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ON A METHOD OF SOLVING ELASTIC PROBLEMS OF PLATES

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In the elastic problems of vertically loaded plates, the normal stress components of the direction of the thickness of the plate are always considered very small in comparison with the other components and neglected generally without any certifications. But we cannot think that, in the stressed states of the plates, these normal stress components in the neighbourhood of the points under loads are negligibly small in comparison with the other components.

In the present report, we treat the problems considering all the stress components and show that the general solutions(1) of the so-called two dimensional elastic problems which have already been published by the author can be easily derived from the general solution obtained above.

1. Introduction. In the usual method of solving the elastic bending problems of vertically loaded plates, the normal stress components of the direction of the thickness of the plates have been set at naught without any certifications as mentioned above, and some boundary conditions at the peripheral sides of the plates have also been left out of consideration. The papers(2) treating the problems considering these imperfections have been proposed lately. But we have not yet appropriate treatment of the methods.

In the present report, using the following necessary conditions, namely (i) the equation of equilibrium, (ii) the identical equations of stress components, (iii) the characters of stress components and displacement ones, (iv) the relations among stress, strain and displacement, mentioned in the next fundamental principle of the theory, we investigate the general solution of the elastic bending problems of the plates with vertical loads satisfying a plane harmonic equation on the upper surface. Next, as the special cases of the problems stated above, we touch the general solutions of the so-called two dimensional elastic problems which have already been proposed by the author.

2. Fundamental principle of the theory. In general, we have the following conditions in the elastic problems, when body force and accelaration can be neglected.

(i) Equation of equiliblium:—

$$\frac{\partial}{\partial x}(\widehat{xx},\ \widehat{xy},\ \widehat{zx}) + \frac{\partial}{\partial y}(\widehat{xy},\ \widehat{yy},\ \widehat{yz}) + \frac{\partial}{\partial z}(\widehat{zx},\ \widehat{yz},\ \widehat{zz}) = 0 \qquad (A).$$

(ii) Identical equations of stress components

$$\nabla^2(\widehat{xx}, \widehat{yy}, \widehat{zz}, \widehat{yz}, \widehat{zx}, \widehat{xy}) = \frac{-1}{1+\sigma}\left(\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}, \frac{\partial^2}{\partial y\,\partial z}, \frac{\partial^2}{\partial z\,\partial x}, \frac{\partial^2}{\partial x\,\partial y}\right)\Theta \qquad \text{(B)},$$

in which σ = poisson's ratio, $\Theta = \widehat{xx} + \widehat{yy} + \widehat{zz}$, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

(iii) Characters of stress and displacement:—

$$\left.\begin{aligned} &\nabla^2(\Theta, \Delta) = 0, \quad \nabla^2(\tilde{\omega}_x, \tilde{\omega}_y, \tilde{\omega}_z) = 0 \\ &\nabla^4(\widehat{xx}, \widehat{yy}, \widehat{zz}, \widehat{yz}, \widehat{zx}, \widehat{xy}) = 0, \quad \nabla^4(u, v, w) = 0 \end{aligned}\right\} \text{(C)},$$

in which $\nabla^4 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^2$, $2\tilde{\omega}_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$, $2\tilde{\omega}_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$,

$$2\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \quad \Delta = e_{xx} + e_{yy} + e_{zz}.$$

(iv) Relations among stress, strain and displacement:—

$$\left.\begin{aligned} &(e_{xx}, e_{yy}, e_{zz}) = \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}\right) = \frac{1}{E}\{(1+\sigma)(\widehat{xx}, \widehat{yy}, \widehat{zz}) - \sigma\,\Theta\} \\ &e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{N}\widehat{xy}, \quad e_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \frac{1}{N}\widehat{yz} \\ &e_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{1}{N}\widehat{zx}, \end{aligned}\right\} \text{(D)},$$

in which E = Young's Modulus, and N = Torsional rigidity.

Hence, the present problems reduce to find stress and displacement satisfying all of the above conditions (A), (B), (C), and (D) considering the boundary conditions.

Now, in the present report, we treat the elastic problems of the plate under vertical loads acting only on its upper surface.

First, let us take the x- and the y-axes perpendicularly to each other in the middle plane of the plate and the z axis in the direction of the loads. Next, we assume that stress component $\widehat{zz}$ can be expressed by the equation

$$\widehat{zz} = p(x\,y)\cdot f(z) \qquad (1),$$

in which $p(x, y)$ denotes the function of distributed loads acting on the upper surface of the plate, h the half length of the thickness of the plate, and

$$f(z) = \frac{3}{4}\left\{\frac{-2}{3} + \frac{z}{h} - \frac{1}{3}\left(\frac{z}{h}\right)^3\right\}.$$

If so, applying the Eq. (1) to the 3rd equation of conditions (B), we have

$$\nabla^2\widehat{zz} = \nabla_1^2 p\cdot f + p\cdot f'' = \frac{-1}{1+\sigma}\frac{\partial^2}{\partial z^2}\Theta,$$

in which $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

Next, let us assume overagain that the load function $p(x, y)$ satisfies a plane harmonic equation. If so, the function Θ can be determined as follows by means of integrating the above equation twice with respect to z and performing the calculation of the operator ∇^2 again under the consideration of the 1st equation of the condition (C),

$$\Theta = \Theta_0 + \varphi_0 z - (1+\sigma) p \cdot f \qquad (2),$$

in which (Θ_0, φ_0, p) = functions of variables x, y only and

$$\nabla_1{}^2(p, \Theta_0) = 0, \quad \nabla_1{}^2 \varphi_0 = -\frac{3}{2}\frac{1}{h}(1+\sigma) p.$$

