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Yamada, Hikoji
Research Institute for Applied Mechanics, Kyushu University

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ON THE SLOW MOTION OF VISCOUS LIQUID PAST A CIRCULAR CYLINDER

By Hikoji YAMADA

Summary.—The two-dimensional problem which determines the flow field of a viscous liquid past a circular cylinder placed perpendicular to the uniform flow has been treated elegantly by S. Tomotika and T. Aoi,[1] by means of the Oseen's linearised equations. We here treat the same problem by the same equations but in somewhat different way, with the view of determining the Reynolds number at which the rear twin-vortices make their appearance. The found value is about 3.02 which is not remote from the experimental one 2.65, but this coincidence is rather contingent and the Oseen's approximation in general seems to be remote from the exact solution of the Navier-Stokes equations except the case of extremely small Reynolds number. This is indicated in the following by the consideration of the pressure.

§1. **Integration of Oseen's Equations.**—A circular cylinder of radius a is placed in a uniform liquid flow of velocity U, density ρ and kinematic viscosity ν, which flows along the x-axis to its positive direction, being the liquid boundless and the cylinder length infinite. The coordinate axes are rectangular and the cylinder axis coincides with the z-axis. The equations of motion of the liquid in the Oseen's approximation are, as are well known,

$$\Delta\Psi = -\Omega, \quad \nu\,\Delta\Omega - U\frac{\partial\Omega}{\partial x} = 0, \tag{1}$$

where Ψ is the stream function and Ω the vorticity. We introduce the non-dimensional quantities

$$\frac{\Psi}{Ua} = \psi, \quad \frac{\Omega}{U/a} = \omega, \tag{2}$$

and the non-dimensional coordinates, rectangular and polar,

$$\left(\frac{x}{a}, \frac{y}{a}\right) = (x_1, y_1) = (r, \theta), \tag{2'}$$

which yield the equations in the form

$$\Delta_1\psi = -\omega, \quad \Delta_1\omega - \frac{Ua}{\nu}\frac{\partial\omega}{\partial x_1} = 0, \tag{1'}$$

(1) S. Tomotika: Studies on mathematical physics (in Japanese), Vol. 1, (1949), pp. 130-150, *Iwanami* book co.; or S. Tomotika and T. Aoi: The steady flow of viscous fluid past a sphere and circular cylinder at small Reynolds numbers, Quat. Jour. Mech. and App. Math., Vol. 3 (1950), pp. 140-161.

and then separate the disturbance due to the presence of the cylinder from the uniform flow, writing ψ as

$$\psi = y_1 + \phi\,, \tag{3}$$

which brings the equations into

$$\Delta_1 \phi = -\omega\,, \quad \Delta_1 \omega - 2R_1 \frac{\partial \omega}{\partial x_1} = 0\,, \tag{1''}$$

where R_1 is a quarter of the Reynolds number R:

$$R_1 = \frac{Ua}{2\nu} = \frac{1}{4} \cdot \frac{2aU}{\nu} = \frac{1}{4} R\,. \tag{4}$$

The solution ω of (1″), which is anti-symmetrical with respect to the x_1-axis and vanishes at infinity, is well known and given by a series of the modified Bessel functions $K_m(z)$, i.e.

$$\omega = e^{R_1 x_1} \sum_{m-1}^{\infty} C_m K_m(R_1 r) \sin m\theta\,, \tag{5}$$

and then the equation which the function ϕ satisfies is the following:

$$\begin{aligned}\Delta_1 \phi &= -e^{R_1 x_1} \sum_{m=1}^{\infty} C_m K_m(R_1 r) \sin m\theta \\ &= -\sum_{n=1}^{\infty} \{\sum_{m=1}^{\infty} C_m K_m(R_1 r) [I_{m-n}(R_1 r) - I_{m+n}(R_1 r)]\} \sin n\theta\,.\end{aligned}$$

Here we expand the solution ϕ into the Fourier series with respect to the variable θ and assume

$$\phi = \sum_{n=1}^{\infty} \phi_n(R_1 r) \sin n\theta\,, \tag{6}$$

then the general expression of ϕ_n can easily be determined as follows:

$$\begin{aligned}\phi_n(\eta) &= (\alpha_n \eta^n + \alpha_{-n} \eta^{-n}) \\ &\quad - \frac{1}{2\pi R_1^2} \sum_{m=1}^{\infty} C_m \int_{R_1}^{\eta} (\eta^n \xi^{1-n} - \eta^{-n} \xi^{1+n}) K_m(\xi) \{I_{m-n}(\xi) - I_{m+n}(\xi)\} d\xi\,;\end{aligned} \tag{7}$$

herein the first term on the right side is the complementary function and the second the particular integral, η being written for $R_1 r$. C_m and α_n, α_{-n} are the undetermined integration constants.

