ON THE CRITICAL TRACTIVE FORCES

KURIHARA, Michinori
Research Institute for Fluid Engineering, Kyushu University

TSUBAKI, Toichiro
Research Institute for Fluid Engineering, Kyushu University

https://doi.org/10.15017/7158053

出版情報：流體工學研究所報告. 6 (2), pp.70-86, 1950-01. 九州大學流體工學研究所

ON THE CRITICAL TRACTIVE FORCES[1]

By

Michinori KURIHARA and Tōichirō TSUBAKI

From Vol. 4. No. 3.

Abstract. The existing tractive-force formulas are various. Taking into account the effects of turbulence all the available experimental results are analyzed and a definite curve is obtained by plotting $\tau_0/\rho gk(\rho-\rho_0)$ (inversely proportional to the turbulence factor) against $k(\tau_0/\rho_0)^{\frac{1}{2}}/\beta\nu$, where τ_0 is the critical tractive force, k and ρ are mean diameter and density of the grains, ρ_0 and ν are density and kinematic viscosity of the fluid, g is the acceleration of gravity, and β is a parameter allowing for the non-uniformity of the grains (Fig. 2). For practical use a similar curve is obtained (Fig. 3) and a set of simple empirical formulas (dimensionally correct) is suggested (equation (3.4)). In either curve there is found a distinct minimum at about $k(\tau_0/\rho_0)^{\frac{1}{2}}/\beta\nu = 25$ or $k = 1$ mm for usual sand grains, which corresponds to a maximum of the turbulence factor.

It is pointed out that the occurrence of the maximum of turbulence factor is attributed to a synchronizing phenomenon between the grain size and scale of turbulence. Taking into account the fluctuations of pressure gradient and using a simple turbulence model the forces acting on the grains are estimated. The theoretical results agree well with the experimental results.

§ 1. **Introduction.** So far as the hydraulic engineer is concerned, the movement of bed-load material in artificial and natural watercourses is of great importance. If the bed-load material consist of particles, they share forces due to shearing stress of the fluid at the bottom i.e. tractive force, with one another. When the tractive force exceeds a certain critical value particles begin to move. It has long been known that the critical tractive force depends alone upon grain characteristics. However the ex-

[1] From "On the Critical Tractive Forces" (in Japanese), M. Kurihara and T. Tsubaki, Rep. Res. Inst. Fluid Engng., Kyūshū Univ. 4 (1948) 1–26, being retouched in some points.

perimental results show considerable scattering and the proposed empirical formulas are various. Much of the discrepancy is undoubtedly due to the complexity of the problem.

Now many efforts have been made to inquire i) the physical mechanism underlying the critical tractive force and ii) the effects of the non-uniformity of grains, regarding as dominant causes of scattering of measured points. But it is clear that the first is essential and the manifoldness of the existing empirical forumlas seems to be preferable as an important key for exploration.

The purpose of the present paper is to re-examine the whole problem and to inquire mainly the physical mechanism of tractive force apart from the question of non-uniformity.

I. Empirical Formulas

§ 2. The existing empirical forumlas and the point in question. In a channel flow with given characteristics the critical tractive force τ_0 must be determined solely by density, mean diameter, shape, porosity, arrangement and size frequency distribution of grains setting apart the question of the criterion for movement.

Research workers have attempted to analyse the observational data considering τ_0 as a function of density $\rho'(=\rho-\rho_0$, where ρ, ρ_0 are density of grains and of fluid), mean diameter k and a parameter which allows for non-uniformity of the other characters of grains. The existing empirical formulas are as follows[1]:

i) *Schoklitsch* tractive-force formula:

$$\tau_0 = c_1\sqrt{\rho' g \xi V}\,, \tag{2.1}$$

where g is the acceleration of gravity, V the volume of a grain, ξ a form factor being 1 for spherical grains and c a constant (the same in followings).

ii) *Krey* tractive-force formula:

---

[1] T. Sakai, "Critical Tractive Forces for the gravels of River Bed," J. Civ. Engng. Soc. Japan, **31** (1946) 1. Y. L. Chang, "Laboratory Investigation of Fluid Traction and Transportation," Amer. Soc. Civ. Engng. **63** (1937) 1701. Trans. Amer. Geophy. Union, Eighteenth annual meeting (1937) 456.

$$\tau_0 = c_2 \rho' g k. \tag{2.2}$$

The formula is based on tests with sands practically uniform in size.

iii) *Kramer* tractive-force formula:

$$\tau_0 = c_3 \frac{\rho' g k}{M}, \tag{2.3}$$

in which M is the "uniformity-modulus" related to size-frequency distribution of grains and being unit for uniform grain sand.

iv) *U.S. Waterways Experiment Station* tractive-force formula:

$$\tau_0 = c_4 \sqrt{\frac{\rho' g k}{M}}, \tag{2.4}$$

the basis of which is the data Kramer used plus the results of tests with eight mixtures conducted in the Experiment Station.

v) *Indri* tractive-force formula:

$$\tau_0 = a \frac{\rho' g k}{M} + b, \tag{2.5}$$

where a, b are constant being different according to $k \gtrless 1$ mm.

vi) *Aki and Sato* tractive-force formula:

$$\tau_0 = c_6 \rho' g k \lambda, \tag{2.6}$$

where λ is a parameter related to size-frequency.

vii) *Sakai* tractive-force formula:

$$\tau_0 = c_7 \rho' g \beta k^{6/5}, \tag{2.7}$$

where β is a parameter connected with Kramer's uniformity modulus M by $\beta = \dfrac{2+M}{1+2M}$.