Here, applying the above results to the 3rd equation of the condition (A), the 4th and the 5th equitions of condition (B) and considering the boundary conditions on the upper and lower surfaces, as to $(\widehat{zx}, \widehat{yz})$ components, we have the equations

$$\left.\begin{aligned} &\frac{\partial \widehat{zx}}{\partial x} + \frac{\partial \widehat{yz}}{\partial y} + p \cdot f' = 0, \quad \nabla^2(\widehat{zx}, \widehat{yz}) = \frac{-1}{1+\sigma}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\{\varphi_0 - (1+\sigma) p \cdot f'\}, \\ &(\widehat{yz})_{z=\pm h} = (\widehat{zx})_{z=\pm h} = 0 \end{aligned}\right\}.$$

Then, we know that, from these equation, $(\widehat{zx}, \widehat{yz})$ components are given by the following formulae,

$$\left.\begin{aligned} \widehat{zx} &= (h^2 - z^2)\frac{\partial}{\partial x}\left\{\frac{\varphi_0}{2(1+\sigma)} - \frac{3p}{4h}\left(1 - \frac{h^2+z^2}{6h^2}\right)\right\} + \frac{\partial H}{\partial y} \\ \widehat{yz} &= (h^2 - z^2)\frac{\partial}{\partial y}\left\{\frac{\varphi_0}{2(1+\sigma)} - \frac{3p}{4h}\left(1 - \frac{h^2+z^2}{6h^2}\right)\right\} - \frac{\partial H}{\partial x} \end{aligned}\right\} \qquad (3),$$

where $\qquad \nabla^2 H = 0, \quad (H)_{z=\pm h} = \alpha_i$ (constant).

And, moreover, adopting the above results to the remaining condition, namely the 1st, the 2nd & the 6th equations of the condition (B), we have

$$\left.\begin{aligned} &\frac{\partial \widehat{xx}}{\partial x} + \frac{\partial \widehat{xy}}{\partial y} = -\frac{\partial \widehat{zx}}{\partial z} = \frac{\partial}{\partial x}\left\{\frac{z}{1+\sigma}\varphi_0 - \frac{3}{2} p \frac{z}{h}\left(1 - \frac{z^2}{3h^2}\right)\right\} - \frac{\partial^2 H}{\partial y\, \partial z} \\ &\frac{\partial \widehat{xy}}{\partial x} + \frac{\partial \widehat{yy}}{\partial y} = -\frac{\partial \widehat{yz}}{\partial z} = \frac{\partial}{\partial y}\left\{\frac{z}{1+\sigma}\varphi_0 - \frac{3}{2} p \frac{z}{h}\left(1 - \frac{z^2}{3h^2}\right)\right\} + \frac{\partial^2 H}{\partial z\, \partial x} \\ &\nabla^2(\widehat{xx}, \widehat{yy}, \widehat{xy}) = \frac{-1}{1+\sigma}\left(\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial x\, \partial y}\right)\{\Theta_0 + \varphi_0 z - (1+\sigma) p f\} \end{aligned}\right\} \qquad (4_1).$$

Hereupon, let us separate the above equations into two parts as follows,

$$\left.\begin{aligned} &\widehat{zx}_0 = \widehat{yz}_0 = \widehat{zz}_0 = 0, \quad \widehat{xx}_0 + \widehat{yy}_0 = \Theta_0, \\ &\frac{\partial}{\partial x}(\widehat{xx}_0, \widehat{xy}_0) + \frac{\partial}{\partial y}(\widehat{xy}_0, \widehat{yy}_0) = 0 \\ &\nabla^2(\widehat{xx}_0, \widehat{yy}_0, \widehat{xy}_0) = \frac{-1}{1+\sigma}\left(\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial x\, \partial y}\right)\Theta_0 \end{aligned}\right\} \qquad (4_2)$$

and $\widehat{zx}_1 = \widehat{zx}, \quad \widehat{yz}_1 = \widehat{yz}, \quad \widehat{zz}_1 = \widehat{zz}$

$$\left.\begin{aligned} &\nabla^2(\widehat{xx}_1, \widehat{yy}_1, \widehat{xy}_1) = \frac{-1}{1+\sigma}\left(\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial x\,\partial y}\right)\{\varphi_0 z - (1+\sigma)\,p\cdot f\} \\ &\frac{\partial\,\widehat{xx}_1}{\partial x} + \frac{\partial\,\widehat{xy}_1}{\partial y} = \frac{\partial}{\partial x}\left\{\frac{z}{1+\sigma}\varphi_0 - \frac{p}{2}\frac{z}{h}\left(3 - \frac{z^2}{h^2}\right)\right\} - \frac{\partial^2 H}{\partial y\,\partial z} \\ &\frac{\partial\,\widehat{xy}_1}{\partial x} + \frac{\partial\,\widehat{yy}_1}{\partial y} = \frac{\partial}{\partial y}\left\{\frac{z}{1+\sigma}\varphi_0 - \frac{p}{2}\frac{z}{h}\left(3 - \frac{z^2}{h^2}\right)\right\} + \frac{\partial^2 H}{\partial z\,\partial x} \end{aligned}\right\} \quad (4_3).$$

If so, we see easily that the former stressed states defined by the Eqs. (4_2) are the well-known plane stress states. Therefore, as well known, these stress components $(\widehat{xx}_0, \widehat{yy}_0, \widehat{xy}_0)$ are expressed by the following formulae

$$\widehat{xx}_0 = \frac{\partial^2 \chi_1}{\partial y^2}, \quad \widehat{yy}_0 = \frac{\partial^2 \chi_1}{\partial x^2}, \quad \widehat{xy}_0 = -\frac{\partial^2 \chi_1}{\partial x\,\partial y}, \quad \widehat{yz}_0 = \widehat{zx}_0 = \widehat{zz}_0 = 0 \quad (5_1),$$

in which

$$\chi_1 = \Psi + \Theta_1 z - \frac{\sigma}{2(1+\sigma)}\Theta_0 z^2, \quad \Psi, \Theta_0, \Theta_1 = \text{functions of } x, y \text{ only}$$

