A METHOD OF APPROXIMATE INTEGRATION OF THE LAMINAR BOUNDARY LAYER EQUATION

YAMADA, Hikoji
Research Institute for Fluid Engineering, Kyushu University

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A METHOD OF APPROXIMATE INTEGRATION OF THE LAMINAR BOUNDARY LAYER EQUATION[1]

By

Hikoji YAMADA

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§ 1. **General equations.**—Here we attempt to improve the degree of approximation of the Kármán-Pohlhansen's solution of the laminar boundary layer equation. The principle of our method has already been given.[2]

The velocity distribution u in y-direction normal to the boundary surface is approximated by a polynomial, whose coefficients are functions of x, distance along the boundary surface, and then introducing the thickness $\delta(x)$ of the boundary layer, it becomes

$$\frac{u}{u_1} = \alpha(x)\eta + \beta(x)\eta^2 + \cdots\cdots \quad (\eta = y/\delta), \tag{1}$$

where $u_1(x)$ is the velocity parallel to the boundary at the outer edge of the boundary layer. Some of the coefficients are determined by the boundary conditions

$$\left.\begin{aligned} &(u)_{\eta=0} = 0, \\ &(u)_{\eta=1} = u_1, \quad \left(\frac{\partial u}{\partial \eta}\right)_{\eta=1} = 0, \end{aligned}\right\} \tag{2}$$

the others and $\delta(x)$ must be determined so as (1) to satisfy the equation of the two-dimensional boundary layer in steady state:

$$\Delta(u;\, x,\, y) = u\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\int_0^y \frac{\partial u}{\partial x}\, dy - u_1 u_1' - \nu\frac{\partial^2 u}{\partial y^2} = 0 \tag{3}$$

as well as possible, where $u_1' = du_1/dx$.

We employ for this purpose the "moment-equations"[2]

$$\int_0^1 \Delta(u;\, x,\, \eta)\,\eta^n\, d\eta = 0, \qquad n = 0,\, 1,\, \cdots\cdots,\, N-1, \tag{4}$$

the number N of which is the same as the unknown functions. By this

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[1] A short account of papers (in Japanese) reported in this REPORTS, Vol. 4, No. 3 (1948), pp. 27–42, Vol. 5, No. 2 (1949), pp. 1–10.

[2] On a method of approximate solution of differential equations (in Japanese), in this REPORTS, Vol. 3, No. 3, pp. 29–35 (1947). A short account in English is given in the present volume.

process the differential expression Δ, instead of being identically zero through the boundary layer, becomes zero in $y(0, \delta)$ N times at least, these zero-points being distributed uniformly in a certain meaning.

If, moreover, we require $\Delta = 0$ at the both ends of the layer, (2) has to be supplemented by two further conditions:

$$\left(\frac{\partial^2 u}{\partial \eta^2}\right)_{\eta=0} = -u_1 \cdot \frac{u_1' \delta^2}{\nu}, \qquad \left(\frac{\partial^2 u}{\partial \eta^2}\right)_{\eta=1} = 0, \tag{2'}$$

and accordingly either the degree of the polynomial is to be increased by two or the number of the moment-equations (4) decreased by two. When the polynomial is of fourth degree (2) and (2′) determine all the coefficients but one, i.e. δ, so that the first moment-equation only is to be used. This first moment-equation is nothing but the momentum equation of v. Kármán, and this method of approximate integration reduces to that of Pohlhausen.

As the increase in number of moment-equations introduces much more increasing labor in numerical calculations here we content ourselves with the formulae for a polynomial of sixth degree. Conditions (2) and (2′) determine the polynomial of the form

$$u/u_1 \equiv f \equiv F(\eta) + \omega G(\eta) + \vartheta H(\eta) + \varphi K(\eta), \tag{5}$$

$$\left.\begin{array}{ll} F(\eta) = 2\eta - 2\eta^3 + \eta^4; & G(\eta) = \eta - 3\eta^2 + 3\eta^3 - \eta^4; \\ H(\eta) = -\eta + 6\eta^3 - 8\eta^4 + 3\eta^5; & K(\eta) = -\eta + 5\eta^3 - 5\eta^4 + \eta^6; \end{array}\right\} \tag{5'}$$

