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LIQUID RISE IN A SMALL TUBE

By

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CONTENTS

INTRODUCTION

When pressure is suddenly applied at the bottom of the liquid column which was initially in equilibrium in a small tube maintained vertically over the reservoir, the column will ascend immediately to its final height where new state of equilibrium will be established. This process is the essential feature of a liquid-manometer which is put to use to measure the pressure in water; the time taken for the liquid column to reach its final position is called usually "the time-lag of the manometer."

In the paper which follows investigations will be made in two parts: in Part I the problem will be reduced to a very simplified one and the theoretical computations will be compared with the experiment. In Part II another set of experiments will be described in which time-lags were measured for each of the six glasswares which can be regarded as the model-manometers of different dimensions. These results will be discussed with a view to find the main factors determining the motion of the liquid column and to give a standard for constructing a manometer of the least time-lag as possible.

At the beginning of these studies the authors' plan was to develop the theory of Part I so as to cover the phenomena experienced in the tests of Part II. But to do so, we realized in the course of our study, we had to make much closer examinations of the instability of flow (the break-down of laminar motion) in every part of the arrangements of various pipes, which, however, has been found impossible by means of our equipments available at present. And so Part I will appear somewhat independent of the subsequent descriptions.

PART I

§ 1—STATEMENT OF THE PROBLEM

A small tube, A (radius a), is mounted vertically over the larger one, B (radius a', length l'), in the equipment shown schematically in Figure 1.

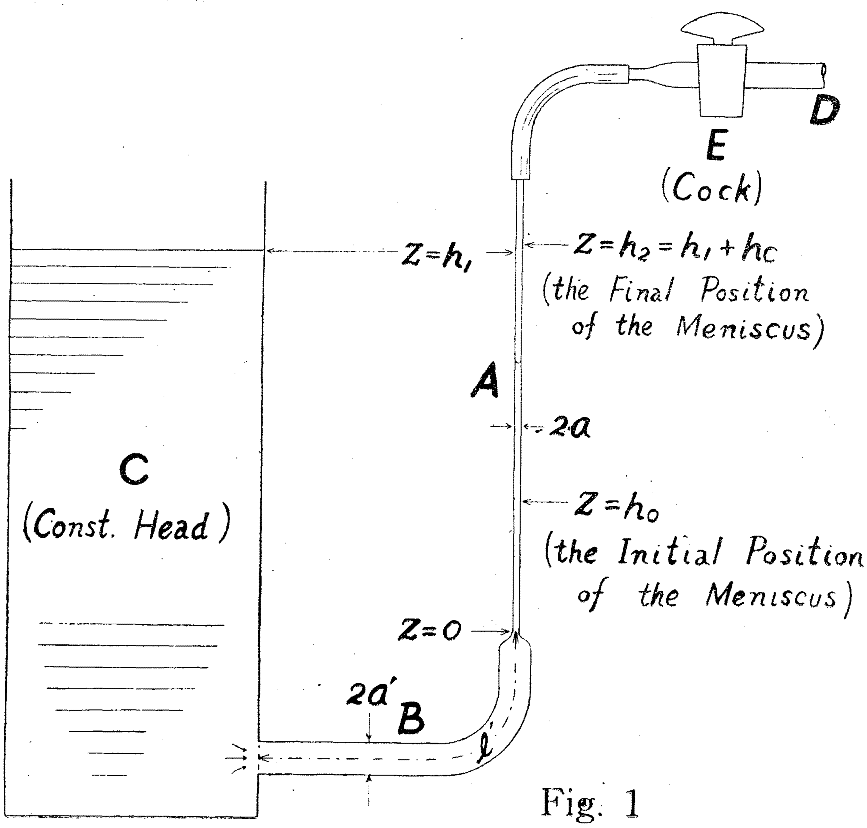


Fig. 1

The former is the manometer in which the motion of the meniscus can be measured, while the latter serves as a conduit which supplies the manometer with the liquid from the constant head, C. The origin of the z-axis is taken at the bottom of A and z is measured vertically upwards. If we denote by h the height of the meniscus at time t above $z=0$, h is a function of t only.

Initially the meniscus of the water has been kept low at $z=h_0$ being pressed down by breath blown through D. E, a cock, was closed when the desired low level had been obtained. At time $t=0$ the cock is suddenly plucked out, thus the pressure over the meniscus being released it begins to ascend instantly. Our problem is to follow the motion of the meniscus.

§2. EXPERIMENT

EQUIPMENTS

(i) Glasswares: After two glass-tubes A and B (*see* Figure 1 of the preceding section) were welded together at $z=0$ the inner radius of the manometer-tube A was measured at intervals of about 5 cm. along its length by filling mercury. The result is shown in Table I below.

Table I

z_i	6.5	10	15	20	25	30	35	40
a_i	0.1252	0.1253	0.1249	0.1246	0.1240	0.1243	0.1246	0.1244
z_i	45	50	55	60	65	70	75	*mean*
a_i	0.1246	0.1253	0.1251	0.1251	0.1249	0.1244	0.1231	0.12465

z_i : the ordinate of the middle point of the mercury filled in the tube (measured from $z=0$ in cm.).

a_i : the radius in cm., being the average value of it in the neighbourhood of $z=z_i$.

The mean radius throughout the tube A, which will be used in the sebsequent numerical calculation, was obtained from the following formula:

$$\frac{1}{(a\ mean)^4}=\frac{1}{N}\sum_{i=1}^{N}\frac{1}{a_i^4}, \tag{2.1}$$

where N is the total number of the samples ($N=15$). In the experiment the tube A was fixed vertically upright adjusted with a suspended weight.

As will be seen from the theoretical discussions later, the radius of the pipe is a very important factor that determines the nature of the motion of the liquid column, therefore much care has been taken in its measurement. In fact, the parameter with which the frictional resistance on the wall can be expressed varies inversely as the fourth power of the inner radius, cf. (3.16). This is the reason why the formula (2.1) was adopted.

(ii) Water Tank: To obtain the desired head-difference a large wooden vessel of rectangular section was used. As its sectional area was about 400 cm^2., i.e. roughly 10^4 times of that of the manometer-tube, the level of the water in the tank could be regarded as practically constant during the motion. This tank has a tubular outlet of brass near the bottom, which is illustrated in Figure 2. The actual views of these equipments are shown in Plates I, II, and III at the end of this paper.

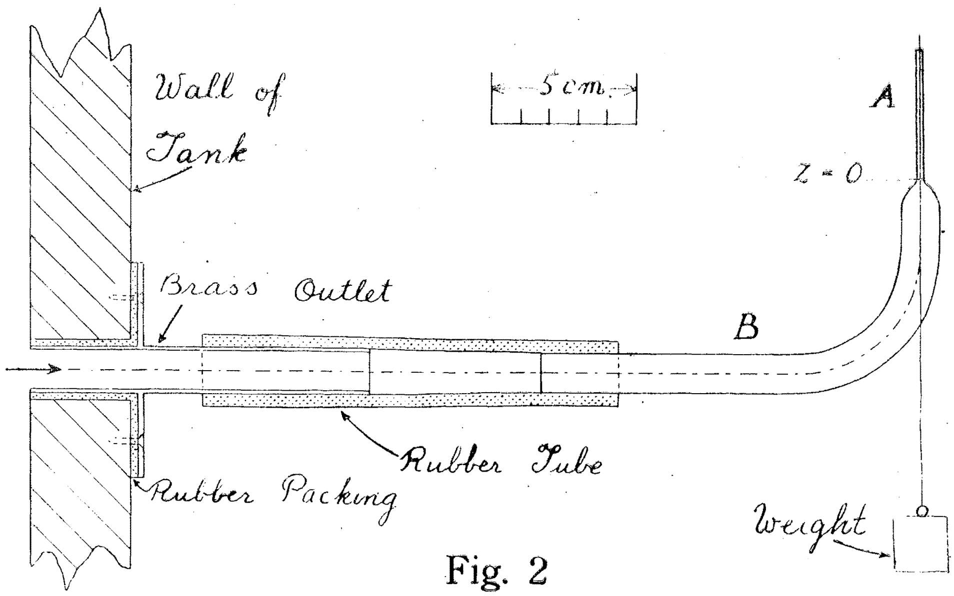


Fig. 2

(iii) Liquid: We used water in the experiments of Part I and of Part II and care was taken to remove small air bubbles in the water, because they frequently adhere on the wall of a small pipe thus possibly make the condition of flow completely different.

METHOD OF MEASUREMENTS

In order to trace the motion of the meniscus we took a 16 mm. motion-picture of the ascending liquid column. The essential points of our method are as follows:

(i) Illumination: The manometer-tube was illuminated from one side by an intense sheet of light through a vertical narrow slit (about 2 mm. wide) and the camera was fixed at the middle height of the initial and the final positions of the meniscus (*see* Figure 3).[1] Then, because the lower part of the glass-tube which was filled with water became less reflective than the upper part filled with air, we could distinguish the instantaneous positions of the meniscus by the contrast of brightness on the negative film. Three electric lamps of 250 watts were used as the light source; this local illumination brought about necessarily the rise of the temperature and hence the decrease of the kinematic viscosity (ν) of the water where this is exposed to the light. In order to know the value of ν as correctly as possible which is rather sensitive to slight changes of temperature, we fixed another test-tube full of water in the light and measured the temperature in it (19.0°C.),

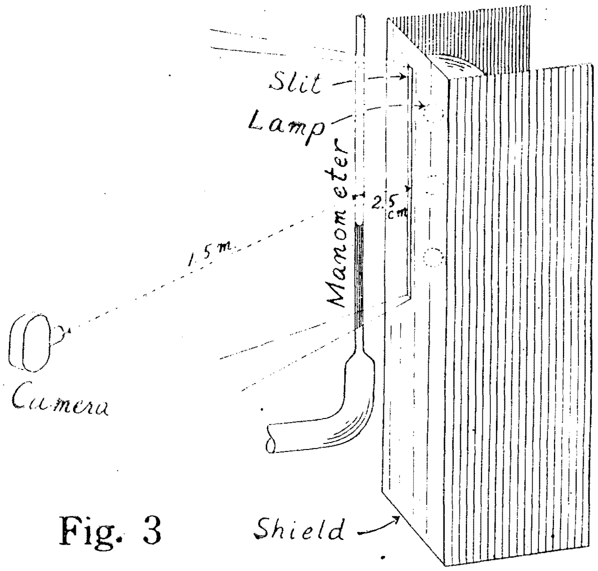


Fig. 3

[1] The horizontal distance between the camera and the manometer was about 1.5 m.

assuming that the temperature would rise by equal degrees in the test-tube as in the manometer, because in the latter any direct measurement could be hardly possible.

(ii) Determination of h: We magnified the negative film by a microscopic comparator and referring to a scale attached preliminarily just behind the manometer we could determine the successive values of h. However, the facts is that owing to the finite time of exposure of the picture (estimated to be about 0.016 sec.) the image of the meniscus was diffused with more or less breadth, especially where the velocity of ascent was the maximum, but by means of plausible interpolation the final result was obtained with accuracy of about ± 1 mm.