viii) *Chang* tractive-force formula:

$$\left.\begin{aligned} \tau_0 &= c_8 \left[\frac{\rho' k}{\rho_0 M}\right] \quad \text{for} \quad \frac{\rho' k}{\rho_0 M} > 2.0, \\ \tau_0 &= c_8' \left[\frac{\rho' k}{\rho_0 M}\right]^{\frac{1}{2}} \quad \text{for} \quad \frac{\rho' k}{\rho_0 M} < 2.0, \end{aligned}\right\}. \tag{2.8}$$

A comparison of the above equations shows that the non-uniformity of form and size of grains is practically important and many efforts have been made to allow it. However these equations must hold for the case of uniform spherical grains. Therefore, if the parameters ξ, M, λ, β are put equal to unit, they must show at least the same law apart from numerical coefficients. But, in fact, only the formulas (2.2), (2.3) and (2.6) which are dimensionally correct give linear law between τ_0 and $\rho' gk$ and the others give different power laws $\tau_0 \infty k^{\frac{1}{2}}$, $k^{\frac{1}{3}}$, $k^{\frac{6}{5}}$. Since the discrepancy can not be attributed to non-uniformity, we may be led to the following speculation: every one of the above formulas is correct and merely the regions of their validity are different according to data used and their objects aimed. In fact, Indri, Chang tractive-formulas seem to point out existence of a critical grain diameter at which (τ_0, $\rho' g k$) curve changes slope more or less abruptly (see Fig. 1). If so, it is natural to infer that the critical grain diameter may presumably be attributed to some peculiar phenomena and that exploration of the mechanism of critical tractive force must based on this point of view.

Although many improvements of empirical formulas have been done to reduce the scattering of measured points by taking account of the non-uniformity effects of grains, the residual is not yet so small. As its causes we can mention i) the difficulty to allow the non-uniformity of grains by a simple device, ii) the non-uniformity of hydrodynamical characteristics of flumes used, iii) various definitions of the critical tractive force adopted, etc.

Among these we can by no means ignore the third, when we analyse systematically the data given by various investigaters.

The definition of the critical tractive force adopted by H. Kramer and U.S. Waterways Experiment Station is such that which effects a general movement of the material up to and including the largest component particles. H. Krey's criterion is beginning of noticeable or lively movement and A. Schoklitsch's is commencement of motion of a single particle resting freely on a bed of material of like size. Thus it is natural that the measured values show systematical differences according to the criterions and we must make allowances for this circumstance in a theoretical investigation.

§3. **Dimensional analysis.** When the flow is turbulent the critical tractive force must be considerably influenced by turbulent motion. If we assume as usual that the turbulent state can be specified by two parameters, namely its intensity v' and its linear dimension λ, then the independent quanitities which enter into consideration are τ_0, ρ, ρ_0, k, g, ν, v', λ and in general the mean velocity v. And there are seven independent non-dimensional quantities ρ/ρ_0, λ/k, $\tau_0/\rho g k$, vk/ν, $v'k/\nu$, $v/\sqrt{kg}$, $v'/\sqrt{kg}$. Since it is not conceivable physically the density and acceleration of gravity to play any rôle other than the gravitational force in the fluid acting on grains, we can safely omit ρ/ρ_0 and Froude's numbers and write

$$\frac{\tau_0}{\rho' g k} = f\left(\frac{vk}{\nu},\ \frac{v'k}{\nu},\ \frac{\lambda}{k}\right). \tag{3.1}$$

The equation holds in general for any flume regarding v, v' and λ as characterizing quantities of the flow near the bed. In the case of a uniform steady flow the state of turbulent flow near the bed can be determined only by τ_0, ρ, ν and k, so that again by dimensional consideration vk/ν, $v'k/\nu$ and λ/k are functions of merely $v_* k/\nu$, where $v_* = \sqrt{\tau_0/\rho}$.

Thus (3.1) becomes $\tau_0/\rho' gk = f(v_* k/\nu)$. Finally introducing the parameter β mentioned in § 2, vii) to allow the non-uniformity of grains we have

$$\frac{\tau_0}{\beta\rho' gk} = f\left(\frac{v_* k}{\beta\nu}\right). \tag{3.2}$$

Now almost all the existing data concerning critical tractive force have been tabulated in the papers of Y. L. Chang[1] and T. Sakai.[2] The measurements of Y. L. Chang himself show systematically greater values than others. This systematic deviation seems to be caused by his use of a accelerated flume, in which turbulence is weaker than in an ordinary uniform flow. Thus although his data are numerous we have regretfully rejected them. Because of a similar reason or considerable scattering C. M. White's[3] and E. I. Indri's are also rejected. The data thus selected are shown in Table I with the calculated values of $\tau_0/\beta\rho' gk$ and $v_* k/\beta\nu$.

With these data, by plotting log τ_0 and $\tau_0/\beta\rho' gk$ against log $\beta\rho' gk$ and log $v_* k/\beta\nu$ respectively we obtain Fig. 1, 2. The later shows the existence of the relation-ship (3.2) and that the slight change of slope in Fig. 1 at about $\beta\rho' gk = 250$ which gives rise to various confusions in establishment of tractive-force formulas turns into a distinct minimum at about $v_* k/\nu\beta = 25$ in Fig. 2.

---

1), 2) Loc. cit.

3) C. M. White, The equilibrium of grains on the bed of a stream, Proc. Roy. Soc. A. **174** (1940) 322.