where ϑ, φ and

$$\omega = u_1'\delta^2/6\nu \tag{6}$$

are functions of x, to be determined by the moment-equations. ω is one sixth of the Pohlhausen's well-known parameter Λ. Be noted that the condition of separation is

$$2 + \omega - \vartheta - \varphi = 0. \tag{7}$$

Introducing (5), (5′) into Δ and making the moment-equations (4), we arrive, after some labor of numerical calculations, at the final expressions:

$$\left(\frac{A_n}{\omega} + a_n\right)\omega' + b_n \vartheta' + c_n \varphi' + \left(\frac{d_n}{6\omega} + B_n\right)\frac{u_1'}{u_1} - A_n \frac{u_1''}{u_1'} = 0, \tag{8}$$

$$(n = 0,\ 1,\ 2).$$

Here are A_n, B_n quadratic, a_n, b_n, c_n, d_n linear with regard to the variables ω, ϑ and φ, their coefficients as given in Table 1 (a) and 1 (b); dash denotes the differentiation with regard to x.

Table 1 (a)

	abs.	ω ω^2 $\omega\vartheta$	ϑ ϑ^2 $\vartheta\varphi$	φ φ^2 $\varphi\omega$
A_0	0.0587302	−0.0031746 −0.0019841 0.0075396	0.0126984 −0.0075036 −0.0159813	0.0146465 −0.0085193 0.0079725
A_1	0.0306746	−0.0040079 −0.0012202 0.0049893	0.0130665 −0.0052057 −0.0111986	0.0149532 −0.0060265 0.0053482
A_2	0.0180303	−0.0032829 −0.0007594 0.0032828	0.0106312 −0.0035588 −0.0077198	0.0122335 −0.0041875 0.0035582
B_0	0.5349206	−0.0626985 −0.0079365 0.0301588	0.1507937 −0.0300144 −0.0639253	0.1657288 −0.0340772 0.0318901
B_1	0.1676984	−0.0310317 −0.0036309 0.0149569	0.0822657 −0.0158227 −0.0340798	0.0919554 −0.0183660 0.0160356
B_2	0.0768976	−0.0176047 −0.0019517 0.0085137	0.0500503 −0.0093820 −0.0203848	0.0566959 −0.0110778 0.0092309

Table 1 (b)

	abs.	ω	ϑ	φ
a_0	−0.0063492	−0.0079365	0.0150793	0.0159451
a_1	−0.0016270	−0.0036309	0.0078573	0.0084868
a_2	−0.0003102	−0.0019517	0.0046464	0.0051059
b_0	0.0253969	0.0150795	−0.0300144	−0.0319626
b_1	0.0103607	0.0070955	−0.0158227	−0.0171718
b_2	0.0047691	0.0038673	−0.0093820	−0.0103425
c_0	0.0292930	0.0159450	−0.0319627	−0.0340772
c_1	0.0124279	0.0075487	−0.0169080	−0.0183660
c_2	0.0059579	0.0041250	−0.0100423	−0.0110778
d_0	−2.0000000	−1.0000000	1.0000000	1.0000000
d_1	−1.0000000	——	——	——
d_2	−0.6000000	0.1000000	−0.2000000	−0.2142857

The system of ordinary simultaneous differential equations of the first order (8), then, replaces the boundary layer equation approximately. When we take $\varphi \equiv \vartheta \equiv 0$ Pohlhausen's equation follows. His approximation have since been well discussed and its utility established except in certain cases, so we can hope that our refinement of his method can secure the utility in these exceptional cases.

In fact in the following examples, although two moment-equations only have been used, we see the results remarkably improved.

§ 2. **Examples of application.**—Several examples have been calculated and discussed in the original, but in this short report their principal results shall be mentioned quite briefly.

(i) *Homologous flow.* The homologous flow being defined by the condition that u/u_1 is a function of η with coefficients independent of x, it follows from (5) that ω, ϑ, φ in this case must be constant with regard to x, and then from (8)

$$\frac{u_1 u_1''}{u_1'^2} = \frac{\frac{d_n}{6\omega} + B_n}{A_n} \equiv \frac{m-1}{m}, \qquad (n = 0,\ 1,\ 2), \qquad (9)$$

where m is a constant with regard to x and n. We see from (9) that the homologous flow can exist only when the outer flow distribution is a power or a exponential function of x, and that the calculation of flow pattern is reduced to the solution of simulataneous algebraic equations.