$$\nabla_1^2 \Psi = \Theta_0, \quad \nabla_1^2(\Theta_0, \Theta_1) = 0.$$

On the other hand, as to the latter stress states defined by the equations (4_3), let us first take the following forms as the solutions of the Eqs. (4_3),

$$\left.\begin{aligned} \widehat{xx}_1 &= \frac{z}{1+\sigma}\varphi_0 - \frac{p}{2}\frac{z}{h}\left(3 - \frac{z^2}{h^2}\right) + 2\frac{\partial^2}{\partial x\,\partial y}\int H\,dz + \frac{\partial^2 \chi_2}{\partial y^2} \\ \widehat{yy}_1 &= \frac{z}{1+\sigma}\varphi_0 - \frac{p}{2}\frac{z}{h}\left(3 - \frac{z^2}{h^2}\right) - 2\frac{\partial^2}{\partial x\,\partial y}\int H\,dz + \frac{\partial^2 \chi_2}{\partial x^2} \\ \widehat{xy}_1 &= \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}\right)\int H\,dz - \frac{\partial^2 \chi_2}{\partial x\,\partial y} \end{aligned}\right\} \quad (5_2).$$

If so, the 7th and the 8th equations of the Eqs. (4_3) are always satisfied without reference to the function χ_2. Because, substituting the Eqs. (5_2) in these equations, we have

$$\left.\begin{aligned} \frac{\partial\,\widehat{xx}_1}{\partial x} + \frac{\partial\,\widehat{xy}_1}{\partial y} &= \frac{\partial}{\partial x}\left\{\frac{z}{1+\sigma}\varphi_0 - \frac{p}{2}\frac{z}{h}\left(3 - \frac{z^2}{h^2}\right)\right\} - \frac{\partial^2 H}{\partial y\,\partial z} \\ \frac{\partial\,\widehat{xy}_1}{\partial x} + \frac{\partial\,\widehat{yy}_1}{\partial y} &= \frac{\partial}{\partial y}\left\{\frac{z}{1+\sigma}\varphi_0 - \frac{p}{2}\frac{z}{h}\left(3 - \frac{z^2}{h^2}\right)\right\} + \frac{\partial^2 H}{\partial z\,\partial x} \end{aligned}\right\}$$

and the above equations are nothing but these equations of the Eqs. (4_3). Next, as to the function χ_2, from the Eqs. (5_2) and the Eqs. (1) & (2), we have first the following condition

$$\nabla_1^2 \chi_2 = -\frac{1-\sigma}{1+\sigma}\varphi_0 z + p\left\{\frac{z}{h}\left(3 - \frac{z^2}{h^2}\right) - (2+\sigma)f\right\} \quad (6_1).$$

Further, applying the above equations and the 1st equation of the Eqs. (5_2) to the 4th equation of the Eqs. (4_2), we have

$$
\begin{aligned}
\nabla^2 \widehat{xx} &= \frac{3}{2}\frac{p}{h^3} z + \frac{1}{1+\sigma}\frac{\partial^2}{\partial y^2}\{\varphi_0 z - (1+\sigma) p f\} \\
&= \frac{z}{1+\sigma}\nabla_1^2 \varphi_0 + \frac{3}{h^3} p z + \frac{\partial^2}{\partial y^2}\left(\nabla_1^2 \chi_2 + \frac{\partial^2 \chi_2}{\partial z^2}\right) \\
&= -\frac{3}{2}\frac{p z}{h^3} + \frac{3}{h^3} p z + \frac{\partial^2}{\partial y^2}\left[-\frac{1-\sigma}{1+\sigma}\varphi_0 z + p\left\{\frac{z}{h}\left(3 - \frac{z^2}{h^2}\right)\right.\right. \\
&\qquad\qquad \left.\left. - (2+\sigma) f\right\} + \frac{\partial^2 \chi_2}{\partial z^2}\right]
\end{aligned}
$$

$$
\therefore \quad \frac{\partial^2}{\partial y^2}\left[\frac{\partial^2 \chi_2}{\partial z^2} - \frac{2-\sigma}{1+\sigma}\varphi_0 z + p\left\{\frac{z}{h}\left(3 - \frac{z^2}{h^2}\right) - (1+\sigma) f\right\}\right] = 0 .
$$

Similarly as the above descriptions, applying the 2nd and the 3rd equations of the Eqs. (5_2) and the Eqs. (6_1) to the 5th & the 6th equations of the Eqs. (4_3), we have

$$
\left(\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial x\, \partial y}\right)\left[\frac{\partial^2 \chi_2}{\partial z^2} - \frac{2-\sigma}{1+\sigma}\varphi_0 z + p\left\{\frac{z}{h}\left(3 - \frac{z^2}{h^2}\right) - (1+\sigma) f\right\}\right] = 0 .
$$

Then, integrating these equations with regard to the variables (x, y) and neglecting the independent terms of stress states, we have

$$
\chi_2 = \frac{2-\sigma}{6(1+\sigma)}\varphi_0 z^3 - p\left\{\frac{z^3}{2h}\left(1 - \frac{1}{10}\frac{z^2}{h^2}\right) - (1+\sigma)\int\!\!\int f (dz)^2\right\} + \varphi_1 z + \varphi_2 ,
$$

in which $\varphi_1, \varphi_2 =$ functions of the variables (x, y) only.