(iii) Time-Record: To complete the measurement it is necessary on the other hand to determine the scale of the time-axis. For this purpose we filmed again a rotating disc driven with a constant speed by an A. C. (58.5 cycles) synchronous motor. A strip of paper on this disc made a sectoral image on the film and the angle between the bisects of these sector-angles of the two consecutive pictures gave the corresponding time interval, Δt, while the time of exposure, $\Delta t'$, could be estimated from the angle of a sector (*see* Plate IV). Thus we obtained

$\Delta t = 0.068 \pm 0.001$ sec.,
and $\Delta t' = 0.016$ sec. (average value of the 16 sets of the samples)

Figure 4 shows schematically the mode of advance of the film in the camera. The results of two independent experiments are embodied by two sets of points in Figure 9 at the end of Part I.

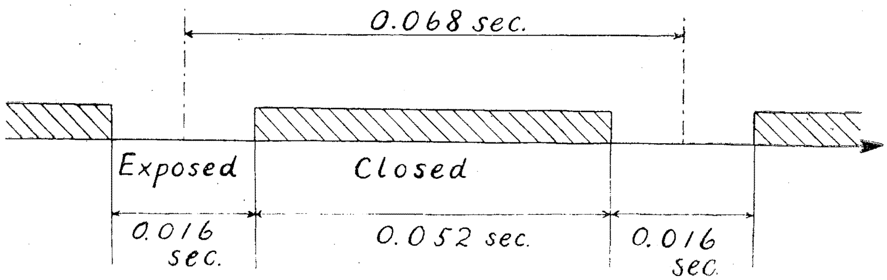


Fig. 4

§3—THEORY

Denoting by w the velocity of the liquid along the z-axis, by p the pressure, ρ the density and ν the kinematic viscosity, the equation of the laminar motion through A can be expressed in terms of the cylindrical co-ordinates (r, z, ϕ) as follows:

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right), \tag{3.1}$$

where g is a constant of gravity (980 cm./sec.^{2}) and an assumption has been made that w is a function of (r, t) and independent of (z, ϕ).

Let us consider next that p may be safely assumed to be independent of r and average the both hand sides of this equation over the cross-section of the tube, then we readily obtain

$$\frac{dw_0}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g - 2\frac{\tau_0}{\rho a}, \tag{3.2}$$

where
$$w_0(t) = \frac{1}{\pi a^2}\int_0^a w(r, t) 2\pi r dr,$$

and
$$\tau_0(t) = -\mu\left(\frac{\partial w}{\partial r}\right)_{r=a},$$

τ_0 being the traction on the wall and $\mu \equiv \rho\nu$. Integrating the both hand sides of (3.2) with respect to z from zero to h (the meniscus at time t), we get

$$\left(\frac{dw_0}{dt} + g + \frac{2\tau_0}{\rho a}\right)h = \frac{1}{\rho}(p_{z=0} - p_{z=h}), \tag{3.3}$$

where $p_{z=0}$ and $p_{z=h}$ are respectively the pressure at the bottom of the tube and that just below the meniscus (cf. Figure 1). If we take as the standard the atmospheric pressure equal to zero, then

$$p_{z=h} = -\rho g h_c; \tag{3.4}$$

h_c denotes the rise of the liquid due to its capillarity, viz.

$$h_c = \frac{2\sigma \cos \alpha}{\rho g a},$$

where σ is the surface tension of the liquid and α is the angle of contact between the surface of the liquid and the wall of the tube. It must be noticed in these expressions above that σ, α and accordingly h_c have been assumed constant in the course of motion of the meniscus. This assumption has been verified partly, of course not proved with certainty, in the experiment, for the meniscus was observed quite steady in motion.

Our next step will be to estimate the value of $p_{z=0}$. If we put the constant head explained in § 1 (*see* Figure 1) equal to h_1, $p_{z=0}$ can be conveniently written as follows:

$$p_{z=0} = \rho g h_1 - L_f - L_v - L_a, \tag{3.5}$$

where L_f, L_v, and L_a stand for the loss of pressure caused in the conduit by friction, velocity and acceleration respectively. Next we are going to discuss these factors separately.

(i) L_f: This is the loss due to friction on the wall of the conduit B. As is easily supposed from the fact that the velocity of the liquid is very small in B (in our experiment the average velocity was estimated less than 3 cm. sec.$^{-1}$ at the maximum), this part of loss cannot be so great and we can evaluate it without making any serious error in the final result of the theory when we assume the *Poiseuille's parabolic law* for the flow pattern, neglecting the effect of the deviation of the velocity profile caused on the one hand by the unsteady character of the flow and on the other by the windings of the path. Then L_f is given by the well-known formula[1]

$$L_f = \frac{8\mu Q l'}{\pi a'^4},$$

where Q is the flux across the section of B while l' and a' are its length and radius respectively, which, if we consider the continuity of the flux, can be readily expressed in terms of a and w_0, viz.

$$Q = \pi a^2 w_0,$$

therefore

$$L_f = \frac{8\mu a^2 l'}{a'^4} w_0.$$

Or for convenience' sake we can put it into the form

$$\left.\begin{aligned} L_f &= \rho k' l' w_0, \\ \text{where} \quad k' &= \frac{8\nu a^2}{a'^4}. \end{aligned}\right\} \tag{3.6}$$

(ii) L_v: The liquid, in its passage through the conduit, suddenly acquires the sensible velocity w_0 when it enters A. A smaller part of

(1) Goldstein, Modern Developments in Fluid Dynamics, vol. I, p. 20 (1938).

pressure, L_v, must be consumed in order to bring about this increase in velocity head. Ignoring for simplicity, however, other factors for the time being Bernoulli's theorem (which is perfectly correct for the steady flow of an inviscid fluid) suggests us it may be fairly reasonable to assume that this correction can be approximated by the quantity $\rho w_0^2/2$, viz.

$$L_v = \frac{1}{2}\rho w_0^2 . \tag{3.7}$$

Before going further we have to notice that another loss of pressure can be caused by the eddies generated at this point of discontinuity of cross-section (the so-called loss of contraction). But in order to minimize it we welded beforehand two pieces of glass-tubes of different diameters (A and B) into one piece of fairly smooth contraction (*see* Plate II), and by this means, we hope, we could reduce it sufficiently to make it legitimate for us to proceed without taking account of it in the following discussions.[1]

(iii) L_a: At the beginning of the motion the liquid mass contained in B is accelerated from rest. Some work must be done on the liquid, thus again some loss, L_a, (or gain when retarded) takes place. In other words, a smaller portion of the mass of the liquid in the reservoir B acts as an additional mass to that ascending in A whose motion we are now computing. When we denote the average velocity of the liquid over the cross-section of B by w_0' and assume it dependent of t only, then the force to accelerate the liquid in B amounts to $(mass)\times(acceleration)=\rho\pi a'^2 l'(dw_0/dt)$, so that per unit area it is $\rho l' dw_0/dt$ which we may regard as an approximate expression for L_a. Making use of continuity of the flux, this can be written

$$L_a = \rho l' \frac{a^2}{a'^2}\frac{dw_0}{dt} . \tag{3.8}$$

Finally when these three factors L_f (3.6), L_v (3.7), and L_a (3.8) are together substituted in (3.5), $p_{z=0}$ is readily found to be embodied in a somewhat lengthy form

[1] According to a text-book of hydraulics, e.g. "Heat Transmission" by McAdams, p. 121 (1933), to which reference was made by Brittin (cf. §5), the loss of contraction is generally expressed by some fraction of $\rho w_0^2/2$ and seems to be at most about $\rho w_0^2/4$ for an extreme case of the sudden remarkable contraction of turbulent nature. In our experiment this loss must be much smaller because of the smooth change of cross-section and the laminar character of flow, so that our final results will not be affected to a great extent by the neglection of this correction.

$$p_{z=0} = \rho g h_1 - \rho k' l' w_0 - \frac{\rho w_0^2}{2} - \rho l' \frac{a^2}{a'^2} \frac{dw_0}{dt}. \tag{3.9}$$

Combining (3.4) and (3.9), (3.3) can be written

$$\left(\frac{dw_0}{dt} + g + \frac{2\tau_0}{\rho a}\right) h = g h_2 - k' l' w_0 - \frac{w_0^2}{2} - l' \frac{a^2}{a'^2} \frac{dw_0}{dt}, \tag{3.10}$$

where h is related to w_0 by the equation

$$\frac{dh}{dt} = w_0,$$

and $h_1 + h_c$ has been written h_2 for brevity. However, when t tends to infinity, $w_0, dw_0/dt$ and τ_0 tend to zero and it follows from the above equation that

$$h_{t\to\infty} = h_2,$$

therefore h_2 can be easily measured in the experiment.

But in order to solve the equation (3.10) it is necessary here to assign a definite value for τ_0 which is left untouched so far; if we assume that the velocity profile in the tube A is always parabolic, then τ_0 is found to be equal to $4\mu w_0/a$, therefore

$$\left.\begin{aligned} 2\frac{\tau_0}{\rho a} &= k_p \frac{dh}{dt}, \\ k_p &\equiv \frac{8\nu}{a^2}. \end{aligned}\right\} \tag{3.11}$$

where [1]

And if we define more generally

$$k = 2\tau_0 / \left\{\rho a \left(\frac{dh}{dt}\right)\right\} \tag{3.12}$$

for any of the velocity profile, then (3.10) can be written

$$\left(h + l' \frac{a^2}{a'^2}\right) \frac{d^2 h}{dt^2} + \frac{1}{2}\left(\frac{dh}{dt}\right)^2 + (kh + k' l') \frac{dh}{dt} + g(h - h_2) = 0, \tag{3.13}$$

which is a differential equation of the second order and represents the non-linear oscillation of the liquid column.

Equation (3.13), however, can be transformed into more accessible form by the following procedure; viz. if we put

---

(1) The suffix p implies the Poiseuille's parabolic flow-pattern.

$$\left(\frac{dh}{dt}\right)^2 = \theta$$

and change the independent variable from t to h, we obtain

$$\frac{d\theta}{dh} = \frac{2g(h_2-h) \mp 2(kh+k'l')\sqrt{\theta} - \theta}{h+l'(a^2/a'^2)}, \tag{3.14}$$

(– ; when ascending ; + ; when descending).
This is a differential equation of the first order and can be solved by means of a method of numerical integration far more easily than its original form (3.13). The time t which is elapsed since the beginning of the motion of the meniscus till it attains to the height $z = h$ can be found from the following formula :

$$t = \int_{h_0}^{h} \frac{dh}{\sqrt{\theta}}. \tag{3.15}$$

The initial condition to which (3.14) is subject is that $\theta = 0$ at $h = h_0$. The values of the constants included in the above equation are

$h_0 = 36.72$ cm., $h_2 = 76.72$ cm.,
$a = 0.12465$ cm., $a' = 0.669$ cm. (mean value),
$l' = 36.6$ cm., $k' = 0.006404$ sec.$^{-1}$,
$\nu = 1.032 \times 10^{-2}$ cm.^{2}/sec., $k_p = 5.314$ sec.$^{-1}$.

