# A CONTRIBUTION TO THE THEORY OF THE SEEPAGE THROUGH AN EARTH DAM

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# A CONTRIBUTION TO THE THEORY OF THE SEEPAGE THROUGH AN EARTH DAM

By Michio OHJI

Some calculating formulas are proposed for the seepage through a two-dimensional earth dam on the impermeable ground, making use of the Cauchy's theorem in the function theory together with other simplifying assumptions. The method seems to be a general one, but only the cases of a rectangular and trapezoidal dams with a horizontal base are calculated here, for which both of the experimental and theoretical data are available. As is seen from the numerical examples, these formulas give at least the correct order of magnitude for the total flux and the height of the seepage surface. Incidentally the legitimacy of the Dupuit-Forchheimer-Muskat's formula for the total flux of a rectangular dam is pointed out.

**1. Introduction.**[1] In most of the practical cases the steady motion of the ground-water can be described by the Darchy's law

$$\mathbf{v} = -\operatorname{grad} \Phi, \quad \Phi = \frac{k}{\mu}(p + \rho g y), \tag{1.1}$$

where $\mathbf{v}$ is the vector of the filtration velocity, $k$ a constant factor called the permeability of the soil, $\mu$ the viscosity, $p$ the pressure, $\rho$ the density, $g$ the gravitational accerelation and $y$ the vertical coordinate measured upward. Further in view of the condition of incompressibility

$$\operatorname{div} \mathbf{v} = 0, \tag{1.2}$$

we have readily Laplace's equation for the velocity potential $\Phi$, that is,

$$\nabla^2 \Phi = 0, \tag{1.3}$$

provided that the permeability $k$ can be considered uniform. Mathematically, therefore, the problem is to find the solution of (1.3) under the appropriate conditions at the boundaries.[2]

Now we shall consider the seepage of water through an earth dam built on the impermeable layer (see Fig. 1 or 2). In this case the straightforward

[1] For the detailed account of this section, see Muskat, M., *The Flow of Homogeneous Fluid through Porous Media*, (1937) McGraw-Hill, especially Chaps. II, III, IV, VI. This book will be simply denoted by [M] hereafter.

[2] A number of contributions in this connection are described in the monograph of P. Ya. Polubarinova-Kochina and S. B. Falkovich, *Theory of Filtration of Liquid in Porous Media*, vol II of "Advances in Applied Mechanics", (1951) Academic Press Inc.

analysis becomes extremely difficult, owing to the fact that there appear a free surface and a surface of seepage[1] determined by the condition that $p=0$ upon them (the atmospheric pressure is taken to be zero), and their geometrical forms can not be given a priori. Indeed for any two-dimensional system such a problem can be rigorously solved, in principle, by means of the method of hodograph or the complex velocity plane, as is done by Hamel and Günther for the case of a dam with vertical faces (rectangular dam)[2] but the laborious work in the numerical computations makes it almost impossible to proceed to more general cases. This is the reason why various attempts have been made by several authors to obtain the approximate theories. Among them the methods of Dupuit-Forchheimer[3] and of Muskat[4] for a rectangular dam, and those of Dachler[5] and of Casagrande[6] for a trapezoidal one (oblique faces) are frequently quoted, but none of them seems to have succeeded in covering all cases.

On the other hand, we have a convenience to use the experimental means. As is well known, the equation (1.3) is identical in the form with that of the electric potential for the steady current or of the Hele-Shaw's potential for the two-dimensional slow motion of a strongly viscous fluid, from which the methods of electrical models[7] on one hand and Hele-Shaw's models[8] on the other have been developed in addition to the direct method of sand models. By proper choice of the apparatus it will be possible in either way to obtain the results with sufficient accuracy for arbitrary two-dimensional systems. However, considerable amounts of time would be naturally required to do so, especially in the cases with free boundaries such as ours. Then is not there any other means which are suitable for the rapid (though somewhat rough) estimation of the important quantities? An attempt to answer this question will be found in the subsequent sections.

