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An Illustrative Example of Solving a Sum Equation by Fourier Transforms

By Jun-ichi OKABE

By a *sum equation* we mean an equation in which discrete summations of an unknown quantity are involved. It stands therefore, so to speak, towards an integral equation just in the same position as a difference equation does towards a differential equation.

Now we shall consider the motion of an infinite number of equal particles of mass μ attached to a long string of density ρ and tension T at equal intervals a. Denoting the lateral displacement of the nth particle by $\psi_n(t)$, let us assume that it is governed by the sum equation

$$\psi_n(t)=\sigma_n(t)+\sum_{m=1}^{\infty}\Big\{\text{the contribution of } \psi_{n-m}\left(t-\frac{m\,a}{c}\right) \text{ and } \psi_{n+m}\left(t-\frac{m\,a}{c}\right)$$
$$\text{towards } \psi_n(t) \text{ through the free vibration of the string}\Big\}, \tag{1}$$

where $\sigma_n(t)$ is a given function and $c\equiv\sqrt{T/\rho}$ which is the speed of propagation of the lateral vibration of the string. We shall assume further that the motion is periodic in the sense that

$$\psi_{n-m}\left(t-\frac{m\,a}{v}\right)=\psi_n(t)=\psi_{n+m}\left(t+\frac{m\,a}{v}\right),$$
$$\text{and}\qquad \sigma_{n-m}\left(t-\frac{m\,a}{v}\right)=\sigma_n(t)=\sigma_{n+m}\left(t+\frac{m\,a}{v}\right), \tag{2}$$

where v represents some characteristic velocity of a constant value. By means of this relation, (1) can be rewritten

$$\psi_n(t)=\sigma_n(t)+\sum_{m=1}^{\infty}\left\|\psi_{n+m}\left(t+\frac{2m\,a}{v}-\frac{m\,a}{c}\right)+\psi_{n+m}\left(t-\frac{m\,a}{c}\right)\right\|, \tag{3}$$

where, and in the following discussions as well, the symbol $\|\Psi\|$ signifies "the contribution of the quantity Ψ to the nth particle, which we shall abbreviate as $<n>$, through the free vibration of the string."

If we write, e.g.,

$$\bar{\psi}_n(p)=\int_{-i\infty}^{i\infty}\psi_n(t)\,e^{-ipt}\,dt,$$

the Fourier transform of $\psi_n(t)$, we get from (3)

$$\bar{\psi}_n(p)=\bar{\sigma}_n(p)+\sum_{m=1}^{\infty}\left\|\bar{\psi}_{n+m}(p)\exp\left\{i\left(2\frac{m\,a}{v}-\frac{m\,a}{c}\right)p\right\}+\bar{\psi}_{n+m}(p)\exp\left(-i\frac{m\,a}{c}p\right)\right\|; \tag{4}$$

in deriving the right-hand side of this equation we have taken account of the possibility of superposition owing to the assumed linearity of the problem. Denoting

$$i\,p\,a/c = \zeta, \quad \text{and} \quad c/v = \lambda,$$

(4) can be readily transformed into

$$\overline{\varphi}_n(\zeta) = \overline{\sigma}_n(\zeta) + \sum_{m=1}^{\infty} \left\| \overline{\varphi}_{n+m}(\zeta)\, \exp\{(2\lambda - 1)\, m\, \zeta\} + \overline{\varphi}_{n+m}(\zeta)\, \exp(-\overset{*}{m}\, \zeta) \right\| . \tag{5}$$

Our next step will be to obtain an explicit expression of each term under summation.