(a) When $u_1 = ax^{-0.0905}$ the flow pattern is that of just separating.[3] By means of the first two equations ($n = 0,\ 1$) of (9), the approximate value of the exponent is -0.0915 (when $\varphi \equiv 0$, i.e. polynomial of fifth degree is assumed) or -0.0907 (when $\vartheta \equiv 0$, i.e. polynomial of sixth degree which lacks the fifth degree term is used). The value -0.1000 of the Pohlhausen's method is much improved by our one step. The value by the Howarth's method[4] is -0.0938.

(b) The Blasius solution of the uniform flow along a flat plate is well known. Our results which assume the fifth degree polynomial ($\varphi \equiv 0$;

[3] D. R. Hartree: On an equation occuring in Falkmer and Skan's approximate treatment of the equations of the boundary layer. Proc. Camb. Phil. Soc. 33 (1937).

[4] L. Howarth: On the solution of the laminar boundary layer equations. Proc. Roy. Soc. London (A). 164 (1938).

$n = 0$, 1) are given in the following Table 2, in comparison with the Blasius' and Pohlhausen's values, where τ_0, δ_1 being the skin friction and the displacement thickness (for the last colume see § 3).

Table 2

	[Blasius]	[Pohlhausen]	[$\varphi \equiv 0$; $n=0$, 1]	[$\varphi \equiv 1.8147$; $n=0$, 1]
$\sqrt{\frac{u_1}{\nu x}} \cdot \delta$	——	5.8356	5.5524	5.2382
$\sqrt{\frac{u_1 x}{\nu}} \cdot \frac{\tau_0}{\rho u_1^2}$	0.3321	0.3427	0.3358	0.3368
$\sqrt{\frac{u_1}{\nu x}} \cdot \delta_1$	1.7208	1.7507	1.7410	1.7630

(c) As an example of accelerating flow the case $u_1 = ax$ was taken up. The first two equations of (9) has no real root in either case of $\varphi \equiv 0$ or $\vartheta \equiv 0$. Instead of taking in, then, the third equation ($n = 2$) we adopted the expedient which takes φ or ϑ in as a controlling parameter. The results are as in the following Table 3, where $[y''(0)]_H$ is a coefficient defined by Hartree[3] in connection with the skin friction.

Table 3

ω	[Hartree]	2.0000	1.8950	1.8000	1.6000	[Pohlhansen]
ϑ		−2.1579	−2.0761	−2.0199	−1.9906	
φ		1.8914	1.8147	1.7657	1.7601	
$[y''(0)]_H$	1.2326	1.2316	1.2326	1.2337	1.2363	1.196

In all these examples velocity distributions calculated are quite similar to the correct ones, and we can judge the improvement of the approximation by the step adopted.

(ii) *Linear retarding flow.*—Another example of the known accurate calculations is the boundary layer of linearly retarding outer flow,[4] i.e.

$$u_1 = a_0 - a_1 x = a_0 (1-\xi), \qquad \xi \equiv a_1 x / a_0 . \tag{10}$$

We content ourselves with the first step of our approximation, and (8) gives

$$\left.\begin{aligned}&\mathfrak{A}_n\,\omega' + b_n\,\omega\vartheta' = (1-\xi)^{-1}\mathfrak{B}_n \qquad (n=0,\ 1),\\ &\varphi \equiv 0,\quad \mathfrak{A}_n = A_n + a_n\,\omega,\quad \mathfrak{B}_n = \frac{d_n}{6} + B_n\,\omega,\end{aligned}\right\}\tag{11}$$