**2. Method of Contour Integrals.** Let us again consider the two-dimensional motion and introduce the complex potential $W$ in the following way:

1) Some physical considerations lead us to the conclusion that the free surfaee, in general, must terminate a little above the outflow level, leaving the surface of seepage at the lower part of the outflow face in accordance with our experiences. See [M], pp. 288-291.

2) Hamel, G., *Z.a.M.M.*, 14 (1934) 129—analytical part; Hamel, G. and Günther, E., *Z.a.M.M.*, 15 (1935) 255—numerical part. An alternative method is demonstrated by Polubarinova-Kochina, (1939), see *ibid.*, pp. 175-180.

3) Forchheimer, Ph., *Hydraulik*, 3d ed., (1930) Verlag Teubner, Chap. III, or [M], §6.17.

4) Muskat, M., *Trans. Amer. Geophys. Union*, (1936) 391. or [M], §6.20.

5) Dachler, R., *Grundwasserströmung*, (1936) Julius Springer, pp. 106-110, or [M], § 6.10.

6) Casagrande, L., *Die Bautechnik*, Heft 15 (1934) 205, or [M], p. 339.

7) The distribution of the electric potential is measured on the relatively high resistance sheet (such as that coated with aquadag, or a graphite colloid) equivalent to the permeable cross-section of the dam. See [M], §4.17 and § 6.6.

8) The Motion of a strongly viscous fluid, syrup for instance, between two glass pletes (set up at an interval of 2-3 mm) is observed. Günther, E., *Wasserkraft u. Wasserwirtschaft*, Heft. 3 (1940) 49.

$$W = \Phi + i\Psi , \tag{2.1}$$

where $\Psi$ is the stream function such that

$$\nabla^2\Psi = 0 , \tag{2.2}$$

and

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y}(= -u) , \quad \frac{\partial \Phi}{\partial y} = -\frac{\partial \Psi}{\partial x}(= -v) , \tag{2.3}$$

$x$ being the horizontal coordinate, $u$ and $v$ the $x$- and $y$-components of $\mathbf{v}$ respectively. Any curve on which $\Psi(x, y) = \text{const.}$ is a streamline; particularly the free surface is specified by the condition that $\Psi = -Q$,[1] while the impermeable base corresponds to the value $\Psi = 0$. Here $Q$ means the total flux through the dam.

Since we are interested only in the non-singular solution inside the boundaries, the complex potential $W$ can be considered as a regular function of the complex variable $z = x + iy$ there because of the Cauchy-Riemann's relation (2.3). Accordingly, for any simple closed curve K within this region we have

$$\oint_K W \, dz = 0 , \tag{2.4}$$

in virtue of the Cauchy's theorem. Furthermore assuming that $W$ can be analytically continued beyond the boundaries, and integrating it along each of the boundary segments, (2.4) is still valid so long as the singular points, which may occur on the boundaries, are excluded. Let such a contour be denoted by G. Then deviding the integral into the real and imaginary parts, we get from (2.4)

$$R \equiv \oint_G \Phi \, dx - \Psi \, dy = 0 , \tag{2.5}$$

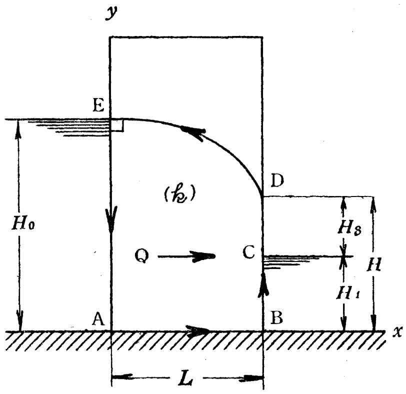
and

$$I \equiv \oint_G \Phi \, dy + \Psi \, dx = 0 , \tag{2.6}$$

respectively. These are the necessary (of course not sufficient) conditions which must be satisfied by the exact harmonic solution. Conversely if only the requirements (2.5) and (2.6) are taken into considerations, we may expect to have an approximation in some sense. This is the principle of our present method.