Table I

Investigator	τ_0 dynes/cm^2	Density of grain[1] ρ gr/cm^3	Mean Dia. k cm	β	v_* [2]	$\frac{v_* k}{\beta\nu}$ [2]	$\frac{\tau_0}{\beta\rho' gk}$	$\left\{102\times10^{-7}\frac{g}{\beta\nu^2}\frac{\rho'}{\rho_0}\right\}^{\frac{1}{3}}\times k$
F. Schaffernak	7.359	2.65	0.1536	1.20	2.71	34.7	2.47×10^{-2}	0.171
A. Schoklitsch	53.40	2.60	0.6520	1.00	7.31	477	5.22×10^{-2}	0.763
	25.84	2.60	0.405	1.00	5.08	206	4.07	0.474
	11.09	2.60	0.2256	1.00	3.33	75.2	3.14	0.264
	4.32	2.60	0.1240	1.00	2.08	25.8	2.22	0.145
	2.75	2.60	0.0916	1.00	1.66	15.2	1.92	0.107
H. Krey	2.35	2.68	0.0376	1.07	1.53	5.40	3.56×10^{-2}	0.436
	2.75	2.61	0.0526	1.05	1.66	8.31	3.15	0.0605
	3.14	2.57	0.0800	1.09	1.77	13.0	2.34	0.0904
Prussian	5.01	2.65	0.1154	1.48	2.24	17.5	1.81×10^{-2}	0.120
Exp. Institute	4.12	2.65	0.0846	1.78	2.03	17.2	1.69×10^{-2}	0.0863
	5.69	2.65	0.0836	1.40	2.39	14.2	3.01	0.0886
	5.49	2.65	0.0744	1.33	2.34	13.1	3.43	0.0797
	2.35	2.65	0.0244	1.30	1.53	2.88	4.60	0.0364
	5.49	2.65	0.0806	1.37	2.34	13.8	3.07	0.0854
	4.91	2.65	0.0686	1.31	2.22	11.6	3.37	0.0741
H. Engels	9.81	2.65	0.1484	1.53	3.13	30.4	2.67×10^{-2}	0.153
G. K. Gilbert	3.83	2.69	0.0576	1.04	1.96	10.9	3.86×10^{-2}	0.0674
	5.99	2.69	0.1906	1.12	2.45	41.6	1.69	0.219
	29.0	2.69	0.3710	1.05	5.38	190	4.49	0.434
	48.1	2.69	0.5296	1.07	6.93	343	5.12	0.615
H. Kramer	5.11	2.70	0.0706	1.38	2.26	11.5	3.15×10^{-2}	0.0756
	3.83	2.70	0.0558	1.28	1.96	8.54	3.22	0.0614
	4.81	2.70	0.0800	1.32	2.19	13.3	2.73	0.0872
U.S. Waterways Exp. Station	4.64	2.65	0.0586	1.46	2.15	8.65	3.35×10^{-2}	0.0609
	4.21	2.65	0.0541	1.30	2.05	8.53	3.70×10^{-2}	0.0585
	4.21	2.65	0.0525	1.22	2.05	8.83	4.06	0.0583
	4.02	2.65	0.0506	1.33	2.01	7.81	3.69	0.0541
	3.83	2.65	0.0483	1.30	1.96	7.28	3.77	0.0521
	2.87	2.65	0.0347	1.15	1.69	5.11	4.45	0.0371
	3.16	2.65	0.0310	1.23	1.78	4.48	5.12	0.0341
	2.25	2.65	0.0205	1.21	1.50	2.54	5.61	0.0227
	27.8	2.65	0.4077	1.20	5.27	179	3.51	0.453
Aki and Satō	2.16	2.70	0.022	1.14	1.47	2.84	5.17×10^{-2}	0.0251
	7.35	2.70	0.070	1.60	2.71	11.9	3.93	0.0714
	9.60	2.70	0.122	1.38	3.10	27.4	3.43	0.131
Ishihara	6.37	2.58	0.071	1.45	2.52	12.4	4.01×10^{-2}	0.0732
	5.19	2.58	0.092	1.35	2.28	15.5	2.70	0.0976

Remarks: [1] Since all the measurements are concerned with water channels, the density has not been distinguished from the specific weight.

[2] In $\rho'=\rho-\rho_0$ the density of water has been put equal to unit. As the molecular viscosity of water ν we have used 0.01 corresponding to 20°C.

