THEORY OF HYDRAULIC JUMP

TSUBAKI, Toichiro
Research Institute for Fluid Engineering, Kyushu University

https://doi.org/10.15017/7158055

出版情報：流體工學研究所報告. 6 (2), pp.99-106, 1950-01. 九州大學流體工學研究所

THEORY OF HYDRAULIC JUMP[1)]

By

Toichiro TSUBAKI

From Vol. 5 No. 2

§ 1. Since the disturbances in an open channel travel relative to the flow with the propagation velocity $\sqrt{gh}$, they can travel up and downstream when $M^2<1$, whereas they travel only downstream when $M^2>1$; where M^2 is the kinetic flow factor of B.A. Bahkmeteff, i.e. $M^2 = 1/h\int_0^h U^2/ghdy$ with usual notations. When a transition from $M^2>1$ to $M^2<1$ occurs, the disturbances accumulate causing the abrupt change of velocities and surface height etc. This phenomenon is well known as the hydraulic jump; the relations between the quantities related to the begining and the end of jump and the length of jump have been studied. The former are obtained theoretically as the discontinious phenomenon, but our knowledge of the latter is confined to only experimental one.

In the present paper, the length and internal physical processes of hydraulic jump are discussed. The hydraulic jump as one of the shocking phenomena accompanies with a definite rate of energy loss caused by the violent turbulence in the surface roller for the perfect jump or by wave motions for the undulating jump. Thus, if we can estimate the nature of turbulence in the jump—and hence of apparent loss of energy—which is characteristic of the phenomenon, it becomes possible to study theoretically the perfect jump considering energy equation as well as the continuity and momentum equation. The notations used in this paper are illustrated in Fig. 1 and it is mentioned here that the suffixes 1, 2 refer to the quantities at the begining and the end of jump respectively and suffix R to the end of roller.

§ 2. Now we shall confine ourselves to the problem of two dimensional flow. If we take the origin of coordinates at the beginning of the jump, x

[1)] Abstract of "Theory of Hydraulic Jump" (in Japanese) Rep. Res. Inst. Fluid Engng. Kyūshū Univ. Vol. 5 No. 2 (1949) pp 16–29.

axis along the bed, y axis upward and denote by U the mean velocity, by (u, v) components of turbulent velocities and by h the depth, then, we have following equations,

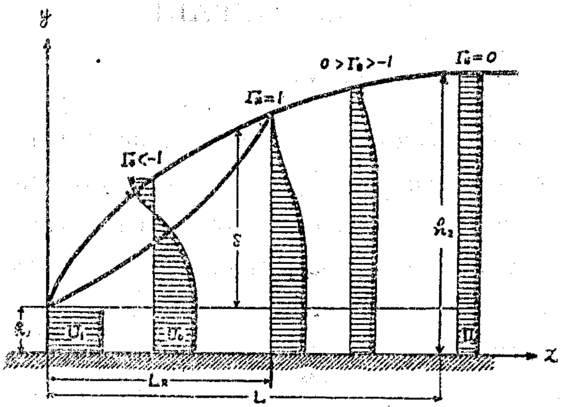

Fig. 1

$$\frac{d}{dx}\int_0^h U\,dy = 0\,, \tag{2.1}$$

$$\frac{d}{dx}\left[\int_0^h \rho U^2\,dy + \frac{\rho g h^2}{2}\right] = 0\,, \tag{2.2}$$

$$\frac{d}{dx}\left[\int_0^h \left(\frac{1}{2}\rho U^3 + \rho g h U\right)dy\right] = \int_0^h \rho\,\overline{uv}\,\frac{\partial U}{\partial y}\,dy, \tag{2.3}$$

with usual approximations. (2.1) and (2.2) are the equations of continuity and momentum. (2.3) is the energy equation showing the rate at which the flow of mechanical energy is decreased by the action of turbulent stress, producing the turbulent energy,

The shearing stress $-\rho\overline{uv}$ at any point in the field is given by

$$-\rho\,\overline{uv} = \rho\, l \mathrm{v} \frac{\partial U}{\partial y}\,, \tag{2.4}$$

where l is mixture length and v the root mean square of the turbulent velocity. In a hydraulic jump the turbulent energy would be generated mainly by the violent turbulent stresses in the roller and then diffuse into the main stream and finally dissipate by molecular viscosity. Since the detailed analysis of these processes is too complicated to treat here, in the following sections we shall introduce suitable assumptions as regard to these turbulent field.