Now for the convenience of our discussion let us imagine a model: a number of particles $\cdots, <n-m>, \cdots, <n-1>$ and $<n>$ attached to a string as in the figure. When we press $<n>$ into the motion defined by $\overline{\varphi}_n(\zeta)$, how much influence will be excited on other particles? If we denote the induced motion of $<n-1>, <n-2>, \cdots$ by $<\overline{\varphi}_{n-1}>, <\overline{\varphi}_{n-2}>, \cdots$, respectively, we can derive a system of an infinite number of the equations among them,

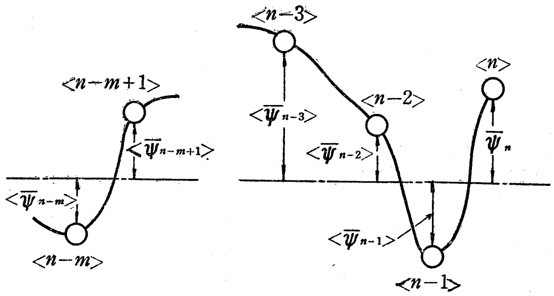

FIG. 1.

viz.

$$\left.\begin{aligned} \overline{\varphi}_n \quad + x <\overline{\varphi}_{n-1}> + <\overline{\varphi}_{n-2}> &= 0, \quad \text{(i)} \\ <\overline{\varphi}_{n-1}> + x <\overline{\varphi}_{n-2}> + <\overline{\varphi}_{n-3}> &= 0, \quad \text{(ii)} \\ <\overline{\varphi}_{n-2}> + x <\overline{\varphi}_{n-3}> + <\overline{\varphi}_{n-4}> &= 0, \quad \text{(iii)} \\ \cdots, & \end{aligned}\right\} \tag{6}$$

where

$$x = -2\cosh\zeta - (\alpha\zeta + \beta)\sinh\zeta,$$

$$\alpha = \frac{\mu}{\rho\, a}, \quad \beta = \frac{\nu\, c}{T}.$$

ν is the damping factor of the motion of the particle owing to the viscosity of the surrounding medium, both α and β being non-dimensional.[1] Since it is obvious that

$$<\overline{\varphi}_{n-\infty}> = 0 \tag{7}$$

owing to the viscosity existing in the medium, let us first assume in (6, i) that $<\overline{\varphi}_{n-2}> = 0$ as the roughest approximation for (7), then we have readily

$$<\overline{\varphi}_{n-1}> = -\frac{1}{x}\overline{\varphi}_n.$$

Taking one more step forward in this approximation by putting $<\overline{\varphi}_{n-3}> = 0$, we shall obtain this time

$$<\overline{\varphi}_{n-1}> = -\frac{x}{x^2-1}\overline{\varphi}_n, \quad \text{and} \quad <\overline{\varphi}_{n-2}> = \frac{1}{x^2-1}\varphi_n.$$

Similarly, corresponding to $<\overline{\varphi}_{n-4}> = 0$, we can arrive at the result

$$<\overline{\varphi}_{n-1}> = -\frac{x^2-1}{x^3-2x}\overline{\varphi}_n, \quad <\overline{\varphi}_{n-2}> = \frac{x}{x^3-2x}\overline{\varphi}_n, \quad \text{and} \quad <\overline{\varphi}_{n-3}> = -\frac{1}{x^3-2x}\overline{\varphi}_n.$$

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[1] Okabe, J. On the Motion of a Trolley-wire, *Rep. Res. Inst. for App. Mech.*, vol. 2, No. 7 (1953), p. 117 (footnote).

Thus we shall be able to form an infinite sequence of such computations. Now if we rewrite in terms of G's defined by (II, 6), we can summarize the above procedure as follows: by putting $<\bar{\varphi}_{n-2}> = 0$,

$$<\bar{\varphi}_{n-1}> = -\frac{G(0)}{G(1)}\bar{\varphi}_n, \tag{8}$$

next by putting $<\bar{\varphi}_{n-3}> = 0$,

$$<\bar{\varphi}_{n-1}> = -\frac{G(1)}{G(2)}\bar{\varphi}_n, \quad \text{and} \quad <\bar{\varphi}_{n-2}> = \frac{G(0)}{G(2)}\bar{\varphi}_n, \tag{9}$$

