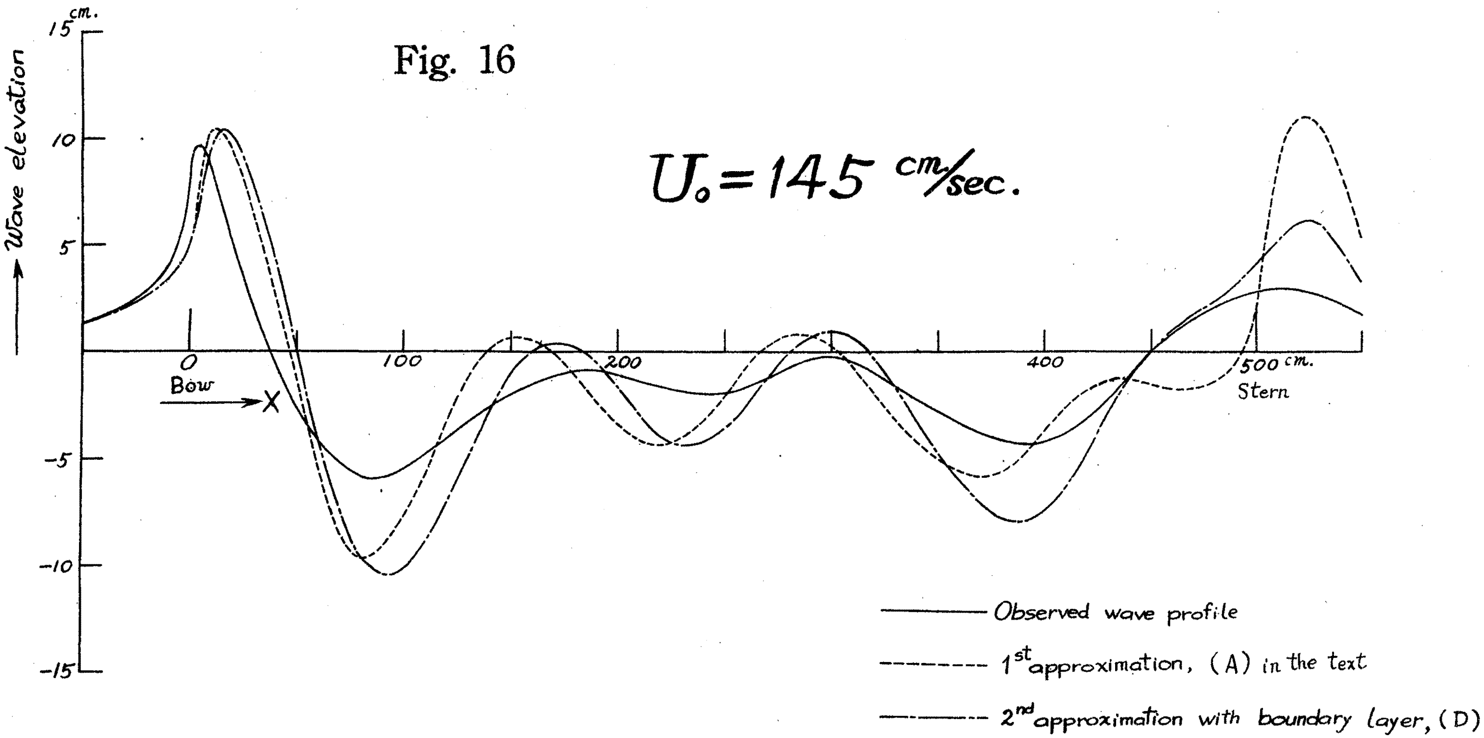


Fig. 13

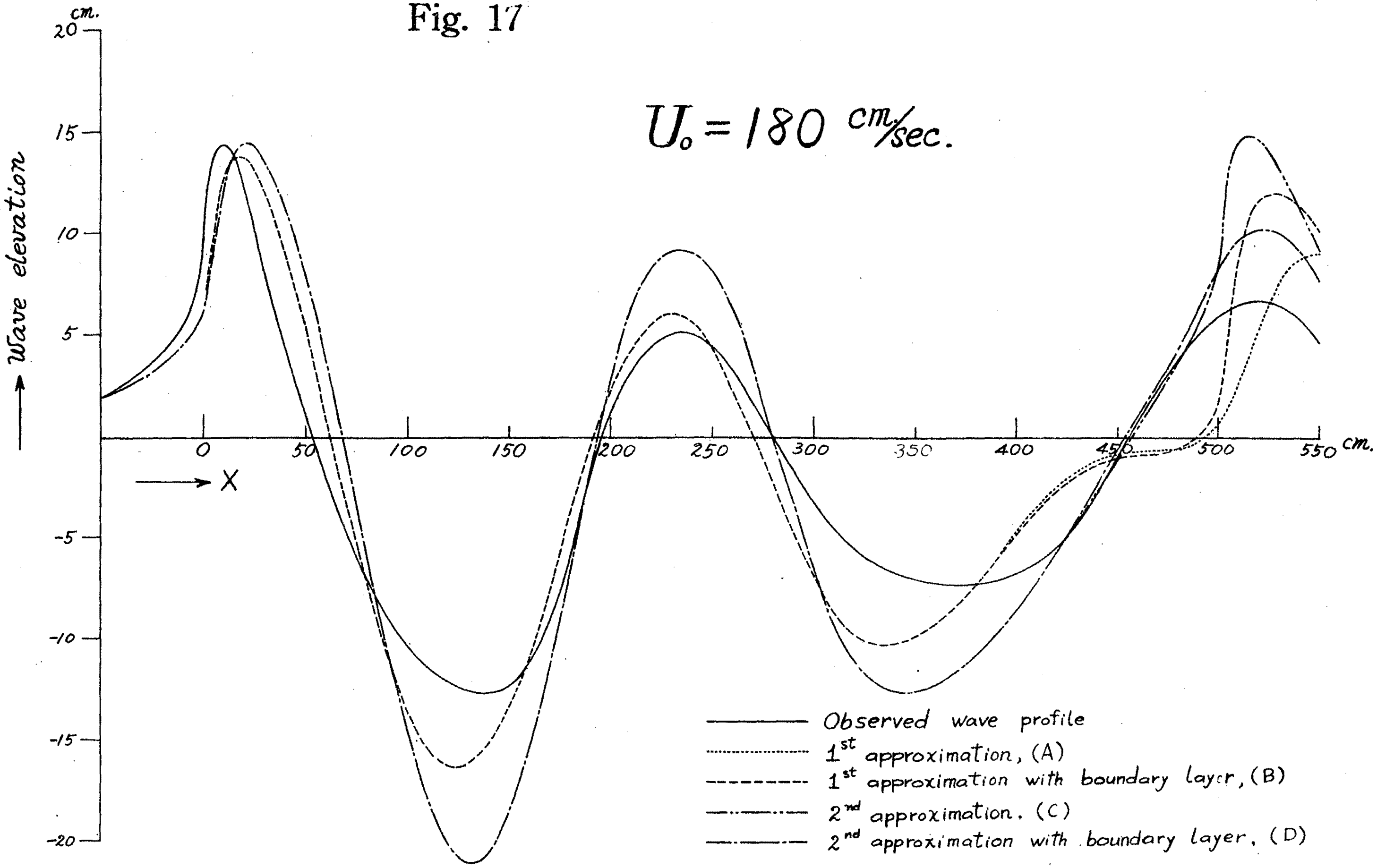
Real and Modified Ship Forms in a Magnified Scale for $U_0 = 180$ cm/sec.

