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https://doi.org/10.15017/7158044

出版情報：流體工學研究所報告. 6 (2), pp.15-21, 1950-01. 九州大學流體工學研究所

A NEW INTERPRETATION OF MIXTURE LENGTH THEORY IN VIEW OF ENERGY RELATION AND TURBULENT FLOW IN A PIPE

By

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From Vol. 3. No. 1

§ **1.** Up to the present, in investigations on the turbulent flows in a pipe or between parallel walls the diffusion of turbulent energy generally has not been taken into consideration. For instance, Prandtl's hypothesis assumes that the turbulent energy is supplied from the energy of the mean flow which exhibits velocity gradient of some degree through the random motion of lumps of fluid and the turbulent velocities are of the same order in magnitude as the excess velocities which the lumps will aquire during their motion in a direction along the velocity gradient through a mixture length. However the turbulent motion are maintained even in a region where the mean flow does not show any appreciable change in its velocity distribution. Hence it seems that the diffusion of turbulent energy from the other regions may be also one of the dominant processes e.g. in a pipe flow. In this paper it is examined, how may be interpreted physically Prandtl's hypothesis in the mixture length theory when the transfer of momentum and turbulent energy are taken into consideration at the same time. By the aid of its consequences the distributions of turbulent energy and velocity of the mean flow in a pipe or between parallel walls have been studied, especially considering the contribution of molecular viscosity.

I. Generation and Dissipation of Turbulent Energy

§ **2.** Let u', v', w' be the components of turbulent velocities and ρE the kinetic energy of turbulence i.e. $E=\frac{1}{2}\sum\overline{u'^2}$. If we are justified in

[1] Abbrivated from "Investigations on Turbulence III, A new Interpretation of Mixture Length Theory in view of Energy Relation and Turbulent Flow in a Pipe" (in Japanese), Rep. Res. Inst. Fluid Engng. Kyūshū Univ. (1946) 21～68.

for the present case.

The value of $\sqrt{\eta}$ can be estimated from the measurements of Prandtl cited in the paper of G. I. Taylor (Statistical Theory of Turbulence III, Proc. Roy. Soc. A, 151 (1935)) as $\sqrt{\eta} = 0.210$.

§ **4.** We shall now examine the behaviours of the functions $\varepsilon(y)$, $\psi(y)$ in two diemensional pressure flow by the aids of equations (8), (8)′ when $\eta(y) = \text{const.}$ is assumed.

At the first place, let us assume Prandtl's hypothesis concerning to mixture length, $l = l_0 \frac{\xi}{h}$. Put $E_0 = \frac{1}{\sqrt{\eta}} \frac{\tau_0}{\rho}$, where the suffix "$o$" refers to the values at the wall. Then neglecting molecular viscosity (9) gives $E = E_0 \frac{y}{h}$. Inserting these relations into (8) and solving for the function $\psi(y)$ we have

$$a\psi(y) = \eta + \left(\frac{l_0}{h}\right)^2 \left\{\frac{1}{2}\xi - y\right\} \cdot \frac{\xi}{y^2} \tag{10}$$

for the case of parallel walls. The function $\varepsilon(y)$ can be obtained by the definition of η, $\varepsilon(y) = \frac{\eta}{a\psi(y)}$. Similar equations hold for the case of a pipe flow.

On the surface where $\varepsilon(y) = 1$ the production and the dissipation are exactly in balance. $\varepsilon(y)$ increases from zero at the centers to unit at $y = \frac{h}{3}$ for parallel walls, $y = \frac{3}{5}a$ for a pipe respectively and then through a maximum tends to again unit at the wall.

