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NOTE

An Approximate Calculation of the Flow of a Viscous Fluid Past a Body Having a Flat End

By Jun-ichi OKABE

1. A few years ago Yamada and the present writer showed that the velocity profile within the boundary layer of a flat plate can be represented approximately by the solution of the equation

$$n^2 U \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{1.1}$$

where $$n^2 = \sqrt{2} - 1 = 0.414 \cdots,$$

which is called the modified Oseen's equation.[1] Namely,

$$u = \frac{2}{\sqrt{\pi}} U \int_0^{\frac{n}{2}\sqrt{\frac{U}{\nu x}}y} e^{-\xi^2} d\xi. \tag{1.2}$$

The notations are as usual. It is the object of this note to show that, starting from this equation, we shall be enabled to know the general features of the flow of a viscous fluid past a body having a flat end for small Reynolds numbers.

As a matter of fact, however, it is more convenient to adopt instead as the basic equation the alternative form

$$n^2 U \frac{\partial \zeta}{\partial x} = \nu \frac{\partial^2 \zeta}{\partial y^2},$$

or

$$\frac{\partial \zeta}{\partial x} = \kappa \frac{\partial^2 \zeta}{\partial y^2}, \tag{1.3}$$

where $$\kappa = \nu / n^2 U,$$

in terms of the vorticity

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{1.4}$$

For in the equation of vorticity we shall not have to take into account the effect of the pressure gradient which takes place necessarily in a flow past a body of finite breadth, since the equation of vorticity has been derived by eliminating the terms of the pressure from the equations of motion.

2. As the preliminaries we shall calculate the wake behind a flat plate of length L placed edgeways to the stream of velocity U washing only one side of the plate. Let us take the origin of the coördinates at the leading edge of the plate, the x-, and the y-axes in parallel and perpendicularly to the plate respectively, the field of flow

[1] Yamada H. and Okabe, J., An Approximate Calculation of the Laminar Wake Behind a Flat Plate. *Rep. Res. Inst. for Fluid Engg.*, vol. 4, No. 1 (1947), 1-13.

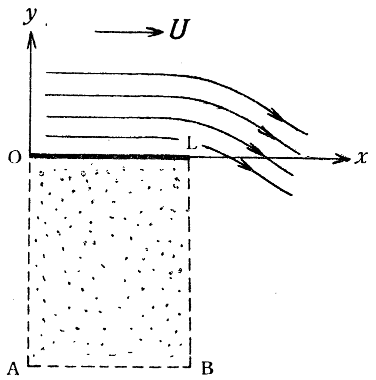

FIG. 1.

of the fluid being chosen on the positive side of y. The value of n^2 given by (1.1) relates to the boundary layer, and so it is open to doubt to assume that this value might be used in the wake as well. Our assumption is supported, however, by the following analogy:— In the boundary layer u varies from $0(y=0)$ to $U(y=\infty)$, and $n^2U \fallingdotseq 0.4\,U$, derived necessarily from the condition of momentum conservation, on the left-hand side of the equation (1.1) can be interpreted as a kind of the mean of these two extreme values. On the other hand in our case of the wake also, u is found to be $0(y=-\infty)$ and $U(y=\infty)$ in the direction of y, and further if we trace the extension of the plate, $u=0(x=L)$ and $U(x=\infty)$ in the direction of x. So we may regard the wake as an analogue of the boundary layer, the *double boundary layer* so to speak extending in two directions x and y. If we reason in this way, the value of n^2 valid in the boundary layer will be supposed to be still appropriate more or less in the wake too.

Since the boundary conditions to be imposed upon ζ are, writing $\varphi(x)$ as an unknown function, ζ_u and ζ_l as ζ for $y>0$ and $y<0$ respectively,

$$
\begin{aligned}
&[\zeta_u]_{x=0} = 0\,, \\
&[\zeta_u]_{y=0} = -\frac{nU}{\sqrt{\pi}}\sqrt{\frac{U}{\nu x}} = -\frac{U}{\sqrt{\pi\kappa x}}\,^{1)} \qquad (L>x>0)\,, \\
&[\zeta_u]_{y=0} = \varphi(x) \qquad (L<x)\,,
\end{aligned}
$$

and

$$
\begin{aligned}
&[\zeta_l]_{x=0} = 0\,, \\
&[\zeta_l]_{y=0} = 0 \qquad (L>x>0)\,, \\
&[\zeta_l]_{y=0} = \varphi(x) \qquad (L<x)\,,
\end{aligned}
$$