**3. A Rectangular Dam.** First, for the sake of comparison with the exact results, the case of a rectangular dam is considered in particular, although it is reduced easily from the case of the next section. Using the notations and the coordinate system of Fig. 1, we shall define the dimensionless quantities as follows:

[1] The negative sign comes from those in the equations (1.1) and (2.3).




$H_0$ : inflow level
$H_1$ : outflow level
$H = H_1 + H_S$ : seepage level
$L$ : width of the dam

Fig. 1.

$$\left.\begin{aligned} &\xi = x/L, \quad \eta = y/L, \quad h_0 = H_0/L, \quad h_1 = H_1/L, \quad h = H/L, \quad h_S = h_S/L, \\ &\phi = \Phi/\bar{k}L, \quad \psi = \Psi/\bar{k}L, \quad q = Q/\bar{k}L, \end{aligned}\right\} \quad (3.1)$$

where

$$\bar{k} = \frac{k\rho g}{\mu},$$

a constant with the dimension of velocity. The boundary conditions for this system are written in the following schema:

<table>
<tr><th colspan="2">Segments</th><th>$\phi$</th><th>$\psi$</th><th rowspan="6">(3.2)</th></tr>
<tr><td colspan="2">inflow face ($\xi = 0$)</td><td>$\phi = h$[1]</td><td rowspan="2">$\frac{\partial \psi}{\partial \xi} = 0$</td></tr>
<tr><td rowspan="2">outflow face ($\xi = 1$)</td><td>$0 \leqq \eta \leqq h_1$</td><td>$\phi = h_1$[1]</td></tr>
<tr><td>$h_1 \leqq \eta \leqq h$</td><td rowspan="2">$\phi = \eta$ ($p = 0$)</td><td>$\frac{\partial \psi}{\partial \xi} = -\bar{k}$</td></tr>
<tr><td colspan="2">free surface</td><td>$\psi = -q$</td></tr>
<tr><td colspan="2">impermeable base ($\eta = 0$)</td><td>$\frac{\partial \phi}{\partial \eta} = 0$</td><td>$\psi = 0$</td></tr>
</table>

Now let us apply the equations (2.5) and (2.6). In order to remove the ambiguity, the contour G is at first considered to be deformed so as to avoid each of the points A, B, C, D and E of Fig. 1 by the circular arc of a small radius. But from the physical considerations it is natural to suppose that both of $\phi$ and $\psi$ are finite and single-valued everywhere on the boundaries, and hence in the limit the contributions from these special points must

[1] These are equivalent to suppose the pressure distributes hydrostatically along these segments (see (1.1)).

vanish with the radii of the circles. Thus we can safely take the integrations along the boundary segments without modifying them. Then under the conditions of (3.2), it turns out immediately that

$$\frac{1}{2}(h_0{}^2 - h_1{}^2) - q = 0\,,$$

for (2.6), from which we have

$$q = \frac{1}{2}(h_0^2 - h_1^2)\,, \qquad \text{or} \qquad Q = \frac{\bar{k}(H_0^2 - H_1^2)}{2\,L}\,. \tag{3.3}$$

This is quite the same as the customary formula of Dupuit-Forchheimer-Muskat. It has been long noticed that in spite of the approximate formulation of these authors, (3.3) gives the numerical value of the total flux obtained from the exact theory within the errors of computations.[1] In the present theory we get again the same result under no other specific assumptions than the validity of the Canchy's theorem (2.4), which may be permissible now. We can, therefore, infer that the relation (3.3) is a correct one, not an approximation. In this respect, it is very interesting that former approximate theories, though quite different in nature, succeeded so far as the flux is concerned.