The integration constants are determined by the boundary conditions. First, on the cylinder surface $r = 1$, ψ and $\partial\psi/\partial r$ are to vanish, i.e. when $\eta = R_1$

$$\begin{aligned}\phi_1 &= -1\,, & \phi_2 &= \phi_3 = \cdots = 0\,, \\ \phi_1' &= -R_1^{-1}\,, & \phi_2' &= \phi_3' = \cdots = 0\,,\end{aligned}$$

prime indicating the differentiation with respect to η. From these equations we obtain

$$\alpha_1 = -R_1^{-1}, \quad \alpha_{-1} = \alpha_2 = \alpha_{-2} = \alpha_3 = \cdots = 0. \tag{8}$$

Next we consider the behavior of ϕ when η tends to infinity. The condition in this case is the finite remain of ϕ. As is easily seen

$$\eta^{-n}\int_{R_1}^{\eta} K_m(\xi)\{I_{m-n}(\xi) - I_{m+n}(\xi)\}\,\xi^{1+n}\,d\xi \sim 0(1),$$

and then by means of the relations (6) and (7) we have to assure the finiteness of the expressions

$$\left.\begin{aligned} &-R_1^{-1}\eta - \frac{1}{2}R_1^{-2}\eta\sum_{m=1}^{\infty} C_m\int_{R_1}^{\eta} K_m(\xi)\{I_{m-1}(\xi) - I_{m+1}(\xi)\}\,d\xi\,, \\ &-\frac{1}{2n}R_1^{-2}\eta^n\sum_{m=1}^{\infty} C_m\int_{R_1}^{\eta} K_m(\xi)\{I_{m-n}(\xi) - I_{m+n}(\xi)\}\,\xi^{1-n}d\xi\,, \\ &\qquad\qquad\qquad\qquad (n \neq 1) \end{aligned}\right\} \tag{9}$$

when η tends to infinity. In these expressions integrals can be accomplished by means of the formulae

$$\begin{aligned} &\int_{R_1}^{\eta} K_m(\xi)\{I_{m-n}(\xi) - I_{m+n}(\xi)\}\xi^{1-n}\,d\xi \\ &= \Big[-\xi^{1-n}\{K_{m-1}(\xi)\,I_{n-m}(\xi) + K_{m-2}(\xi)\,I_{n-m+1}(\xi) + \cdots \\ &\qquad + K_{-m}(\xi)\,I_{n+m-1}(\xi)\}\Big]_{R_1}^{\eta}, \qquad (n = 1, 2, 3, \cdots) \end{aligned} \tag{10}$$

and[2] the condition of finiteness results in the simultaneous linear equations of the remaining integration constants C_m's:

$$\begin{aligned} &\sum_{m=1}^{\infty} C_m\{K_{m-1}(R_1)\,I_{n-m}(R_1) + K_{m-2}(R_1)\,I_{n-m+1}(R_1) + \cdots + K_{-m}(R_1)I_{n+m-1}(R_1)\} \\ &\qquad = \begin{cases} -2R_1 & (n = 1), \\ 0 & (n \geqq 2), \end{cases} \end{aligned} \tag{11}$$

or writing

$$C_m = 2R_1 C_m' \tag{12}$$

the equations read:

$$\begin{aligned} &\sum_{m=1}^{\infty} C_m'\{K_{m-1}(R_1)\,I_{n-m}(R_1) + K_{m-2}(R_1)\,I_{n+m-1}(R_1) + \cdots + K_{-m}(R_1)\,I_{n+m+1}(R_1)\} \\ &\qquad = \begin{cases} -1 & (n = 1), \\ 0 & (n \geqq 2). \end{cases} \end{aligned} \tag{11'}$$

[2] Changing the sign of n we obtain the integrated expressions of $\int_{R_1}^{\eta} K_m(\xi)\{I_{m-n}(\xi) - I_{m+n}(\xi)\}\,\xi^{1+n}\,d\xi\,(n = 1, 2, \cdots)$, which appear in the expressions of ϕ_n's, but which we do not require in this paper.

At this point we take into consideration the fact that $C_m{}'s$ appear usually as the products with the factors $K_m(R_1 r)'$s. When the argument is small the value of the function K_m increases rapidly with increasing m, and then it is rather preferable for the numerical calculation to obtain the values of the products $C_m{}' K_m(R_1)$ than the values of $C_m{}'$ themselves. For this purpose we rewrite (11′) as follows:

$$C_m{}' K_m(R_1) = x_m \,, \tag{13}$$

$$\sum_{m=1}^{\infty} x_m \left\{ \frac{K_{m-1}(R_1)}{K_m(R_1)} I_{n-m}(R_1) + \frac{K_{m-2}(R_1)}{K_m(R_1)} I_{n-m+1}(R_1) + \cdots + \frac{K_{-m}(R_1)}{K_m(R_1)} I_{n+m-1}(R_1) \right\}$$
$$= \begin{cases} -1 & (n = 1)\,, \\ 0 & (n \geqq 1)\,, \end{cases} \tag{14}$$

and these are the final equations[3] for the determination of the integration constants.