When the velocity profile is parabolic k becomes naturally equal to k_p and is constant. But as is easily supposed, at the beginning of its motion the velocity profile of the liquid is much flatter than parabolic (Figure 5) and the value of k defined above by (3.12) is possibly greater than k_p, and on the contrary when the flow is retarded k will become smaller than k_p (cf. Figure 11). Therefore in such a non-steady flow as ours, k must be a function of $w_0(= dh/dt)$ and accordingly of h. In order to know whether this is the case or not and how far its effect is we make use of the theory proposed recently by Professor Yamada of our Research Institute. It is a linearized theory on the ascent of liquid assuming that the deviation of the length of the liquid column from its mean value is so small that the higher order of this quantity can be neglected in the course of its motion (and in the case of a U-tube in which the length of the liquid column is kept con-

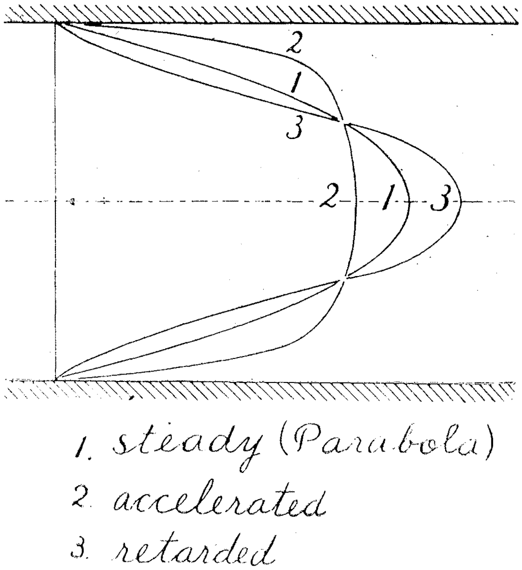


Fig. 5

stant it gives an exact solution of the problem). The details of this theory has been described by himself in the present volume under the title "Note on the Liquid Motion in U-Tube"; the following is the résumé of it.

LINEARIZED THEORY

Under the above assumption the rise of the liquid column can be determined in terms of the two non-dimensional parameters

$$\lambda = \frac{L\nu^2}{ga^4} \quad \text{and} \quad \tau = \frac{\nu t}{a^2}, \text{(1)} \tag{3.16}$$

where ν is the kinematic viscosity of the liquid, t the time, a the radius of the tube and L the length characteristic to the problem, e.g. the mean length of the liquid column. If $\lambda < 0.0847 \cdots$ the motion is of oscillatory nature, while when $\lambda \geqq 0.0847 \cdots$ it is not, but aperiodic. In our problem the change of the length of the liquid column is not so small as to permit neglecting the higher order and strictly speaking it is beyond the limit of the linearized theory. But in order to adapt the theory somehow in an

(1) τ is a non-dimensional time and τ_0 stands for the surface traction; they are utterly different.

admissible way for the present case, we take as the most reasonable value of L the length $(h_2+h_0)/2$. Then λ is found to be $0.02553\cdots$, and the motion is of oscillatory type. $h-h_0$, dh/dt and τ_0 are given from the theory by the following formulae:

$$\left.\begin{aligned} \frac{h-h_0}{\varepsilon} &= 1+\frac{1}{3}\sum_{n=1}^{\infty}\frac{\lambda j_n^4}{\Lambda_n}e^{-j_n^2\tau}\,, \\ \frac{dh}{dt}\bigg/\frac{\varepsilon\nu}{a^2} &= -\frac{1}{3}\sum_{n=1}^{\infty}\frac{\lambda j_n^6}{\Lambda_n}e^{-j_n^2\tau}\,, \\ \tau_0\bigg/\left\{\rho a\varepsilon\left(\frac{\nu}{a^2}\right)^2\right\} &= -\frac{1}{6}\sum_{n=1}^{\infty}\frac{(1+\lambda j_n^4)j_n^4}{\Lambda_n}e^{-j_n^2\tau}\,, \\ \text{where}\qquad \Lambda_n &= 1+(1+\lambda j_n^4)\left\{\frac{j_n^2}{12}(1+\lambda j_n^4)-1\right\}; \end{aligned}\right\} \tag{3.17}$$

ε is the impressed head (assumed constant for simplicity and put equal to h_2-h_0), and j's are the roots of the equation

$$J_2(j)-\lambda j^4 J_0(j) = 0\,, \tag{3.18}$$

where J_0 and J_2 denote the Bessel functions. The successive values of the roots corresponding to $\lambda = 0.02553\cdots$ are given in the table; when $n\geqq 6$ they are identical to the zeros of J_0 with sufficient accuracy.

Table II

j_1	$2.041+i\,1.0821$	j_4	8.6521
j_2	$2.041-i\,1.0821$	j_5	11.7912
j_3	5.5051	j_6	14.9309

The roots of the equation (3.18).

The values of t, $h-h_0$, dh/dt, τ_0 and k are tabulated in Table III for various values of τ and are reproduced in Figure 6.

The abnormal value of k with an asterisk corresponds to the maximum or minimum of h: this was caused evidently by dividing a finite value of τ_0 by a very small value of dh/dt. It may appear quite absurd at first sight that in all the sequence of k only these two figures are negative while all others are positive. But it is not impossible, because near the maximum position of its ascent the liquid may have a wavy velocity profile; it may have an upward motion on the whole ($w_0>0$) while τ_0 is negative and likewise possibly $w_0<0$ with $\tau_0>0$, cf. Figure 7. We have not closely

Table III

τ	$h-h_0$ (cm.)	$\frac{dh}{dt}$ (cm./sec.)	$\tau_0 \Big/ \left\{ \rho a \varepsilon \left(\frac{\nu}{a^2} \right)^2 \right\}$	k(sec.$^{-1}$)[(1)]	t(sec.)
0	0	0	0	∞	0
0.01	0.08	8.898	4.217	16.727	0.02
0.02	0.27	16.561	5.825	12.414	0.03
0.03	0.59	23.28	6.968	10.562	0.05
0.04	0.97	29.69	7.911	9.404	0.06
0.05	1.54	35.34	8.655	8.642	0.08
0.075	3.03	47.42	9.771	7.272	0.11
0.10	5.00	56.81	10.981	6.822	0.15
0.15	9.78	68.84	11.773	6.036	0.23
0.20	15.18	73.44	11.590	5.570	0.30
0.25	20.70	72.35	10.733	5.236	0.38
0 30	24.34	67.13	9.440	4.963	0.45
0.40	34.83	49.60	6.281	4.469	0.60
0.50	40.79	29.60	3.231	3.853	0.75
0.60	43.89	12.411	0.8788	2.499	0.90
0.70	44.78	0.3720	− 0.5970	−56.65 *	1.05
0.80	44.27	− 6.320	− 1.2829	7.164	1.20
0.90	43.10	− 8.673	− 1.3913	5.662	1.36
1.00	41.80	− 8.153	− 1.1600	5.022	1.51
1.20	39.97	− 3.777	− 0.4215	3.938	1.81
1.40	39.43	− 0.17263	0.05624	−11.498*	2.11
1.60	39.61	1.0198	0.16645	5.761	2.41
1.80	39.90	0.7631	0.09908	4.582	2.71
2.00	40.05	0.2243	0.018845	2.965	3.01
2.50	40.01	− ·0.09431	− 0.012393	4.638	3.76
3.00	39.99	0.01389	0.0018345	4.660	4.52
4.00	40.00	− 0.0004324	− 0.00008806	7.187	6.02

examined whether this is the case or not in our present problem, but for convenience' sake this uncertainty has been removed from the subsequent com-

(1) k can be calculated from (3.12).

putations. For in order to be able to make use of these data in the equation (3.14), we must re-tabulate k as a function of h. However, because the oscillatory motion represented by (3.14) and that approximated by the linearized theory are not necessarily of the same amplitude and phase, the existence of these abnormal values of k is very embarassing attended with danger of misleading the computation altogether. And on the other hand after the motion has been retarded, the variation of k will be scarcely important when multiplied by w_0 into the form of $\tau_0 = \rho a k w_0/2$ (the surface traction), so we take the following expedient: at the start of the motion while k is greater than k_p we adopt the value of k calculated from the linearized theory, but once after k becomes equal to k_p we use this constant value invariably during the rest of the motion.

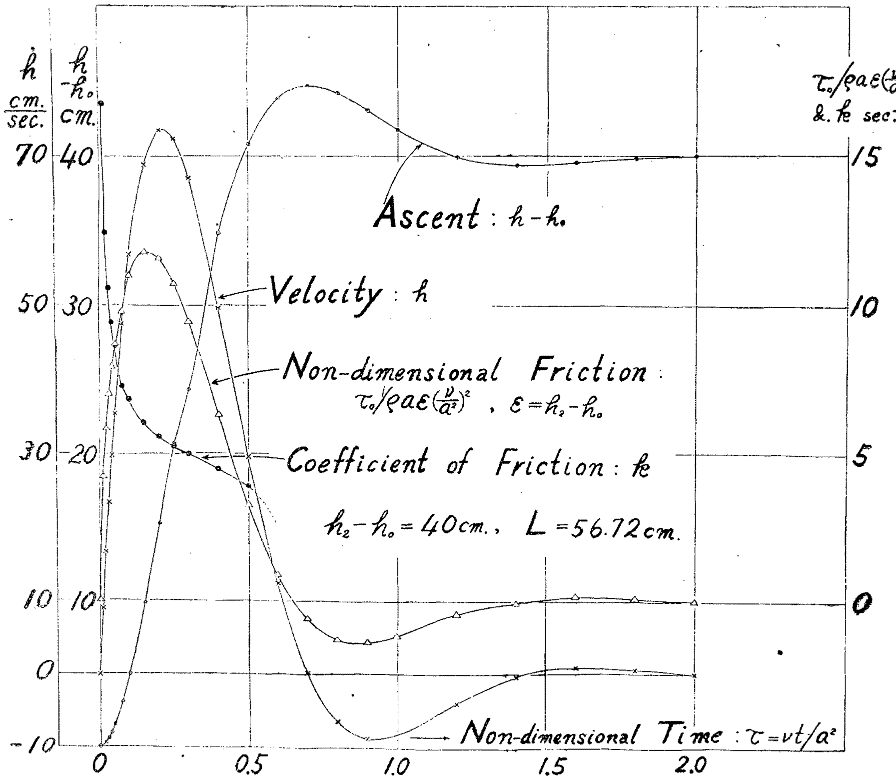


Fig. 6

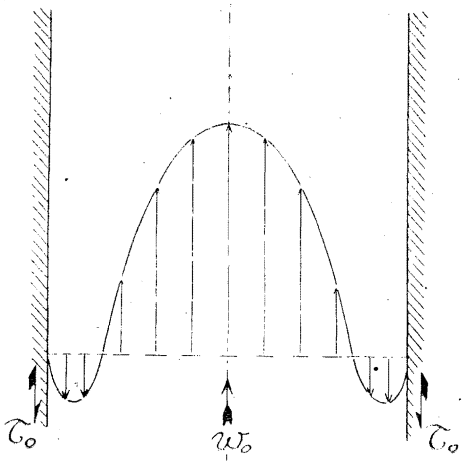


Fig. 7

§ 4—DISCUSSIONS

(I) COMPARISON WITH EXPERIMENT

The results of the preceding calculations are embodied in Figures 8 and 9 together with Table IV. In Figure 9 two sets of points

Table IV

$h-h_0$ (cm.)	θ	$dh/dt=\sqrt{\theta}$ (cm./sec.)	$h-h_0$ (cm.)	θ	$\frac{dh}{dt}=\sqrt{\theta}$ (cm./sec.)
0	0	0	29	3220.33	56.75
0.5	796.51	28.22	33	1975.08	44.44
1.0	1452.92	38.12	35	1399.18	37.41
2.0	2560.10	50.60	37	878.62	29.64
3.0	3488.19	59.06	39	434.97	20.86
4.0	4257.28	65.25	41	101.94	10.10
5.0	4880.10	69.86	41.8	14.41	3.80
7.0	5752.63	75.85	41.99	0	0
9.0	6287.62	79.29	41.69	7.52	− 2.74
11	6589.51	81.18	41.39	9.59	− 3.10
13	6679.91	81.73	41.09	9.12	− 3.02
17	6354.07	79.71	40.79	7.13	− 2.67
21	5570.92	74.64	40.49	4.37	− 2.09
25	4472.25	66.87	40.19	1.58	− 1.26

The results of the numerical integration of the equation (3.14) (*see* Figure 8).

obtained from two independent experiments A and B are reproduced for ready comparison with the theory. In plotting the experimental points, however, it should be noticed that from the nature of the cinematographic method the origin of the time-axis, $t=0$ (the instant of the start of the ascent), cannot be determined exactly without any uncertainty. In other words, we can only assert that the beginning of the motion took place *within* 0.052 sec. (the mean time between two consecutive exposures, cf. Figure 4) prior to the first frame of picture where the ascent of the liquid column was caught sensibly. But taking into consideration the fact that the features of both motions obtained theoretically and experimentally are quite similar, especially in its initial part where the tangents to these curves are all parallel, we adjusted the origins of the experimental curves so that these three curves may be coincident over the broad range of their earlier developments.[1] The considerable discrepancy between the theory and the experiment in the later stage of the motion will be discussed in (III) of this section.