Fig. 16



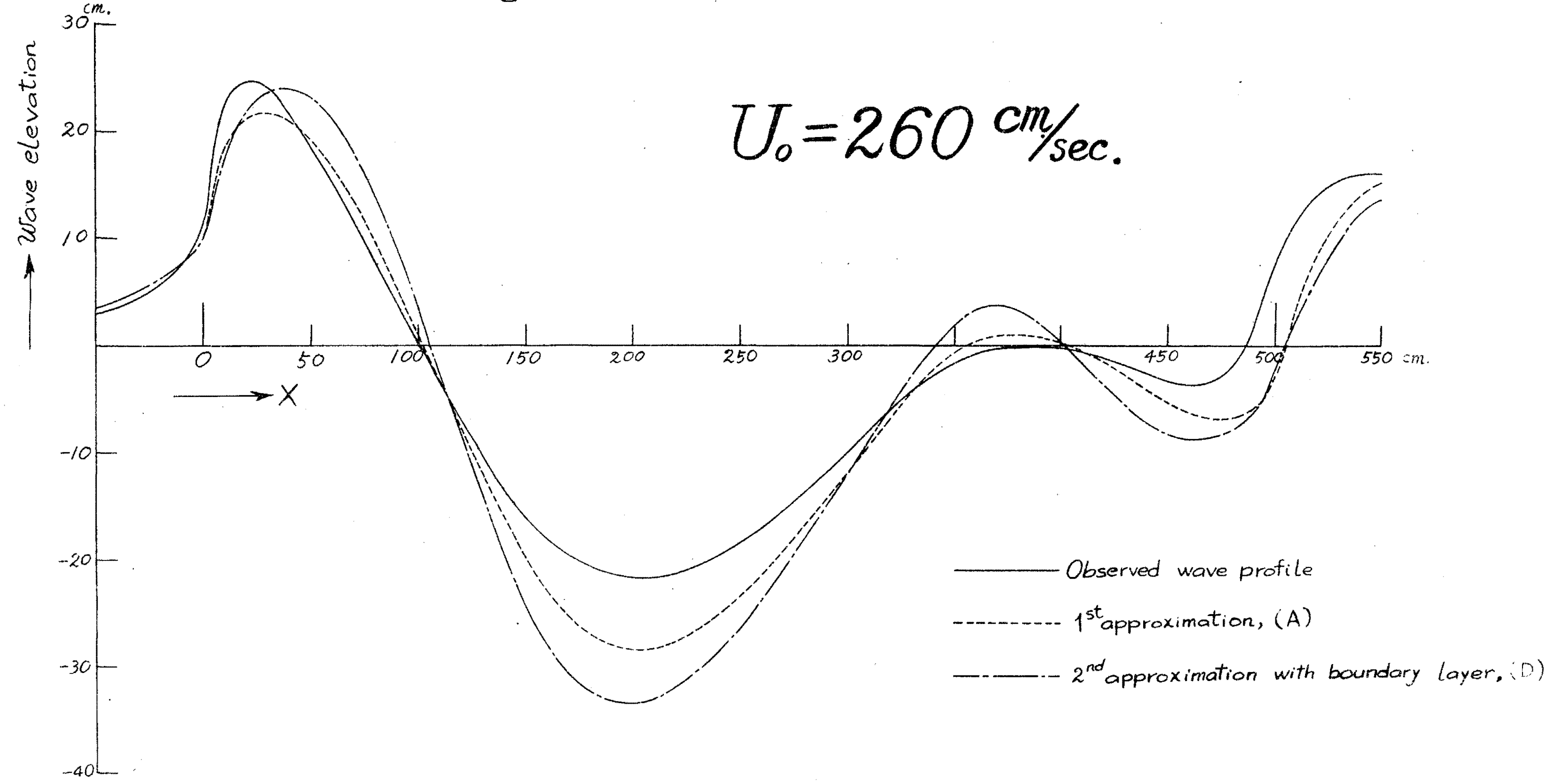
$U_0 = 145$ cm/sec.
Wave elevation
15 cm.
10
5
-5
-10
-15
0
Bow
X
100
200
400
500 cm.
Stern
Observed wave profile
1st approximation, (A) in the text
2nd approximation with boundary layer, (D)