For any given R_1 we calculate $x_m{}'s$ from the equations (14) and then all field quantities are determined for the Reynolds number $R = 4 R_1$. Especially the vorticity (non-dimensional) on the cylinder surface is

$$(\omega)_{r=1} = 2 R_1 e^{R_1 \cos\theta} \sum_{m=1}^{\infty} x_m \sin m\theta \,, \tag{15}$$

and in the followings use is made of this quantity only. For instance, the existence of the rear twin-vortices appears as the existence of a pair of separation points on the cylinder surface and the positions of these separation points are determined by the condition of vanishing vorticity: $(\omega)_{r=1} = 0$.

When we go through a separation point along the cylinder surface the vorticity has to change the sign. Physically this is evident, for the flow along the cylinder surface reverses its direction at this point. Mathematically this is the branching point of the locus $\psi(r, \theta) = 0$. Expanding ψ at a certain point $(1, \vartheta)$ on the surface and taking into consideration the relations

$$\psi(1, \vartheta) = \left(\frac{\partial\psi}{\partial\theta}\right)_{1,\vartheta} = \left(\frac{\partial^2\psi}{\partial\theta^2}\right)_{1,\vartheta} = \left(\frac{\partial^3\psi}{\partial\theta^3}\right)_{1,\vartheta} = 0\,,$$
$$\left(\frac{\partial\psi}{\partial r}\right)_{1,\vartheta} = \left(\frac{\partial^2\psi}{\partial\theta\,\partial r}\right)_{1,\vartheta} = \left(\frac{\partial^2\psi}{\partial\theta^2\partial r}\right)_{1,\vartheta} = 0\,,$$

the expansion reads:

$$\psi(1 + \Delta r, \vartheta + \Delta\theta) = \frac{1}{2}\left(\frac{\partial^2\psi}{\partial r^2}\right)_{1,\vartheta} \Delta r^2$$
$$+ \frac{1}{6}\left\{\left(\frac{\partial^3\psi}{\partial r^3}\right)_{1,\vartheta} \Delta r^3 + 3\left(\frac{\partial^3\psi}{\partial\theta\,\partial r^2}\right)_{1,\vartheta} \Delta r^2 \Delta\theta\right\} + \text{higher terms.}$$

[3] As is easily seen (14) is the same in its contents as the corresponding equations of the papers quoted in the footnote (1).

From this expression we see the fact that through only the point $(1, \vartheta)$ for which $(\partial^2\psi/\partial r^2)_{1,\vartheta}$ just vanishes the locus $\psi = 0$ has two directions:

$$\Delta r^2 = 0, \quad \left(\frac{\partial^3\psi}{\partial r^3}\right)_{1,\vartheta} \Delta r + 3\left(\frac{\partial^3\psi}{\partial\theta\,\partial r^2}\right)_{1,\vartheta} \Delta\theta = 0,$$

and then the condition of a separation point is $(\partial^2\psi/\partial r^2)_{1,\vartheta} = 0$. This condition, by means of the relation $\Delta_1\psi = -\omega$, is equivalent to the condition $(\omega)_{1,\vartheta} = 0$.

§ 2. **Birth of Twin-Vortices.**—The equations (14) are solved for several small values of R_1 and the results are tabulated in Table 1.

TABLE 1.

R_1	R	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
0.1	0.4	−3.42460	0.92877	−0.04727	0.00118	−0.00004	——	——	——
0.2	0.8	−2.24086	0.93303	−0.09367	0.00475	−0.00016	——	——	——
0.3	1.2	−1.81875	0.95017	−0.14012	0.01064	−0.00082	——	——	——
0.4	1.6	−1.60844	0.97609	−0.18731	0.01889	−0.00126	——	——	——
0.6	2.4	−1.42065	1.04838	−0.28618	0.04276	−0.00420	——	——	——
0.8	3.2	−1.36408	1.14447	−0.39439	0.07726	−0.01029	0.00103	−0.00008	——
1.0	4.0	−1.37120	1.26357	−0.51570	0.12371	−0.02042	0.00257	−0.00027	0.00000

In the table the mark —— indicates the neglect of these higher terms, i.e. the assumed zeros at the outset.