(II) STABILITY OF FLOW

Might not there happen an instability of flow, and the friction actually taking place, wasn't it of turbulent nature? So far we have not mentioned explicitly about the stability of flow and formulated the various equations upon the assumption of laminar motion. But we must notice here that if we pay attention to the mean motion exclusively, the fundamental equation (3.14) is no less valid when the motion is turbulent. However, when turbulence sets in, the numerical evaluation of τ_0 and accordingly of k must be very different from those which were described in the preceding section. And on the contrary if we take an appropriate value for k, we can develop the same line of computation as before for the ascent of the liquid of turbulent nature.

In order to decide whether our assumption of laminar motion was legitimate or not, let us calculate the maximum Reynolds number attained

(1) The origins of time were chosen at 0.040 sec. for A and 0.036 sec. for B prior to the *first* frame. They lie, of course, within the limit above-mentioned. The fact that the two experiments can be brought into almost complete coincidence seems to prove the legitimacy of this procedure.

in the tube A; by numerical differentiation of the experimental curve of Figuer 9 it is found that w_0 was roughly 85 cm./sec. at its maximum, and accordingly

$$Re_{max} = \frac{w_0 max \times 2a}{\nu} = 2{,}053.$$

On the other hand it is well-known that the measured values of the critical Reynolds number when disturbances are introduced at the entry are round about 2,000 and further for no initial disturbances does the critical Reynolds number fall below about 2,000,[1] (or after Prandtl,[2] "··· gibt es wie die verschiedensten Experimente gezeigt haben, einen unteren Grenzwert der kritischen Reynoldsschen Zahl, der ungefähr bei $\bar{u}r/\nu = 1{,}000$ oder etwas darüber liegt. ···" $\bar{u}$ is the mean velocity $= w_0$ in our notation, r the radius $= a$.) Thus the value of our Re_{max} lies just in the critical region and no definite conclusion can be drawn out about the stability; besides we must notice that the above quoted data relate essentially to the steady state and in our case the flow is much accelerated. But perhaps we may safely conjecture that turbulence did not take place in our experiment, or even if it did, the period over which it could be maintained was quite short and passed without producing any serious consequences upon the general features of the motion.

(III) ESTIMATION OF k

When deriving the equation (3.14) we put, cf. (3.12),

$$k = \frac{2\tau_0}{\rho a(dh/dt)},$$

and remarked that k would become equal to $k_p = 8\nu/a^2 =$ const., if we assume that the velocity profile in the tube A is always parabolic during the motion. In fact when the radius is very small and the motion is very slow this simplification yields no discrepancy with the experiment (cf. e.g. Brittin's paper cited in the supplementary note). But in our case the velocity of the ascending liquid column is not so small and the radius of the pipe which was 0.05 cm. or 0.01777 cm. in his computation is in our experiment about 0.12 cm. Therefore the estimation of the friction on the wall should be more precise.

(1) Goldstein, *op. cit.* vol, I, p. 71 (1938).
(2) Prandtl-Tietjens, Aero-und Hydromechanik, Bd. II. S. 37 (1931).

We have not examined, however, how much difference can be resulted in the final results between the simple assumption of the Poiseuille's flow pattern and the more exact computation based upon the line-

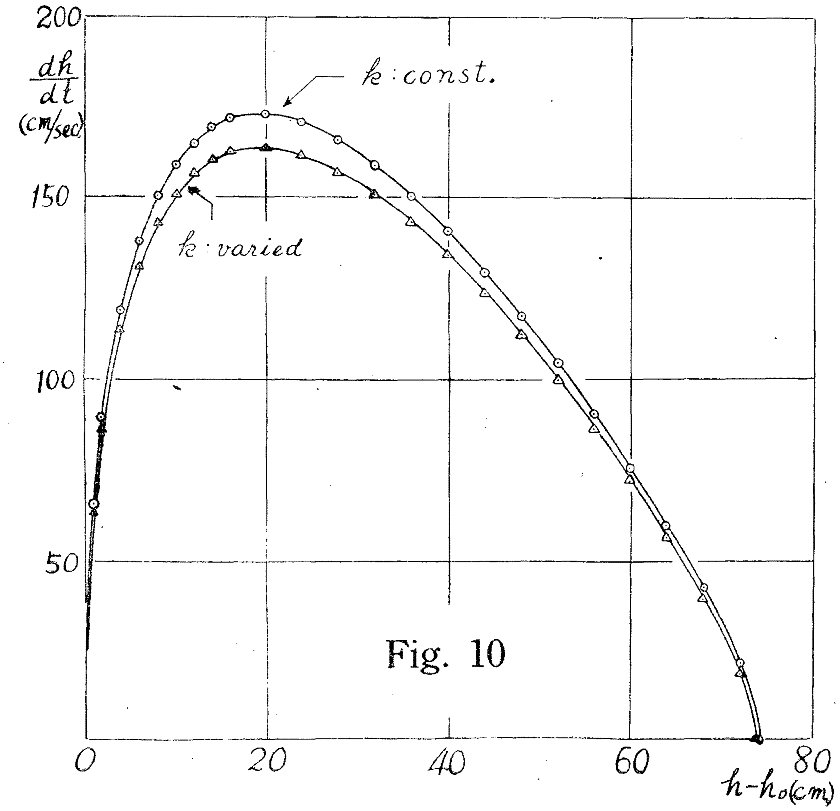


Fig. 10

arized theory, but let us quote here another example:——it is an analysis of just the same kind of experiment (unpublished) only with different dimensions, viz.

$h_0 = 15.7$ cm., $h_2 = 85.7$ cm.,
$a = 0.1212$ cm., $a' = 0.445$ cm.,
$\nu = 0.01032$ cm.^{2}/sec. (19°C.), $l' = 174$ cm.,
$k_p = 5.62$ sec.$^{-1}$, $k' = 0.0309$ sec.$^{-1}$;

the parameter λ, cf. (3.16), where L was taken equal to $(h_0+h_2)/2$ as before is 0.02553 $\cdots$. In Figure 10 the calculated values of dh/dt (cm./sec.) from both of these methods are shown. In this figure the curve denoted by (k:

const.) refers to the Poiseuille's flow pattern and (k: *varied*) to the improved theory. We think the difference is of the same order in the previous example.

Let us return to Figure 9. The considerable discrepancy between the theory and the experiment in the later stage of the motion, where is this to be ascribed?

In the computations of §3 we had to retabulate the values of k calculated preliminarily from the linearized theory as a function of h. But in the circumstances already stated, we took the expedient assumption illustrated by Figure 11. The transfer from the improved value of k obtained from the linearized theory to the constant k_p took place at $h-h_0 \doteqdot 20$ cm. and is indicated by the arrows in Figure 9.

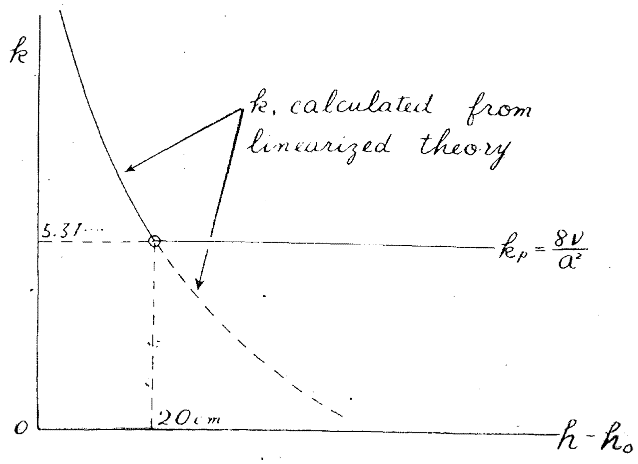


Fig. 11

If we compare these two curves, theoretical and experimental, bearing in mind the above reasons, we readily recognize that the discrepancy between them are chiefly due to the over-estimation of the value of k in the range of h greater than h_0+20 cm. In order to improve the agreement some measures may be conceivable but we shall not enter into details.

(IV) RELAXATION-TIME OF PRESSURE AFTER THE COCK IS PLUCKED OUT

While the cock, E in Figure 1, is kept closed and the meniscus of the water in the tube A remains low at $z = h_0$ before the motion, the pressure over the meniscus must be greater than the atmospheric pressure, p_0, by the amount

$$\Delta p_1 = g(h_2 - h_0). \tag{4.1}$$

When the cock is plucked out instantly, air discharges through the aperture and the difference of pressure above-mentioned begins to diminish until it becomes zero. Because this process will take a short but finite time, T, the preceding calculation of § 3 in which this relaxation-time was utterly neglected may be modified to some extent. In the case of our experiment, however, T is found to be quite negligible in reality and the correction is not necessary out of the following considerations.