Fig. 17



$U_0 = 180$ cm/sec.
Wave elevation
cm.
20
15
10
5
-5
-10
-15
-20
0
50
100
150
200
250
300
350
400
450
500
550 cm.
X
Observed wave profile
1st approximation, (A)
1st approximation with boundary layer, (B)
2nd approximation. (C)
2nd approximation with boundary layer. (D)

Fig. 18



$U_o = 260$ cm/sec.
Wave elevation
30 cm.
20
10
-10
-20
-30
-40
0
50
100
150
200
250
300
450
500
550 cm.
X
Observed wave profile
1st approximation, (A)
2nd approximation with boundary layer, (D)

Fig. 20

(Havelock)

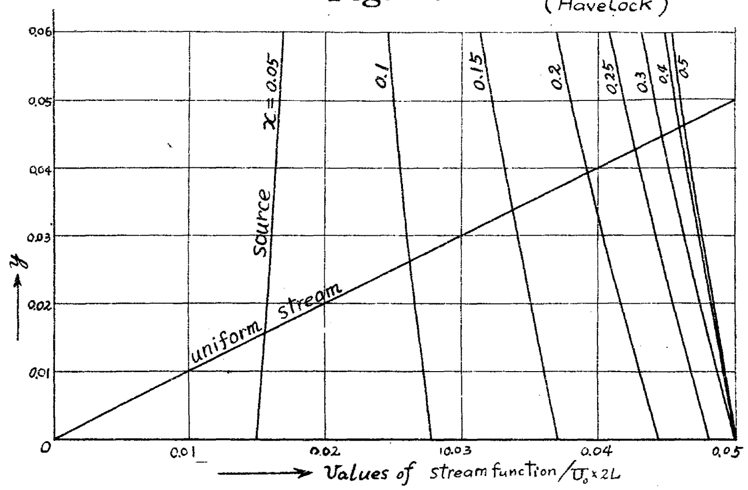


Fig. 21

(1st Approximation)

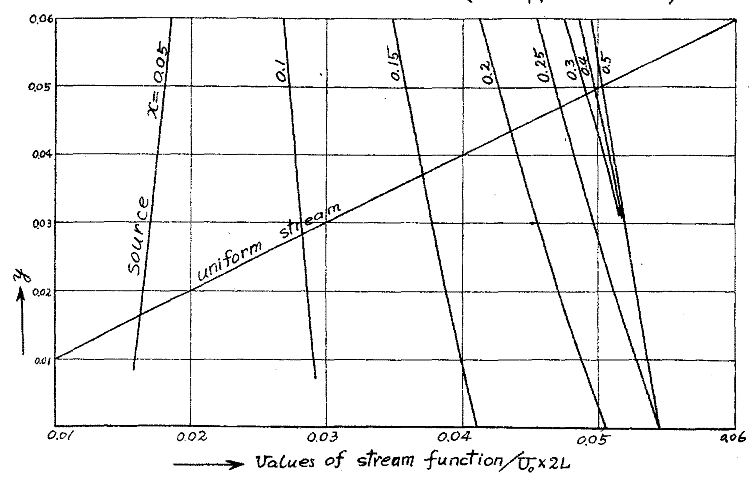


Fig. 22

(3rd Approximation)