and by putting $<\bar{\varphi}_{n-4}> = 0$,

$$<\bar{\varphi}_{n-1}> = -\frac{G(2)}{G(3)}\bar{\varphi}_n, \ <\bar{\varphi}_{n-2}> = \frac{G(1)}{G(3)}\bar{\varphi}_n, \quad \text{and} \quad <\bar{\varphi}_{n-3}> = -\frac{G(0)}{G(3)}\bar{\varphi}_n, \tag{10}$$

etc. From (8), (9) and (10) we can infer the conclusion,—— if we denote $\lim_{n\to\infty} G(n-1)/G(n)$, $\lim_{n\to\infty} G(n-2)/G(n)$, $\cdots$, symbolically by $G(\infty-1)/G(\infty)$, $G(\infty-2)/G(\infty)$, $\cdots$, respectively, the solution of the equation (6) compatible with the boundary condition (7) will be finally

$$\left.\begin{aligned}
<\bar{\varphi}_{n-1}> &= -\frac{G(\infty-1)}{G(\infty)}\bar{\varphi}_n \equiv \mathfrak{G}(1)\bar{\varphi}_n, \\
<\bar{\varphi}_{n-2}> &= \frac{G(\infty-2)}{G(\infty)}\bar{\varphi}_n \equiv \mathfrak{G}(2)\bar{\varphi}_n, \\
<\bar{\varphi}_{n-3}> &= -\frac{G(\infty-3)}{G(\infty)}\bar{\varphi}_n \equiv \mathfrak{G}(3)\bar{\varphi}_n, \\
&\cdots.
\end{aligned}\right\} \tag{11}$$

As shown in (II, 6), $G(n)$ is a polynomial of the nth order with regard to x, and accordingly

$$\mathfrak{G}(1) = O(x^{-1}), \ \mathfrak{G}(2) = O(x^{-2}), \ \mathfrak{G}(3) = O(x^{-3}), \ \cdots. \tag{12}$$

The solution embodied by (11) is the motion induced on $<n-1>$, $<n-2>$, $\cdots$ when $<n>$ is prescribed to move according to $\bar{\varphi}_n(\zeta)$; on the contrary the contribution transferred to $<n>$ when $<n+j>$ $(j \geqq 1)$ is forced to move (let us denote it by $<\bar{\varphi}_n>_{n+j}$ for shortness) can be readily written down, viz.

$$\left.\begin{aligned}
<\bar{\varphi}_n>_{n+1} &= \mathfrak{G}(1)\bar{\varphi}_{n+1}, \\
<\bar{\varphi}_n>_{n+2} &= \mathfrak{G}(2)\bar{\varphi}_{n+2}, \\
&\cdots.
\end{aligned}\right\} \tag{13}$$

Hence $\sum_{m=1}^{\infty}\left\|\bar{\varphi}_{n+m}(\zeta)\exp\{(2\lambda-1)\,m\,\zeta\}\right\| = \sum_{m=1}^{\infty}\mathfrak{G}(m)\,\bar{\varphi}_{n+m}(\zeta)\exp\{(2\lambda-1)\,m\,\zeta\}$

and $\sum_{m=1}^{\infty}\left\|\bar{\varphi}_{n+m}(\zeta)\exp(-m\,\zeta)\right\| = \sum_{m=1}^{\infty}\mathfrak{G}(m)\,\bar{\varphi}_{n+m}(\zeta)\exp(-m\,\zeta)$.

But owing to the relation (2) we can prove for each value of m that

$$\bar{\varphi}_{n+m}(\zeta) = \bar{\varphi}_n(\zeta)\exp(-m\,\lambda\,\zeta),$$

and making use of these for the equation (5) we get

$$\bar{\varphi}_n(\zeta) = \frac{\bar{\sigma}_n(\zeta)}{1 - \sum_{m=1}^{\infty} \mathfrak{G}(m) \left[\exp\{-(1+\lambda)\, m\,\zeta\} + \exp\{-(1-\lambda)\, m\,\zeta\}\right]}, \tag{14}$$

or by expanding the right-hand side we can arrive at the result

$$\bar{\varphi}_n(\zeta) = \bar{\sigma}_n(\zeta)(1 + \omega + \omega^2 + \cdots),$$

$$\text{where} \qquad \omega(\zeta) = \sum_{m=1}^{\infty} \mathfrak{G}(m) \left[\exp\{-(1+\lambda)\, m\,\zeta\} + \exp\{-(1-\lambda)\, m\,\zeta\}\right]. \tag{15}$$