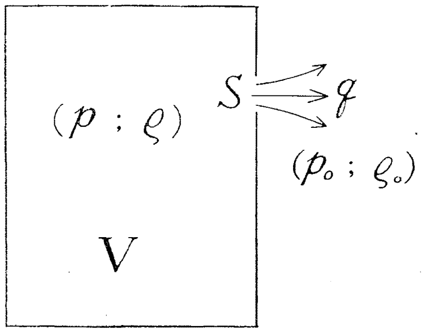


Fig. 12

Now as a model of discussion let us study the efflux of air from a vessel of volume V through a small opening of area S. Denoting by q the velocity of discharge, by p the pressure and by ρ the density of the air, the condition of conservation of mass gives

$$-\frac{d\rho}{dt} = \frac{S\rho_0}{V} q, \tag{4.2}$$

where the suffix 0 indicates the corresponding quantities in the outer field of normal condition. Under the assumption of adiabatic change the pressure and the density are related in the following way:

$$p = \kappa\rho^\gamma, \tag{4.3}$$

where $\kappa = p_0/\rho_0^\gamma$ and γ is the ratio of the specific heats of air, i.e. $\gamma = c_p/c_v$. Combining (4.2) with (4.3) and putting $\alpha = 1-\gamma^{-1}$, we have

$$-\frac{p^\alpha}{q}\frac{dp}{dt} = \frac{S\gamma}{V} p_0^{1-\alpha}. \tag{4.4}$$

On the other hand if we neglect the motion of the air in the vessel and assume among p, ρ and q Bernoulli's theorem of the steady state for each moment of efflux, then q is expressed by the formula

$$q = \sqrt{\frac{2\kappa^{1-\alpha}}{\alpha}(p^\alpha - p_0^\alpha)}. \tag{4.5}$$

Substituting (4.5) in (4.4), we obtain the differential equation

$$\frac{dp}{dt} = \beta p^{\alpha}\sqrt{p^{\alpha}-p_0^{\alpha}}\ , \tag{4.6}$$

where

$$\beta = -\frac{S\gamma}{V}\,p_0^{1-\alpha}\left(\frac{2\chi^{1-\alpha}}{\alpha}\right)^{\frac{1}{2}} = \text{const.} \tag{4.7}$$

But when $p-p_0 = \Delta p$ is very small compared with p_0, (4.6) can be readily integrated and the relaxation-time T stated above is found to be

$$T = \frac{\sqrt{2\rho_0}}{p_0\gamma}\,\frac{V}{S}\sqrt{\Delta p_1} - \frac{\alpha}{3p_0}\sqrt{(\Delta p_1)^3}\,. \tag{4.8}$$

The values of the constants in the normal state of air are

$$p_0 = 1.01\times 10^6\ \text{dyne/cm}^2.,$$

and

$$\rho_0 = 1.3\times 10^{-3}\ \text{gr./cm.}^3.$$

Then the second term of (4.8) can be neglected in comparison with the first one and we can write approximately

$$T \doteqdot 3.6\times 10^{-8}\frac{V}{S}\sqrt{\Delta p_1}\quad (\text{in C.G.S.}). \tag{4.9}$$

In our cases as it is that $V/S \doteqdot 10$ cm. and $h_2-h_0 = 40$ cm., (4.9) gives

$$T \doteqdot 3.6\times 10^{-8}\times 10\times\sqrt{980\times 40} \doteqdot 7.2\times 10^{-5}\ \text{sec.},$$

which can be safely neglected in our approximation.

§5—SUPPLEMENTARY NOTE

In the course of our work we were informed of the paper "Liquid Rise in a Capillary Tube" by E. W. Brittin (Journal of Applied Physics, vol. 17, No. 1, 1946). In this paper the rise of liquid under the action of surface tension (no pressure being impressed) is treated. By consideration of the change of momentum in the capillary tube the author derived the differential equation of motion of the meniscus which is naturally quite similar to our (3.13) and expressed its formal solution in terms of a Dirichlet series.(1) He showed a good agreement between the theory and the experiment (carried out by Rense, 1944). We hope that our work, although very different from his in formulation and line of thought, can be successfully utilized in the above study as well.

(1) According to the opinion of the present authors $\frac{5}{4}\left(\frac{dZ}{dt}\right)^2$ in his equation (1), p. 38, *loc. cit.* should be $\frac{1}{4}\left(\frac{dZ}{dt}\right)^2$, but in his numerical example no sensible error was caused because of very low speed of ascent of the liquid.

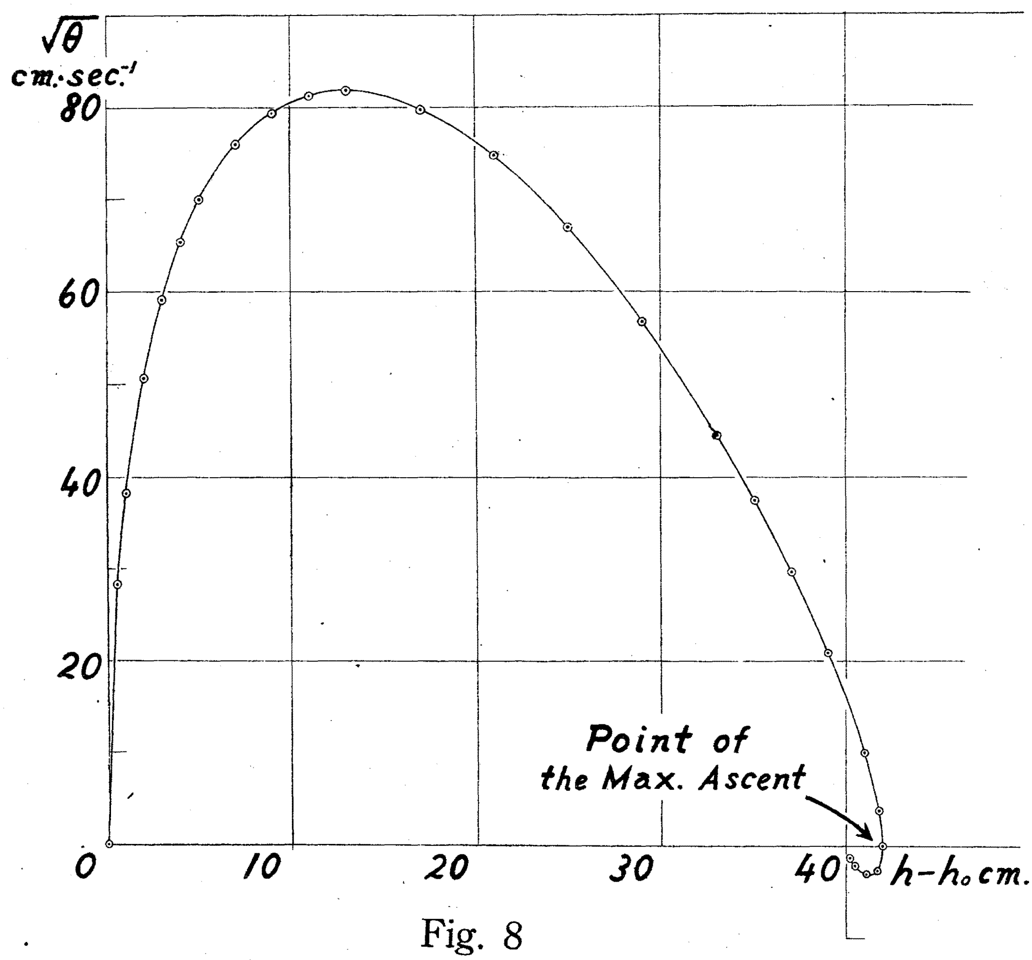


Fig. 8

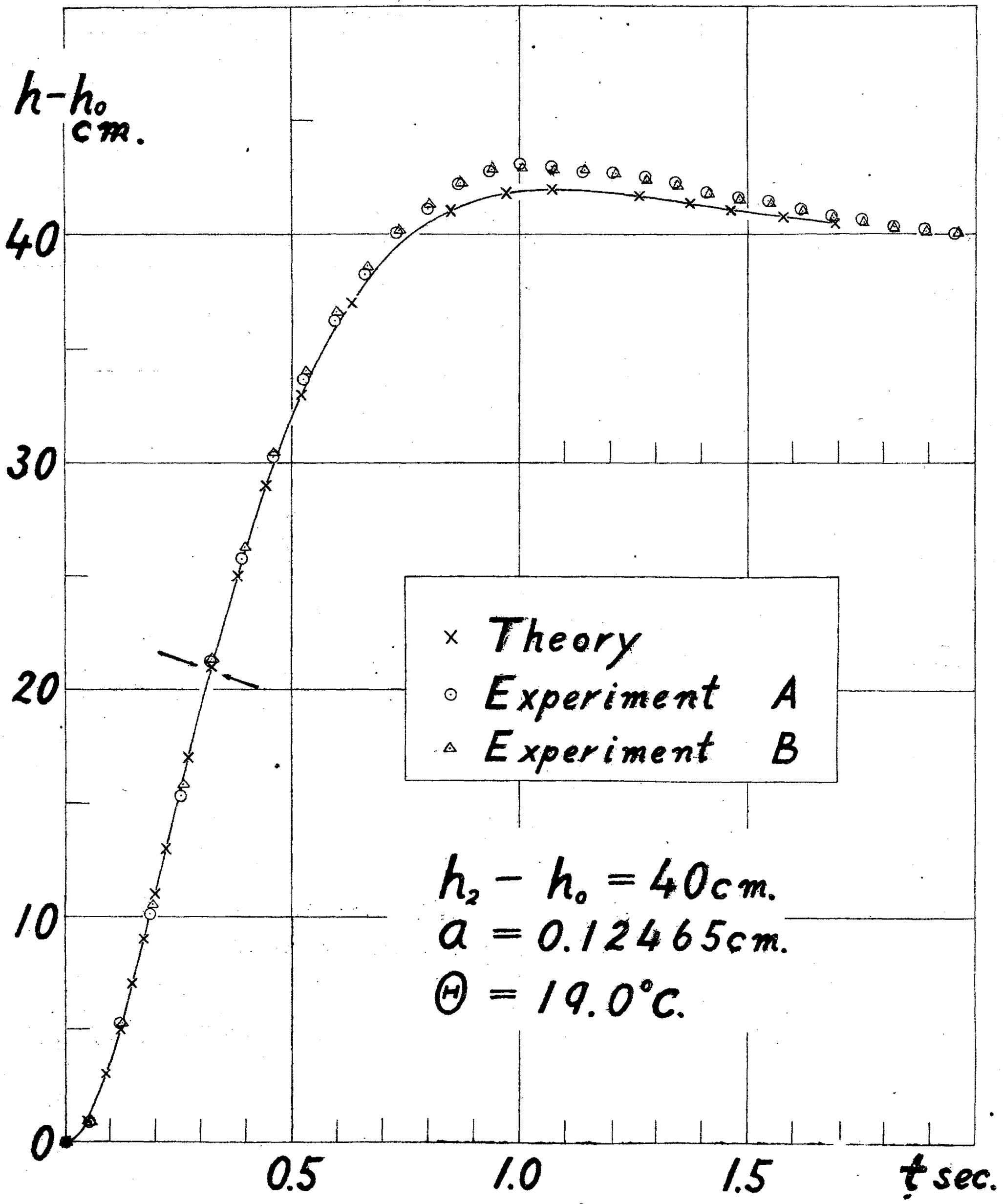


Fig. 9 Θ is the observed temperature of water (*see* P. 19).

PART II

§ 1—PURPOSES OF EXPERIMENT

The theoretical treatment described in Part I is concerned mainly with a very simplified problem and in fact they have little bearing on the actual manometer. A typical construction of a manometer, e.g. that which is used in the model-ship experiments in water-tanks, is shown in Figure 13: an aperture is bored in the flank of a body, through which the external pressure prevailing at P is transmitted to the manoneter, and by observing the motion of the meniscus this pressure can be read at once. The purposes of our experiment are to measure the time required for the meniscus to be settled in a new position of equilibrium when the manometer is set to work suddenly at any instant and to find out the factors which are most important in determining the motion of the liquid column.