Moreover, performing the calculation of the operator (∇_1^2) for the above equation and using the Eqs. (6_1), the function χ_2 can be determined as follows,

$$
\chi_2 = \frac{2-\sigma}{6(1+\sigma)} z^3 \varphi_0 - \frac{p z^2}{4}\left\{(1+\sigma) + \frac{3-\sigma}{2}\frac{z}{h}\left(1 - \frac{1}{10}\frac{z^2}{h^2}\right)\right\} + \varphi_1 z + \varphi_2 \quad (6_2),
$$

in which

$$
\nabla_1^2 \varphi_1 = \frac{3}{4}(2-\sigma)\frac{p}{h} - \frac{1-\sigma}{1+\sigma}\varphi_0 , \qquad \nabla_1^2 \varphi_2 = \left(1 + \frac{\sigma}{2}\right) p .
$$

After all, from the above treatments, the stress components $(\widehat{xx}, \widehat{yy}, \widehat{xy})$ are found by simply summing up both of the above stress states expressed by the Eqs. (5_1) and (5_2) respectively and the results are as follows

$$
\left.
\begin{aligned}
\widehat{xx} &= \frac{z}{1+\sigma}\varphi_0 - \frac{p}{2}\frac{z}{h}\left(3 - \frac{z^2}{h^2}\right) + 2\frac{\partial^2}{\partial x\, \partial y}\int H\, dz + \frac{\partial^2 \chi}{\partial y^2} \\
\widehat{yy} &= \frac{z}{1+\sigma}\varphi_0 - \frac{p}{2}\frac{z}{h}\left(3 - \frac{z^2}{h^2}\right) - 2\frac{\partial^2}{\partial x\, \partial y}\int H\, dz + \frac{\partial^2 \chi}{\partial x^2} \\
\widehat{xy} &= \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}\right)\int H\, dz - \frac{\partial^2 \chi}{\partial x\, \partial y}
\end{aligned}
\right\} \quad (5_3),
$$

in which

$$\chi = \Psi + \varphi_2 + z(\Theta_1 + \varphi_1) - \frac{z^2}{6(1+\sigma)}\{3\sigma\Theta_0 - (2-\sigma)\varphi_0 z\}$$
$$- \frac{1}{4} p z^2 \left\{(1+\sigma) + \frac{3-\sigma}{2}\frac{z}{h}\left(1 - \frac{1}{10}\frac{z^2}{h^2}\right)\right\},$$

$$\nabla_1^2 \Psi = \Theta_0, \quad \nabla_1^2 \varphi_0 = -\frac{3}{2}(1+\sigma)\frac{p}{h^3}, \quad \nabla_1^2 \varphi_1 = \frac{3}{4}(2-\sigma)\frac{p}{h} - \frac{1-\sigma}{1+\sigma}\varphi_0,$$

$$\nabla_1^2 \varphi_2 = \frac{1}{2}(2+\sigma)p, \quad \nabla_1^2(\Theta_0, \Theta_1) = 0.$$

In conclusion, from the above descriptions, we know that the stress states of the present problems are given by the following formulae.—

$$\left.\begin{aligned}
\widehat{xx} &= \frac{z}{1+\sigma}\varphi_0 - \frac{p}{2}\frac{z}{h}\left(3 - \frac{z^2}{h^2}\right) + 2\frac{\partial^2}{\partial x\,\partial y}\int H\,dz + \frac{\partial^2 \chi}{\partial y^2} \\
\widehat{yy} &= \frac{z}{1+\sigma}\varphi_0 - \frac{p}{2}\frac{z}{h}\left(3 - \frac{z^2}{h^2}\right) - 2\frac{\partial^2}{\partial x\,\partial y}\int H\,dz + \frac{\partial^2 \chi}{\partial x^2} \\
\widehat{xy} &= \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}\right)\int H\,dz - \frac{\partial^2 \chi}{\partial x\,\partial y} \\
\widehat{zx} &= (h^2 - z^2)\frac{\partial}{\partial x}\left\{\frac{\varphi_0}{2(1+\sigma)} - \frac{3p}{4h}\left(1 - \frac{h^2+z^2}{6h^2}\right)\right\} + \frac{\partial H}{\partial y} \\
\widehat{yz} &= (h^2 - z^2)\frac{\partial}{\partial y}\left\{\frac{\varphi_0}{2(1+\sigma)} - \frac{3p}{4h}\left(1 - \frac{h^2+z^2}{6h^2}\right)\right\} - \frac{\partial H}{\partial x} \\
\widehat{zz} &= \frac{3}{4}p\left\{\frac{-2}{3} + \frac{z}{h} - \frac{1}{3}\left(\frac{z}{h}\right)^3\right\}
\end{aligned}\right\} \quad (7),$$

in which

$$\chi = (\Psi + \varphi_2) + \varphi_1 z - \frac{z^2}{6(1+\sigma)}\{3\sigma\Theta_0 - (2-\sigma)\varphi_0 z\}$$
$$- \frac{1}{4} p z^2 \left\{(1+\sigma) + \frac{3-\sigma}{2}\frac{z}{h}\left(1 - \frac{1}{10}\frac{z^2}{h^2}\right)\right\},$$

$$\nabla_1^2 \Psi = \Theta_0, \quad \nabla_1^2 \varphi_0 = -\frac{3}{2}(1+\sigma)\frac{p}{h^3},$$

$$\nabla_1^2 \varphi_1 = \frac{3}{4}(2-\sigma)\frac{p}{h} - \frac{1-\sigma}{1+\sigma}\varphi_0, \quad \nabla_1^2 \varphi_2 = \frac{2+\sigma}{2}p,$$

$\nabla_1^2(p, \Theta_0, \Theta_1) = 0, \quad \nabla^2 H = 0, \quad (H)_{z=\pm h} = (\alpha_1, \alpha_2)$ (constant),

$(p, \Theta_0, \Theta_1, \Psi, \varphi_0, \varphi_1, \varphi_2)$ = functions of variables x, y only,

p = the function of distributed loads.