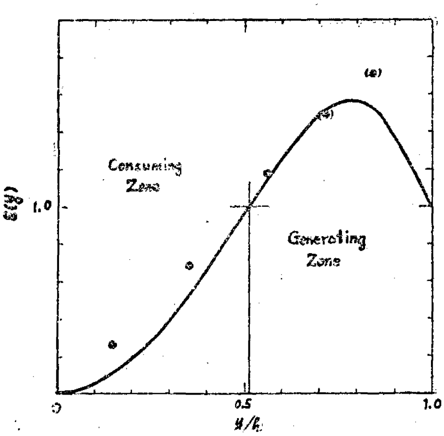

Fig. 1

Secondly if we assume Okaya's phypothesis (Proc. Phys.-Math. Soc. Japan. **22** (1940) 146) about the mixture length, $\frac{l}{h} = l_* - (1 - e^{-a' \frac{\xi}{h}})$, where l_* and a' are non-dimensional constants depending slowly upon Reynolds number, we have similar results and the surfaces on which $\varepsilon(y)$ becomes unit are given by $y = 0.510h$ or $0.696a$. The results of calculations for the flow between parallel walls are given in Fig. 1 with the observed $\varepsilon(y)$ calculated from the data given by Taylor in his investigation on distribution of dissipation of energy in a pipe over its cross-section (Table II, Statistical Theory of Turbulence III, loc. cit.). If we remember that our hypothesis i.e. $\eta(y) = \text{const.}$ or equation (8) has been introduced orginally for the purpose of studies of turbulent flow near a wall, but not in a central region and that on the contrary Taylor's data become inaccurate near the wall, it seems that the hypothesis explains well the truth for large part of the domain.

It is here noted that the outer region between the surface $\varepsilon(y) = 1$ and the wall may be called the generating zone, in which the rate of production of turbulent energy is greater than that of dissipation and similarly the central region up to $\varepsilon(y) = 1$ the consuming zone, its turbulent energy being supplied from the generating zone by diffusion.

II. Velocity Distribution

§ 5. **Velocity Distribution in the generating Zone.** With the assumption $\eta = a\epsilon\psi = \text{const.}$ we can treat the problem of turbulent flow near a wall taking into account the contribution of molecular viscosity.

Following Prandtl, if we assume near a wall $l = l_0 \frac{\xi}{h}$ and put for simplicity $U_\tau = \sqrt{\frac{\tau_0}{\rho}}$, $\frac{U_\tau}{\sqrt[4]{\eta}} = \sqrt{E_0}$, then we have from (7), (9) for the ratio of molecular viscosity to turbulent viscosity and the turbulent energy

$$\theta(y) = \frac{\nu}{\sqrt{E_0}\, l_0} \sqrt{\frac{E_0}{E}}\, \frac{h}{\xi}, \quad \frac{E}{E_0} = \frac{1}{1+\theta} \cdot \frac{y}{h}, \tag{11}$$

where E_0 denotes the supposed turbulent energy at the wall when molecular viscosity is neglected.

Introducing a non-dimensional length χ defined as

$$\chi = 2K \frac{U_\tau \xi}{\nu}, \quad K \equiv \frac{l_0}{h} \cdot \frac{1}{\sqrt[4]{\eta}} \tag{12}$$

and eliminating θ from (11) we have

$$\sqrt{\frac{E}{E_0}} = \left\{ -\frac{1}{\chi} \sqrt{\frac{h}{y}} + \sqrt{1 + \frac{1}{\chi^2} \cdot \frac{h}{y}} \right\} \cdot \sqrt{\frac{y}{h}}. \tag{13}$$

As we shall see later $K = 0.4$. Since so χ becomes considerable when

$\frac{h}{y}-1$ slightly increases, we can safely put for the entire domain $\frac{h}{y}=1$ in the bracket of (13). Thus we have from (11) and (13)

$$\sqrt{\frac{E}{E_0}} = \Psi(\chi)\sqrt{\frac{y}{h}}, \quad \theta = \frac{1}{\chi\Psi(\chi)}\cdot\sqrt{\frac{y}{h}}, \tag{14}$$

where

$$\Psi(\chi) = -\frac{1}{\chi}+\sqrt{1+\frac{1}{\chi^2}}. \tag{15}$$

Solving (15) for χ

$$\chi = \frac{2\Psi}{1-\Psi^2}. \tag{16}$$

We see by (14), (15), (16) that very near the wall $\chi\sim 2\Psi$, i.e. turbulent energy increases linearly with the distance from the wall, and $\theta\sim\frac{4}{\chi^2}$, i.e. the ratio of molecular viscosity to turbulent viscosity decreases proportionally to the inverse square of the distance. Numerical values of Ψ and θ for various values of χ are given in Table I.