§ 3. We shall now introduce the following assumptions about the velocity profile as shown in Fig. 1:

(i) For the velocity distributions in the sections at the begining of jump and the sufficiently far distance downstream from the end of jump, we assume uniform velocities U_1, U_2 respectively.

(ii) Violent turbulence produced in the roller could not give any appreciable effects to the layer near the bed, so that we may assume in the layer from the bed to h_1 (the water height at the begining of jump) the uniform velocity U_0 which is a function of x only.

(iii) In the upper part of roller the adverse flow takes place, accompanied with the region of velocity defect between the end of roller and the end of jump. Accordingly we may assume the velocity distribution in these regions above h_1,

$$\left.\begin{aligned} \frac{U}{U_0} &= 1+(3\eta^2-2\eta^3)\Gamma_*\,, \\ \Gamma_* &= \Gamma-1 \end{aligned}\right\} \tag{4.1}$$

where η is the non-dimensional height from h_1 defined by $\eta=\{(y-h_1)/\delta\}$ (δ is the height of water surface measured from h_1), Γ the ratio of the surface velocity U_s to the base velocity U_0 and Γ_* is the velocity profile parameter. It is clear that $\Gamma_* < -1$ in the roller, $\Gamma_* = -1$ at the end of roller, $0 > \Gamma_* > -1$ in the region of recovering velocity and $\Gamma_* = 0$ at end of jump. Prandtl's assumption concerning the turbulent velocity $\mathrm{v} = l(dU/dy)$ is valid strictly in the state of equilibrium between the generation and the dissipation of turbulence[2]. But for the turbulent field in the jump (especially near the toe of roller), convection and diffusion-processes appear to be more important than dissipation. So that for the very complicated flow as the jump it is natural to assume a constant turbulent velocity throughout the jump region, which is proportional to some velocity characteristic to the jump. We may take as such a velocity the base velocity U_{0c} at the point $M^2=1$. After Prandtl mixture length l is supposed to be proportional to δ. Thus we have for the turbulent stress

$$-\rho\overline{uv} = \rho l\mathrm{v}\frac{\partial U}{\partial y} = \alpha\rho\delta U_{0c}\frac{\partial U}{\partial y}\,, \tag{3.2}$$

[2] M. Kurihara. Rep. Res. Inst. Fluid Engng. Kyūshū Univ. Vol. 3. No. 1. (1946)

where α denotes an unknown constant determined experimentally. Of course, it is not allowed that the value of α constradicts with our knowledge on turbulence (see § 6).

If we put $h_2-h_1=h_j$, $x/h_j=\xi$, $\delta/h_j=\delta_*$, $h_j/h_1=\zeta$, $\zeta\delta^*=\vartheta$, $U_0/U_1=U_*$, $U_{0c}/U_1=U_{*c}$ and $M_1^2=U_1^2/gh_1$ (c.f. Fig. 1), then inserting (3.1) and (3.2) into (2.1), (2.2) and (2.3) we obtain

$$U_*\left\{1+\vartheta\left(1+\frac{1}{2}\Gamma_*\right)\right\}=1, \tag{3.3}$$

$$U_*^2\left\{1+\vartheta\left(\frac{13}{35}\Gamma_*^2+\Gamma_*+1\right)\right\}+\frac{1}{M_1^2}\left(\vartheta+\frac{1}{2}\vartheta^2\right)=1, \tag{3.4}$$

$$\frac{d}{d\xi}\left\{\frac{1}{2}U_*^2+\frac{1}{2}U_*^3\vartheta\left(\frac{43}{140}\Gamma_*^3+\frac{39}{35}\Gamma_*^2+\Gamma_*\right)\right\}+\frac{1}{M_1^2}\frac{d\vartheta}{d\xi}$$
$$=-1.2\,\alpha\zeta\,U_*^2U_{*c}\Gamma_*^2. \tag{3.5}$$