By means of this table we first write down the expression $(\omega)_{r=1}$ for $R = 0.4$, and know easily that $(\omega)_{r=1}$ has only two zero-points $\theta = 0$ and $\theta = \pi$, between these points the value of $(\omega)_{r=1}$ being decidedly negative. This means that the liquid flows arround the cylinder from fore- to aft-stagnation point throughly and has no separation point between them. In other words, the twin-vortices do not exist. The same is true up to $R = 2.4$. As an example of this vorticity distribution along the cylinder surface the case of $R = 2.4$ is shown in Table 2 and plotted in Figure 1. But when the Reynolds number grows past a certain number between 2.4 and 3.2, the matter is changed. In this case there appears a pair of zero points of $(\omega)_{r=1}$, i.e. separation points between the stagnation points, as the case $R = 4$, for example, which is tabulated in Table 2 and plotted in Figure 4 clearly proves. We know that the existence or non-existence of the separation points depends on the sign of the gradient along the cylinder surface $(r = 1)$ of the vorticity at the rear stagnation point $(\theta = 0)$, and the Reynolds number at which the twin-vortices make their appearance is characterised by the condition of vanishing of this gradient.

This gradient at the rear stagnation point is given by

$$\left(\frac{\partial \omega}{\partial \theta}\right)_{1,0} = 2R_1 e^{R_1} \sum_{m=1}^{\infty} m\, x_m\,, \tag{16}$$

TABLE 2.

θ degree	$(\omega)_{r=1}$ $R=2.4$	$(\omega)_{r=1}$ $R=4.0$	θ degree	$(\omega)_{r=1}$ R = 2.4	$(\omega)_{r=1}$ R = 4.0	θ degree	$(\omega)_{r=1}$ R = 2.4	$(\omega)_{r=1}$ R = 4.0
0	0.00000	0.00000	70	−0.82415	−0.94240	140	−1.67227	−2.42206
10	−0.01504	0.01108	80	−1.09585	−1.33696	150	−1.38712	−2.02972
20	−0.04612	0.00204	90	−1.36640	−1.75130	160	−0.99277	−1.46308
30	−0.10826	−0.04732	100	−1.60570	−2.13794	170	−0.51776	−0.76626
40	−0.21379	−0.15618	110	−1.78169	−2.44346	180	0.00000	0.00000
50	−0.36986	−0.33920	120	−1.86470	−2.61678			
60	−0.57632	−0.60310	130	−1.83211	−2.61738			

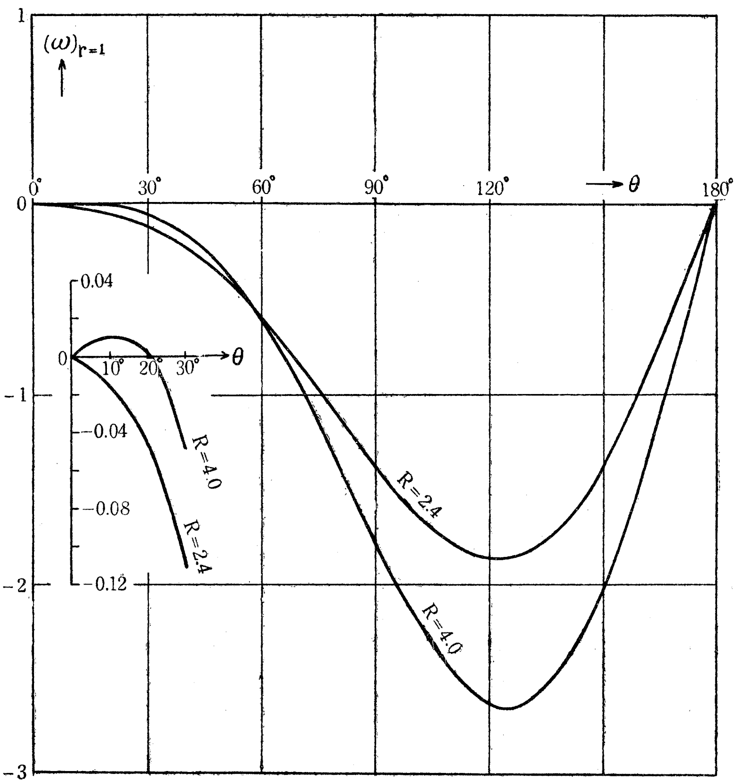

FIG. 1.

and can easily be calculated by means of Table 1. The results are given in Table 3 and plotted in Figure 2. From the figure we know the vanishing gradient realised at the Reynolds number about 3.02. Then the twin-vortices appear at this Reynolds number in the immediate neighborhood of the rear stagnation point, and enlarge their domain of existence with the Reynolds number.

TABLE 3.