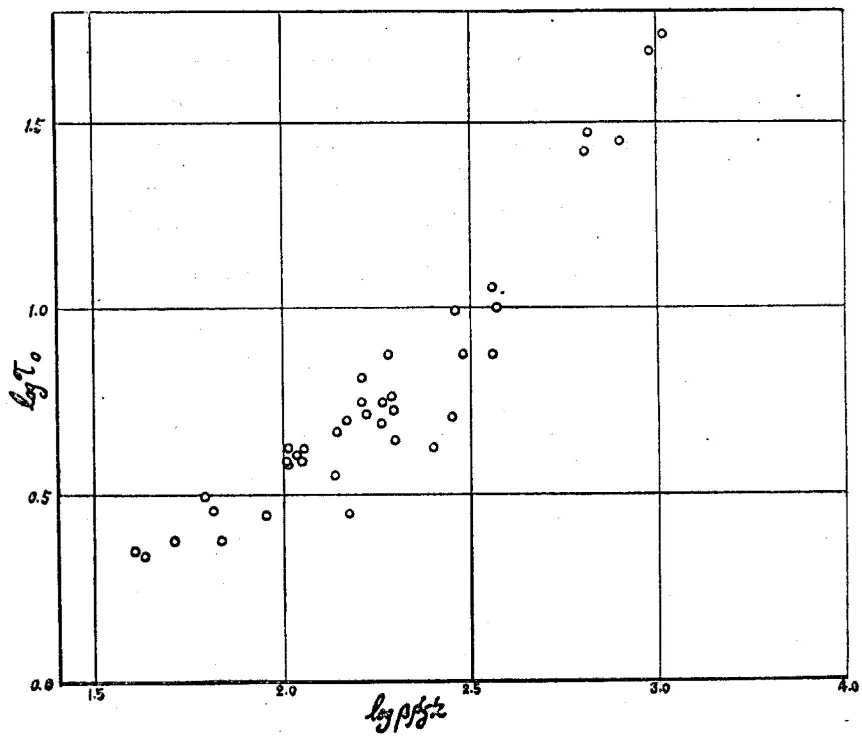
1.5
1.0
0.5
0.0
1.5
2.0
2.5
3.0
4.0

Fig. 1

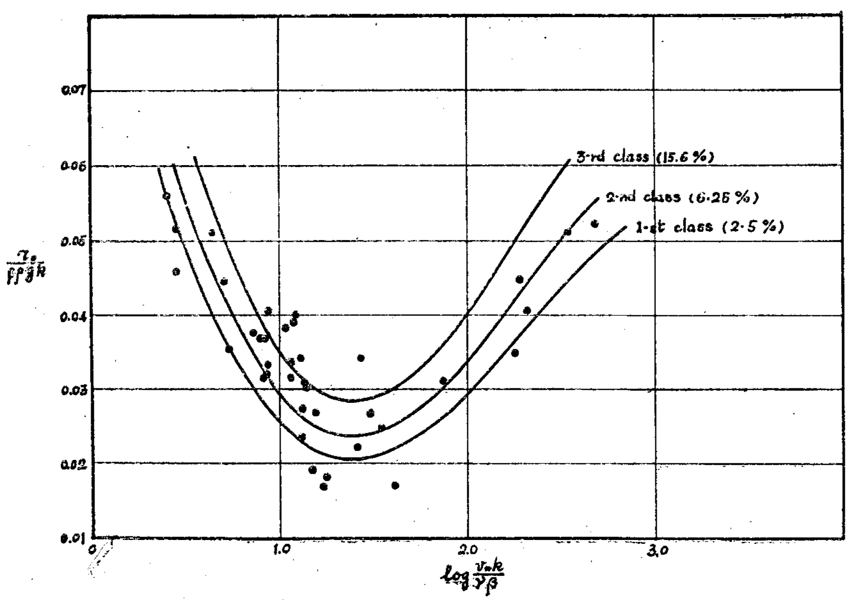
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
1.0
2.0
3.0
3-rd class (15.6 %)
2-nd class (6.25 %)
1-st class (2.5 %)

Fig. 2

By C. M. White's Theory, in a laminar flow $\tau/\beta\rho' gk$ is a constant determined by surface arrangement and angle of repose of grains, but in a turbulent flow it is reduced by $1/T$, where T is called as "turbulence factor." Accordingly the minimum of $\tau_0/\beta\rho' gk$ corresponds to a maximum of turbulence factor. Fig. 2 shows thus existence of some peculiar phenomenon in which turbulence plays an important rôle and any theory concerning the critical tractive forces must give a satisfactory explanation of this peculiarity.

Equation (3.2) can be rewritten in a convenient form for practical use by a transformation, $\left(\frac{\tau_0}{\beta\rho' gk}, \frac{v_* k}{\nu\beta}\right)$ to $\left(\frac{\tau_0}{\beta\rho' gk}, \mu = \left(\frac{v_* k}{\nu\beta}\right)^{\frac{2}{3}}\left(\frac{\tau_0}{\beta\rho' gk}\right)^{-\frac{1}{3}}\right)$, namely

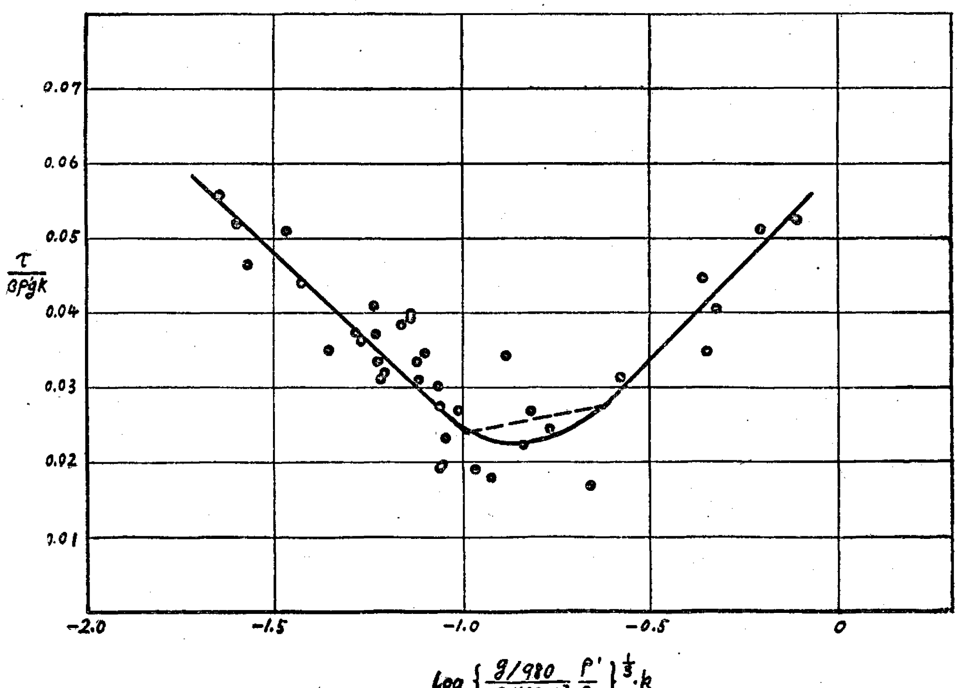

Fig. 3

$$\frac{\tau_0}{\beta\rho' gk} = \psi\left(\sqrt[3]{\frac{g\rho'}{\beta\nu^2\rho_0}} \cdot k\right). \tag{3.3}$$