Next, let us consider the displacement components. These displacement formulae can be easily derived from the stress equatinos (7) obtained above and the conditions (D) as follows. That is, applying the stress equation (7) to the 1st, the 2nd and the 3rd equations of the conditions (D), we have

$$\left.\begin{aligned} E\frac{\partial u}{\partial x} &= \Theta_0 + (1+\sigma)\left\{p + 2\frac{\partial^2}{\partial x\,\partial y}\int H\,dz - \frac{\partial^2 \chi}{\partial x^2}\right\} \\ E\frac{\partial v}{\partial y} &= \Theta_0 + (1+\sigma)\left\{p - 2\frac{\partial^2}{\partial x\,\partial y}\int H\,dz - \frac{\partial^2 \chi}{\partial y^2}\right\} \\ E\frac{\partial w}{\partial z} &= (1+\sigma)^2 p f - \sigma(\Theta_0 + \varphi_0 z) \end{aligned}\right\} \quad (8).$$

Then, integrating each of these equations with respect to variables (x, y, z) respectively and using the remaining equations of the conditions (D) and, moreover, neglecting the terms expressing the displacements of the whole plate as a rigid body, we have

$$\left.\begin{aligned} E\,u &= \xi_0 - (1+\sigma)\left\{\frac{\partial \chi}{\partial x} - 2\frac{\partial}{\partial y}\int H\,dz - \xi_1\right\}, \\ E\,v &= \eta_0 - (1+\sigma)\left\{\frac{\partial \chi}{\partial y} + 2\frac{\partial}{\partial x}\int H\,dz - \eta_1\right\}, \\ E\,w &= \frac{1}{16}(1+\sigma)^2 p\,z\left\{-8 + 6\frac{z}{h} - \left(\frac{z}{h}\right)^3\right\} - \sigma z\left(\Theta_0 + \frac{1}{2}\varphi_0 z\right) \\ &\qquad + h^2\varphi_0 + (1+\sigma)\varphi_1 - \frac{5}{4}(1+\sigma)h\cdot p \end{aligned}\right\} \quad (9),$$

in which

$$\frac{\partial}{\partial x}(\xi_0, \xi_1) = \frac{\partial}{\partial y}(\eta_0, \eta_1) = (\Theta_0, p), \quad \frac{\partial}{\partial y}(\xi_0, \xi_1) = -\frac{\partial}{\partial x}(\eta_0, \eta_1),$$

χ = the function defined by the Eqs. (7).

These equations, namely, are the required displacement formulae.

In conclusion, the results described above, namely the Eqs. (7) and (9), are the general solutions of the present problems of the plate in such cases as the vertical loads acting on its upper surface satisfy a place harmonic equation. Thereupon, in the practical calculations, we have only to determine the unknown functions of these formulae, namely the stress equations (7) and the displacement equation (9), using the following binded conditions at the periphery of the plate. That is to say, taking the s-axis in the direction of the touching line to the intersecting curve of the $x-y$ plane and the periphery plane of the plate, the n-axis in the direction of the loads, these binding conditions are as follows:—

In such problems as the external forces acting on the periphery plane of the plate are known, as to the binding conditions, we have

$$\left.\begin{aligned} \widehat{nn} &= (\widehat{nn}), & \widehat{ns} &= (\widehat{ns}), & \widehat{nz} &= (\widehat{nz}), \\ M_n &= (M_n), & T_{ns} &= (T_{ns}), & T_{zn} &= (T_{zn}) \end{aligned}\right\} \quad (10_1),$$

in which the mark ( ) denotes the boundary value and

$$M_n = \int_{-h}^{h} \widehat{nn} \cdot z\,dz\,, \quad T_{ns} = \int_{-h}^{h} \widehat{ns} \cdot z\,dz\,, \quad T_{zn} = \int_{-h}^{h} \widehat{zn} \cdot z\,dz\,.$$

In such problems as the displacements occuring in the periphery plane are known, the binding conditions are as follows

$$\left.\begin{array}{lll} u_n = (u_n)\,, & v_s = (v_s)\,, & w_z = (w_z)\,, \\ \tilde{\omega}_n = (\tilde{\omega}_n)\,, & \tilde{\omega}_s = (\tilde{\omega}_s)\,, & \tilde{\omega}_z = (\tilde{\omega}_z)\,. \end{array}\right\} \quad (10_2)\,.$$

Further, as the binding conditions, we have often such cases as some parts of the combined conditions of the Eqs. (10_1) and (10_2) stated above are given.

Next, as the special case of the above descriptions, from the above results (7) and (9), let us derive the general solutions of the so-called two dimensional elastic problems composed of the problems of (i) the generalized plane stress, (ii) the mean stress of the generalized plane stress, (iii) the plane stress, (iv) the mean stress of the plane stress and (v) the plane strain, which have already been proposed by the author.

(i) Generalized plane stress.

We call the stress states ocurred in such a plane "generalized plane stress" as the plate is acted by external forces in the direction of the plate only and have not any force on its upper and lower surfaces.

First, in these cases, from the conditions stated above, we know easily that the functions defined by the Eqs. (7) are as follows

$$\left.\begin{array}{l} p = 0\,, \quad \chi = \Psi + \varphi_1 z - \dfrac{z^2}{6(1+\sigma)}\{3\,\sigma\,\Theta_0 - (2-\sigma)\,\Theta_1 z\}\,, \\ \nabla_1^2 \varphi_1 = -\dfrac{1-\sigma}{1+\sigma}\,\Theta_1\,, \quad \Theta_1 + i\,G_1 = f(x+i\,y) \\ \nabla_1^2 \Psi = \Theta_0\,, \quad \nabla_1^2(\Theta_0, \Theta_1) = 0\,, \quad \nabla^2 H = 0\,, \quad (H)_{z=\pm h} = \alpha_i \end{array}\right\} \quad (11)\,.$$