Incidentally, we are now enabled to discuss the physical meanings of each term on the right-hand side of (15). For this purpose it is necessary to notice that $\mathfrak{G}(m)$ signifies the rate of reduction of the induced motion compared with the original disturbance when a particle is put in motion by that free vibration of the string which has travelled m spans $(m\,a)$ from the particle where it was originated. We shall next examine, e.g., $\bar{\sigma}_n \mathfrak{G}(m) \exp\{-(1+\lambda)\, m\,\zeta\}$ and $\bar{\sigma}_n \mathfrak{G}(m) \exp\{-(1-\lambda)\, m\,\zeta\}$ as the two representatives of the second term $(\bar{\sigma}_n\,\omega)$ of the right-hand side. Making use of (2) we can readily find that they are equivalent to $\bar{\sigma}_{n+m} \mathfrak{G}(m) \exp(-m\,\zeta)$ and $\bar{\sigma}_{n-m} \mathfrak{G}(m) \exp(-m\,\zeta)$ respectively. These motions of $< n >$ are thus found to be those due to the disturbances $\bar{\sigma}$ generated at $< n+m >$ and $< n-m >$, respectively. The factor $\exp(-m\,\zeta)$ indicates the elapsing of time by m units. Since time is measured with the scale a/c, c.f. $i\,p\,a/c = \zeta$, this is the time necessary for a disturbance to pass through m spans.

But on the other hand the true motions of $< n+m >$ and $< n-m >$ are $\bar{\varphi}_{n+m}$ and $\bar{\varphi}_{n-m}$ respectively, and not $\bar{\sigma}_{n+m}$ and $\bar{\sigma}_{n-m}$. The terms coming after the second terms of the right-hand side may be regarded as the corrections resulting from this fact. As an example of the third term $(\bar{\sigma}_n\,\omega^2)$ let us proceed to take, e.g.,

$$\bar{\sigma}_n \mathfrak{G}(1)^2 \left[\exp\{-(1+\lambda)\, 2\zeta\} + 2\exp\{-2\zeta\} + \exp\{-(1-\lambda)\, 2\zeta\}\right];$$

from the relation (2) this is found to be identical to

$$\bar{\sigma}_{n+2} \mathfrak{G}(1)^2 \exp(-2\,\zeta) + 2\bar{\sigma}_n \mathfrak{G}(1)^2 \exp(-2\zeta) + \bar{\sigma}_{n-2} \mathfrak{G}(1)^2 \exp(-2\zeta). \tag{16}$$

Owing to the disturbance $\bar{\sigma}_{n+2}$ of $< n+2 >$ the motion of $< n+1 >$ will deviate from $\bar{\sigma}_{n+1}$ by the quantity $\bar{\sigma}_{n+2} \mathfrak{G}(1) \exp(-\zeta)$, which, again on its part, will be transferred to $< n >$ yielding the final contribution $\bar{\sigma}_{n+2} \mathfrak{G}(1)^2 \exp(-2\,\zeta)$. This process may be illustrated by the schema

$$\bar{\sigma}_{n+2},\ < n+2 > \rightarrow \bar{\sigma}_{n+2} \mathfrak{G}(1) \exp(-\zeta),\ < n+1 > \rightarrow \bar{\sigma}_{n+2} \mathfrak{G}(1)^2 \exp(-2\zeta),\ < n >.$$

