ON A METHOD OF APPROXIMATE SOLUTION OF DIFFERENTIAL EQUATIONS

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バージョン：

ON A METHOD OF APPROXIMATE SOLUTION OF DIFFERENTIAL EQUATIONS[1]

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§ 1. Method.—If a function $f(x)$ of x, being continuous in the interval $x(a, b)$, satisfies the conditions

$$\int_a^b x^n f(x)\,dx = 0, \qquad n = 0,\ 1,\ \cdots\cdots,\ N-1, \tag{1}$$

then $f(x)$ changes its sign N times, i.e., it has N separate roots at least in that interval. When N is infinite $f(x)$ is identically zero.

This well-known theorem on moments is here applied to the solution of *non-linear*[3] differential equations

$$A(y;\ x) = 0 \tag{2}$$

in the interval $x(a, b)$, y being the required function. If a function $g(x)$ inserted for y makes the left-hand side of (2) vanish identically, then $y = g(x)$ is an *exact* solution, and evidently hold

$$\int_a^b x^n A(g(x);\ x)\,dx = 0, \qquad n = 0,\ 1,\ \cdots\cdots\to\infty. \tag{3}$$

Inversely, a function $y = g(x)$ which satisfies (3) is, by reason of the moments theorem, an exact solution of (2); the differential equation (2) and the moments equations (3) is identical.

Taking up the problem in the form of (3), we write down

$$\int_a^b x^n A(y;\ x)\,dx = 0, \qquad n = 0,\ 1,\ \cdots\cdots,\ N-1, \tag{4}$$

as its approximate form. For any one of solutions $g(x)$'s of this set of equations (4) $A(y;\ x)$ changes its sign N times at least; instead of being identically zero $A(y;\ x)$ vanishes N times at least in the interval $x(a, b)$. When (4) is taken up with the end conditions which are assigned for

[1] A short account of a paper (in Japanese) reported in this REPORTS, vol. 3, No. 3, pp. 29-35 (1947).

[2] When one or both of a, b tend to infinity a certain additive condition is necessary to secure this conclusion.

[3] It is evident that the method applies to non-linear as well as linear equations.

solution, then among $g(x)$'s there will be one which is approximate to the exact solution, and the degree of approximation will be the more exact when the larger N becomes.

We find $g(x)$ in the following manner: A function which satisfies the end conditions and contains N undetermined constants is inserted for y in (4), which gives a system of N algebraic equations for N unknown constants, thus the solution $g(x)$ being determined. When $A(y; x)$ is not linear the algebraic equations are not linear also, and then solution is not unique. But one which is appropriate to the problem can earsily be determined by inspection or by certain numerical methods, e.g., by the deviation of $A(y; x)$ from zero. The degree of approximation depends on the assumed form for $g(x)$ and on the number of used moments N. Selection of these must be done by the physical interpretation of the problem in hand. The changes in the calculated values of constants, which occur when we go from the Nth approximation to the $N+1$th, are of the order of Nth moment, i.e.

$$\int_a^b x^N A(g(x); x)dx,$$

where $g(x)$ being the Nth approximation; this Nth moment may be seemed as something indicating the degree of the Nth approximation.

§ 2. Examples.—Omitting the applications to the ordinary non-linear equation and to the eigenvalue problem given in the original, applications to two partial non-linear problems will be mentioned.

(1) *Application to the boundary layer equation.*—In the two-dimensional boundary layer equation[4)]

$$\Delta(u; x, y) = u\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\int_0^y \frac{\partial u}{\partial x}dy - u_1\frac{du_1}{dx} - \nu\frac{\partial^2 u}{\partial y^2} = 0, \qquad (5)$$

the velocity distribution in y-direction is approximated by a four-degree polynomial, which satisfies the boundary conditions,

$$\left.\begin{aligned} \frac{u}{u_1} &= \{(2\eta - 2\eta^3 + \eta^4) + \frac{1}{6}\Lambda\eta(1-\eta)^3\}, \\ \eta &= y/\delta, \quad \Lambda = \frac{du}{dx}\cdot\frac{\delta^2}{\nu}, \end{aligned}\right\} \qquad (6)$$

4) cf. S. Goldstein: Modern Developments in Fluid Dynamics. vol. 1, §§ 52, 60. Notation is the same in there.

where δ is the only unknown coefficient (function of x).

The equation (5) has to hold in the domain $\eta(0, 1)$ and there is only one unknown coefficient $\delta(x)$, so that one moment equation, i.e.,

$$\int_0^1 A(u; x, \eta)\, d\eta = 0 \tag{7}$$

is sufficient to determine the solution.