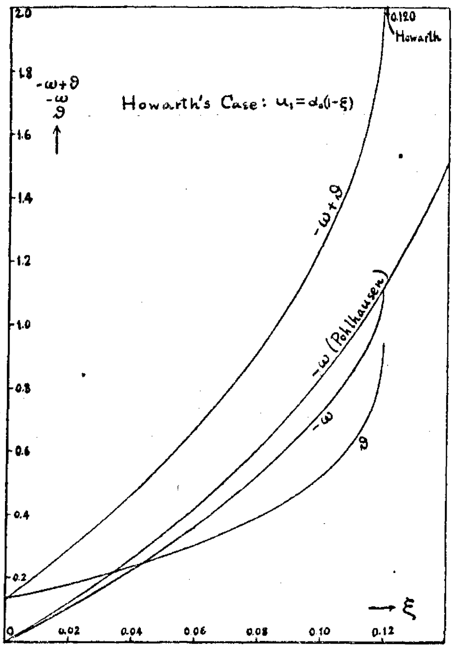

Fig. 1

where dashes denote the differentiation with regard to ξ. The leading edge condition is $\omega = 0$, and then from (11) $\vartheta = 0.13563$, $\omega' = -5.1518$, $\vartheta' = 2.1750$. With these initial values (11) were integrated step by step by the Runge-Kutta's method, the results as given in Fig. 1. At the separation point $-\omega+\vartheta$ is 2, which occurred at $\xi = 0.11925$, where also $d\omega/d\xi$, $d\vartheta/d\xi$ became infinite; these properties coinciding well with the Howarth's results. Velocity profiles coincide also, but all the details must be omitted here except the following comparison table (Table 4).

(iii) *Schubauer's elliptic cylinder.*—In the preceding examples we have seen that the approximation $[\varphi \equiv 0;\ n = 0,\ 1]$ is sufficient for ordinary purposes, so also the elliptic cylinder of Schubauer[5] was tried in this approximation. Inserting $\varphi \equiv 0$ in (8) and changing the variable ω in $u_1'\zeta$ ($\zeta = \delta^2/6\nu$), we have the convenient form

$$\left.\begin{aligned}-\zeta' &= \frac{1}{u_1}\,\frac{\mathfrak{B}_0 b_1 - \mathfrak{B}_1 b_0}{\mathfrak{A}_0 b_1 - \mathfrak{A}_1 b_0} + u_1''\,\zeta^2\,\frac{a_0 b_1 - a_1 b_0}{\mathfrak{A}_0 b_1 - \mathfrak{A}_1 b_0},\\ -\vartheta'\,\zeta &= \frac{1}{u_1}\,\frac{\mathfrak{A}_0\mathfrak{B}_1 - \mathfrak{A}_1\mathfrak{B}_0}{\mathfrak{A}_0 b_1 - \mathfrak{A}_1 b_0} + u_1''\,\zeta^2\,\frac{\mathfrak{A}_0 a_1 - \mathfrak{A}_1 a_0}{\mathfrak{A}_0 b_1 - \mathfrak{A}_1 b_0}.\end{aligned}\right\}\tag{12}$$

The used outer flow is as in Table 5, and the results as given

[5] G. B. Schubauer: Air flow in a separating laminar boundary layer, N.A.C.A., Tech. Rep., No. 527 (1935).

Table 4

ξ	$\sqrt{\frac{\nu}{a_1 a_0^2}} \cdot \left(\frac{\partial u}{\partial y}\right)_{y=0}$		$\sqrt{\frac{a_1}{\nu}} \cdot \delta_1$	
	[Howarth]	[$\varphi\equiv 0$; $n=0, 1$]	[Howarth]	[$\varphi\equiv 0$; $n=0, 1$]
0.00	∞	∞	0.000	0.0000
0.05	1.011	1.0257	0.447	0.4542
0.10	0.315	0.3215	0.794	0.8126
Sp. Pt.	0.000	0.0000	1.110	1.1352

in that table and in Fig. 2. In these I denotes the case where our approximation is connected to that of Pohlhausen at the pressure minimum $x = 1.30$ (where $\omega = 0$, $\vartheta = 0$; $\delta^2/\nu = 23.3$ sec.), II the case where the connection was done at $x = 0.5$ in the accelerating region