ζ_u and ζ_l, both of which are the solutions of (1.3), can be readily given in the following forms:[2]

$$
\left.
\begin{aligned}
\zeta_u &= \frac{y}{2\sqrt{\pi\kappa}}\int_0^L\left(-\frac{U}{\sqrt{\pi\kappa l}}\right)\exp\left[-\frac{y^2}{4\kappa(x-l)}\right]\frac{dl}{(x-l)^{3/2}} \\
&\quad + \frac{y}{2\sqrt{\pi\kappa}}\int_L^x \varphi(l)\exp\left[-\frac{y^2}{4\kappa(x-l)}\right]\frac{dl}{(x-l)^{3/2}}\,, \\
\zeta_l &= -\frac{y}{2\sqrt{\pi\kappa}}\int_L^x \varphi(l)\exp\left[-\frac{y^2}{4\kappa(x-l)}\right]\frac{dl}{(x-l)^{3/2}}\,.
\end{aligned}
\right\} \qquad (2.1)
$$

v, the component of velocity in the direction of y, being very small, we have approximately

$$\zeta = -\frac{\partial u}{\partial y} \qquad (2.2)$$

1) From the equations (1.2) and (1.4).

2) Carslaw, H. S., *Introduction to the Mathematical Theory of the Conduction of Heat in Solids*, 2nd edition (1921), p. 47.

from (1.4). Integrating (2.2) with regard to y from $-\infty$ to ∞, we have

$$-U = \int_{-\infty}^{0} \zeta_l \, dy + \int_{0}^{\infty} \zeta_u \, dy .$$

So that

$$\int_{L}^{x} \frac{\varphi(l)}{\sqrt{x-l}} dl = \frac{U}{2}\sqrt{\frac{\pi}{\kappa}}\Big(\frac{1}{\pi}\sin^{-1}\frac{2L-x}{x} - \frac{1}{2}\Big).$$

This is Abel's integral equation and we can write down the solution quite easily, viz.

$$\varphi(l) = -\frac{U}{2\sqrt{\pi\kappa l}} . \tag{2.3}$$

Note that this is one half of the value of ζ_u at $y=0$. Substituting (2.3) for $\varphi(x)$ in (2.1), it is not difficult to prove that these two solutions, ζ_u and ζ_l, can be expressed in a unified manner by

$$\zeta = -\frac{U}{2\sqrt{\pi\kappa x}} \exp\Big(-\frac{y^2}{4\kappa x}\Big)\Big\{1 + \Phi\Big[\frac{\sqrt{L}\, y}{2\sqrt{\kappa x(x-L)}}\Big]\Big\}, \tag{2.4}$$

where $\Phi(\gamma)$ denotes the error integral, i.e.

$$\Phi(\gamma) = \frac{2}{\sqrt{\pi}} \int_0^{\gamma} e^{-\xi^2} d\xi .$$

The velocity profile in the wake can be calculated by means of the vorticity from the formulas

$$U = u_u - \int_y^{\infty} \zeta_u \, dy ,$$

and

$$u_l = 0 - \int_{-\infty}^{y} \zeta_l \, dy ,$$

u_u and u_l representing u for $y > 0$ and $y < 0$ respectively. From (2.4) by integrating we are led to

$$u = \frac{U}{2}\Big\{1 + \Phi\Big(\frac{y}{2\sqrt{\kappa x}}\Big) - \frac{2}{\pi} \int_0^{\tan^{-1}\sqrt{L/(x-L)}} \exp\Big(-\frac{y^2}{4\kappa x}\sec^2\omega\Big) d\omega\Big\} \tag{2.5}$$

both for $y > 0$ and $y < 0$. In order to find out v, we write approximately

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\kappa \frac{\partial^2 u}{\partial y^2},$$

hence

$$v = -\kappa \int_{-\infty}^{y} \frac{\partial^2 u}{\partial y^2} dy = \kappa \zeta ,$$

therefore

$$v = -\frac{\sqrt{\kappa}\, U}{2\sqrt{\pi x}} \exp\Big(-\frac{y^2}{4\kappa x}\Big)\Big\{1 + \Phi\Big[\frac{\sqrt{L}\, y}{2\sqrt{\kappa x(x-L)}}\Big]\Big\}. \tag{2.6}$$

It is worth noticing that, since ζ vanishes on every side of the rectangle OABL which is infinitely long downwards, from the uniqueness of the solution ζ is found to be identically zero everywhere in the infinite rectangle. From this fact we know both u and v vanish on and in that rectangle; let us remember in particular that $u = v = 0$ on the straight line LB; see Figure 1.

3. Having finished preliminaries, the main part of the note is very brief. If we place two flat plates *vis-à-vis* at $y = \pm c$ in such a manner as each one side of the plates is washed by the flow, then we can reproduce approximately the flow past a body having the flat end of breath $2c$, because the boundary conditions on that broken line in the figure which corresponds to the bottom are satisfied automatically, cf. the end of 2. The velocity distribution can be obtained by superposing the results of the preceding section, viz.