Equation (7) is nothing but the momentum equation of v. Kármán and the method of integration here adopted is no more than the one well known by the name of Kármán-Pohlhausen. From our point of view the cause of the occasional failure of their method is the insufficiency in number of the used moment-equations. The improved calculations are given in another paper.[5]

(2) *Progress of freezing limit.*[6]—The surface temperature of a bottomless liquid which, at the time $t = 0$, is held throughly at the freezing temperature (0, say), is decreased according to the function $\varphi(t)$. Then freezing occurs and the depth of solid $\varepsilon(t)$ increases with time. Denoting the temperature, density, conductivity, capacity and the heat of melting of solid by $\theta(x, t)$, ρ, k, c and L, we have

$$\frac{\partial\theta}{\partial t} = \varkappa\frac{\partial^2\theta}{\partial x^2}; \quad \theta(0, t) = \varphi(t), \quad \theta(\varepsilon, t) = 0, \tag{8}$$

and

$$\varepsilon(0) = 0, \quad \left(-\frac{\partial\theta}{\partial x}\right)_{x=\varepsilon} = a\frac{d\varepsilon}{dt}, \tag{9}$$

where $\varkappa = k/\rho c$, $a = \rho L/k$ and x is directed to the depth, origin being at the surface.

The second equation of (9), i.e., the condition at the freezing limit, is non-linear, and by this difficulty exact solution is the only one due to Stefan, where $\varphi(t) = \Phi$ a negative constant. His solution is

$$\theta = \Phi\int_{x/2\sqrt{\varkappa t}}^{\beta} e^{-z^2} dz \Big/ \int_0^{\beta} e^{-z^2}\, dz, \quad \varepsilon(t) = 2\beta\sqrt{\varkappa t}, \tag{10}$$

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[5] A method of approximate integration of laminar boundary layer equation (in Japanese), in the REPORTS, vol. 4, No. 3, (1948) pp. 27–42, vol. 5, No. 2, (in press); a short account in English is given in the present volume.

[6] cf. Riemann-Weber: Differentialgleichungen der Physik. Bd. II, Kap. 6, §3 (1925).

where β is to be determined by

$$\beta e^{\beta^2}\int_0^\beta e^{-z^2}\,dz = -\frac{\Phi c}{2L}. \tag{11}$$

When $\varphi(t)$ is a monotone slowly changing function the temperature distribution in the solid will be well expressed by a certain low-degree polynomial in x, for example by a fourth degree polynomial, its coefficients being determined by the moment-equations. From the end condition it follows

$$\theta = (1-2\xi+\xi^2)\varphi+(-\xi+\xi^2)\frac{1}{2}a\lambda'+(\xi-2\xi^2+\xi^3)\alpha+(2\xi-3\xi^2+\xi^4)\beta, \tag{12}$$

where $\xi = x/\varepsilon$, $\lambda = \varepsilon^2$, $\lambda' = d\lambda/dt$, and α, β are functions of t. λ, α and β are unknown function of t, which have to be determined such that θ satisfies the equation (8) approximately. Then the moment-equations to be used are three:

$$\int_0^\varepsilon x^n\left(\frac{\partial\theta}{\partial t}-\kappa\frac{\partial\theta}{\partial x^2}\right)dx = 0, \qquad n = 0,\ 1,\ 2 \tag{13}$$

Using (12) into (13), we have three simultaneous differential equations of first order with respect to λ', α and β. The solution of this simultaneous equations for the given function $\varphi(t)$ determines the depth of solid and the temperature distribution in it, so that the problem is reduced to this ordinary equations of first order. Especially in the Stefan's case all is reduced to the solution of a single algebraic equation

$$\begin{aligned}\lambda'^4+\left(280\,\kappa-8\frac{\Phi}{a}\right)\lambda'^3+\left(7040\,\kappa^2+1200\,\kappa\frac{\Phi}{a}\right)\lambda'^2\\ +\left(134400\,\kappa^3-30720\,\kappa^2\frac{\Phi}{a}\right)\lambda'+268800\,\kappa^3\frac{\Phi}{a} = 0.\end{aligned} \tag{14}$$

To see the degree of approximation, we take an example of ice: $\rho = 0.92$, $k = 0.005$, $c = 0.63$ and $L = 80$ (in C. G. S. and degree °C.). Assuming $\Phi = -10$°C (14) gives $\lambda' = 1.3243\times10^{-3}$, and then

$$\varepsilon(t) = 0.03639\sqrt{t}\ \text{cm} \quad (t \text{ in sec.}),$$

which is just the value the Stefan's solution gives, and by the simultaneous equations $\alpha = -0.2743$, $\beta = 0.0747$. With these values in (12) the temperature distribution in ice can be calculated, which is compared with the Stefan's solution in the following table.

ξ	θ_{correct}	$\theta_{\text{approximate}}$
0.0000	−10.000	−10.000
0.0722	−9.269	−9.268
0.1444	−8.539	−8.536
0.2165	−7.809	−7.806
0.3609	−6.351	−6.347
0.5052	−4.900	−4.896
0.6496	−3.457	−3.453
0.7939	−2.024	−2.021
0.8661	−1.312	−1.311
0.9383	−0.603	−0.602
1.0000	0.000	0.000

(3 Apr. 1950)