Table I.

χ	0	2	4	6	8	10	15	20	30	50	100
$\Psi\left(=\sqrt{\frac{E}{E_0}}\right)$	0	0.618	0.781	0.847	0.882	0.905	0.936	0.951	0.967	0.980	0.990
$\theta\left(=\frac{\nu}{\nu}\right)$	∞	1.618	0.640	0.393	0.283	0.220	0.142	0.105	0.068	0.040	0.020

Molecular viscosity is dominant in the regions $\chi \gtrsim 5$ and then rapidly decreases and becomes 9.1% of the total viscosity at $\chi=20$.

§ **6.** Now near the wall neglecting the gradient of shearing stress, i.e. putting $\frac{y}{h}=1$, we can write the equation of motion for the mean flow as $\frac{d}{dx}\left(\frac{U}{U_\tau}\right)=\frac{1}{K}\cdot\frac{1}{2+\chi\Psi}$. Transforming the independent variable χ to Ψ we can easily obtain its solution with the boundary condition $U=0$ for $\Psi=0$,

$$\frac{U}{U_\tau} = \frac{1}{K}\left\{-\Psi(\chi)+\log\frac{1+\Psi(\chi)}{1-\Psi(\chi)}\right\}. \tag{17}$$

This solution gives for $\chi \gg 1$ the logarithmic velocity distribution. The comparison with the measurements of Nikuradse shows that in our theory turbulent viscosity is over estimated in the immediate neighbourhood of the wall, where the wall and the molecular viscosity strongly restrict the turbulent motion. So that it is natural to introduce as an approximation a pure laminar sub-layer. Let its thickness be Δ and the corresponding value of χ be χ_1. Then as the complete solution of the problem we get

$$\left.\begin{aligned} \frac{U}{U_\tau} &= \frac{\chi}{2K} \quad \text{for} \quad \chi \leq \chi_1, \\ \frac{U}{U_\tau} &= \frac{1}{K}\left\{\frac{\chi_1}{2} - \Psi(\chi-\chi_1) + \log\frac{1+\Psi(\chi-\chi_1)}{1-\Psi(\chi-\chi_1)}\right\} \quad \text{for} \quad \chi \geqq \chi_1 \end{aligned}\right\}. \tag{18}$$

(18) gives for $\chi \ll \chi_1$ $\quad \dfrac{U}{U_\tau} = \dfrac{1}{K}\left\{\dfrac{\chi_1}{2} - 1 + \log 4K + \log\dfrac{\xi U^\tau}{\nu}\right\}$.