R	0.4	0.8	1.2	1.6	2.4	3.2	4.0
$\left(\frac{\partial \omega}{\partial \theta}\right)_{1,0}$	−0.37672	−0.31150	−0.24323	−0.17772	−0.07082	+0.01745	+0.08193

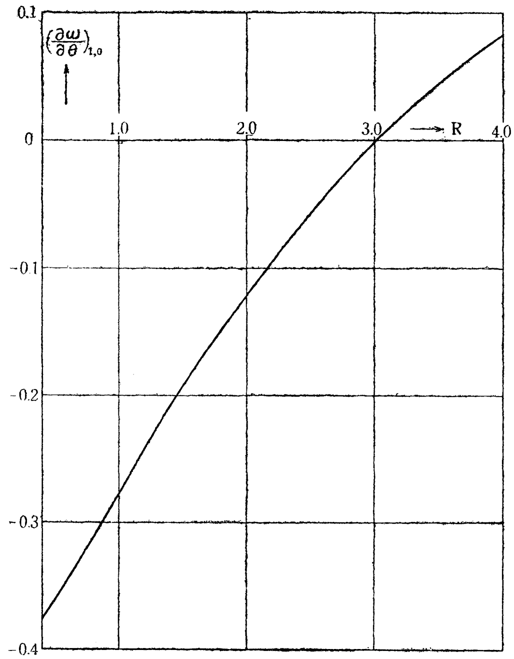

FIG. 2.

§ 3. **Consideration of Pressure and Resistance.**— The critical value 3.02 of the Reynolds number found above is not remote from the experimental one[4] 2.65. In this respect the Oseen's approximation seems good enough for the practical purpose. Then up to what Reynolds number the approximation holds? To answer this question we consider the pressure.

[4] C.f. H. Nisi and A. W. Porter: On eddies in air, Phil. Mag. (6), 46 (1923), pp. 754-768.

We take up the Oseen's equations in the form:

$$\left.\begin{aligned} U\frac{\partial \mathbf{V}}{\partial x} = -\frac{1}{\rho}\,\mathrm{grad}\,P + \nu\,\Delta\mathbf{V}\,, \\ \mathrm{div}\,\mathbf{V} = 0\,, \end{aligned}\right\} \tag{17}$$

and introduce the non-dimensional quantities:

$$\frac{\mathbf{V}}{U} = \mathbf{v}\,, \qquad \frac{P}{\rho U^2} = p\,, \tag{18}$$

and the non-dimensional polar coordinates (r, θ), the equations becoming then into the form:

$$\left.\begin{aligned} \frac{\partial p}{\partial r} &= -\frac{1}{2R_1}\frac{\partial \omega}{r\,\partial\theta} - \frac{\partial u}{\partial r}\cos\theta + \frac{\partial u}{r\,\partial\theta}\sin\theta - \frac{v}{r}\sin\theta\,, \\ \frac{\partial p}{r\,\partial\theta} &= \frac{1}{2R_1}\frac{\partial\omega}{\partial r} - \frac{\partial v}{\partial r}\cos\theta + \frac{\partial v}{r\,\partial\theta}\sin\theta + \frac{u}{r}\sin\theta\,, \end{aligned}\right\} \tag{18'}$$

where (u, v) are the components of the non-dimensional velocity $\mathbf{v}$ in the polar coordinates, therefore the relations

$$u = \frac{\partial\psi}{r\,\partial\theta}\,, \qquad v = -\frac{\partial\psi}{\partial r}$$

have to hold between (u, v) and ψ in section 1. Other notations are as before or self-evident. The well known relation

$$\Delta_1\mathbf{v} = -\mathrm{curl}\,\mathrm{curl}\;\mathbf{v} = \left(-\frac{\partial\omega}{r\,\partial\theta}\,, \frac{\partial\omega}{\partial r}\right) \tag{19}$$

are used in the derivation of (18').

When we integrate the first equation of (18') along the initial line from $r=1$ to ∞, and the second equation along the cylinder surface from $\theta=0$ to θ, we have

$$p_0 - p_\infty = 1 + \frac{1}{2R_1}\int_1^\infty \frac{1}{r}\left(\frac{\partial\omega}{\partial\theta}\right)_{\theta=0} dr\,, \tag{20}$$

$$p_\theta - p_0 = \int_0^\theta \left(\frac{1}{2R_1}\frac{\partial\omega}{\partial r} - \omega\cos\theta\right)_{r=1} d\theta\,, \tag{21}$$

denoting by p_0, p_∞, p_θ the pressures at the rear stagnation point, at infinity and at the point $(1, \theta)$ on the cylinder surface respectively. The integrals on the right sides of these equations require a little consideration and can be evaluated as follows.