§ 4. Elliminating U_* between (3.3) and (3.4), we get

$$\frac{1}{M_1^2}\left(1+\frac{1}{2}\vartheta\right)\left\{1+\vartheta\left(1+\frac{1}{2}\Gamma_*\right)\right\}^2=\left(1-\frac{13}{35}\Gamma_*^2\right)+\left(1+\frac{1}{2}\Gamma_*\right)^2\vartheta, \tag{4.1}$$

which gives the relation between ϑ and Γ_* at any cross section. As the particular cases, we can deduce the following relations:

(a) If we put the conditions at the end of jump $\Gamma_*=0$, and $\vartheta=\zeta$, (4.1) gives the well known relation between the depth at the beginning and end of jump, that is

$$\zeta=\frac{h_2-h_1}{h_1}=\frac{-3+\sqrt{1+8M_1^2}}{2}. \tag{4.2}$$

(b) By the condition at the begining of jump $\delta_*=0$,

$$\Gamma_{*I}=-\sqrt{\frac{35}{13}\left(1-\frac{1}{M_1^2}\right)}. \tag{4.3}$$

(c) By the condition at the end of roller $\Gamma_*=-1$, we get the following equation for ϑ_R;

$$\frac{1}{M_1^2}\left(1+\frac{1}{2}\vartheta\right)^3=\frac{22}{35}+\frac{1}{4}\vartheta. \tag{4.4}$$

It is interesting that this equation has no possitive root, in other words, the roller can not exist physically within $1.6\geqq M_1^2\geqq 1$. Comparing this limit with the observed condition for perfect jump $M_1^2\geqq 3$, we can infer that at low values of $M_1^2(1\leqq M_1^2\leqq 3)$, the undulating motions are more liable to occur than the roller as the energy absorbing mechanism, since the hydraulic jump must be necessarily accompanied by the loss of energy. The calculated values of ζ, Γ_{*1} and δ_{*R} are shown in Fig. 2.

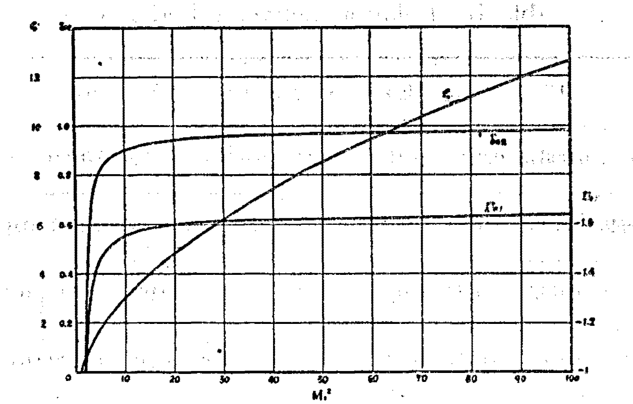

Fig. 2

§ **5.** After some calculations, the equation of surface profile is obtained from (3.3), (3.4) and (3.5) by considering the condition $\delta_* = 0$ at $\xi = 0$,

$$\int_0^{\delta_*} \frac{-g(\delta_*)}{U_*^2 U_{*c} \Gamma_*^2} d\delta_* = 1.2\, a\xi, \tag{5.1}$$

where

$$g(\delta_*) = -U_*^3 \left\{ \left(1+\frac{1}{2}\Gamma_*\right) + \frac{1}{2}\vartheta \frac{d\Gamma_*}{d\vartheta} \right\} \left\{ 1+\frac{3}{2} U_* \vartheta A(\Gamma_*) \right\}$$

$$+ \left\{ \frac{1}{2} U_*^3 A(\Gamma_*) + \frac{1}{M_1^2} \right\} + \frac{1}{2} U_*^3 \vartheta \frac{dA(\Gamma_*)}{d\Gamma_*} \cdot \frac{d\Gamma_*}{d\vartheta}, \tag{5.2}$$

$$A(\Gamma_*) = \frac{43}{140}\Gamma_*^3 + \frac{39}{35}\Gamma_*^2 + \Gamma_* .$$

In (5.1) the integrand is a function of δ_* only; since from (3.3) and (3.4) we have the relation $\Gamma_* \sim \delta_*$, $U_* \sim \delta_*$ and $d\Gamma_*/d\vartheta \sim \delta_*$ and hence by (5.2) g is expressed in terms of δ_*.