Table 5

x	u_1	u_1'	u_1''	(I) ω	(I) ϑ	(II) ω	(II) ϑ
0.5 0.55	1.215 1.231	0.353 0.287	−1.52 −1.15			0.3800	0.00000
0.6 0.65	1.244 1.254	0.233 0.191	−0.915 −0.750			0.3336	−0.08171
0.7 0.8	1.262 1.274	0.155 0.1094	−0.620 −0.420			0.2743	−0.08171
0.9 1.0	1.282 1.288	0.0734 0.0449	−0.285 −0.215			0.1791	−0.05434
1.1 1.2	1.292 1.295	0.0240 0.0089	−0.180 −0.175			0.0754	−0.01448
1.3 1.4	1.295 1.294	−0.0054 −0.0240	−0.210 −0.255	0.0000	0.0000	−0.0211	0.00987
1.5 1.6	1.290 1.284	−0.0500 −0.0750	−0.260 −0.255	−0.2180	0.0229	−0.2394	0.01319
1.7 1.8	1.275 1.264	−0.1000 −0.119	−0.220 −0.140	−0.5448	0.0821	−0.5890	0.06985
1.9 2.0	1.252 1.240	−0.126 −0.116	−0.005 −0.170	−0.8745	0.2819	−0.9408	0.2777
2.1	1.230	−0.096	−0.260	−0.8283	0.6245	−0.9050	0.6758

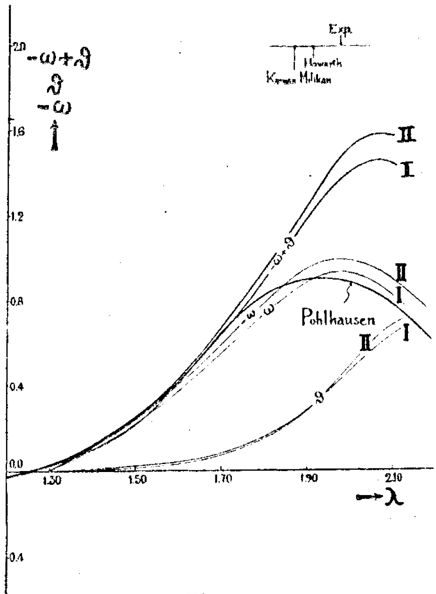

Fig. 2

(where $\omega=0.380$, $\vartheta=0$). In both the cases separation does not reached, contrary to the Howarth's or to the Kármán-Millikan's approximation, and above all, to the experimental result. But here we are reminded of the fact that the exact solution of the boundary layer equation is not always conform to the experiment near the separation point.

(iv) *Hiemenz's circular cylinder.*—Between the well known Blasius-Hiemenz solution and the Görtler's numerical solution[6] of a circular cylinder there is a discrepancy in the velocity distribution near the separation point. Our approximation $[\varphi\equiv0;\ n=0,\ 1]$ connected to that of Pohlhausen at the point of pressure minimum conformed to the Görtler's, contrary to our expectations, as is seen from Fig. 3. Our separation point 80.5° is also near to his value 80° rather than 82° of Hiemenz, and also of Pohlhausen's. The fact that our improved values of Pohlhausen's results lie on the side of Görtler's will stand by the Görtler's solution.

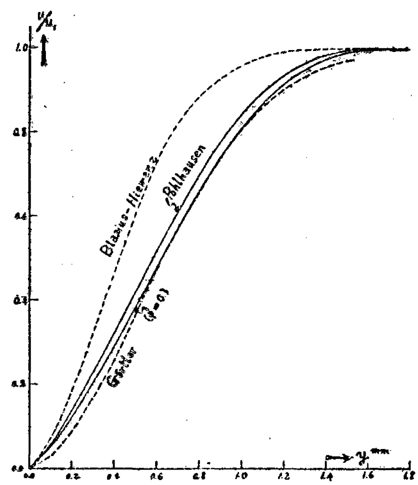

Fig 3.

§ 3. **Singular points of general equations etc.**—It is well known that when the singular points $\omega=2.00,-2.96$ (i.e. $\Lambda=12,-17.76$) of the Pohlhausen's calculation lie in the integration interval his approximation breaks down. Now in our calculation $[\varphi\equiv0;\ n=0,\ 1]$

[6] H. Görtler: Z.A.M.M., Vol. 19 (1939), pp. 129-140.

the variables are two and singularities make their appearance as lines in the ω, ϑ-plane. From (12) these lines are defined by

$$\mathfrak{A}_0 b_1 - \mathfrak{A}_1 b_0 = 0, \qquad (13)$$

whose solutions are given in Table 6 and inscribed in Fig. 4 as the lines (ω', $\varphi' = \infty$).