Substituting the expression (5) of ω into (20) we have

$$\begin{aligned} p_0 - p_\infty &= 1 - \sum_{m=1}^\infty C_m' \int_{R_1}^\infty e^\xi\{K_m'(\xi) + K_{m-1}(\xi)\}\,d\xi \\ &= 1 + \sum_{m=1}^\infty C_m' \left[e^{R_1}K_m(R_1) + \int_{R_1}^\infty e^\xi\{K_m(\xi) - K_{m-1}(\xi)\}\,d\xi\right], \end{aligned}$$

and taking account of the recurrence formula

$$\int_{R_1}^{\infty} e^{\xi}\{K_m(\xi) - K_{m-1}(\xi)\}\, d\xi = 2\, e^{R_1} K_{m-1}(R_1) + \int_{R_1}^{\infty} e^{\xi}\{K_{m-1}(\xi) - K_{m-2}(\xi)\}\, d\xi$$

the integral can be evaluated resulting into the expression:

$$p_0 - p_\infty = 1 + e^{R_1} \sum_{m=1}^{\infty} C_m'\{K_m(R_1) + 2\, K_{m-1}(R_1) + \cdots + 2\, K_1(R_1) + K_0(R_1)\}. \tag{20'}$$

For the second integral (21) the substitution of ω brings it into

$$p_\theta - p_0 = \sum_{m=1}^{\infty} C_m' \int_0^{\theta} e^{R_1 \cos\theta}\{-R_1 \cos\theta\, K_m(R_1) \sin m\theta + R_1 K_m'(R_1) \sin m\theta\} d\theta .$$

We replace $R_1 K_m'(R_1)$ of this expression by $-m\, K_m(R_1) - R_1 K_{m-1}(R_1)$ and partially integrate the term which contains $-m\, K_m(R_1)$, then it becomes

$$\int_0^{\theta} e^{R_1 \cos\theta}\{-R_1 \cos\theta\, K_m(R_1) \sin m\theta + R_1\, K_m'(R_1) \sin m\theta\}\, d\theta$$

$$= \Big[e^{R_1 \cos\theta} K_m(R_1) \cos m\theta \Big]_0^{\theta}$$

$$- \int_0^{\theta} e^{R_1 \cos\theta}\{R_1 K_m(R_1) \sin \overline{m-1}\,\theta + R_1 K_{m-1}(R_1) \sin m\theta\}\, d\theta ,$$

and here taking account of the recurrence formula

$$\int_0^{\theta} e^{R_1 \cos\theta}\{R_1 K_m(R_1) \sin \overline{m-1}\,\theta + R_1 K_{m-1}(R_1) \sin m\theta\}\, d\theta$$

$$= -\Big[2\, e^{R_1 \cos\theta} K_{m-1}(R_1) \cos \overline{m-1}\,\theta \Big]_0^{\theta}$$

$$+ \int_0^{\theta} e^{R_1 \cos\theta}\{R_1 K_{m-1}(R_1) \sin \overline{m-2}\,\theta + R_1 K_{m-2}(R_1) \sin \overline{m-1}\,\theta\}\, d\theta$$

the integration can be completed; the result is as follows:

$$\begin{aligned} p_\theta - p_0 = \sum_{m=1}^{\infty} C_m' e^{R_1 \cos\theta}\{&K_m(R_1) \cos m\theta + 2\, K_{m-1}(R_1) \cos \overline{m-1}\,\theta \\ &+ \cdots + 2\, K_1(R_1) \cos\theta + K_0(R_1)\} \\ -\sum_{m=1}^{\infty} C_m' e^{R_1} \{&K_m(R_1) + 2\, K_{m-1}(R_1) + \cdots + 2\, K_1(R_1) + K_0(R_1)\}. \end{aligned} \tag{21'}$$

Combining (20′) and (21′) the pressure arround the cylinder is finally

$$p_\theta - p_\infty = 1 + e^{R_1 \cos\theta} \sum_{m=1}^{\infty} C_m' \left\{ \begin{matrix} K_m(R_1) \cos m\theta + 2\, K_{m-1}(R_1) \cos \overline{m-1}\,\theta \\ + \cdots\cdots\cdots\cdots + 2\, K_1(R_1) \cos\theta + K_0(R_1) \end{matrix} \right\}, \tag{22}$$

which is the expression just equivalent to the one already known.[5]

[5] C.f. the papers quoted in the footnote (1).

As the pressure around the cylinder is thus given the coefficient[6] of resistance due to pressure C_p can easily be obtained:

$$C_p = -\frac{\pi}{2}\sum_{m=1}^{\infty} C_m' \left\{ \begin{array}{l} K_0(R_1)\cdot\overline{I_{-1}(R_1)+I_1(R_1)}+2K_1(R_1)\cdot\overline{I_0(R_1)+I_2(R_1)} \\ +\cdots\cdots\cdots +2K_{m-1}(R_1)\cdot\overline{I_{m-2}(R_1)+I_m(R_1)} \\ \qquad +K_m(R_1)\cdot\overline{I_{m-1}(R_1)+I_{m+1}(R_1)} \end{array} \right\}. \quad (23)$$