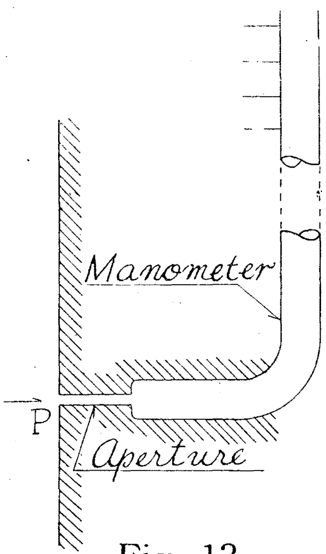


Fig. 13

We have in contemplation an experiment in a water-tank, in which the distribution of pressure around a moving body must be measured with the least time-lag as possible. We were at first induced to the experiments of Part I and II by the idea that the measurements which will be described would enable us to decide the rough dimensions of the suitable manometers and to predict the time-lag accompanied.

§ 2—METHOD AND APPARATUS

The general plan of the experiment is essentially similar to that of Part I and can be summarized in the following items:

I) Glasswares: With a view to reproduce the phenomena in the most simple way as possible we constructed six kinds of glasswares of different dimensions whose precise values are shown in Table V and Figure 14. The contractions (l) of three of these pipes are about 15 cm. long and in the other three $l \doteqdot 2$ cm. (Plate V). The effect of the length of the aperture bored in the flank of a body upon the time-lag of a monometer was expected to manifest itself by examining these two extreme cases of l, while that of the different diameters would be made clear by varying $2a$ among 1, 2 and 3 mm.

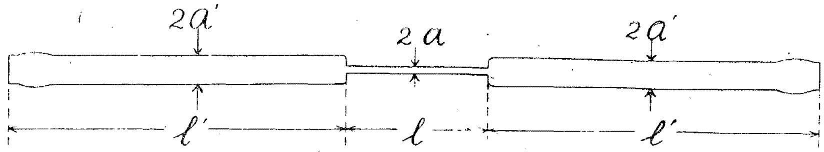


Fig. 14

II) Arrangement: The general arrangement of the pipes are shown schematically in Figure 15 together with the necessary dimensions in Table VI. The contraction was set horizontally and the manometer vertically with sufficient accuracy for present purpose, and they were connected with a flexible rubber-tube as in the actual manometer.(1) The observed changes of the level of water stored in the wooden tank were retained under 1 mm. i.e. $\frac{1}{8}\%$ and $\frac{1}{4}\%$ of the impressed heads and were regarded negligible throughout the measurement.

III) Measurements: In order to trace the motion of the meniscus several auxiliary points were marked on the manometer and after the cock was plucked out suddenly at $t=0$, times were measured between various pairs of these marks with an ordinary stop-watch. After repeated measurements from five to ten times for each interval, the results were averaged to eliminate the casual errors. Interchanging the contractions among six of them while the other conditions were kept as constant as possible, we measured the motions of the meniscus and plotted them into the $h \sim t$

(1) The radius of our manometer-tube was about 0.38 cm. and by far larger than the contraction. It was readily supposed, therefore, that the frictional resistance on the liquid would act chiefly at the aperture in contrast with the preceding experiment in which the flow was retarded by the manometer-tube itself (in fact this conjecture was verified in the later measurements and will be discussed in (ii) of § 3). But if we succeed to clarify other conditions now remaining ambiguous, we shall be able to compute the problem of Part II in a unified manner theoretically with the method developed in Part I by introducing some necessary assumptions at the points of discontinuity of cross-section.

curves shown in Figures 16—21, in each of which the initial head impressed at $t=0$ were 80 cm. (A) and 40.5 cm. (B).

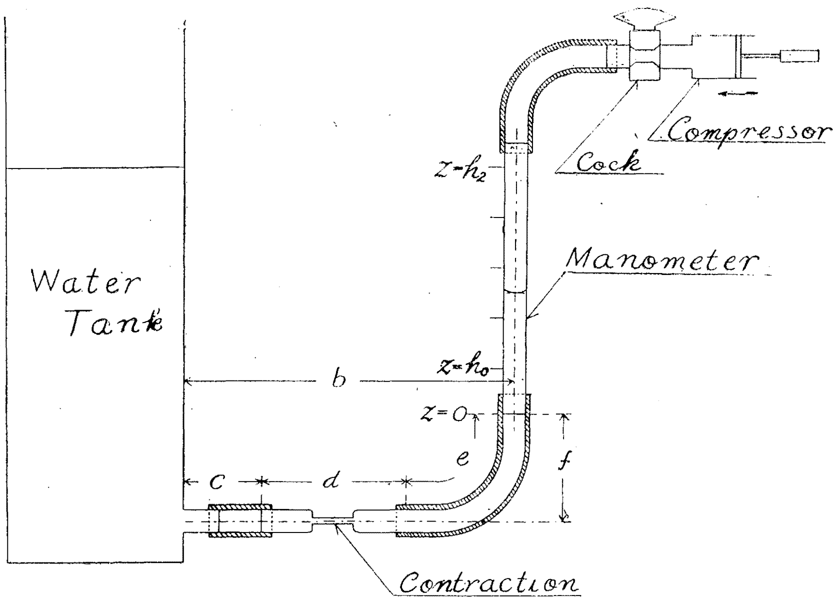


Fig. 15

Table V

Test	a(cm.)	l(cm.)	a'(cm.)	l'(cm.)
No. 1	0.168	1.95	0.45	11.8
No. 2	0.167	14.65	0.45	11.6
No. 3	0.103	2.13	0.445	11.45
No. 4	0.102	15.10	0.455	11.6
No. 5	0.051	2.02	0.425	11.7
No. 6	0.054	14.95	0.40	11.65

Dimensions of the Glasswares (*see* Figure 14).

These notations are independent of those of Part I.

Table VI

Test	b(cm.)	c(cm.)	d(cm.)	e(cm.)	f(cm.)
No. 1	52.0	10.0	25.6	32.5	23.0
No. 2	66.3	10.0	37.9	34.9	22.3
No. 3	52.0	10.0	25.05	32.5	23.0
No. 4	66.3	10.3	38.3	33.4	23.3
No. 5	51.5	10.1	25.4	33.1	24.0
No. 6	66.3	10.0	38.4	35.6	23.0

Characteristic Dimensions of Various Parts of Apparatus (*see* Figure 15).
The mean radius of the manometer tube is 0.38 cm.

§ 3—RESULTS AND DISCUSSIONS

In order to clarify the nature of the motion of the water in a manometer-tube, we tabulated in Table VII the values of some characteristic quantities observed or calculated therefrom: viz. the time-lag, the maximum velocity of ascent, the maximum Reynolds numbers attained (in the contraction and in the manometer) together with the types of motion. From these and the corresponding $h \sim t$ curves (Figures 16—21) we can arrive at some qualitative conclusions which may be summarized as follows:

(i) The motion is of oscillatory or non-oscillatory nature according to the dimensions of the contraction. But the type of motion is not likely influenced sensibly, when the magnitude of the impressed pressure or the initial head difference, i.e. $h_2 - h_0 \equiv \Delta h$, is reduced to about one half (from 80 cm. to 40.5 cm.) under the otherwise same conditions.

(ii) The greater part of the frictional force acting towards retarding the motion is concentrated at the contraction. For by a slight change of its dimension there can be established an entirely different type of motion. We can notice further that the friction is far more influenced by the inner diameter of the aperture (or contraction) than by its length.[1] There will exist, of course, certain limits within which this conclusion can hold, but since there may be few possibilities in water-tank experiments when an

[1] Cf. the parameter λ in § 3, Part I (p. 26). But it is open to doubt whether λ can be as valid as ever in the case of turbulent flow without any modification in its form.

aperture is designed larger than 3 mm. in diameter or longer than 15 cm., we hope we can cover the region of practical use.

(iii) The time-lag, on the other hand, varies roughly in proportion to $\sqrt{\Delta h}$ in cases of larger contractions, viz. it decreases in the ratio of about $1/\sqrt{2} \doteqdot 0.7$ when Δh is halved. The less validity of this simple relation for smaller contractions will be ascribed probably to the greater contribution of frictional resistance in the latter cases.

(iv) As is seen from Table VII, in almost all cases the maximum Reynolds number exceeded the critical value (*ca.* 2,000) for break-down of steady flow,[1] so that the flow must have become more or less turbulent at least for some period of the motion. But in view of both of the facts that

Table VII

Test		Initial Head (cm.)	Mean Temp. (°C.)	Type* of Motion	Time** Lag (sec.)	Max. Velocity in Manometer (cm./sec.)	Re_{max} attained in Contraction	Re_{max} attained in Manometer
No. 1	A	80	26.8	O	7.0	110.4	22,000	9,600
	B	40.5	26.55	O	5.0	64.6	12,500	5,600
No. 2	A	80	22.45	O	8.0	73.1	13,000	5,800
	B	40.5	23.45	O	6.0	55.6	10,000	4,500
No. 3	A	80	23.15	O	7.5	22.5	6,600	1,800
	B	40.5	23.3	O	5.0	21.3	6,300	1,700
No. 4	A	80	22.55	N	11.2	20.8	6,100	1,650
	B	40.5	23.1	N	8.0	18.3	5,400	1,500
No. 5	A	80	23.55	N	36.5	5.58	3,300	450
	B	40.5	24.75	N	28.0	3.55	2,200	300
No. 6	A	80	23.95	N	102.0	3.57	2,100	300
	B	40.5	24.4	N	90.0	2.04	1,200	170

* The signs O and N indicate the motion of the type *oscillatory* and *non-oscillatory* respectively.

** The time-lag was defined

(a) in the *oscillatory* motion as the time from the start of the ascent till the amplitude has diminished practically negligible.

(b) in the *non-oscillatory*, from the start till the column attains to 99.5% of the final height.

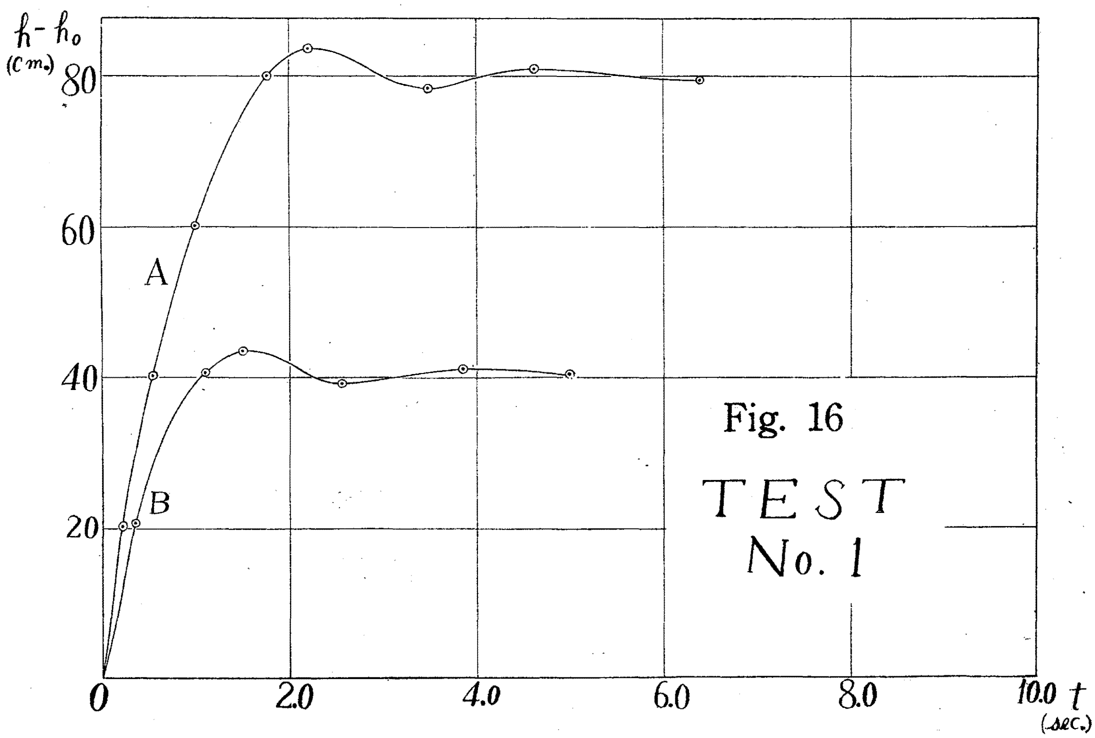
the observed motion was much accelerated and that the conduit was composed of the tubes of various diameters and was liable to produce addi-

[1] Cf. the discussions in (ii) of § 4, Part I (p. 32).

tional disturbances at their junctions, we cannot decide here definitely whether or not this value (2,000) can be adopted in our problem without further investigations as really critical for the transition to turbulence. This uncertainty, as stated before in the Introduction of this paper, made it impossible for us to treat this problem mathematically.