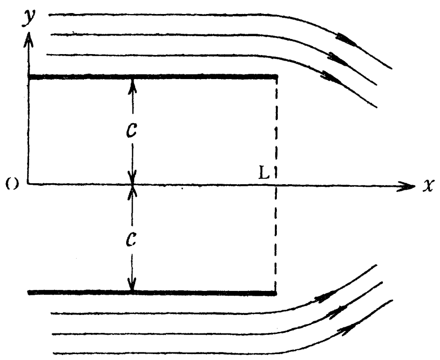

FIG. 2.

$$\left.\begin{aligned} \frac{u}{U} &= 1 + \frac{1}{2}\left\{\Phi\left(\frac{y-c}{2\sqrt{\kappa x}}\right) - \Phi\left(\frac{y+c}{2\sqrt{\kappa x}}\right)\right\} \\ &\quad - \frac{1}{\pi}\int_0^{\tan^{-1}\sqrt{L/(x-L)}} \left\{\exp\left[-\frac{(y-c)^2}{4\kappa x}\sec^2\omega\right] + \exp\left[-\frac{(y+c)^2}{4\kappa x}\sec^2\omega\right]\right\} d\omega, \\ \frac{v}{U} &= -\frac{\sqrt{\kappa}}{2\sqrt{\pi x}}\left\{\exp\left[-\frac{(y-c)^2}{4\kappa x}\right]\left[1 + \Phi\left(\frac{\sqrt{L}(y-c)}{2\sqrt{\kappa x(x-L)}}\right)\right]\right. \\ &\qquad \left. - \exp\left[-\frac{(y+c)^2}{4\kappa x}\right]\left[1 - \Phi\left(\frac{\sqrt{L}(y+c)}{2\sqrt{\kappa x(x-L)}}\right)\right]\right\}. \end{aligned}\right\} \quad (3.1)$$

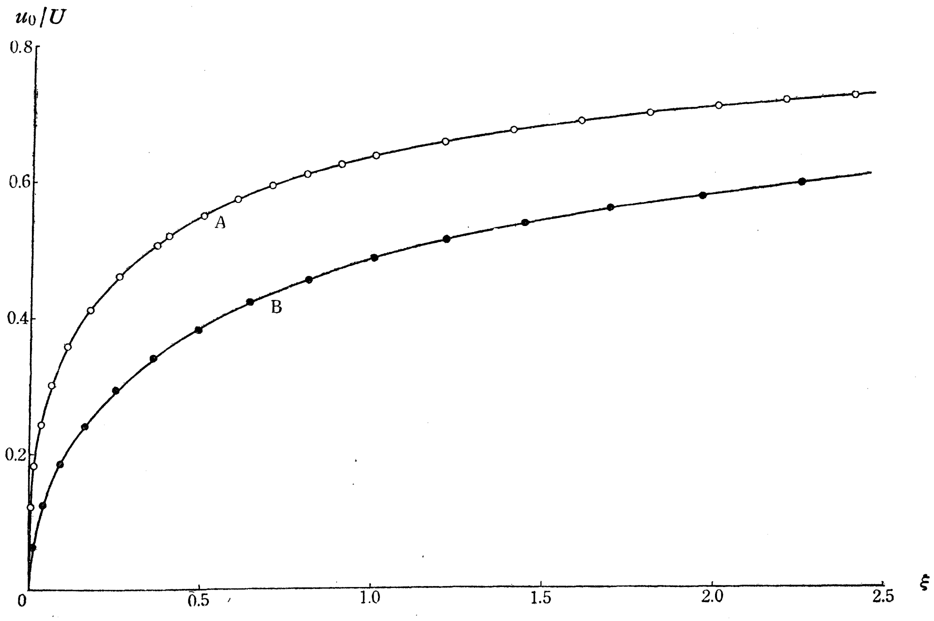

FIG. 3.