U_{*c} are listed in Table I and the integration of (5.1) is easily performed

Table I Values of U_{*c}

M_1^2	4	7	10	30	50	100
U_{*c}	0.776	0.713	0.675	0.555	0.502	0.432

by quadrature. Thus, the relation between $1.2\,\alpha\xi$ and nondimensional depth δ_* is obtained as in the following Table and in Fig. 3.

Table II Relation between $12\alpha\xi$ and δ_*

M_1^2 \ δ_*	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.94	0.98
4	0	0.00421	0.00845	0.0127	0.0176	0.0225	0.0285	0.0372	0.0499	0.0749	0.0947	0.150
7	0	0.00648	0.0125	0.0184	0.0245	0.0306	0.0374	0.0455	0.0566	0.0766	0.0898	0.127
10	0	0.00750	0.0145	0.0211	0.0275	0.0339	0.0409	0.0489	0.0591	0.0767	0.0895	0.1196
30	0	0.0102	0.0185	0.0258	0.0325	0.0388	0.0449	0.0519	0.0599	0.0723	0.0805	0.0991
50	0	0.0111	0.0196	0.0268	0.0333	0.0392	0.0451	0.0510	0.0582	0.0688	0.0761	0.0920
100	0	0.0128	0.0212	0.0279	0.0340	0.0395	0.0448	0.0502	0.0561	0.0650	0.0709	0.0828

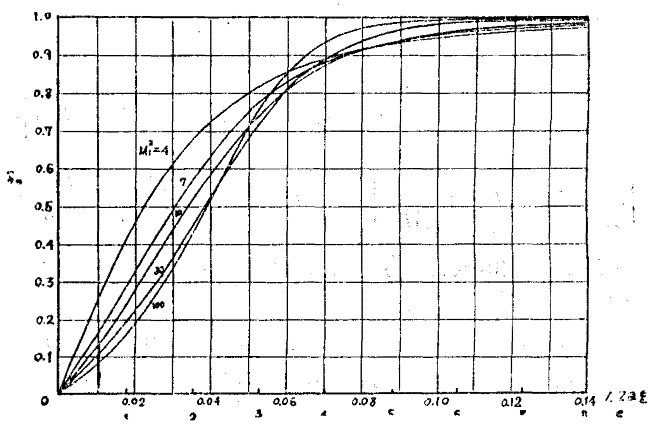

Fig. 3

Using δ_{*R} obtained from Fig. (2) we can easily find the length ξ_R. On the other hand, the length of jump ξ_L is indeterminate owing to the asymptotic approach of the surface profile to 1, so that we more or less arbitrarily define it as ξ corresponding to $\delta_* = 0.98$ in our theory. Finally, the numerical values of $1.2\,\alpha\xi_R$ and $1.2\,\alpha\xi_L$ are given in Table III.

Table III Values of $1.2\,\alpha\xi_R$ and $1.2\,\alpha\xi_L$

M_1^2	4	7	10	30	50	100
$1.2\alpha\xi_R$	0.0455	0.0692	0.0771	0.0862	0.0853	0.0822
$1.2\alpha\xi_L$	0.150	0.127	0.1196	0.0991	0.0920	0.0828

§6. The comparison between the calculated and experimental curves measured by K. Safranez,[3] B. A. Bahkmeteff[4] and E. T. Mavis[5] are shown in Fig. 4 with $\alpha = 0.0145$. The agreement is good, particularly with Bahkmeteff's result. This value of α is physically plausible, since if we assume $l/\delta \sim 0.1$ and $v/U_{0c} \sim 0.1$ according to view of turbulence, we have $\alpha \sim 0.01$ by (3.2).