Next we consider the real part (2.5). It becomes at once

$$\int_0^1 (\phi_{AB} - \phi_{ED})\, d\xi + \int_0^{h_0} \psi_{AE}\, d\eta - \int_0^{h} \psi_{BD}\, d\eta + q(h_0 - h) = 0\,, \qquad (3.4)^{2)}$$

but contrary to the foregoing case, we cannot proceed beyond this without the knowledge of the true solution, unless we introduce some kind of assumption or approximation here.

So let us approximate most simply each of the actual distribution of the integrands in (3.4) by a straight line: for instance, since $\psi(A) = 0$ and $\psi(E) = -q$ we assume $\psi_{AE} = -q\,\eta/h_0$ accordingly $\int_0^{h_0} \psi_{AE}\, d\eta = -q\, h_0/2$, etc. Of course, this linearity assumption must be very crude, but, for the time being, we are going to examine to what extent such a rough simplification can be effective. Thus the equation (3.4) simply reduces to

$$\frac{1}{2}(h_1 - h) + \frac{q}{2}(h_0 - h) = 0\,,$$

which gives

$$h = \frac{q\, h_0 + h_1}{q + 1}\,, \tag{3.5}$$

[1] [M], p. 317, p. 363 and p. 380, also see Table 1.

[2] $\phi_{AB}$ means the value of the potentials on the segment A B, and so on.

or in view of (3.3)

$$h_S = \frac{q\,\Delta h}{q+1} = \frac{(h_0^2 - h_1^2)\,\Delta h}{(h_0^2 - h_1^2) + 2}, \tag{3.6}$$

where

$$\Delta h = h_0 - h_1 .$$

(3.5) or (3.6), together with (3.3) are the calculating formulas of a rectangular dam. In Table 1 a few examples of application are shown in comparison with the exact values, where it is seen that the present formulas give the values of $h_S$ in the correct order of magnitude. Remembering the roughness of the linearity assumption we may content ourselves with these accuracies.

TABLE 1. Examples of Rectangular Dams.

| Case | | I | II | III | IV | V | VI |
|---|---|---|---|---|---|---|---|
| $h_0$ | | 1.988 | 1.509 | 2.043 | 1.802 | 1.419 | 1.077 |
| $h_1$ | | 0.519 | 0.356 | 0 | 0 | 0 | 0 |
| $q$ | from (3.3) | 1.842 | 1.075 | 2.09 | 1.624 | 1.007 | 0.580 |
| | from hodograph theory | 1.846 | 1.081 | 2.09 | 1.618 | 1.008 | 0.581 |
| $h_S$ | from (3.6) | 0.952 | 0.597 | 1.381 | 1.115 | 0.712 | 0 395 |
| | from hodograph theory | 0.753 | 0.455 | 1.307 | 1.072 | 0.713 | 0.425 |

These cases correspond to those tabulated in Table 14, p. 314 of [M].

## 4. A Trapezoidal Dam.

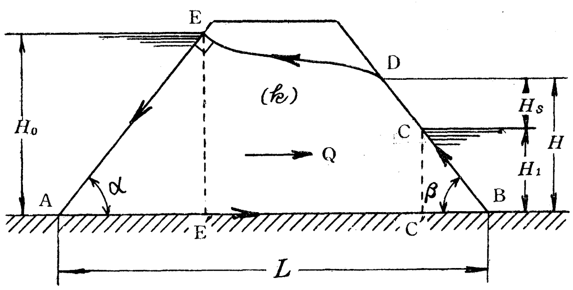



Fig. 2.

The above calculation can be extended to the case of a trapezoidal dam without special difficulties. Let the angles of inclination of the faces be denoted by $\alpha$ and $\beta$ in the sense of Fig. 2, and let us use the other notations as before. Here we suppose always $0 < \alpha,\ \beta \leq 90°$. The boundary conditions are:

| Segments | | $\phi$ | $\psi$ |
|---|---|---|---|
| inflow face, AE | | $\phi = h_0$ | $\dfrac{\partial\psi}{\partial n} = 0$ |
| outflow face, BD | along BC | $\phi = h_1$ | |
| | along CD | $\phi = \eta$ | $\dfrac{\partial\psi}{\partial n} = -\bar{k}\sin\beta$ |
| free surface, DE | | $(p = 0)$ | $\psi = -q$ |
| impermeable base, AB | | $\dfrac{\partial\phi}{\partial\eta} = 0$ | $\psi = 0$ |