In the same figure is inscribed the separation line, which expresses the condition of separation (7), and the point Tg.-Ent. which expresses the

Table 6

ϑ	ω_1	ω_2	ω_3
2.5	1.156	11.987	−7.025
2.25	0.768	11.004	−7.649
2.0	0.416	10.002	−8.300
1.75	0.066	9.004	−8.951
1.5	−0.276	8.000	−9.608
1.25	−0.620	7.000	−10.264
1.0	−0.963	6.005	−10.924
0.75	−1.302	5.001	−11.583
0.5	−1.640	4.003	−12.245
0.25	−1.973	3.000	−12.910
0.0	−2.306	1.999	−13.575
−0.25	−2.643	1.001	−14.240
−0.5	−2.978	0.003	−14.908
−0.75	−3.308	−0.998	−15.577
−1.0	−3.638	−1.998	−16.247
−1.25	−3.966	−2.999	−16.917

condition of tangential entrance of main flow ($\omega = 0$, $\vartheta = 0.13563$). At the pressure minimum, where the connection to the Pohlhausen's being intended, $\omega = 0$, $\vartheta = 0$ and from this point on to the separation there is no singularity to disturb the integration of the equation. One branch of singularity curves which lies near the separation line seems to coincide with it, if the approximation were further advanced, and then, when integration curve reaches it beforehand of the separation line, this point of encounter may be taken as an approximate position of separation point.

The dotted lines in Fig. 4 are several integration curves which have been obtained in the precedings. Their forms differ considerably from each other and we can see the reason why Pohlhausen's approximation, which forces every solution to lie on the ω-axis, fails sometimes. Howarth's method, which uses his fundamental solution as the basis of the approximation, can be interpreted from our point of view as forcing every solution on the line "Howarth" in Fig. 4. Our method itself forces all the solutions on the ω, ϑ-plane, or, in other words, to the fifth degree polynomials, but the region of its applicability has considerably been enlarged.

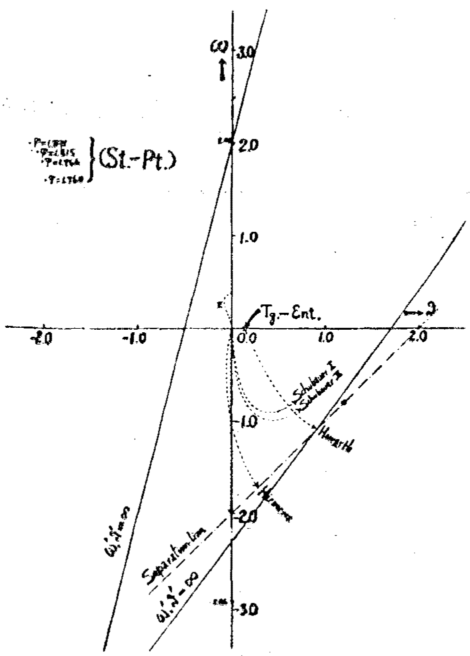

Fig. 4

For the application of the method from up to the stagnation point, the stagnation condition in necessary. At this point $u_1 = 0$, and then from (12), as this point must not be a singular point of them,

$$\mathfrak{B}_0 = \mathfrak{B}_1 = 0. \tag{14}$$

(14) are due to determine values of ω, ϑ of the stagnation point, but these are the very equations in § 2 (c) and have no real solution. That is, the approximation $[\varphi \equiv 0;\ n = 0,\ 1]$ fails to be that. Leaving φ as a parameter, as there, (14) specify a certain space curve in the ω, ϑ, φ-space, which does not intersect with the ω, ϑ-plane but does intersect with certain planes parallel to it. Table 3 is nothing but the examples of the coordinates of these intersection points; the projections of these points to the ω, ϑ-plane are inscribed in Fig. 4.

With a certain value φ_0 of φ the approximation $[\varphi \equiv \varphi_0;\ n = 0,\ 1]$ can be used from the forward stagnation point to the separation point throughly, provided that the singular lines do not disturb the integration. For a trial $\varphi_0 = 1.8147$ was assumed, and the results are as given in Fig. 5, all the numericals being here omitted. The selection of φ_0 happaned to be a little too small, resulting in one branch of singular line very near to the stagnation point. However. this example is sufficient to indicate the existence of the one with desired properties. The degree of approximation also seems to be preserved. For example, the exponent of §2 (a) is -0.0901, and for a plate edgewise placed in uniform flow it is as seen in the last colume of Table 2, according to the approximation [$\varphi \equiv 1.8147$; $n=0, 1$].