This relation is shown in Fig. 3 using the data in Table 1. The minimum corresponding to that at $v_* k/\nu\beta = 25$ in Fig. 2 appears at about

$\log\left\{\frac{g/980}{\beta(100\nu)^2}\cdot\frac{\rho'}{\rho_0}\right\}^{\frac{1}{3}}\cdot k=\bar{1}{\cdot}15$ which gives for a usual condition ($\beta \doteqdot 1.0$, $\rho'/\rho_0 \doteqdot 1.6$, $\nu \doteqdot 0.1$) $k \doteqdot 0.12$ cm.

Since the argument of ψ in (3.3) involves β and ρ'/ρ_0 as cubic root and ν as 2/3 power, it is approximately a function of only k, the grain diameter, and that proportional. Thus we may propose a system of empirical formulas obtainable from Fig. 3,

$$\left.\begin{aligned}
\frac{\tau_0}{\beta\rho' gk} &= -0.047\log_{10}X-0.023, && \log_{10}X<-1.0,\\
\frac{\tau_0}{\beta\rho' gk} &= 0.01\log_{10}X+0.034, && -1.0<\log_{10}X<-0.6,\\
\frac{\tau_0}{\beta\rho' gk} &= 0.0517\log_{10}X+0.057, && -0.6<\log_{10}X,
\end{aligned}\right\} \qquad (3.4)$$

where

$$X=\left\{1.02\times10^{-7}\frac{g}{\beta\nu^2}\cdot\frac{\rho'}{\rho_0}\right\}^{\frac{1}{3}}\cdot k \doteqdot \sqrt[3]{\frac{\rho'}{\rho_0}}\cdot k$$

It is here mentioned that in the last approximate expression of X k must be measured in cm.

II. Theoretical Considerations

§ **4.** We shall now consider physical meaning of the occurrence of the sharp minimum in the critical tractive force diagram at about $v_* k/\nu\beta = 25$.

By observing other turbulent phenomena[1] we may conclude the occurence of the minimum to be intimately connected with the fact that as Reynolds number decreases molecular viscosity become dominant compared with turbulent viscosity, in other words, the inertia effects becomes smaller than the effects of pressure gradient and molecular viscosity.

Thus following Taylor's discussion[2] we are led to a conjecture that the forces acting on grains are primarily due to the transverse pressure

[1] For example: i) The resistance of rough pipes show the just noticeable change from quadratic law at $v_* k/\nu \doteqdot 25$ when Reynolds number decreases, ii) the form drag of a cylinder becomes important abruptly at Reynolds number $\doteqdot 50$ as it increases, iii) in a smooth pipe molecular viscosity becomes 10% of turbulent viscosity at the distance $v^* y/\nu = 32$ when the wall is approached, etc.

[2] G. I. Taylor, Note on the Distribution of Turbulent Velocities in a Fluid near a Solid Wall, Proc. Roy. Soc. A, **135** (1932).

gradient and that the occurrence of its maximum effect may be caused by a synchronizing phenomenon between the grain size and the scale of turbulence which produces the pressure gradient.

In the turbulent flow through a channel with roughness k not so large the distribution of velocity defect is the same as the case of a smooth channel. So that in the large part of the whole region except the small region near the walls the turbulence is mainly produced by the action of mixing processes, the influences of roughness being negligible. Then we can provisionally divide the whole region into the inner and outer layer. The former is a small region including grains its turbulent energy being produced by roughness and by influences of the outer turbulence. The later is a large region outside of the inner layer where the turbulent energy being supplied from the energy of mean flow and the influences of roughness being negligible.

Though this definition is rather arbitrary, following the discussion of Taylor we can infer that because of negligible vertical component of turbulent velocities the fluctuations of pressure gradient near the grains are produced by that of the outer layer the directions of which are nearly transverse to the mean flow. Thus we can consider a equivalent layer which represents the outer layer in such a way that the characteristics of pressure fluctuations in it are equivalent to that of the effective pressure fluctuations of the outer layer acting on the inner layer.

The bigger the grains, the greater the distance of the equivalent layer from the wall is. On the other hand, if k is less than the thickness of the laminar sub-layer Δ the height of the equivalent layer must be the same as that for Δ. So that we may put in general as an approximation.

$$\frac{v_*\xi}{\nu} = a\frac{v_*\Delta}{\nu} + b\frac{v_*k}{\nu}, \tag{4.1}$$

where ξ denotes the height of the equivalent layer measured from the virtual laminar sub-layer and a, b are contants.

Let us take the x-axis along the wall in the direction of mean flow, the y-axis perpendicularly to the wall and the z-axis transversely to the mean flow. According to Taylor's discussion the scale of turbulence in x-direction is far greater than that in z-direction. Further the forces due to pressure gradient acting on a grain are negligible when the scale of turbulence is very large. So that we may be satisfied by considering merely the linear dimension of turbulence in z-direction, M. On the other hand it might be expected that the scales of turbulence in y- and z-direction become comparable as ξ increases. Then we may assume M to be same order of the mixing length,

$$\frac{v_* M}{\nu} = b' \frac{v_* \xi}{\nu}, \tag{4.2}$$

since as seen later ξ/Δ is not small.