Therefore, applying these results to the Eqs. (7), we have

$$\left.\begin{array}{l} \widehat{xx} = \dfrac{1}{1+\sigma}\,\Theta_1 z + 2\,\dfrac{\partial^2}{\partial x\,\partial y}\displaystyle\int H\,dz + \dfrac{\partial^2 \chi}{\partial y^2}\,, \\ \widehat{yy} = \dfrac{1}{1+\sigma}\,\Theta_1 z - 2\,\dfrac{\partial^2}{\partial x\,\partial y}\displaystyle\int H\,dz + \dfrac{\partial^2 \chi}{\partial x^2}\,, \\ \widehat{xy} = \left(\dfrac{\partial^2}{\partial y^2} - \dfrac{\partial^2}{\partial x^2}\right)\displaystyle\int H\,dz - \dfrac{\partial^2 \chi}{\partial x\,\partial y}\,, \qquad \widehat{zz} = 0 \\ \widehat{zx} = \dfrac{\partial}{\partial y}\left\{H + \dfrac{1}{2(1+\sigma)}(h^2 - z^2)\,G_1\right\}, \\ \widehat{yz} = -\dfrac{\partial}{\partial x}\left\{H + \dfrac{1}{2(1+\sigma)}(h^2 - z^2)\,G_1\right\}, \end{array}\right\} \quad (12)\,,$$

and these equations are the required stress formulae.

Next, applying the Eqs. (11) to the Eqs. (9), we have easily

$$\left.\begin{aligned} E\,u &= \xi_0 - (1+\sigma)\left\{\frac{\partial \chi}{\partial x} - 2\frac{\partial}{\partial y}\int H\,dz\right\}, \\ E\,v &= \eta_0 - (1+\sigma)\left\{\frac{\partial \chi}{\partial y} + 2\frac{\partial}{\partial x}\int H\,dz\right\}, \\ E\,w &= h^2\Theta_1 + (1+\sigma)\,\varphi_1 - \sigma z\,(\Theta_0 + \frac{1}{2}\Theta_1 z) \end{aligned}\right\} \quad (13),$$

in which $\dfrac{\partial \xi_0}{\partial x} = \dfrac{\partial \eta_0}{\partial y} = \Theta_0, \ \dfrac{\partial \eta_0}{\partial y} = -\dfrac{\partial \xi_0}{\partial x}$.
That is, these equations are the required displacement formulae.

(ii) Mean stress of generalized plane stress.

To simplify the caluculation stated above, we substitute sometimes the arithmentical mean values of the stress and the displacement obtained in the previous part (i) for the thickness of the generalized plane stress approximately. We call these states "mean stress of the generalized plane stress."

In such cases, we have only to execute the above calculations about the functions of the Eqs. (11) and (12). Thus, the required stress formulea are as follows

$$\left.\begin{aligned} &\overline{xx} = \frac{\partial^2 \overline{\chi}}{\partial y^2}, \quad \overline{yy} = \frac{\partial^2 \overline{\chi}}{\partial x^2}, \quad \overline{xy} = -\frac{\partial^2 \overline{\chi}}{\partial x\,\partial y} \\ &\overline{zx} = \frac{\partial}{\partial y}\left\{\overline{H} + \frac{h^2}{3(1+\sigma)}G_1\right\}, \quad \overline{yz} = -\frac{\partial}{\partial x}\left\{\overline{H} + \frac{h^2}{3(1+\sigma)}G_1\right\}, \\ &\widehat{zz} = 0 \end{aligned}\right\} \quad (14),$$

in which

$$\overline{\chi} = \Psi - \frac{\sigma h^2}{6(1+\sigma)}\Theta_0 - 2\,\Theta_1 \equiv \Psi, \quad \nabla_1^2\Psi = \Theta_0,$$

$$\overline{H} = \frac{1}{2h}\int_{-h}^{h} H\,dz, \quad \nabla^2 H = 0, \quad (H)_{z=\pm h} = \alpha_i, \quad \Theta_1 + i\,G_1 = f(x+i\,y).$$

And similarly, from the above results and the Eqs. (13), the displacement formulae are derived as follows

$$\left.\begin{aligned} &E\,\overline{u} = \xi_0 - (1+\sigma)\frac{\partial \overline{\chi}}{\partial x}, \quad E\,\overline{v} = \eta_0 - (1+\sigma)\frac{\partial \overline{\chi}}{\partial y} \\ &E\,\overline{w} = h^2\varphi_0 + (1+\sigma)\,\varphi_1 - \frac{\sigma}{6}h^2\Theta \equiv (1+\sigma)\,\varphi_1 \end{aligned}\right\} \quad (15),$$

in which

$$\frac{\partial \xi_0}{\partial x} = \frac{\partial \eta_0}{\partial y} = \Theta_0, \quad \frac{\partial \eta_0}{\partial x} = -\frac{\partial \xi_0}{\partial y}, \quad \chi = \Psi,$$

$$\nabla_1^2\Psi = \Theta_0, \quad \nabla_1^2\varphi_1 = -\frac{1-\sigma}{1+\sigma}\varphi_0, \quad \nabla_1^2(\Theta_0, \varphi_0) = 0.$$

(iii) Plane stress.

We call the following stress states "Plane stress"

$$\widehat{zx} = \widehat{yz} = \widehat{zz} = 0 .$$

Therefore, in this case, applying the above definition to the Eqs. (12), we have

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\left\{H + \frac{1}{2(1+\sigma)}(h^2 - z^2) G_1\right\} = 0 .$$

Then, integrating these equations with respect to x and y, the function H is expressed by the equation

$$H = F(z) - \frac{1}{2(1+\sigma)}(h^2 - z^2) G_1 ,$$

$F(z)$ being an integral function of the variable z only.

