A NEW DEFINITION OF TURBULENCE BASED ON THE THEORY OF PROBABILITY AND THE FUNDAMENTAL EQUATIONS

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A NEW DEFINITION OF TURBULENCE BASED ON THE THEORY OF PROBABILITY AND THE FUNDAMENTAL EQUATIONS.[1]

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The author succeeded to give a new interpretation to the turbulent phenomena based on the theory of probability; i.e. by the turbulence we understand such a state, in which each distribution function of the physical quantities assigned to the fluid fluctuates with certain probability in a corresponding group of functions, of which characteristic properties are to be determined by the physical laws prevailing there and then these probabilities may be thought as the mathematical representation of the turbulence. To make clear this conception a new idea of "representative particles" analogous to molecular partieles in the kinetic theory of gas is introduced.

We can carry out the practical formulation of this idea by expressing the distribution functions in the neighbourhood of a given point through either the values of the physical quantities and their successive derivatives at the point or the values at enumerable many points suitably chosen in the region. In both cases by means of the representative particles the fundamental equations for the probability function in a non-steady turbulence are obtained. From them the fundamental equations for the diffusion of the mean values of products of the physical quantities at N-points are derived.

$\lambda_{(k)}(k = 1, 2, \ldots)$ be the physical quantities (inclusive of velocity) assigned to the fluid and let $\dfrac{D\lambda^{(k)}}{Dt} = -\Lambda^{(k)}$, where $\Lambda^{(k)}$ are functions of λ and their successive derivatives $\lambda_{\alpha,\beta,} \cdots$. Let $f(x^i\, t\,;\, \lambda^{(k)}_{\alpha,\beta,} \ldots)$ be the probability for the appearance of $\lambda^{(k)}_{\alpha,\beta,} \cdots$ at a point $P(x^i)$ and time t. Then following the method of Boltzmann (cf. J.H. Jeans, The Dynamical Theory of Gases (1916), Camb.) we have one of the fundamental equations;

[1] Abstract of "A New Definition of Turbulence based on the Theory of Probability and the fundamental Equations" (in Japanese), Michinori Kurihara, Rep. Res. Inst. Fluid Engng., Kyushu Univ. **1** (1942), 1–20.

$$\frac{\partial f}{\partial t}+u_\sigma\frac{\partial f}{\partial x_\sigma}=u^\sigma_\sigma f-\sum_{k,\alpha,\beta,\dots}\frac{\partial}{\partial\lambda^{(k)}_{\alpha,\beta,\dots}}\left[\frac{\Delta\lambda^{(k)}_{\alpha,\alpha,\dots}}{\Delta\lambda}\cdot f\right], \tag{1}$$

where

$$\frac{\Delta\lambda^{(k)}_{\alpha,\beta,\dots\tau}}{\Delta t}=-\left\{(u^\sigma\lambda^{(k)}_\sigma)_{\alpha,\beta,\dots\tau}-u^\sigma\lambda^{(k)}_{\alpha,\beta\dots\tau,\sigma}+A_{\alpha,\beta,\dots\tau}\right\}.$$

Next let $\lambda^{(k)}_{(j)}$ ($j=1, 2, \dots N$) be the values of the physical quantities at N-selected points in the neighbourhood of a point $P_1(x_1^i)$. If we denote by $F_N(x_1^i,\ \xi_j^i,\ t;\ \lambda_{(j)})$, where ξ_j^i means the coordinate of P_j relative to P_1, the probability that $\lambda_{(j)}$ take preassigned values and by F it's limiting function as N becomes infinity. Then we can easily obtain from (1) the fundamental equation,

$$\frac{\partial F}{\partial t}+u^\sigma_{(1)}\frac{\partial F}{\partial x_1^\sigma}=(u^\sigma_\sigma)_{(1)}F+\sum_{k,j}\frac{\partial}{\partial\lambda^{(k)}_{(j)}}\Big[\{(A^{(k)})_{(j)}+(\lambda^{(k)}_\sigma u^\sigma)_{(j)}-(\lambda^{(k)}_\sigma)_{(j)}(u^\sigma)_{(1)}\}\cdot F\Big], \tag{2}$$

where it is assumed that $(A)_{(j)}$, $(\lambda_\sigma u^\sigma)_{(j)}$, $(\lambda_\sigma)_{(j)}$ can be given as functions of $\lambda_{(i)}$'s.