Rewriting this expression by means of the relation (11′) ($n=2$) we obtain

$$C_p = -\frac{\pi}{R_1}\sum_{m=1}^{\infty} m\, x_m\, I_m(R_1)\,; \quad (24)$$

but this same quantity, making use of the well known relation

$$K_n(R_1)\, I_{n+1}(R_1) + K_{n+1}(R_1)\, I_n(R_1) = R_1^{-1}$$

in the expression (23), can be expressed in another form:

$$C_p = -\frac{\pi}{R_1}\sum_{m=1}^{\infty} m\, C_m' + \frac{\pi}{R_1}\sum_{m=1}^{\infty} m\, x_m\, I_m(R_1)\,, \quad (25)$$

and from these two expressions it results finally[7]

$$C_p = -\frac{\pi}{2R_1}\sum_{m=1}^{\infty} m\, C_m'\,. \quad (26)$$

At this point we turn to the Navier-Stokes equations:

$$\left. \begin{array}{l} (\mathbf{V}\nabla)\,\mathbf{V} = -\dfrac{1}{\rho}\,\mathrm{grad}\, P + \nu\,\Delta\mathbf{V}\,, \\ \mathrm{div}\,\mathbf{V} = 0\,, \end{array} \right\} \quad (27)$$

and introduce the vorticity. Then denoting by **k** the unit vector of z-direction the equations become

$$\left. \begin{array}{l} \mathrm{grad}\left(\dfrac{P}{\rho} + \dfrac{1}{2}\mathbf{V}^2\right) = \nu\,\Delta\mathbf{V} + \Omega[\mathbf{V}, \mathbf{k}]\,, \\ \mathrm{div}\,\mathbf{V} = 0\,, \end{array} \right\} \quad (27')$$

and expressed non-dimensionally in the polar coondinates:

$$\left. \begin{array}{l} \dfrac{\partial}{\partial r}\left(p + \dfrac{1}{2}\mathbf{v}^2\right) = -\dfrac{1}{2R_1}\dfrac{\partial\omega}{r\,\partial\theta} + v\,\omega\,, \\ \dfrac{\partial}{r\,\partial\theta}\left(p + \dfrac{1}{2}\mathbf{v}^2\right) = \dfrac{1}{2R_1}\dfrac{\partial\omega}{\partial r} - u\omega\,. \end{array} \right\} \quad (28)$$

[6] Definition of C_p: $2a\rho U^2 C_p = \int_0^{2\pi} a\,d\theta\,(-P)_{r=1}\cos\theta$.

[7] As the coefficient of resistance due to friction C_f (similarly defined as C_p) is $C_f = -\pi/R_1 \sum_{m=1}^{\infty} m x_m I_m(R_1)$, it results the well known formula of *S.* Tomotika and T. Aoi: $C_p = C_f$; c.f. the papers quoted in the footnote (1).

These are the equations which correspond to (18′), and consequently it follows in the same way as in there the relations which correspond to (20) (21):

$$p_0 - p_\infty = \frac{1}{2} + \frac{1}{2R_1}\int_1^\infty \frac{1}{r}\left(\frac{\partial\omega}{\partial\theta}\right)_{\theta=0} dr\,, \tag{29}$$

$$p_\theta - p_0 = \frac{1}{2R_1}\int_0^\theta \left(\frac{\partial\omega}{\partial r}\right)_{r=1} d\theta\,. \tag{30}$$

These relations between the pressure and vorticity are the correct ones, for they are the direct results from the Navier-Stokes equations, contrary to the approximate nature of (20) (21), due to the approximation adopted in the original Oseen's equations.

But when the approximation is good enough, i.e. when (u, v) and ω treated in section 1 are near to the solution of the Navier-Stokes equations, we can use this Oseen's approximation in (29) (30) and the resulting pressure distribution must be near to the real pressure. On the other hand the Oseen's pressure (20) (21), or (22), is expected to be a good approximation. Thus the two pressure distributions have to coincide approximately, and this coincidence is the assurance of a good approximation to the real of the Oseen's solution. When the coincidence, however, is not good we have to expect the failure of the approximation. In this way we can have a qualitative knowledge of the degree of approximation.

Now when we use the same ω-value the differences between the expressions (29) (30) and (20) (21) are

$$\left.\begin{aligned} \delta(p_0 - p_\infty) &= -\frac{1}{2}\,, \\ \delta(p_\theta - p_0) &= \int_0^\theta (\omega)_{r=1}\cos\theta\, d\theta\,. \end{aligned}\right\} \tag{31}$$