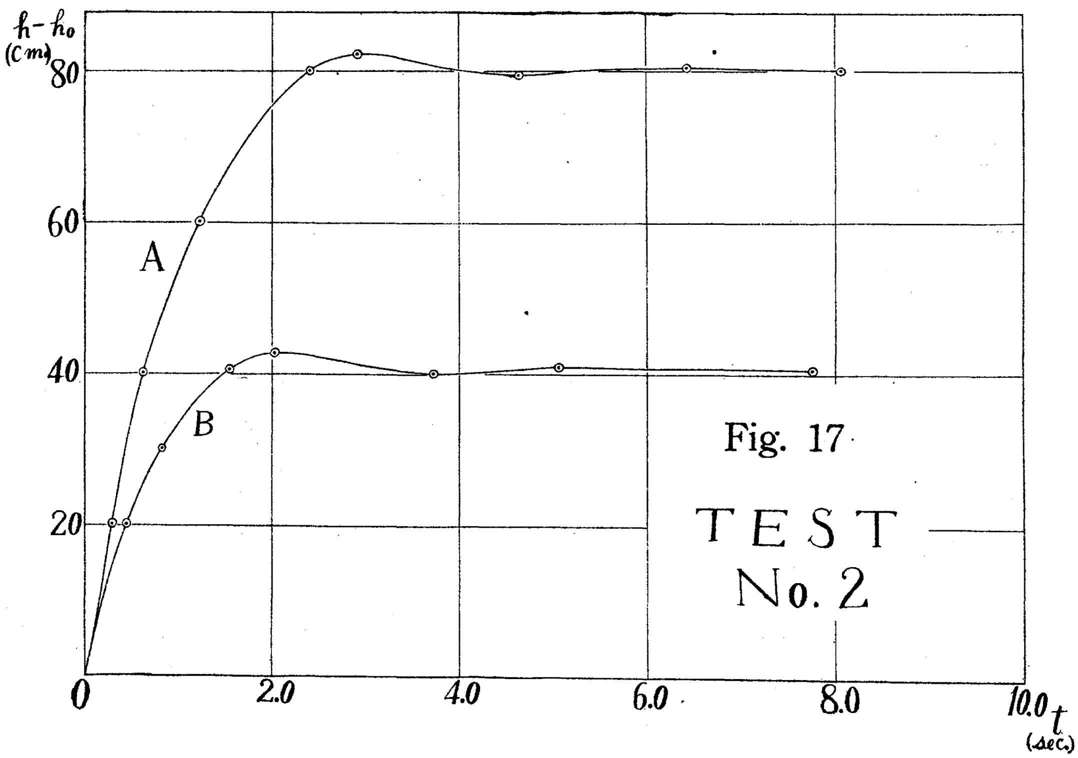
(v) But leaving the mathematical solution out of account for the time being, the results of the series of our experiments seem to teach us that the aperture of about 2 mm. in diameter will be the most suitable one for our future use in a water-tank. That is to say, the ascending motion is much retarded in smaller aperture, and on the contrary in larger one the time-lag increases again accompanied by the oscillatory tail of the motion.

(vi) As mentioned before we measured the motion of the meniscus with an ordinary stop-watch. This measurement is of course somewhat less accurate than the cinematographic method in Part I, but the relative errors are found to lie within fairly small limits because the durations of motions are considerably longer than in the previous experiment.



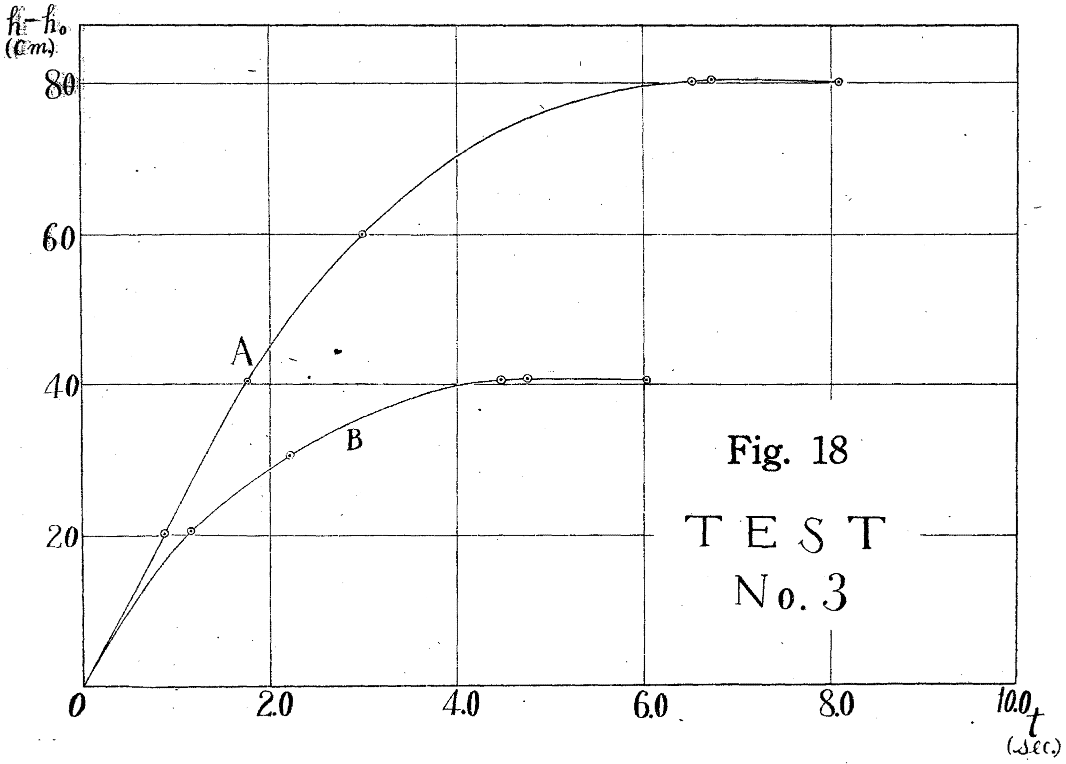
h- h₀
(cm.)
80
60
40
20
0
2.0
4.0
6.0
8.0
10.0 t
(sec.)
A
B
TEST
No. 1

Fig. 16



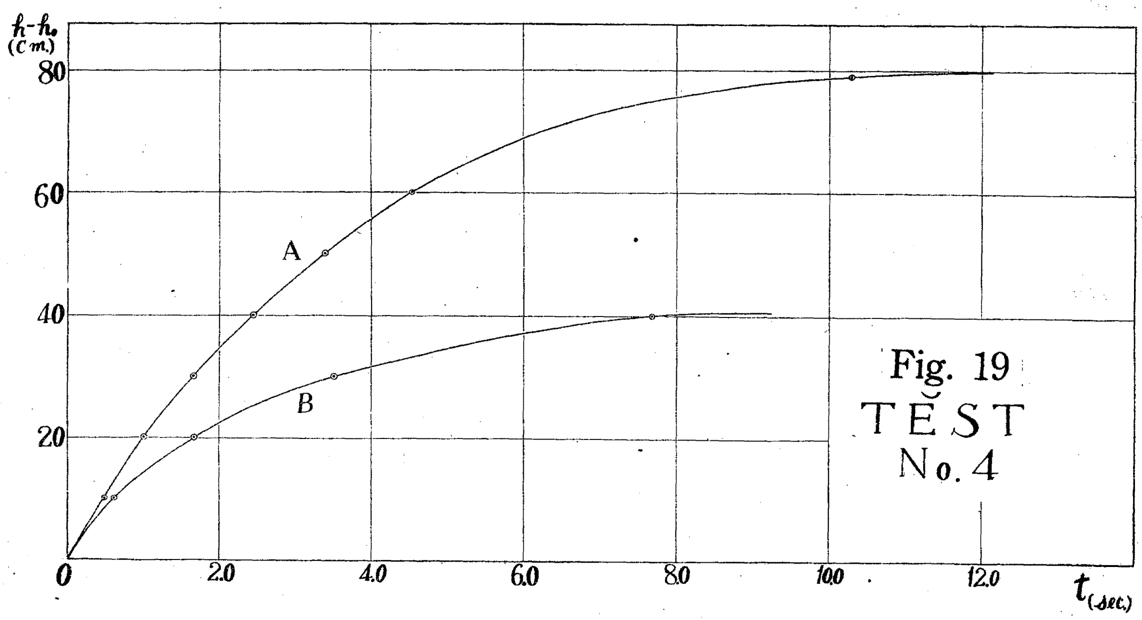
h- h₀
(cm)
80
60
40
20
0
2.0
4.0
6.0
8.0
10.0 t
(sec.)
A
B
TEST
No. 2