Inserting (4.2) to (4.1) we obtain

$$M = c_1 \Delta + c_2 k, \quad c_1 = ab', \quad c_2 = bb'. \tag{4.3}$$

Now the resultant force exerted by fluctuating pressure gradient on a grain is very small in either cases of $M/k \gg 1$ or $\ll 1$ and only considerable when M is comparable with k. If consequently c_2 is suitably small, the effects of turbulence increases and then decrease through a maximum as k increases from zero. In the following sections we shall examine this idea some-what in detail using a simple turbulence model.

§ **5.** Neglecting the x- component of fluctuating gradient we assume a sinuous variation of pressure fluctuation P' in the z-direction $P' = P \sin 2\pi(z+\delta)/M$, where δ denotes phase proportional to time. Further we assume the spherical form of diameter k for grains. Then the force acting on a grain in the z-direction becomes

$$\Delta F_1 = MP \cos 2\pi \frac{\delta}{M} \left[k \cos \pi \frac{k}{M} - \frac{M}{\pi} \sin \frac{\pi k}{M} \right].$$

Therefore the root of mean square of the force acting on unit cross section of the grain Δf_1 is given by

$$\left.\begin{aligned} \Delta f_1 &= 2\sqrt{2}\, P f_1(\zeta) = 4\, \Delta P f_1(\zeta), \\ f_1(\zeta) &= \frac{\sin \zeta - \zeta \cos \zeta}{\zeta^2}, \quad \zeta = \frac{\pi k}{M} \end{aligned}\right\}, \tag{5.1}$$

where ΔP is the root of mean square of pressure fluctuations in the equivalent layer.

As mentioned in the preceding section we can apply the theory of turbulent flow in a smooth pipe to the equivalent layer. So that by the paper I[1] the turbulent energy E at a point not far distant from the wall is given by

$$E = E_0 \Psi^2(\chi), \quad \chi = 2K \frac{v_* \xi}{\nu}, \tag{5.2}$$

where

$$\left. \begin{aligned} v_*^2 &= \frac{\tau}{\rho} = \sqrt{\eta}\, E_0, \quad \sqrt{\eta} = 0.210, \\ \Psi(\chi) &= -\frac{1}{\chi} + \sqrt{1 + \frac{1}{\chi^2}} \end{aligned} \right\}, \tag{5.3}$$

in which τ denotes the shearing stress at the wall and $K = 0.40$.

On the other hand G. I. Taylor[2] has shown

$$\Delta P = \alpha \rho E, \tag{5.4}$$

where α is a numerical constant depending on the mode of fluctuations. As it is nearly the case, when the turbulence is two dimensional and moreover the one component is dominant, we have $\alpha = \sqrt{2}$. Thus we obtain by (5.1), (5.2), (5.3) and (5.4)

$$\Delta f_1 = 4\alpha \frac{\tau}{\sqrt{\eta}} f_1(\zeta)\, \Psi^2(\chi), \quad \alpha \doteqdot \sqrt{2}. \tag{5.5}$$

Next we must estimate the drag due to the turbulent motion in the inner layer considered as being produced by the outer turbulence, since that part of the turbulent motion which is caused by grains would not give appreciable influences, its scale being very small compared with grain size. Let u', w' be the velocity components of such a virtual turbulent motion which is equivalent to that near a smooth wall and Δu, Δw the roots of mean square of u' w'. Then the root of mean square of u', w' on the surface of a grain are given by $\Delta u \{(\sin \zeta/2)/\zeta/2\}$, $\Delta w \{(\sin \zeta/2)/\zeta/2\}$, since the wave length of velocity fluctuations are nearly equal to twice of pressure fluctuations (u' $w' \infty \sin \pi(z+\delta)/M$). So that neglecting the variations of the resistance coefficients, we obtain for the drag per unit area of the bed surface

$$\Delta f_2 \doteqdot \tau \left\{ (1 + \frac{\Delta u}{u} f_2(\zeta))^2 + \left(\frac{\Delta w}{u} f_2(\zeta) \right)^2 \right\} \cdot \frac{u^2}{u^2 + (\Delta u f_2(\zeta))^2}, \tag{5.6}$$

[1] M. Kurihara, Rep. Res. Inst. Fluid Engng. Kyūshū Univ. 3 (1946) 21, its abbrivated account will be seen in this report.

[2] G. I. Taylor, Proc. Camb. Phil. Soc. 32 (1936).

its direction θ being given by $tg\,\theta = \{(\Delta w)/u\} f_2(\zeta)/[1+\{(\Delta u)/u\} f_2(\zeta)]$.

In the equilibrium theory of White the drag in the direction of the mean flow acting on a grain and the gravitational force are just in balance at the commecemcent of movement. But grains may begin their movements by comparatively small tractive force, when they lose their stability by any agency, so that we can define the critical tractive force such that at which the grains are in equilibrium under the actions of the resultant drag and gravity. Thus we have by similar way as in White's theory

$$\left\{\left(\frac{\pi\varepsilon}{4}\Delta f_1+(\Delta f_2)_z\right)^2+(\Delta f_2)^2_x\right\}^{\frac{1}{2}} = \varepsilon\frac{\pi}{6}\rho' gk\, tg\,\varphi, \tag{5.7}$$

where ε denotes the packing coefficient defined as number of grains $=\varepsilon \times \text{area}/k^2$, φ the angle of repose.