By means of (15), which is the expression of $(\omega)_{r=1}$, the second of the above differences becomes

$$\delta(p_\theta - p_0) = 2R_1 \sum_{m=1}^\infty x_m \int_0^\theta e^{R_1\cos\theta}\sin m\theta\,\cos\theta\, d\theta$$

$$= R_1 \sum_{m=1}^\infty x_m \sum_{n=1}^\infty \left\{\begin{aligned} &I_{m+1-n}(R_1) - I_{m+1+n}(R_1) \\ &+ I_{m-1-n}(R_1) - I_{m-1+n}(R_1)\end{aligned}\right\}\cdot \int_0^\theta \sin n\theta\, d\theta\,,$$

and finally

$$\delta(p_\theta - p_0) = R_1 \sum_{n=1}^\infty \frac{1-\cos n\theta}{n} \sum_{m=1}^\infty x_m \cdot \left[\begin{aligned} &\{I_{m-\overline{n-1}}(R_1) - I_{m+\overline{n-1}}(R_1)\} \\ +&\{I_{m-\overline{n+1}}(R_1) - I_{m+\overline{n+1}}(R_1)\}\end{aligned}\right]. \tag{32}$$

The difference in the coefficients of pressure resistance can also be easily evaluated by means of the relation (32), thus:

$$\delta C_p = \frac{1}{2a}\int_0^{2\pi} -\delta(p_\theta - p_0)\cos\theta\, a\, d\theta = \frac{\pi R_1}{2}\sum_{m=1}^{\infty} x_m\{I_{m-2}(R_1) - I_{m+2}(R_1)\}. \tag{33}$$

These differences we have calculated for the two cases: $R = 0.8$ and 4.0. For the pressure distribution we found:

$$R = 0.8: \quad \delta(p_\theta - p_\infty) = -0.52882 - 0.14163 \cos\theta + 0.21705 \cos 2\theta - 0.04719 \cos 3\theta + 0.00058 \cos 4\theta + 0.00001 \cos 5\theta,$$

$$R = 4.0: \quad \delta(p_\theta - p_\infty) = -0.23906 - 0.57679 \cos\theta + 0.49201 \cos 2\theta - 0.19075 \cos 3\theta + 0.01364 \cos 4\theta + 0.00091 \cos 5\theta + 0.00004 \cos 6\theta,$$

and the difference in the coefficients of resistance follows from these values directly. Table 4 and Figure 3 show the results. The values of the Oseen's approximation are to be calculated by the formulae (22) and (26), but here we borrowed them from the paper by S. Tomotika and T. Aoi[8] for the sake of shortness; the curves in the figure indicate the pressure distributions of the two sorts.

TABLE 4.

θ degree	$R = 0.8$		$R = 4.0$	
	* $(P_\theta - P_\infty)/\frac{1}{2}\rho U^2$	$\delta\left(\frac{P_\theta - P_\infty}{\frac{1}{2}\rho U^2}\right)$	* $(P_\theta - P_\infty)/\frac{1}{2}\rho U^2$	$\delta\left(\frac{P_\theta - P_\infty}{\frac{1}{2}\rho U^2}\right)$
0	−2.924	−1.000	−0.834	−1.000
30	−3.028	−1.087	−1.021	−1.000
60	−2.842	−1.323	−1.401	−1.178
90	−1.416	−1.491	−1.158	−1.435
120	1.430	−1.039	0.245	−0.789
150	4.448	−0.596	2.194	1.001
180	5.758	−0.245	3.111	2.067
	* C_p, Oseen	δC_p	* C_p, Oseen	δC_p
	3.392	0.222	1.461	0.906

N. B. The values of the asterisked columns are borrowed from the paper of S. Tomotika and T. Aoi.

From these results we know that the approximation of the Oseen's equations is only good when the Reynolds number is extremely small.

[8] C.f. the footnote (1).

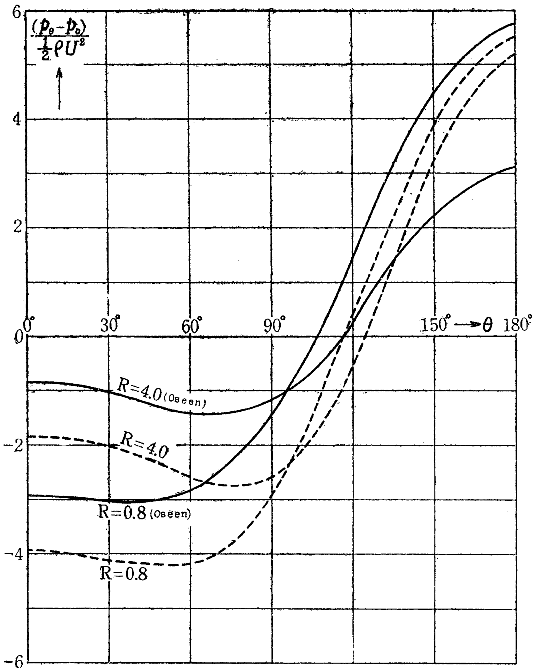

FIG. 3.

In the case of $R=0.8$ the differences above stated seem to be admissible, but when $R=4.0$ the differences are so much considerable that we are forced to think of the approximation one step more advanced to meet practical requirements.

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