Fig. 17



h-h₀
(cm.)
80
60
40
20
0
2.0
4.0
6.0
8.0
10.0
t
(sec.)
A
B
TEST
No. 3

Fig. 18



h-h₀
(cm.)
80
60
40
20
0
2.0
4.0
6.0
8.0
10.0
12.0
t
(sec.)
A
B
TEST
No. 4

Fig. 19

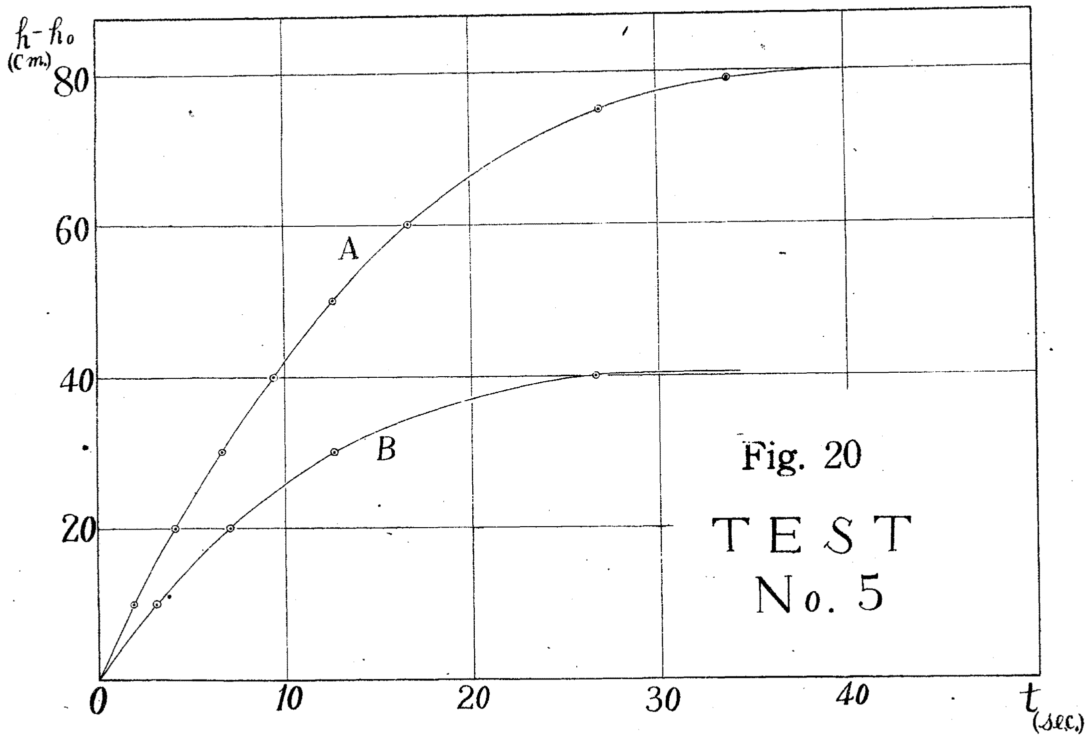


Fig. 20

TEST
No. 5

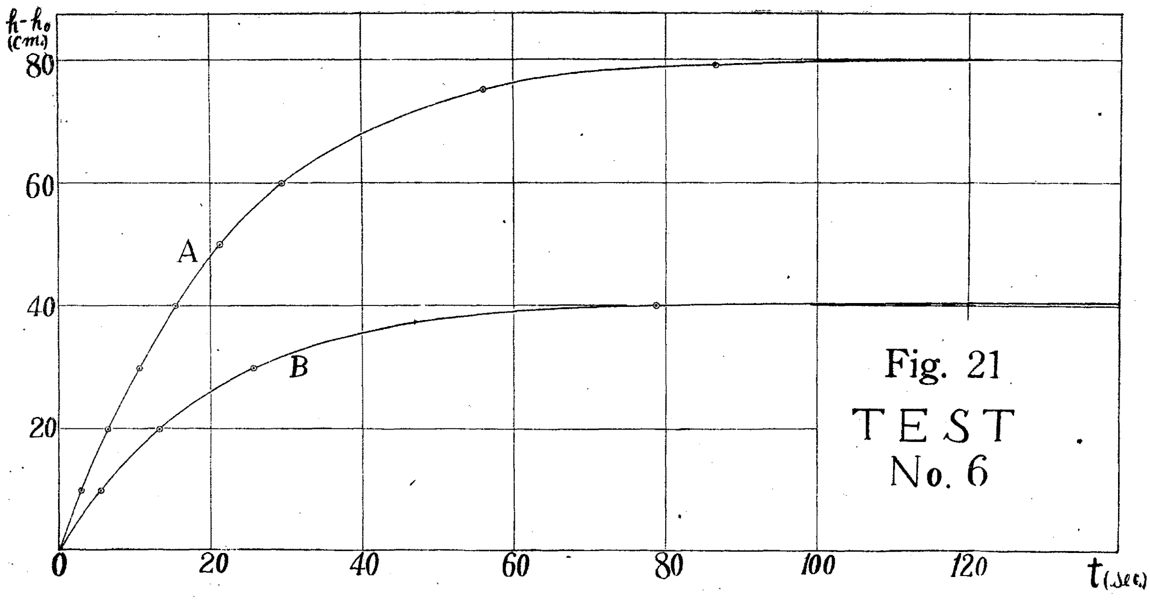


Fig. 21

TEST
No. 6

SUMMARY

In Part I of this paper the theoretical treatment is developed for the ascent of the liquid column in the small tube when pressure is suddenly applied at the bottom. Comparison with the experiment is described together with some supplementary discussions.

Part II is devoted to the explanation of the experiment carried out with a practical object in view to measure the time-lags of the manometers of various dimensions.

ACKNOWLEDGEMENTS

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(December 20, 1949)

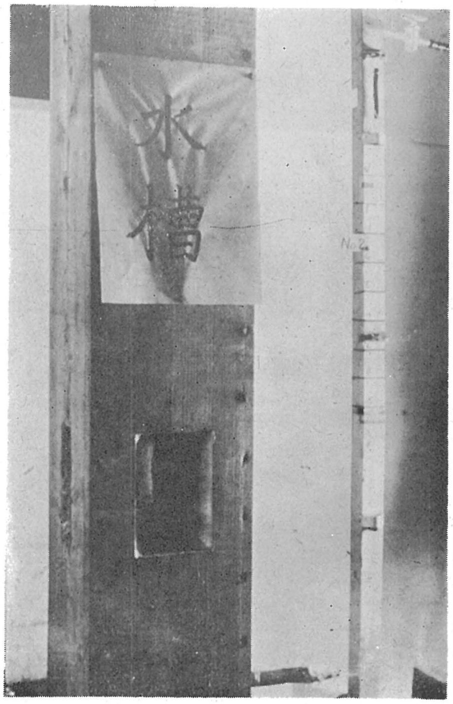

Plate I

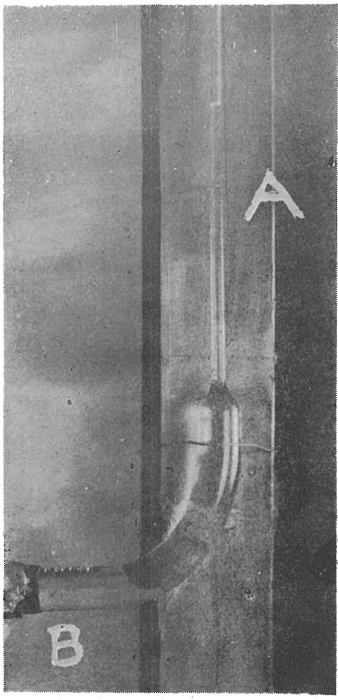

Plate II

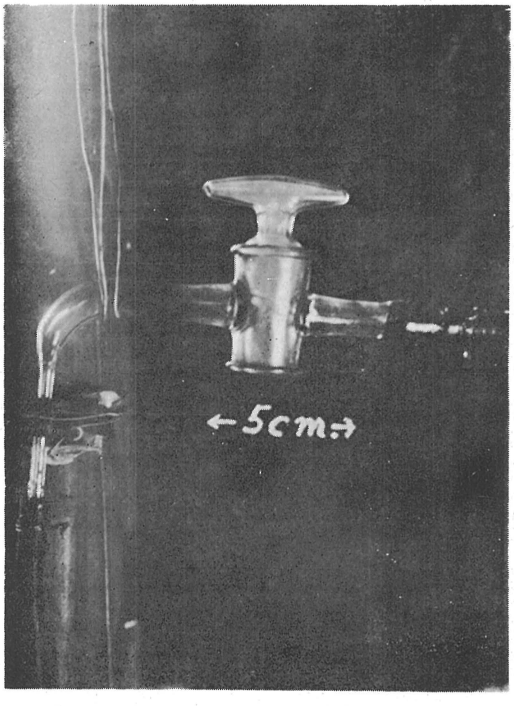

Plate III

Plate I: General view of the experimental apparatus (*see* Figures 1 and 2).

Plate II: The lower part of the manometer-tube, showing the contraction at $z = 0$ (*see* Figure 1).

Plate III: The stop-cock (*see* Figure 1).

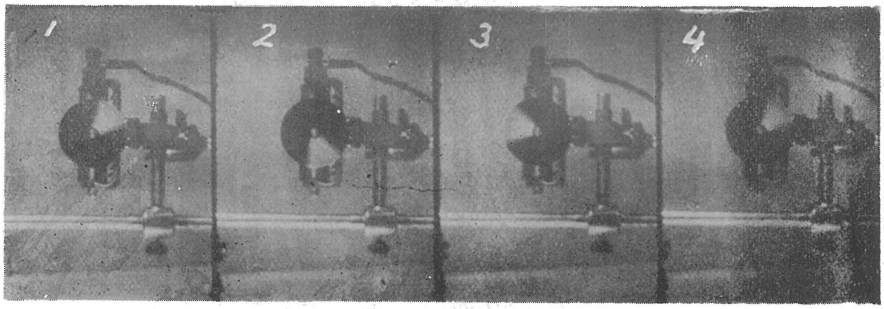

Plate IV

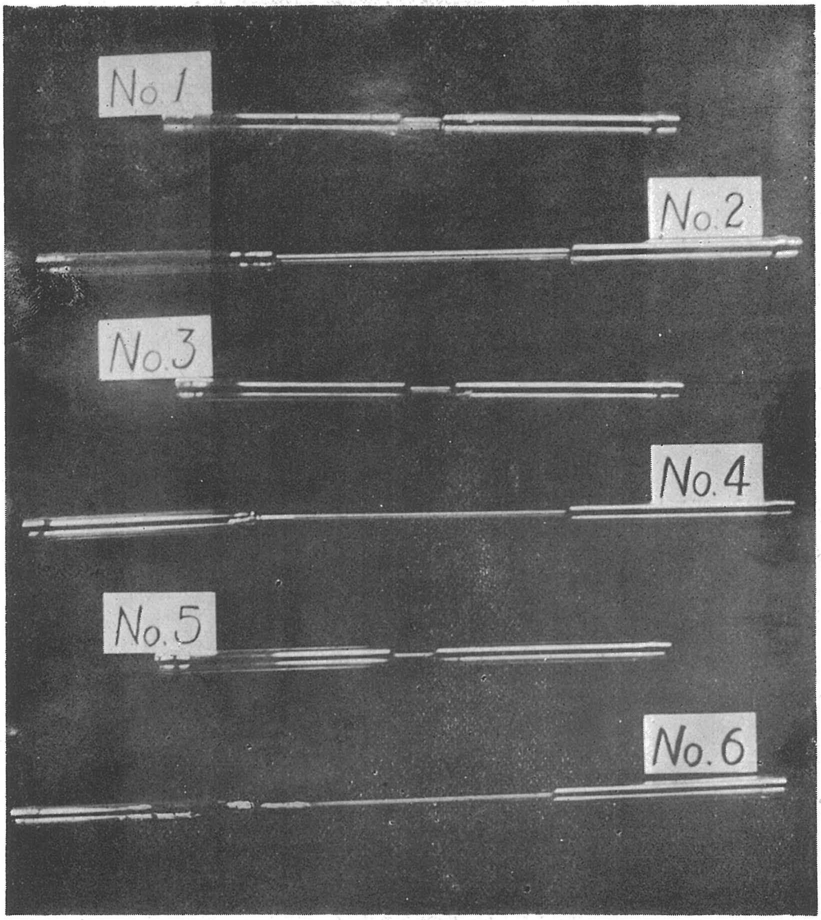

Plate V

Plate IV: An example of a sequence of time-record. In these four frames (1–4) the rotating disc completes about two revolutions (Part I, p. 20).

Plate V: Test-tubes, No. 1 to No. 6 (*see* Figure 14 and Table V).