Comparing with the empirical formula of Nikuradse

$\dfrac{U}{U_\tau} = 5.5 + 5.75 \log_{10}\dfrac{U_\tau \xi}{\nu}$, we obtain

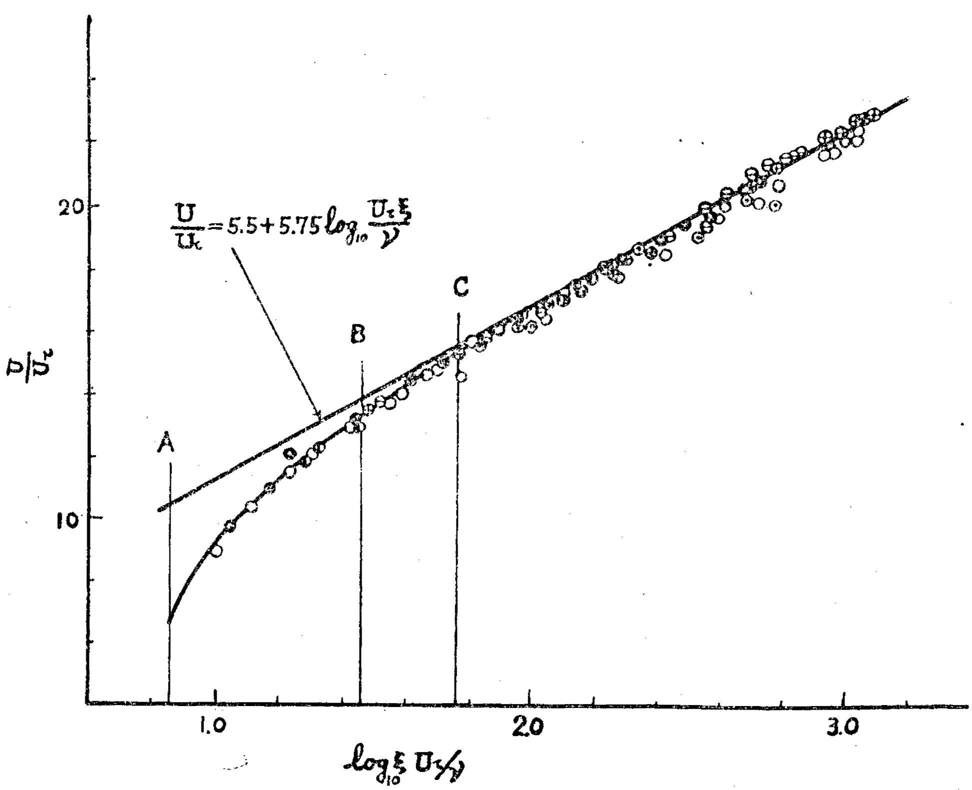

Fig. 2. Velocity Distribution near Wall.

A: Pure laminar sub-layer, B: $\frac{\nu}{\nu} = 0.10$, C: $\frac{\nu}{\nu} = 0.05$.

$$K = 0.40\,, \qquad \chi_1 = 5.46\,, \tag{19}$$

which give $\frac{U_\tau \xi_1}{\nu} = 6.83$, $\frac{U_1 \Delta}{\nu} = 45$ for the pure laminar sub-layer.

The theoretical velocity distribution with the numerical values of (19) is shown in Fig. 2, in which also Nikurade's measurements are plotted. The agreement especially in the immediate neighbourhood of the wall is beautiful. This seems to ascertain our assumptions, i.e., $\eta = \alpha\epsilon\psi = \text{const.}$ and the rapid decrease of mixing length towards the wall or the existence of a pure laminar sub-layer.

§ 7. Distribution of turbulent energy near a wall. In virtue of equation (14) we can discuss the distribution of turbulent velocity in a pipe except its central region. The turbulent velocity has its maximum value at a point ξ_{max} near the wall. Its relation to Reynolds number of the mean flow is shown in Fig. 3. Again experimental results of Fage, Townend and Taylor seem to confirm the theoretical result.

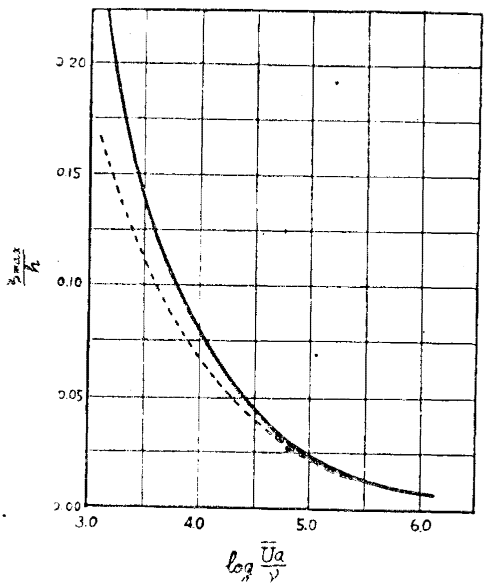

Fig. 3

§ 8. Velocity distribution in a pipe. In the central region of a pipe, i.e. the consuming zone, our theory does not hold in the strict sense. So we assume a hypabolic distribution of turbulent energy instead of that of given by (13) and a constant mixing length for this region. Then we can calculate the velocity distribution through a cross section of the pipe. The results of calculation are as follows:

$r \leq r_3$ (the consuming zone)

$$\frac{U_c - U}{U_\tau} = \frac{1}{K} \cdot \frac{r_3}{\xi_3} \cdot \sqrt{\frac{2r_3}{a}} \cdot \left\{ -1 + \sqrt{1 + \left(\frac{r}{r_3}\right)^2} \right\}, \tag{20}$$

$r \geq r_3$ (the generating zone)

$$\frac{U_c-U}{U_\tau}=\frac{1}{K}\left\{\frac{\xi-\xi_3}{2a}+0.585\frac{r_3}{\xi_3}\sqrt{\frac{r_3}{a}}+\log\frac{\xi_3}{\xi}\right\}. \tag{21}$$

Comparison with the measurements of Nikuradse and Stanton are given in Fig. 4.

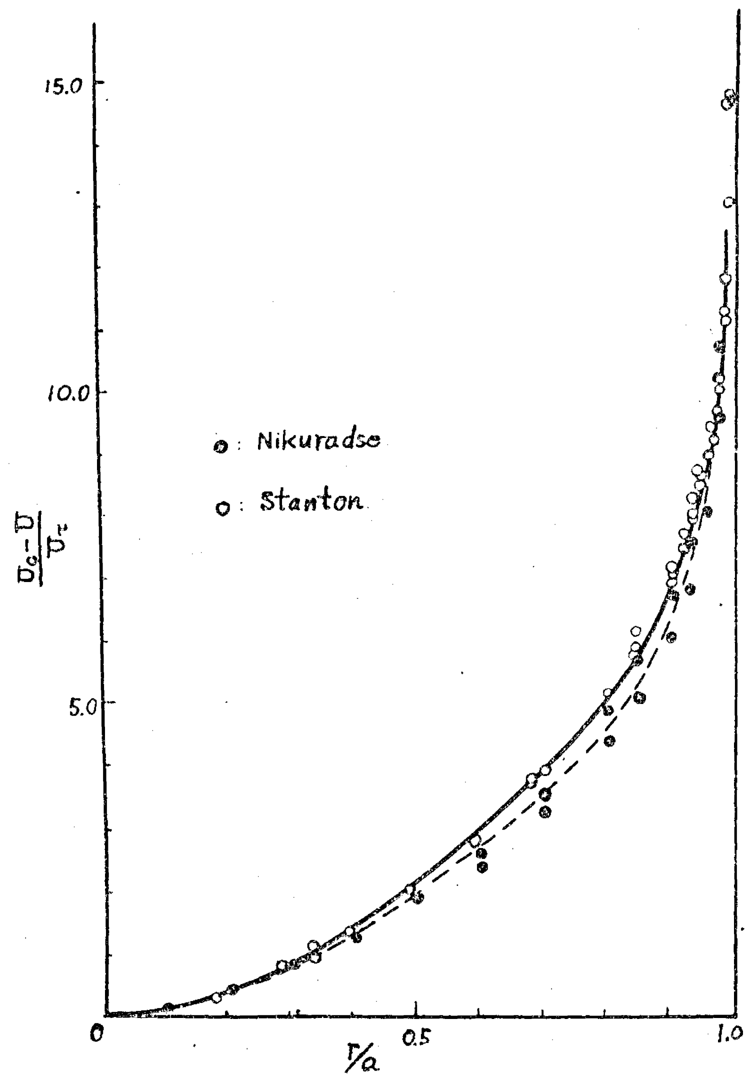

Fig. 4. Velocity Distribution in a smooth Pipe.

Full line: $\xi_3/a=0.21$, $K=0.4$. Dotted line: $\xi_3/a=0.25$, $K=0.38$.