THE TURBULENT FIELD IN A WAKE BEHIND A SYMMETRICAL CYLINDER

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THE TURBULENT FIELD IN A WAKE BEHIND A SYMMETRICAL CYLINDER[1)]

By

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§ 1. In the present paper the intensity and extent of the turbulent field in a wake behind a symmetrical cylinder are considered comparing with those of the field of velocity defect.

If U_0 is the undisturbed stream velocity and $U_0 - U = u$ the velocity defect and (u', v', w') the components of turbulent velocity in the wake, then using usual approximations and notations we have for the mean flow:

$$\left.\begin{aligned} \tau &= -\rho\overline{u'v'} = \rho\varkappa\frac{dU}{dy}, \\ \rho U_0\frac{\partial U}{\partial x} &= \frac{\partial\tau}{\partial y} = \rho\frac{\partial}{\partial y}\left(\varkappa\frac{\partial U}{\partial y}\right) \end{aligned}\right\}. \tag{1}$$

If the turbulence is nearly isotropic and so we neglect the contribution of pressure fluctuations to production of turbulent energy[2)], then we have with the same order of approximation in (1)

$$U_0\frac{\partial c^2}{\partial x} = -2\overline{u'v'}\frac{\partial U}{\partial y} - \frac{\partial}{\partial y}\overline{v'(u'^2+v'^2+w'^2)} + 6\nu\overline{u'\nabla^2 u'} \tag{2}$$

where c denotes the root-mean-square speed of turbulence, $c^2 = \sum\overline{u'^2} \fallingdotseq 3\overline{u'^2}$.

As regards the turbulent viscosity and mixing length we assume the following relations[3)]:

$$\varkappa = \frac{1}{3}cl, \quad \frac{l^2}{\lambda^2} = \frac{\sqrt{3}}{40}\cdot\frac{l\sqrt{\overline{u'^2}}}{\nu}, \tag{3}$$

where λ is the diameter of smallest eddies.

If we can put the diffusion coefficient of turbulent energy equal to $\varkappa$, then (2) becomes

1) Abstract of "Investigations on Turbulence V, The Turbulent Field in a Wake" (in Japanese), Rep. Res. Inst. Engng. 4 (1947) 21-34.

2) According to the recent work of A.A. Townsend this term seems too great to neglect without further words. "Momentum and Engergy· Diffusion in the Turbulent Wake of a Cylinder," Proc. Roy. Soc. A, **197** (1949) 124-140.

3) M. Kurihara, "On Turbulent Viscosity" in this issue.

$$U_0 \frac{\partial c^2}{\partial x} = 2\varkappa\left(\frac{\partial U}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left(\varkappa \frac{\partial c^2}{\partial y}\right) - \frac{c^4}{12\varkappa}. \tag{4}$$

Now we shall search after Prandtl a permanent type of solutions of (1), (4) at a sufficient distance downstream. In the present case, if we choose the independent variable η so as to make equations (1), (4) ordinary differential equations, we can easily find

$$\left.\begin{aligned} &\eta = \frac{y}{\sqrt{x}}, \quad \frac{u}{U_0} = \frac{f(\eta)}{\sqrt{x}}, \quad \frac{c}{U_0} = \frac{\varphi(\eta)}{\sqrt{x}}, \\ &l = \sqrt{x} \times \text{function of } \eta, \\ &\varkappa = \varkappa_0 \psi(\eta), \quad \psi(0) = 1 \end{aligned}\right\}, \tag{5}$$

where $\varkappa_0$ is constant. Then (1), (4) become

$$\frac{1}{2} \frac{d}{d\eta}(\eta f(\eta)) + \frac{d}{d\eta}\left(k \frac{df}{d\eta}\right) = 0, \tag{6}$$

$$\frac{1}{2\eta} \frac{d}{d\eta}(\eta^2 \Phi) + \frac{d}{d\eta}\left(k \frac{d\Phi}{d\eta}\right) - \frac{\Phi^2}{12k} + 2k\left(\frac{df}{d\eta}\right)^2 = 0, \tag{7}$$

where

$$k = \frac{\varkappa}{U_0} = k_0 \psi(\eta), \quad k_0 = \frac{\varkappa_0}{U_0}, \quad \Phi = \varphi^2. \tag{8}$$

§ 2. In the turbulent field in a wake the diffusion processes have a strong tendency to make the distribution of turbulent energy uniform. So that the intensity of turbulence takes a maximum value at the enter of the wake and then gradually decreases to zero when one goes outwards. On the contrary it is likely in all probability that the mixing length gradually increases outwards. Thus it is reasonable, especially when we are concerned with the velocity defect, to make a assumption $\varkappa =$ constant in a cross section of the wake therefore throughout the wake by (5), instead of Prandtl's assumption $l =$ constant in a cross section.