In the above equation, considering that H is a harmonic function, we see easily that H, G_1, φ_0, φ and χ are given respectively as follows

$$\left.\begin{aligned} &H = \frac{1}{2}\{(\alpha_1 + \alpha_2) h^2 + (\alpha_1 - \alpha_2) h z\} , \quad G_1 = \gamma \text{ (constant)}, \\ &\varphi_0 = \beta \text{ (constant)}, \quad \varphi_1 = \Theta_1 - \frac{1-\sigma}{4(1+\sigma)} \beta (x^2 + y^2) , \\ &\chi = \Psi + \left\{\Theta_1 - \frac{1-\sigma}{4(1+\sigma)} \beta (x^2 + y^2)\right\} z \\ &\qquad - \frac{z^2}{6(1+\sigma)}\{3\sigma\Theta_0 - (2-\sigma)\beta z\} \end{aligned}\right\} \quad (16).$$

Thereupon, applying the above equations to the Eqs. (12) and arranging some terms of these equations and, moreover, neglecting the independent terms of the stress states, we have the stress formulae as follows

$$\left.\begin{aligned} &\widehat{xx} = \frac{\partial^2 \chi}{\partial y^2} , \quad \widehat{yy} = \frac{\partial^2 \chi}{\partial x} , \quad \widehat{xy} = -\frac{\partial^2 \chi}{\partial x \, \partial y} , \\ &\widehat{zx} = \widehat{yz} = \widehat{zz} = 0 \end{aligned}\right\} \quad (17),$$

in which

$$\chi = \Psi + \left\{\Theta_1 + \frac{\beta}{4}(x^2 + y^2)\right\} z - \frac{\sigma}{2(1+\sigma)} \Theta_0 z^2 ,$$

$$\nabla_1^2 \Psi = \Theta_0 , \quad \nabla_1^2 (\Theta_0, \Theta_1) = 0 .$$

And from the Eqs. (16) and (13), the displacement formulae are easily derived as follows

$$\left.\begin{aligned} &E u = \xi_0 + \beta z x - (1+\sigma)\frac{\partial \chi}{\partial x} , \quad E v = \eta_0 + \beta y z - (1+\sigma)\frac{\partial \chi}{\partial y} \\ &E w = (1+\sigma)\Theta_1 - \frac{1}{4}(1-\sigma)\beta(x^2 + y^2) - \sigma z\left(\Theta_0 + \frac{1}{2}\beta z\right) \end{aligned}\right\} \quad (18),$$

in which

$$\frac{\partial \xi_0}{\partial x} = \frac{\partial \eta_0}{\partial y} = \Theta_0\,, \quad \frac{\partial \xi_0}{\partial y} = -\frac{\partial \eta_0}{\partial x}\,, \quad \beta = \text{constant}, \quad \nabla_1^2 (\Theta_0, \Theta_1) = 0\,.$$

(iv) Mean stress of plane stress

To simplify the calculation stated above, we substitute sometimes the arithmetical mean values of the stresses and the displacements obtained in the previous part (iii) for the direction of the thickness of the plate approximately instead of the plane stress. We call these stress states "mean stress of plane stress".

In such cases, from the Eqs. (17) and (18) the stress formulae and the displacement ones are directly derived as follows.

Namely, first, as to the χ of the Eqs. (17), we have

$$\bar{\chi} = \frac{1}{2h}\int_{-h}^{h} \chi\, dz = \Psi - \frac{\sigma h^2}{6(1+\sigma)}\Theta_0 \equiv \Psi \qquad (19),$$

in which $\nabla_1^2 \Psi = \Theta_0\,.$

Therefore, applying the above equations to the Eqs. (17), we know that the stress formulae are expressed by the following equations

$$\left.\begin{aligned} &\overline{xx} = \frac{\partial^2 \bar{\chi}}{\partial y^2}\,, \quad \overline{yy} = \frac{\partial^2 \bar{\chi}}{\partial x^2}\,, \quad \overline{xy} = -\frac{\partial^2 \bar{\chi}}{\partial x\, \partial y} \\ &\widehat{zx} = \widehat{yz} = \widehat{zz} = 0 \end{aligned}\right\} \qquad (20_1),$$

in which $\nabla_1^2 \bar{\chi} = \Theta_0\,, \quad \nabla_1^2 \Theta_0 = 0$

and from the Eqs. (18) the displacement formulae are as follows

$$\left.\begin{aligned} &E\,\bar{u} = \xi_0 - (1+\sigma)\frac{\partial \bar{\chi}}{\partial x}\,, \quad E\,\bar{v} = \eta_0 - (1+\sigma)\frac{\partial \bar{\chi}}{\partial y}\,, \\ &E\,\bar{w} = (1+\sigma)\,\Theta_1 - \frac{1}{4}(1-\sigma)\,\beta\,(x^2+y^2) \end{aligned}\right\} \qquad (21):$$

Further, let us rewrite the Eqs. (20_1) in the other form. First, putting the $\bar{\chi}$ as the form

$$\bar{\chi} \equiv \xi_1 + \frac{1}{2}\xi_0 x\,,$$

in which $\nabla_1^2(\xi_0, \xi_1) = 0\,, \quad \dfrac{\partial \xi_0}{\partial x} = \dfrac{\partial \eta_0}{\partial y} = \Theta_0\,, \quad -\dfrac{\partial \xi_0}{\partial y} = \dfrac{\partial \eta_0}{\partial x} = G_0\,,$

and adopting the new functions (ξ, η) such that

$$\left.\begin{aligned} &\frac{\partial^2 \xi_1}{\partial x^2} = -\frac{\partial^2 \xi_1}{\partial y^2} = \frac{\partial^2 \eta_1}{\partial x\, \partial y} \equiv \eta - \frac{1}{2}\Theta_0 \\ &-\frac{\partial^2 \eta_1}{\partial y^2} = \frac{\partial^2 \eta_1}{\partial x^2} = \frac{\partial^2 \xi_1}{\partial x\, \partial y} \equiv \xi + \frac{1}{2}G_0\,, \end{aligned}\right\}$$

we know easily that the Eqs. (20_1) are also rewritten in the forms

$$\left.\begin{aligned} &2\,\overline{xx} = \Theta_0 - x\frac{\partial \Theta_0}{\partial x} - 2\eta\,, \qquad \overline{zx} = \overline{yz} = \overline{zz} = 0 \\ &2\,\overline{yy} = \Theta_0 + x\frac{\partial \Theta_0}{\partial x} + 2\eta\,, \\ &2\,\overline{xy} = -\left(2\xi + x\frac{\partial \Theta_0}{\partial x}\right), \end{aligned}\right\} \qquad (20_2),$$

in which $\qquad \xi + i\eta = f(x+iy)\,, \quad \nabla_1^2\,\Theta_0 = 0\,.$

(v) Plane strain

As well known, the stress states of plane strain are defined as follows

$$\left.\begin{aligned} &u,\ v = \text{functions of variables } x,\ y \text{ only,} \\ &w = c_0 + c_1 x \end{aligned}\right\} \qquad (22),$$

c_0, c_1 being constants.