(4.1)

in which $n$ is the dimensionless length along the inward normal. In this case we have to notice that the velocity should be zero at the lower corners A and B (if $H_1 = 0$, then at A only), where the equipotential lines and streamlines cross obliquely. These stagnation singularities, however, do not influence the validity of the Cauchy's theorem, because they produce no residue, neither $\phi$ nor $\psi$ being infinite there.

Thus in accordance with (2.5) and (2.6), we have

$$\left.\begin{aligned}\int_0^1 \phi_{AB}\,d\xi - \int_{h_0\cot\alpha}^{1-h\cot\beta} \phi_{DE}\,d\xi + \int_0^{h_0} \psi_{AE}\,d\eta - \int_0^{h} \psi_{BD}\,d\eta - \frac{1}{2}(h^2 + h_1^2)\cot\beta \\ - h_0^2\cot\alpha + q(h_0 - h) = 0,\end{aligned}\right\} \tag{4.2}$$

for $R$, and

$$\int_1^{1-h\cot\beta} \psi_{BD}\,d\xi - \int_0^{h_0\cot\alpha} \psi_{AE}\,d\xi + \frac{1}{2}(h_1^2 - h_0^2) + q(1 - h\cot\beta - h_0\cot\alpha) = 0, \tag{4.3}$$

for $I$. In contrast with the previous case, the integrals of the unknown functions now appear in both of these. Again we shall make use of the linearity assumption here, but owing to the existence of the stagnation points stated above, a little modification seems to be necessary. Namely, in the vicinities of these points the motion will be quite slow and accordingly the velocity potential must be stationary there.

Thus, instead of approximating by a single linear distribution, we had better suppose the potential distribution at the base in the following way:

$$\phi_{AB}:\left\{\begin{array}{ll}\phi_{AE'} = h_0\,, & \text{for } 0 \leqq \xi \leqq h_0\cot\alpha\,, \\ \phi_{E'C'} \text{ linearly decreases from } h_0 \text{ to } h_1\,, & \\ & \text{for } h_0\cot\alpha \leqq \xi \leqq 1 - h_1\cot\beta\,, \\ \phi_{C'B} = h_1\,, & \text{for } 1 - h_1\cot\beta \leqq \xi \leqq 1\,,\end{array}\right\} \tag{4.4}$$

where E′ and C′ are the projection of E and C on AB (see Fig. 2). With this assumption the first integral of (4.2) becomes

$$\int_0^1 \phi_{AB}\, d\xi = h_0^2 \cot\alpha + h_1^2 \cot\beta + \frac{1}{2}(h_0 + h_1)(1 - h_0 \cot\alpha - h_1 \cot\beta).$$

Though somewhat contradictorily, evaluating the other integrals under the linearity assumption as before, we get finally from (4.2)

$$q\,\Delta h = h_S[q + 1 - h_0(\cot\alpha + \cot\beta)], \tag{4.5}$$

and from (4.3)

$$q(2 - h_0 \cot\alpha - h_1 \cot\beta - h_S \cot\beta) = h_0^2 - h_1^2, \tag{4.6}$$

which are the simultaneous algebraic equations for the unknowns $q$ and $h_S$. The solutions can be written as follows:

$$q = \frac{a}{1+\sigma_1}(\pm\sqrt{1+b} - 1), \quad (a \neq 0) \tag{4.7}$$

$$h_S = \frac{q\,\Delta h}{q + \sigma_1}, \tag{4.8}$$

where

$$\left.\begin{aligned} a &= \frac{1}{2}[\sigma_1(\sigma_1+\sigma_2) - (h_0^2 - h_1^2)], \\ b &= \frac{\sigma_1(1+\sigma_1)(h_0^2 - h_1^2)}{a^2}, \\ \sigma_1 &= 1 - h_0\lambda, \\ \sigma_2 &= 1 + (\Delta h)\cot\beta, \\ \lambda &= \cot\alpha + \cot\beta, \end{aligned}\right\} \tag{4.9}$$