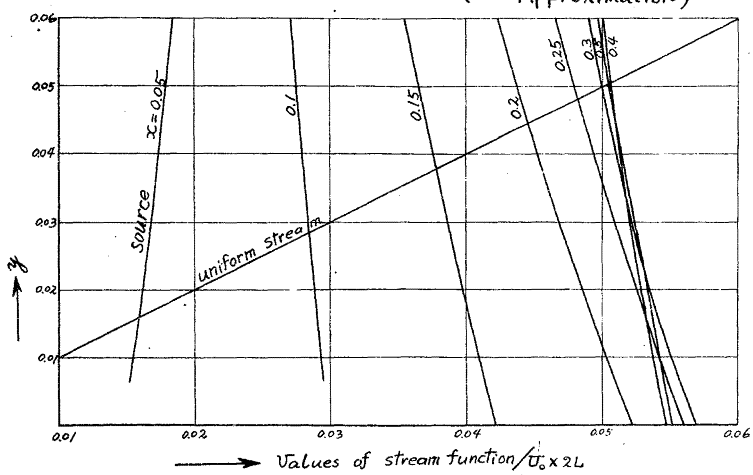


Fig. 23

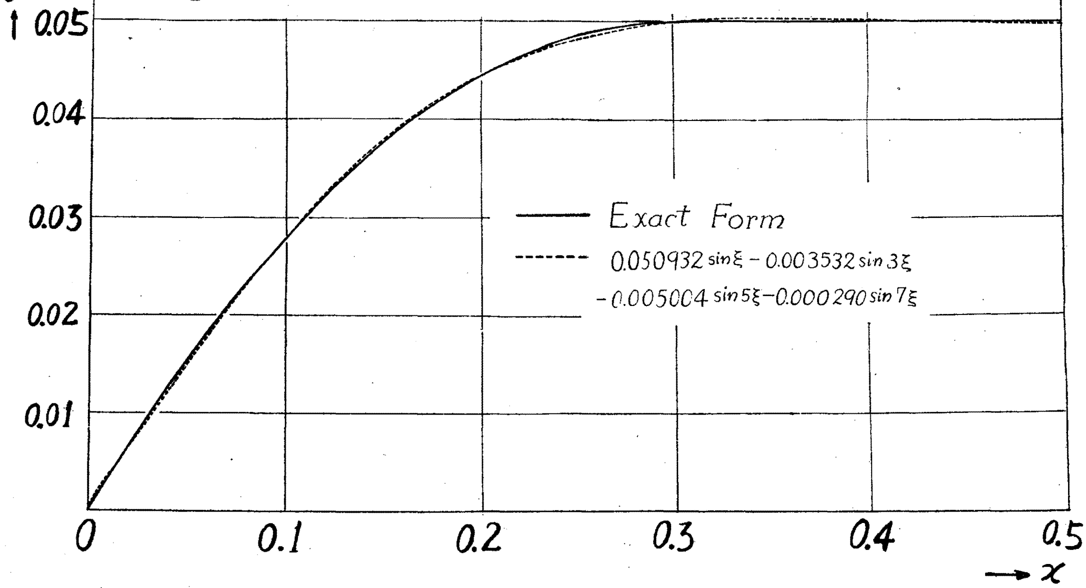


Fig. 24

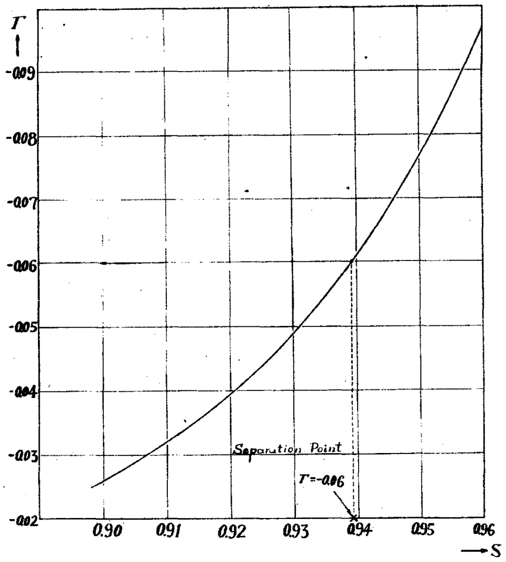


Fig. 25