In all the foregoings the degrees of approximation were estimated by comparison with known exact solutions. Applying the method to a new problem, such comparison fails, and the need of estimation is for this very case. However, such an estimation of our method is difficult, as in usual cases. In general, we may regard the solutions satisfying the boundary conditions sufficiently well, so that the deviations of $\Delta(u; x, y)$ from identical zero can haply be a certain index for the degree of approximation. As a standard example we took the Howarth's case (§2 (ii)), and with our solution, which agreed well with the Howarth's one, the following table (Table 7) is constructed. $\xi=0.08$ is an ordinary point in the retarding part, and $\xi=0.119$ the representative of points just before the separation.

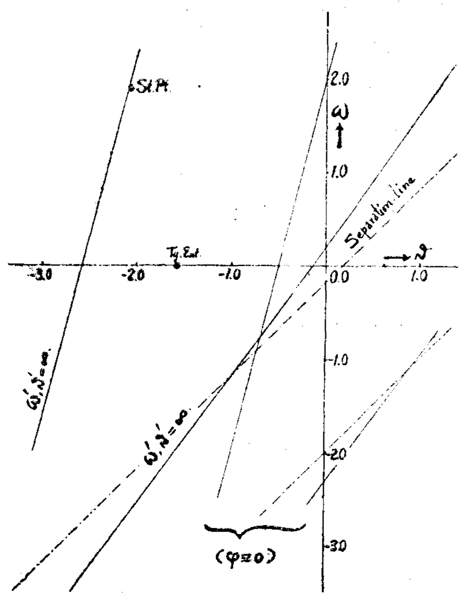

Fig. 5

One of the neglected terms in the boundary layer equation is also estimated with the same solution, and compared with Δ in the last colume of Table 7. In there R is the Raynolds' number defined by $a_0^2/a_1\nu$, and the Reynolds' number refered to the displacement thickness $u_1\delta_1/\nu$ is, in this case, almost equal to $\sqrt{R}$. Some laborious calculations of the estimation are here completely omitted. The second derivative of an approximate solution is haply too rough to take it for the representative of the deviations of the boundary layer equation itself from the Navier-Stokes'. But it indicates that, an approximate solution of the boundary layer equation is not always that of the Navier-Stokes', especially near the separation point of a flow with moderate Reynolds' number, so that the separation point which the former gives is not expected to fall on the experimental one.

This well-known deviation cannot be avoided even when the experimental pressure distribution $p(x, t)$ be used; the neglection of $\nu\partial^2u/\partial x^2$ have serious effect, which indicates that the flow state near the separation point is affected as well by the upstream state as the downstream state, this latter influence being not covered by the adoption of actual pressure distribution.

Table 7

	$\xi=0.08$			$\xi=0.119$			
η	$-\frac{1}{\rho}\frac{\partial p}{\partial x}$	$\nu\frac{\partial^2 u}{\partial y^2}$	Δ	$-\frac{1}{\rho}\frac{\partial p}{\partial x}$	$\nu\frac{\partial^2 u}{\partial y^2}$	Δ	$\nu\frac{\partial^2 u}{\partial x^2}$
0.0	−0.920	0.920	0.000	−0.881	0.881	0.000	$0.000\times10^4\times R^{-1}$
0.1	〃	0.591	0.151	〃	0.806	0.024	−2.450
0.2	〃	0.324	0.166	〃	0.614	0.065	−4.533
0.3	〃	−0.041	0.057	〃	0.348	0.077	−5.984
0.4	〃	−0.397	−0.121	〃	0.049	0.020	−6.651
0.5	〃	−0.702	−0.273	〃	−0.238	−0.085	−6.492
0.6	〃	−0.913	−0.303	〃	−0.470	−0.181	−5.582
0.7	〃	−0.990	−0.157	〃	−0.606	−0.152	−4.111
0.8	〃	−0.891	0.110	〃	−0.602	0.006	−2.386
0.9	〃	−0.576	0.295	〃	−0.414	0.177	−0.834
1.0	−0.920	0.000	0.000	−0.881	0.000	0.000	0.000

N.B. All the numericals are to be multiplied by a factor $\alpha_0\alpha_1$; $R=\alpha_0^2/\alpha_1\nu$.

(14 Apr. 1950)