and the double sign in (4.7) should be taken positive when $a > 0$ and negative when $a < 0$. If $a$ becomes zero or small enough

$$q = \sigma_1\sqrt{\frac{\sigma_1+\sigma_2}{\sigma_1+1}} \tag{4.7'}$$

have to be used in place of (4.7). It is readily seen that (4.7), (4.7′) and (4.8) reduce to (3.3) and (3.6) when $\alpha = \beta = 90°$.

As is stated before, there is no rigorous solution of a trapezodal dam at present, and so we have to refer to the experimental work or existing approximate evaluations, in order to check the reliabilities of these formulas. Unfortunately, however, the author knows only few examples in this respect, of which the following two are mentioned here.[1]

Example 1.

$$\alpha = \beta = 30°,$$
$$h_0 = 0.270, \quad h_1 = 0:$$

[1] [M], pp. 322, 323 and §6.10.

$$\Delta h = 0.270\,, \quad h_0^2 - h_1^2 = 0.07290\,,$$
$$\cot\alpha = \cot\beta = 1.732\,, \quad \lambda = 3.464\,,$$
$$\sigma_1 = 1 - 0.270 \times 3.464 = 0.0647\,, \quad 1 + \sigma_1 = 1.0647\,,$$
$$\sigma_2 = 1 + 0.270 \times 3.464 = 1.468\,,$$
$$\sigma_1 + \sigma_2 = 1.533\,,$$
$$a = \frac{1}{2}(0.0647 \times 1.533 - 0.07290) = 0.01315 > 0\,,$$
$$b = \frac{0.0647 \times 1.0647 \times 0.0729}{(0.01315)^2} = 29.05\,,$$
$$\sqrt{1+b} - 1 = 4.48\,,$$
$$q = \frac{0.01315 \times 4.48}{1.065} = \underset{\sim\sim}{0.0553}\,,$$
$$h_S = \frac{0.270 \times 0.0553}{0.0553 + 0.0647} = \underset{\sim\sim}{0.124}\,.$$

These are to be compared with

$$\left.\begin{aligned} q &= 0.0689\,, \\ h_S &= 0.189\,, \end{aligned}\right\} \quad \text{(electrical method)},$$

and $q = 0.0429$, (Dachler's method).[1]

Example 2.

$$\alpha = \beta = 45°,$$
$$h_0 = 0.377\,, \quad h_1 = 0:$$

We have in the similar way

$$q = \underset{\sim\sim}{0.0957}\,,$$
$$h_S = \underset{\sim\sim}{0.105}\,,$$

which are to be compared with

$$\left.\begin{aligned} q &= 0.106\,, \\ h_S &= 0.170\,, \end{aligned}\right\} \quad \text{(electrical method)},$$

and $q = 0.0897$, (Dachler's method).[1]

**5. Concluding Remarks.** Although the numerical examples of last section alone are not sufficient to draw the general conclusions, we may imagine that the proposed formulas can predict the seepage through any given trapazoidal dam qualitatively. Furthermore the same method would be also applicable to more complicated cases. Still, the principal defects of the present method are: (1) that no information of the shape of free surface is obtained, (2) that the conditions of the derivatives, e.g., $\partial\phi/\partial\eta = 0$ along the base etc. are not taken into considerations and (3) that the numerical results are sometimes greater and sometimes less than

[1] $h_S$ cannot be determined from Dachler's method.

the true values (see the values of $h_S$ in table 1 and § 4). In these connections some improvements would not be impossible, however apart from the mathematical interests, it will be more convenient in practice to introduce the correction-factors for $q$ and $h_S$, which are to be determined from the systematic experiments.

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