The remaining two members of (16) can be shown in the same fashion as

$$\bar{\sigma}_n,\ < n > \begin{matrix} \nearrow \bar{\sigma}_n \mathfrak{G}(1) \exp(-\zeta),\ < n+1 > \searrow \\ \searrow \bar{\sigma}_n \mathfrak{G}(1) \exp(-\zeta),\ < n-1 > \nearrow \end{matrix} 2\bar{\sigma}_n \mathfrak{G}(1)^2 \exp(-2\zeta),\ < n >,$$

and

$$\bar{\sigma}_{n-2},\ < n-2 > \rightarrow \bar{\sigma}_{n-2} \mathfrak{G}(1) \exp(-\zeta),\ < n-1 > \rightarrow \bar{\sigma}_{n-2} \mathfrak{G}(1)^2 \exp(-2\zeta),\ < n >.$$

Similar interpretations will be possible for the fourth and the subsequent terms.

Thus we have succeeded in attributing a definite meaning to each term of (14) when expanded in the form of an infinite series (15). This expansion, however, is not always necessary to the computation of the motion. Our next step will be to evaluate the expression (14) or (15) in a form more accessible.

We have defined by (11) the infinite sequence of the influence functions $\{\mathfrak{G}(m)\}$ $(m = 1, 2, 3, \cdots)$ as

$$\mathfrak{G}(m) \equiv (-)^m \lim_{n\to\infty} \frac{G(n-m)}{G(n)} \tag{17}$$

for $m \geqq 1$, where $G(n)$ is a polynomial of the nth order with regard to x, whose explicit form was shown in (II, 6). If we write for convenience

$$\mathfrak{G}(0) \equiv 1, \tag{18}$$

then (17) is valid for all integral values of m from 0 to infinity. Now as was mentioned in (II, 3), $G(n)$ satisfies the recurrence formula

$$G(n) = x\, G(n-1) - G(n-2) \tag{19}$$

for $n \geqq 2$. By dividing the both hand sides by $G(n-1)$ and taking the limit of $n \to \infty$, we have readily the equation

$$-\mathfrak{G}(1)^{-1} = x + \mathfrak{G}(1),$$

whose solution is obviously

$$\mathfrak{G}(1) = \frac{1}{2}(-x + \sqrt{x^2-4});$$

x defined by (6) is a complex quantity and $\sqrt{x^2-4}$ on the right-hand side is not single-valued. But if we use a convention that in the following discussions $\sqrt{x^2-4}$ signifies that branch of the function which tends to x as x increases indefinitely, the conditions (12) specifies uniquely the value of $\mathfrak{G}(1)$, viz.

$$\mathfrak{G}(1) = \frac{1}{2}(\sqrt{x^2-4} - x), \tag{20}$$

which is equal to

$$-\frac{1}{x} - \frac{1}{x^3} - \frac{4}{x^5} - \cdots \tag{21}$$

as easily verified.

Since we have by definition, cf. (17),

$$\mathfrak{G}(m+1) = (-)^{m+1} \lim_{n\to\infty} \frac{G(n-m-1)}{G(n)},$$

and
$$\mathfrak{G}(m+2) = (-)^{m+2} \lim_{n\to\infty} \frac{G(n-m-2)}{G(n)},$$

using the relation (19) among G's we can arrive at the recurrence formula among $\mathfrak{G}$'s viz.

$$\mathfrak{G}(m+2) = -x\, \mathfrak{G}(m+1) - \mathfrak{G}(m) \tag{22}$$

for $m = 0, 1, 2, \cdots$. Starting from the values of $\mathfrak{G}(0)$ and $\mathfrak{G}(1)$ already given by (18) and (20), we can construct one after another the whole sequence of $\mathfrak{G}(m)$ $(m \geqq 2)$ according to this equation. By mathematical induction we can prove easily that $\mathfrak{G}(m)$ is given in the form

$$\mathfrak{G}(m) = \frac{1}{2^m}(\sqrt{x^2-4} - x)^m \tag{23}$$

for $m = 0, 1, 2, \cdots$.