Let τ_0 be the critical tractive force and T the turbulence factor defined by White, then inserting (5.5), (5.6) in (5.7) we obtain

$$\tau_0 = \frac{1}{T}\varepsilon\frac{\pi}{6}\rho' gk\, tg\,\varphi, \tag{5.8}$$

$$T = \left[X^2+\left(\frac{\pi\varepsilon\alpha}{\sqrt{\eta}}f_1(\zeta)\,\Psi(\chi)+Z\right)^2\right]^{\frac{1}{2}}, \tag{5.9}$$

where

$$\left.\begin{aligned} X &= \left\{\left(1+f_2(\zeta)\frac{\Delta u}{u}\right)^2+\left(\frac{\Delta w}{u}f_2(\zeta)\right)^2\right\}^{\frac{1}{2}}\cdot\left\{1+\left(\frac{\Delta u}{u}f_2(\zeta)\right)^2\right\}^{-1} \\ &\qquad\times\left\{1+\frac{\Delta u}{u}f_2(\zeta)\right\}, \\ Z &= \left\{\left(1+f_2(\zeta)\frac{\Delta u}{u}\right)^2+\left(\frac{\Delta w}{u}f_2(\zeta)\right)^2\right\}^{\frac{1}{2}}\cdot\left\{1+\left(\frac{\Delta u}{u}f_2(\zeta)\right)^2\right\}^{-1} \\ &\qquad\times\frac{\Delta w}{u}f_2(\zeta) \end{aligned}\right\}. \tag{5.10}$$

These equations refer to the mean state of turbulent flow near the wall. As mentioned above the grains can move by actions of some greater forces which appear with not vanishing probabilities. Therefore we must introduce certain coefficients γ_1, γ_2, γ_3 which depend on the criterion of the commencement of grain movement in (5.9), (5.10) as follows:

$$T = \left[X^2+\left(\frac{\pi\varepsilon\alpha}{\sqrt{\eta}}\gamma_1 f_1(\zeta)\,\Psi^2(\chi)+Z\right)^2\right]^{\frac{1}{2}}, \tag{5.11}$$

$$\left.\begin{aligned} X &= \left\{\left(1+\gamma_2\frac{\Delta u}{u}f_2(\zeta)\right)^2+\left(\gamma_3\frac{\Delta w}{u}f_2(\zeta)\right)^2\right\}^{\frac{1}{2}}\cdot\left\{1+\left(\frac{\Delta u}{u}f_2(\zeta)\right)^2\right\}^{-1} \\ &\qquad\times\left\{1+\gamma_2\frac{\Delta u}{u}f_2(\zeta)\right\}, \\ Z &= \left\{\left(1+\gamma_2\frac{\Delta u}{u}f_2(\zeta)\right)^2+\left(\gamma_3\frac{\Delta w}{u}f_2(\zeta)\right)^2\right\}^{\frac{1}{2}}\cdot\left\{1+\left(\frac{\Delta u}{u}f_2(\zeta)\right)^2\right\}^{-1} \\ &\qquad\times\gamma_3\frac{\Delta w}{u}f_2(\zeta) \end{aligned}\right\}. \tag{5.12}$$

III Numerical Considerations

§ 6. **Determination of coefficients c_1, c_2.** Though at present it is impossible to determine theoretically the coefficients a, b, b' in (4.1), (4.2) accordingly c_1, c_2 in (4.3), we can easily estimate the order of magnitude of them.

For the case of $k/\Delta < 1$ the equivalent layer must be defined as that the turbulence at which represents the turbulence in the outer layer for the laminar sub-layer as regards to pressure fluctuations. So that it might be expected that at the equivalent layer the turbulence would not be much influenced by the wall, in other words, the molecular viscosity would be negligible compared with the turbulent viscosity. If we assume temporally molecular viscosity of 5%, we have $a = 7.3$ (see Fig. 4 in I)). On the other band, for the case of $k/\Delta > 1$, the equivalent layer would lie at the distance of ths same order as k above the inner layer, accordingly we get $b \sim 2$.

The mixing length l may be interpreted as one fourth of the wave length when the velocity fluctuations are represented by a sinuous form. Then we have $b' \doteqdot 0.4$, since $l \doteqdot 0.2\,\xi$. By these values we obtain finally

$$c_1 \sim 2.9, \qquad c_2 \sim 0.8. \tag{6.1}$$

The value of $c_2 \sim 0.8$ is consistent with the fact that the theoretical least value of c_2 deduced from the nature of function $f_1(\zeta)$ is 0.699 and c_2 must take a value a little larger than this limiting value to make Δf_1 reasonably small when k tends to practically infinity. But we have no trustworthy support for $c_1 \sim 2.9$. Accordingly we will so determine c_1 as to make agree the theoretical maximum point of the turbulence factor with the observed. As an approximation, if we make the maximum point $\zeta=2.082$ of $f_1(\zeta)$ correspond to $(v_*k)/\nu=25$, anticipating (6.3) and accepting $c_2=0.8$ we obtain $c_1=2.6$, which is in good agreement with the predicted value $c_1\sim 2.9$. Thus we shall use in followings

$$c_1=2.6, \qquad c_2=0.8. \tag{6.2}$$

As seen from the order of magnitude of the height of the equivalent layer χ is much greater than 1 so we can safely put $\Psi(\chi)=1$ in (5.11). Then $f_1(\zeta)$ and $f_2(\zeta)$ which are shown in Fig. 4 represent directly the con-