First, from the above definition and the conditions (D), the following equations are derived,

$$\widehat{zz} = E\,c_1 + \sigma\,\Theta_0\,, \quad \widehat{yz} = \widehat{zx} = 0\,, \quad \widehat{xy} = N\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad (23),$$

in which $\Theta_0 = (\widehat{xx} + \widehat{yy}) =$ plane harmonic function.
Therefore, let us separate the stress states into two parts of $(\widehat{xx}, \widehat{yy}, \widehat{xy})$ states and $\widehat{zz}$ ones. Next, finding the former stress states, we know that the latter stress states can be easily found from the results $\Theta_0 = (\widehat{xx} + \widehat{yy})$ and the Eqs. (23).

On the other hand, the former stress states coincide with the ones described in the part (iv) as to the charcters of these definitions. Accordingly we know that the stress formulae are directly found from the Eqs. (20_1) as follows

$$\left.\begin{aligned} &\widehat{xx} = \frac{\partial^2 \chi}{\partial y^2}\,, \quad \widehat{yy} = \frac{\partial^2 \chi}{\partial x^2}\,, \quad \widehat{xy} = -\frac{\partial^2 \chi}{\partial x\,\partial y}\,, \\ &\widehat{yz} = \widehat{zx} = 0\,, \quad \widehat{zz} = E\,c_1 + \sigma\,\Theta_0 \end{aligned}\right\} \qquad (24_1),$$

in which $\nabla_1^2\,\chi = \Theta_0$, $\chi =$ function of variables x, y only,
or from the Eqs. (20_2)

$$\left.\begin{aligned} &2\,\widehat{xx} = \Theta_0 - x\frac{\partial \Theta_0}{\partial x} - 2\eta\,, \qquad \widehat{zx} = \widehat{yz} = 0\,, \\ &2\,\widehat{yy} = \Theta_0 + x\frac{\partial \Theta_0}{\partial x} + 2\eta\,, \qquad \widehat{zz} = E\,c_1 + \sigma\,\Theta_0\,, \\ &2\,\widehat{xy} = -\left(2\xi + x\frac{\partial \Theta_0}{\partial x}\right), \end{aligned}\right\} \qquad (24_2),$$

in which $\qquad \xi + i\eta = f(x+iy)\,, \quad \nabla_1^2\,\Theta_0 = 0\,.$

Next, as to the displacements, from the 1st, the 2nd, and the 3rd equations of the conditions (D) and the definitions (22) we have

$$\left.\begin{aligned} 2N\frac{\partial u}{\partial x} &= (1-\sigma)\frac{\partial \xi_0}{\partial x} - 2\sigma N c_1 - \frac{\partial^2 \chi}{\partial x^2} \\ 2N\frac{\partial v}{\partial y} &= (1-\sigma)\frac{\partial \eta_0}{\partial y} - 2\sigma N c_1 - \frac{\partial^2 \chi}{\partial y^2} \end{aligned}\right\} .$$

Hereupon, integrating the above equations with respect to x and y and applying the remaining equations, namely the 4th, the 6th of the Eqs. (D), for these results we have

$$\left.\begin{aligned} 2Nu &= (1-\sigma)\xi_0 - 2\sigma N c_1 x - \frac{\partial \chi}{\partial x} \\ 2Nv &= (1-\sigma)\eta_0 - 2\sigma N c_1 y - \frac{\partial \chi}{\partial x} \\ w &= c_1 z \end{aligned}\right\} \quad (25),$$

in which

$$\nabla_1^2 \chi = \Theta_0 , \quad \frac{\partial \xi_0}{\partial x} = \frac{\partial \eta_0}{\partial y} = \Theta_0 , \quad \frac{\partial \xi_0}{\partial y} = -\frac{\partial \eta_0}{\partial x} , \quad c_1 = \text{constant}.$$

These equations are the required displacement formulae.

After all, the above descriptions tell us that all of the general solutions of the so-called two dimensional elastic problems are easily derived from the general solutions of the elastic bending problems of the plate treated in the present report.

3. Conclusion. In the present report, we find the general solution of the elastic bending problems of the plate under the vertical loads acting on its upper surface and being a plane harmonic function. As stated above, in the present formulae, the normal stress component $\widehat{zz}$ neglected usually and the boundary conditions at the periphery set aside in the well-known formulae, such as the conditoins ($u_n = u_s = 0$, $\tilde{\omega}_n = \tilde{\omega}_z = 0$) of the Eqs. ($10_2$) mentioned above when the plate is built in the frame at the periphery, are also taken into the consideration. Therefore, in carrying out the numerical calculations, the present method will be found some more complex than the usual method. But we think that these complex treatment must be considered an inevitable consequence when we need to find more accurate results than obtained by the usual method, or to clarify douftful points of the usual solutions or to point out the degree of the errors involved in the usual solutions.

Next we show that the general solutions of the so-called two dimensional elastic problems which have already been proposed by the author[(3)] can be easily derived from the general solutions of the elastic bending problems of plates obtained above.

Last, in the report, as stated above, we treat the problems under the restriction that the loads acting on the upper surface of the plate have to satisfy a plane harmonic equation. Therefore, the application of the present method are confined within the narrow limits. Thus we are still in the course of research of getting rid of the above restriction about the loads.

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