Putting $k = k_0$ ($=$ constant) we obtain the symmetrical solution of (6):

$$f(\eta) = f_0 e^{-\frac{\eta^2}{4k_0}}, \quad f_0 \equiv f(0). \tag{9}$$

Now in order to compare the theoretical result with the measurements we introduce non-dimensional length ξ measured by the observed width of the wake, b, namely $\xi = \frac{y}{b}$. Then we obtain from (9)

$$\frac{f(\eta)}{f_0} = \frac{u}{u_0} = e^{-\frac{\xi^2}{\sigma^2}}, \quad \sigma^2 \equiv \frac{4k_0 x}{b^2}, \tag{10}$$

where u_0 means the central value of the velocity defect.

Let the curve $e^{-\frac{\xi^2}{\sigma^2}}$ agree with the measured curve of A. Fage and V.M. Falkner[1] at $\xi = 0.5$, then we have

$$\sigma^2 = 0.266, \qquad \sigma = 0.515. \tag{11}$$

The theoretical curve with this value of σ is shown in Fig. 1 with the measured curve and Schlichting's theoretical result.

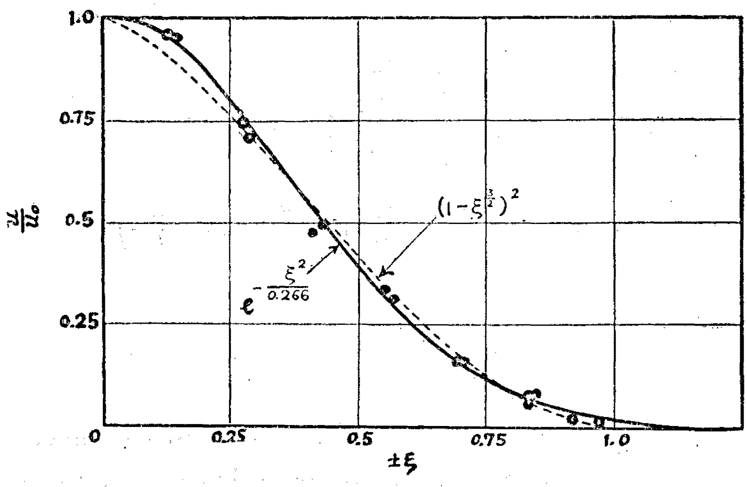

Fig. 1 Velocity Distribution. Full line: the present theory. Dotted line: Schlichting's theory. A. Fage, V.M. Falkner's Measurements,

$$\frac{u_0}{U_0} = 0.139, \quad \frac{2b}{D} = 9.00, \quad \frac{x}{D} = 36.0,$$

$$D = 5/8'', \quad U_0 = 51.7 \text{ ft sec.}, \quad \frac{U_0 D}{\nu} \doteqdot 16900.$$

Our result agrees satisfactory with the measurements somewhat better than Schlichting's.

§ 3. Turbulent Field. We shall search an approximate solution of (7) by Yamada's method[2].

Multipling (7) first by 1 and secondly η and then integrating from 0 to ∞ we obtain with the same assumption $k = k_0$ as before

[1] G. I. Taylor, Proc. Roy. Soc. A. **156** (1935).

[2] H. Yamada. Rep. Res. Inst. Fluid Engng. Kyūshū Univ. **3** (1947) 29-35, an abstract of which will be seen in this issue.

$$\frac{1}{2}\int_0^\infty \Phi\, d\eta - \frac{1}{12\,k_0}\int_0^\infty \Phi^2\, d\eta + 2\,k_0\int_0^\infty \left(\frac{df}{d\eta}\right)^2 d\eta = 0, \tag{12}$$

$$k_0\,\Phi_0 - \frac{1}{12\,k_0}\int_0^\infty \eta\,\Phi^2\, d\eta + 2\,k_0\int_0^\infty \eta\left(\frac{df}{d\eta}\right)^2 d\eta = 0, \tag{13}$$

where $\Phi_0 \equiv \Phi(0)$ and $\left(\frac{d\Phi}{d\eta}\right)_0 = 0$, $\lim \eta^n \Phi = 0$, n being any positive integer.