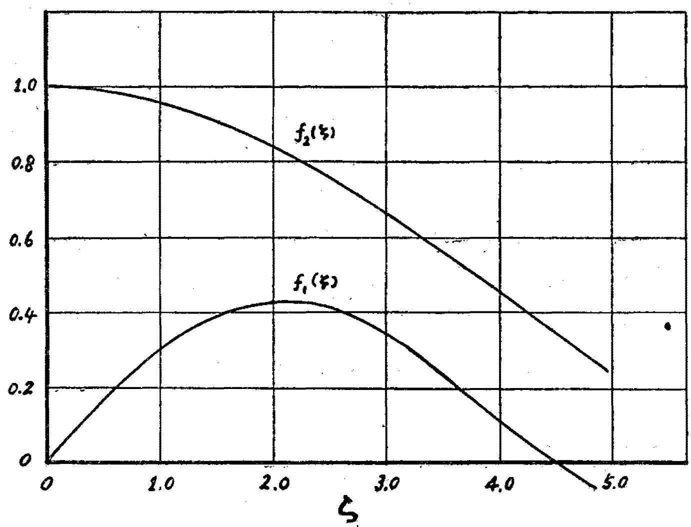

Fig. 4

tributions of fluctuations of pressure and of velocities respectively. The argument can be written by (4.3), (5.1)

$$\zeta=\pi\frac{k}{M}=\pi\frac{k/\Delta}{c_1+c_2\,k/\Delta}, \tag{6.3}$$

where $v_*\Delta/\nu=6.83$.

Thus we can calculate $f_1(\zeta)$ and $f_2(\zeta)$ against the non-dimensional grain-diameter v_*k/ν by (6.3) and Fig. 4.

§ 7. **Coefficients** γ_s. The pressure and components of velocity in the inner layer fluctuate according to certain probability laws characteristic to the turbulence. If we assume for these fluctuating quantities Gaussian law

$P = \{1/\sqrt{2\pi}\,\sigma\}\, e^{-\{x^2/2\sigma^2\}}$, we obtain $\overline{|x|}$ for $|x|>x_1$ with a given probability. For example

$$\left.\begin{aligned} 1-\int_{-x_1}^{x_1} P\,dx &= 0.01\ , & x_1/\sigma &= 2.53, & \overline{|x|}/\sigma &= 2.89, \\ &= 0.025\ , & &= 2.24, & &= 2.58, \\ &= 0.0625, & &= 1.87, & &= 2.24, \\ &= 0.156\ , & &= 1.42, & &= 1.87\ \cdot \end{aligned}\right\}. \qquad (7.1)$$

It is clear that adoption of the commencement of movement of a few grains as the definition of the critical tractive force corresponds to observations of movements by large forces with small probabity of occurrence and general movement of bed material corresponds to small forces with large probability. Thus there is a correspondence between the criterions and the probabilities.

H. C. H. Townend[1] has shown the observed maximum velocity fluctuations nearly equal to three times of the root-mean-square values. So that we might identify the most precise criterion (e. g. due to photographic observation as in the case of Townend) with the probability of 1% by (7.1). Then rather arbitrarily we define the lower order criterions by steps of 2.5 times as in the magnitude of fixed stars.

Naturally γ_s in (5.11), (5.12) must be determined consistently with the probability of occurrence of resultant forces. For this purpose it is necessary to know the correlations between the fluctuations of velocity components and pressure. But on account of lack of our knowledge putting stress on the dominancy of pressure fluctuations we assume provisionally a simple relation $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$, that is to say, we shall consider such cases in which the fluctuations of velocity cooperate farvourably with that of pressure (the most effective factor) to produce maximum resultant forces.

§ 8. **Comparison of the theory with the observations.** Since we have determined the values of coefficients c_1, c_2 and γ_s we are now able to compare the theoretical results with the observations. Namely, using White's measured values of $\varepsilon = 0.4$ and $tg\,\varphi = 1$ for grains, $u_1/u = 0.1$, $w_1/u = 0.6$ estimated from Fage and Townend's measurements[2] of the maximum velocity

[1] H. C. H. Townend, Proc, Roy. Soc. A. **145** (1934).
[2] A. Fage and H. C. H. Townend, Aero. Res. Ctee. Rep. Mem. No. 1474 (1932), H. C. H. Townend, loc. cit. Fig. 8.

fluctuations u_1, w_1 in a square pipe and further the theoretical values $a = \sqrt{2}$, $\sqrt{\eta} = 0.210$ we can calculate the turbulence factor and then the critical tractive force against $(v_* k/\nu)$ by (5.8), (5.11), (5.12) (6.3) and Fig. 4. The results of calculations are shown in Table 2 and in Fig. 2 with full lines.

Table 2

$\frac{k}{\Delta}$	$\log \frac{v_* k}{\nu}$	1-st class (2.5%)		2-nd class (6.25%)		3rd class (15.6%)	
		T	τ_0	T	τ_0	T	τ_0
0.25	0.235	2.738	0.0763	2.416	0.0865	2.087	0.1001
0.50	0.536	4.546	0.0460	3.960	0.0528	3.344	0.0625
1.00	0.836	6.689	0.0313	5.802	0.0360	4.864	0.0430
2.00	1.138	9.339	0.0224	8.089	0.0258	6.767	0.0309
4.00	1.439	10.114	0.0207	8.768	0.0238	7.335	0.0285
6.00	1.615	9.536	0.0219	8.267	0.0253	6.921	0.0302
10.00	1.837	8.199	0.0255	7.118	0.0294	5.966	0.0350
20.00	2.138	6.303	0.0332	5.484	0.0381	4.610	0.0453
50.00	2.537	4.684	0.0446	4.088	0.0511	3.457	0.0605
100.00	2.837	4.056	0.0515	3.552	0.0588	3.018	0.0693
200.00	3.138	3.734	0.0560	3.271	0.0639	2.794	0.0748

The figure shows that in spite of the crudeness of our theory its results explain fairly well the main character of the observed facts. This agreement seems to confirm our conjecture that the fluctuations of pressure gradient play a rôle of paramount importance for the problem of tractive force.