Thus $\omega(\zeta)$ defined by (15) becomes

$$\omega(\zeta) \equiv \sum_{m=1}^{\infty} \mathfrak{G}(m) [\exp\{-(1+\lambda) m \zeta\} + \exp\{-(1-\lambda) m \zeta\}]$$
$$= -\frac{(x - \sqrt{x^2-4})\, e^{-\zeta} + 2\cosh \lambda \zeta}{x \cosh \zeta + \sqrt{x^2-4} \sinh \zeta + 2\cosh \lambda \zeta}.$$

Putting this expression in (14), we can arrive, after some manipulations, at

$$\bar{\varphi}_n(\zeta) = \bar{\sigma}_n(\zeta) \frac{x \cosh \zeta + \sqrt{x^2-4} \sinh \zeta + 2\cosh \lambda \zeta}{x \cosh \zeta + \sqrt{x^2-4} \sinh \zeta + (x - \sqrt{x^2-4})\, e^{-\zeta} + 4\cosh \lambda \zeta},$$

where, cf. (6),

$$x = -2\cosh \zeta - (\alpha \zeta + \beta) \sinh \zeta, \tag{24}$$

$$\alpha = \frac{\mu}{\rho a}, \quad \beta = \frac{\nu c}{T}, \quad \text{and} \quad \lambda = \frac{c}{v}.$$

Finally, the inversion from $\bar{\varphi}_n(\zeta)$ to $\varphi_n(t)$ is enabled by the formula

$$\varphi_n(\tau) = \frac{1}{2\pi i} \frac{c}{a} \int_{-i\infty}^{i\infty} \bar{\varphi}_n(\zeta)\, e^{\tau\zeta}\, d\zeta, \tag{25}$$

where $\tau = c\,t/a$, which is non-dimensional time whose origin should be defined in reference to the form of $\sigma_n(t)$. Namely, we obtain from (24)

$$\varphi_n(\tau) = \frac{1}{2\pi i} \frac{c}{a} \int_{-i\infty}^{i\infty} \bar{\sigma}_n(\tau) \frac{(x \cosh \zeta + \sqrt{x^2-4} \sinh \zeta + 2\cosh \lambda\zeta) \exp(\tau \zeta)\, d\zeta}{x \cosh \zeta + \sqrt{x^2-4} \sinh \zeta + (x - \sqrt{x^2-4}) \exp(-\zeta) + 4 \cosh \lambda\zeta}. \tag{26}$$

If we confine ourselves to the case when τ is small, by the well-known procedure[1] we can make use of the following approximations:

$$x \approx -\alpha \zeta \sinh \zeta,$$

and
$$\sqrt{x^2-4} \approx x \approx -\alpha \zeta \sinh \zeta.$$

Therefore,

$$\varphi_n(\tau) = \frac{1}{2\pi i} \frac{c}{a} \int_{-i\infty}^{i\infty} \bar{\sigma}_n(\zeta) \frac{-\alpha \zeta \exp(\zeta) \sinh \zeta + 2\cosh \lambda \zeta}{-\alpha \zeta \exp(\zeta) \sinh \zeta + 4\cosh \lambda \zeta}\, e^{\tau\zeta}\, d\zeta. \tag{27}$$

We can next prove easily that the function

$$\frac{-\alpha \zeta \exp(\zeta) \sinh \zeta + 2\cosh \lambda \zeta}{-\alpha \zeta \exp(\zeta) \sinh \zeta + 4\cosh \lambda \zeta}$$

has no singular points on the imaginary axis of the ζ-plane. And if $\bar{\sigma}_n(\zeta)$ has a simple expression, it will not be so difficult to evaluate the integral (27) by means of an appropriate contour integration.

These three notes are the by-products from the writer's calculation of the motion of a trolley-wire.[2] Although fragmental and only *mathematical* in the worst sense of the word, they are published as a series, because all of them are related essentially to the same polynomial $G(N)$.

(Received July 1, 1953)

[1] McLachlan, N. W. *Complex Variable & Operational Calculus with Technical Applications* (1939), e.g., 14. 2, p. 241.

[2] Okabe, J. *op. cit.* See the footnote on page 156.