Now if we integrate (1) respect to all the differential coefficients higher order than N over the whole domain, then we have for the probability function $f_N(x^i,\ t;\ \lambda^{(k)}_{a_1,a_2,\cdots a_l})$, $l\leq N$, related with the differential coefficients up to N-th order in $P(x^i)$,

$$\frac{\partial f_N}{\partial t}+u^\sigma\frac{\partial f_N}{\partial x^\sigma}=u^\sigma_\sigma f_N-\sum_{k,a}\frac{\partial}{\partial\lambda^{(k)}_{a_1,a_2,a_l}}\left\{\left(\overline{\frac{\Delta\lambda^{(k)}_{a_1,\cdots a_l}}{\Delta t}}^{p}\right)\cdot f_N\right\}, \tag{3}$$

where $\left(\overline{\frac{\Delta\lambda_{a_1,a_2,\cdots a_l}}{\Delta t}}^{p}\right)$ means the partial mean of $\frac{\Delta\lambda_{a_1,\cdots}}{\Delta t}$ when the differential coefficients up to N-th order are maintained constant.

Similarly through the integration with respect to all $\lambda_{(n)}$, $n>N$ we obtain for the probability function $F_N(\lambda_{(1)}, \lambda_{(2)}, \cdots \lambda_{(N)})$ related to N-points from (2)

$$\frac{\partial F_N}{\partial t}+u^\sigma_{(1)}\frac{\partial F_N}{\partial x_1^\sigma}=\overline{(u^\sigma_\sigma)_{(1)}}^{p}F_N+\sum_n\sum_{j=1}^N\frac{\partial}{\partial\lambda^{(n)}_{(j)}}\Big[\{\overline{(A^{(n)})_{(j)}}^{p}+\overline{(\lambda^{(n)}_\sigma u^\sigma)_{(j)}}^{p}-\overline{(\lambda^{(n)}_\sigma)_{(j)}(u^\sigma)_{(1)}}^{p}\}\cdot F_N\Big]. \tag{4}$$

We are now able to deduce the fundamental equations for the mean values from these equations. In the first instance, let Q be any function of the physical quantities and their derivatives. Multiplying (1) with Q

and integrating about all the quantities $\lambda_{\alpha\ \beta,\dots}$ over their whole regions we have after the manner of Maxwell

$$\frac{D\overline{Q}}{Dt}=\bar{u}^{\sigma}\frac{\partial\overline{Q}}{\partial x^{\sigma}}-\frac{\partial\overline{u^{\sigma}Q}}{\partial x^{\sigma}}+\overline{u^{\sigma}_{\sigma}Q}+\sum\frac{\partial Q}{\partial\lambda^{(k)}_{\alpha,\beta,\dots}}\cdot\frac{\Delta\lambda_{\alpha,\beta,\dots}}{\Delta t}\,,\tag{5}$$

which is nothing but the general equation of transfer.

For example, if we put $Q=u^{i}$ or $\frac{1}{2}\sum(u^{i})^{2}$, equation (5) gives the equation of turbulent flow or of turbulent energy respectively.

Lastly, if we multiply (4) with $\overset{i}{\Pi}Q_{(i)}(\lambda^{(\alpha)}_{(i)})$, where $Q_{(i)}$ is any function of $\lambda^{(\alpha)}$ at the point $P_{(i)}$, and integrate, we obtain after some modification

$$\frac{\partial}{\partial t}(\overline{\overset{i}{\Pi}Q_{(i)}})+\sum_{i,\alpha}\overline{(\lambda_{\sigma}u^{\sigma})_{(1)}\frac{\partial Q_{(i)}}{\partial\lambda^{(\alpha)}_{(i)}}\overset{j}{\underset{j\neq i}{\Pi}}Q_{(j)}}=-\sum\overline{\Lambda^{(\alpha)}_{(i)}\frac{\partial Q_{(i)}}{\partial\lambda^{(\alpha)}_{(i)}}\overset{j}{\underset{j\neq i}{\Pi}}Q_{(1)}}\,,\tag{6}$$

which is regarded as the diffusion equation of the product-mean function. For instance, putting $Q_{(1)}=u^{i}{}_{(1)}$, $Q_{(2)}=u^{\sigma}{}_{(2)}$ in (6), we can derive the fundamental equation of the double correlation function (Karman, Proc. Roy. Soc. London A, 164 (1938)).