We now assume for the distribution of turbulent velocity the same functional form as that of velocity defect and put

$$\Phi(\eta) = \Phi_0\, e^{-\frac{\eta^2}{2a}}. \tag{14}$$

Then (12) and (13) become

$$\left.\begin{aligned} &\sqrt{\frac{a}{k_0}}\,\frac{\Phi_0}{k_0} - \frac{1}{6\sqrt{2}}\sqrt{\frac{a}{k_0}}\left(\frac{\Phi_0}{k_0}\right)^2 + \left(\frac{f_0}{\sqrt{k_0}}\right)^2 = 0\,, \\ &\frac{\Phi_0}{k_0} - \frac{1}{24}\cdot\frac{a}{k_0}\left(\frac{\Phi_0}{k_0}\right)^2 + \left(\frac{f_0}{\sqrt{k_0}}\right)^2 = 0 \end{aligned}\right\}. \tag{15}$$

Eliminating $(\Phi_0/k_0)^2$ between them

$$\frac{\Phi_0}{f_0^2} = -\frac{\left\{\frac{\sqrt{2}}{4}\sqrt{\frac{a}{k_0}} - 1\right\}}{\left\{\frac{\sqrt{2}}{4}\cdot\frac{a}{k_0} - 1\right\}}. \tag{16}$$

Since Φ_0, f_0^2 are positive, (16) gives

$$\frac{4}{\sqrt{2}} > \sqrt{\frac{a_0}{k_0}} > \frac{2}{\sqrt{2}}. \tag{17}$$

Namely the breadth of turbulent field is always greater than that of velocity and their ratio is limited between 1.68 and 2.83 $\left(\frac{u}{u_0} = e^{-\frac{\eta^2}{4k_0}},\ \frac{c}{c_0} = e^{-\frac{\eta^2}{4a}}\right)$, whereas the ratio of the turbulent velocity to the velocity defect at the center, $\frac{\sqrt{\Phi_0}}{f_0}$, varies from zero to infinity.

Considering (17) we obtain from (15), (16)

$$\frac{f_0^2}{k_0} = 6\sqrt{2}\,\frac{\left(\sqrt{\frac{a}{k_0}} - 1\right)\left(\frac{\sqrt{2}}{4}\,\frac{a}{k_0} - 1\right)}{\sqrt{\frac{a}{k_0}}\left(\frac{\sqrt{2}}{4}\sqrt{\frac{a}{k_0}} - 1\right)^2}. \tag{18}$$

$f_0/\sqrt{k_0}$ and $\Phi_0/\sqrt{a}$ are constants concerning with the distribution of velocity defect u and turbulent velocity c respectively and may be taken as measures of concentration of u, c towards the central region. So that we shall call them the concentration of velocity, and of turbulence respectively.

§ 4. Numerical considerations. If the velocity distribution in the section at a distance x is given, then we can calculate by (5), (9) and (10) the concentration of velocity,

$$\frac{f_0}{\sqrt{k_0}} = \frac{2x}{\sigma b} \cdot \frac{u_0}{U_0}. \tag{19}$$

Then with this value we can obtain by (16), (18) the ratio of breadths $\sqrt{\frac{a}{k_0}}$ and the ratio of turbulent velocity to the velocity defect at the central plane $\sqrt{\Phi_0}/f_0\left(=\frac{c_0}{u_0}\right)$. But it is convenient for general use to tabulate the the values of $f_0/\sqrt{k_0}$ and $\sqrt{\Phi_0}/f_0$ against $\sqrt{a}/k_0$. Results of calculations are given in the following table and Fig. 2.

Table

$\sqrt{a/k_0}$	$\frac{2}{\sqrt[4]{2}}=1.682$	1.7	1.8	1.9	2.0	2.1	2.2	2.4	$\frac{4}{\sqrt{2}}=2.828$
$f_0/\sqrt{k_0}$	0	0.69	2.01	3.21	4.53	6.12	8.17	15.0	∞
$\sqrt{\Phi_0/a}$	$\sqrt{3}=1.732$	1.74	1.79	1.84	1.91	1.98	2.08	2.38	∞
$\sqrt{\Phi_0}/f_0=\frac{c_0}{u_0}$	∞	4.26	1.61	1.09	0.84	0.68	0.56	0.38	0