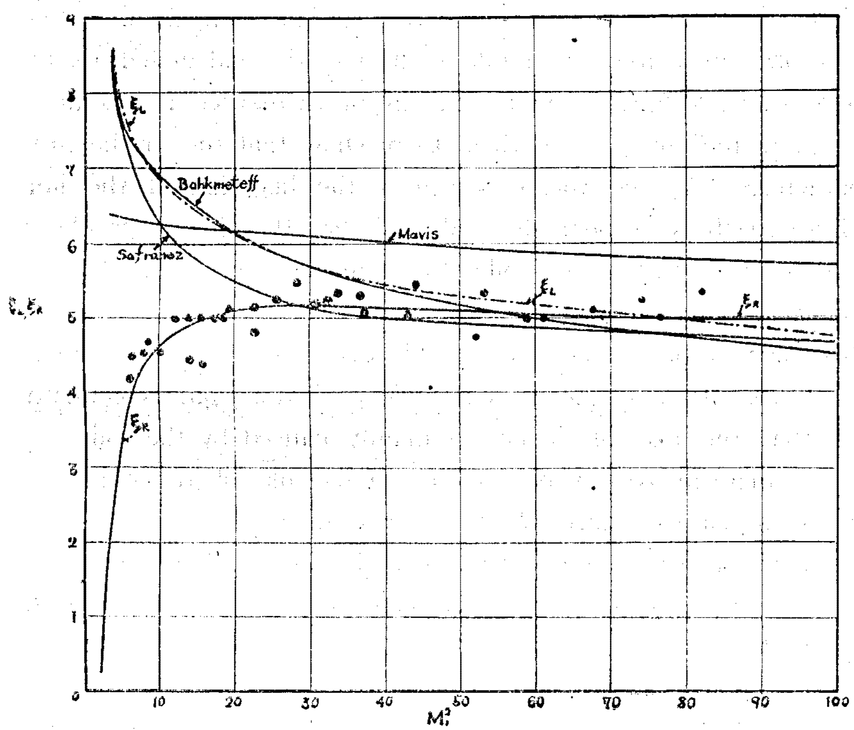

Fig. 4

[3] K. Safranez, Wasserkraft und Wasserwirtschaft. (1933)
[4] B. A. Bahkmeteff and A. E. Matzke, Trans. A.S.C.E. Vol. 101. (1936) pp 630–680.
[5] E. T. Mavis and A. Luksch, ditto. pp 669–672.

The length of roller was measured by E. T. Mavis and A. Luksch (shown in Fig. 4). Besides for $M_1^2 \geqq 50$ the length of jump may be regarded as the length of roller, since they are almost identical for large values of M_1^2 see Fig. 1). Then, the observed values of ξ_R increase with M_1^2 and through the very slight maximum at about $M_1^2 \doteqdot 30$ approaches to a constant ($\xi_R \doteqdot 5$) for $M_1^2 \geqq 50$, showing the good coincidence with the calculated result corresponding to $\alpha = 0.0139$. Although the values found for α in ξ_L and ξ_R are slightly different owing to the accuracy of theory and experiments, the agreement of the average quantities such as the length ξ_L and ξ_R is generally good.

Assuming that the turbulent velocity is constant throughout the jump and proportional to U_{0c}, we could not expect such a good agreement between the theory and the observations about the local quantities as about the average. Neverthless, we may say that the comparison between the surface profile measured by Bahkmeteff and the calculated results with $\alpha = 0.0145$ shows a fairly good agreement of them (see Fig. 3 and Fig. 8 in the Bahkmeteff's paper). Both of them show that the profile are generally characterized by the abrupt rise near the begining of the jump and gradual approach to 1 near the end, and that the rise of profile are promient in the head part of the roller for small M_1^2 and in the rear part for large M_1^2 respectively.

This success on the surface profile makes us possible to discuss the change of velocity and mechenical energy. After some calculations, we can see that the loss of energy is mainly caused by the roller in which energy is transformed to turbulent one. This rôle of roller is seen most clearly for the case of large M_1^2.

In conclusion, the author takes great pleasure in thanking professor M. Kurihara for his suggestion of the problem and his helpful advice throughout this work.