4. In order to test the accuracy of the present computation, let us consider an extreme case when $c=0$. Then the situation reduces to the ordinary wake behind a flat plate of length L. The solution of this kind of flow was calculated accurately and in detail by Goldstein.[1] We have, on the other hand, putting $c=0$ and $y=0$ in (3.1)

$$u_0 = \frac{2}{\pi} U \tan^{-1}\sqrt{\frac{x-L}{x}}. \tag{4.1}$$

In Fig. 3 u_0/U is compared between these two solutions, where we write

$$x - L = \xi L.$$

Unfortunately the agreement is not so satisfactory, and we must expect the same order of the error for the result of (3.1) also. In Fig. 4 the distribution of u in the section of $\xi = 0.259$ is shown for the flow characteristics such that

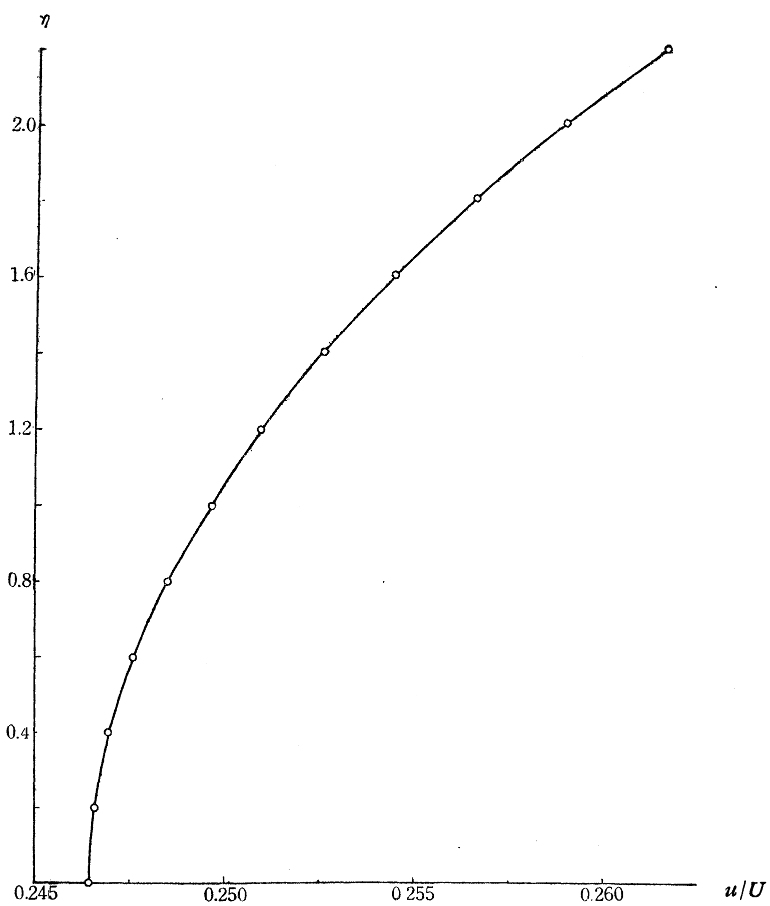

Fig. 4.

[1] Goldstein, S., On the Two-Dimensional Steady Flow of a Viscous Fluid Behind a Solid Body. -I. *Proc. Roy. Soc.* vol. 142 (1933), 545-560.

$$\frac{n}{2}\sqrt{\frac{U}{\nu}\frac{c}{L}\frac{c}{}}\Big/\sqrt{\xi}=\frac{1}{20}. \tag{4.2}$$

η, the ordinate, stands for y/c.

Finally as x approaches to L in (3.1), we should have the velocity profile of the boundary layer, viz.

$$\frac{u}{U}=\Phi\left(\frac{n}{2}\sqrt{\frac{U}{\nu L}}y\right)$$

derived from (1.2) by putting $x=L$. That this is actually so can be verified in virtue of the integral formula

$$\int_0^{\pi/2}\exp\left(-k^2\sec^2\omega\right)d\omega=\frac{\pi}{2}\{1-\Phi(k)\}. \tag{4.3}$$

Proof of the formula:

i) when $k=0$, (4.3) is evidently true.

ii) Differentiating the both hand sides with respect to k, we have $-\sqrt{\pi}\,\exp(-k^2)$ from both of them. *Q.E.D.*

The foregoing note was written several years ago under the guidance of Professor Yamada but remained unpublished. The writer hopes the calculation in this note will be improved some day to yield a better approximation by another person or by himself.

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Page 116, line 6, insert "their" between "of" and "Fourier."

Page 116, denominator of (3.9), for "p^{ω}" read "p^2."

Page 122, line 1 from the bottom, and Page 125, line 8 from the bottom, for "$\mathit{\Gamma}$" and "$\mathit{\Gamma}'$" read "Γ" and "Γ'," respectively.

Page 126, (7.6), for "l_2" read "l^2."

Page 130, line 6 from the bottom, insert "a" between "by" and "graphical."

Page 137, (9.7), left-hand side, for "ζ" read "ζ_2."

Page 152, (16), for "si$_n$h" read "sinh."

Page 153, (19), for "2_λ" read "2^λ."

Page 156, line 3 from the bottom, for "φ_n" read "$\overline{\varphi}_n$."

(*J. O.*)