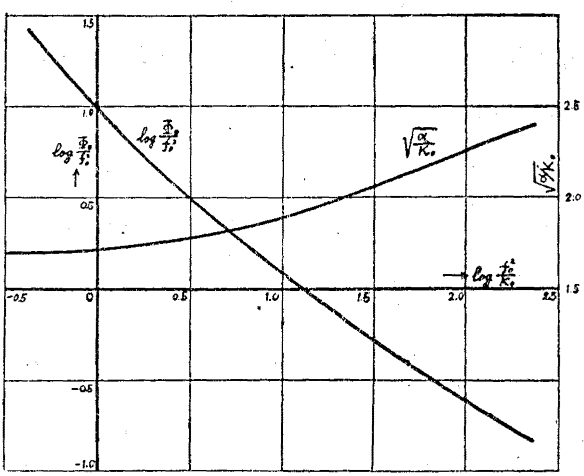

Fig. 2

While, as mentioned in § 3, $\sqrt{a/k_0}$ increases slightly with $f_0/\sqrt{k_0}$ the turbulent velocity exhibits a considerable change, decreasing from ∞ to 0. Its physical meaning is as follows; for very small Reynolds numbers the intensity of turbulence in the wake must be weak and the turbulent viscosity also be small. Hence the large velocity defect produced just behind the obstacle may be maintained a considerable distance downstteam without being equalized with the surrounding. As Reynolds number increases the intense turbulence makes sufficient the turbulent viscosity to equalize the velocity defect and to convert the energy of mean flow into the turbulent energy. Thus we can expect large values of $f_0/\sqrt{k_0}$ and small values of c_0/u_0 for very low Reynolds numbers and the contrary for very high Reynolds numbers.

As an example we consider the case of the turbulent wake behind a cylinder measured by H.L. Dryden and A.M. Kuethe[1] which seems to correspond to a high Reynolds number. Measurements were conducted at two cross sections, namely just behind the cylinder and 4.2 D downstream from the rear stagnation point, where D is the diameter of the cylinder.

The distance 4.2 D is too small to apply our theory. But the Karman vortex system is dissipated and the distribution of speed fluctuations shows a smooth curve at this section. So that it is interesting to see in what degree the results of theory and measurements agree.

We obtain from Fig. 19 and 20 in their paper:

Position of the cross section: $x/D = 4.2+0.5 = 4.7.$

Distribution of mean speed:

$$2\,b/D = 2.6, \qquad u_0/U_0 = 0.1, \qquad y/D = 0.8 \text{ for } u/u_0 = 0.5.$$

Distribution of speed fluctuations:

$$(\varDelta U)_0/U_0 = 0.56, \qquad y/D = 1.55 \text{ for } \varDelta U/(\varDelta U)_0 = 0.5,$$

where $\varDelta U$ denotes double amplitude of speed fluctuations.

Since $c = \dfrac{\sqrt{3}}{2\sqrt{2}}\varDelta U$, we have from above data

$$\frac{c_0}{u_0} = \sqrt{\varPhi_0/f_0^2} = \frac{0.56}{0.1}\cdot\frac{\sqrt{3}}{2\sqrt{2}} = 3.43, \quad \sqrt{\frac{a}{k_0}} = \frac{1.55}{0.8} = 1.94.$$

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[1] H. L. Dryden and A. M. Kuethe, N.A.C.A. Tech. Rep. No. 342 (1930).

On the other hand (11) (19) gives $f_0^2/\sqrt{k_0} = 1.96$. Hence from Fig. 2 we get

$$\sqrt{\Phi_0/f_0^2} = 2.21, \qquad \sqrt{a/k_0} = 1.75.$$

Next, let us consider the experiments of A. Fage[1], which were conducted at small Reynolds numbers. He measured the velocity profils and speed fluctuations at various distances downstream in the wakes of a long triangular prism, its breadth being s:

$$u_1/U = 0.51, \qquad U/U_0 = 0.77 \text{ at } \frac{x}{s} = 30,$$

$$u_1/U = 0.34, \qquad U/U_0 = 0.83 \text{ at } \frac{x}{s} = 50,$$

where u_1 denotes measured maximum velocity fluctuation which is nearly equal to $3 \times \sqrt{\overline{u'^2}}$[2]. Hence the ratio of velocity fluctuations and velocity defects at two sections are 0.72 and 0.74 respectively. Since now the square root of distances is 0.775, these figures seem to ascertain equation (5), namely $\frac{u}{U_0}, \frac{c}{U_0} \sim \frac{1}{\sqrt{x}}$.

Again from Fig. 4 in his paper, we obtain $u_0/U_0 = 0.2$ $(U/U_0 = 0.8)$ $b/s = 6$, $u_1/U = 0.4$ (mean of three components) for $x/s = 35$. So that $f_0/k_0 = 20.4$ by (11) (19). Thus from Fig. 2 we read off the theoritical values $\sqrt{\Phi_0/f_0^2} = 0.84$, $\sqrt{a/k_0} = 2.0$, whereas the observed value of $\sqrt{\Phi_0/f_0^2}$ is 0.93 ($\sqrt{a/k_0}$ can not be determined from his data).

[1] A. Fage, Tech. Report, A.R.C. No. 1520 (1932).
[2] H. C. H. Townened, Proc. Roy. Soc. A. **145** (1934) 180.