On a Forced Lateral Vibration of a Number of Particles Attached to a String at Equal Intervals

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NOTES

On a Forced Lateral Vibration of a Number of Particles Attached to a String at Equal Intervals

By Jun-ichi OKABE

Let us consider a number of, say N, equal particles attached to a massless string at equal intervals a; see the figure. If we denote by μ the mass of each particle and by T the tension, which is assumed constant spatially as well as temporally, of the string, then $\psi_n(n=1,2,\cdots,N)$, the lateral displacement of the nth particle from the position of equilibrium, is governed by the equation

$$\mu\ddot{\psi}_n = Ta^{-1}\{(\psi_{n+1}-\psi_n)+(\psi_{n-1}-\psi_n)\}+Q_n e^{ipt}, \tag{1}$$

so long as the disturbances are assumed small; the last term on the right-hand side represents the external periodic force imposed upon the particle. In order to take out that part of motion which corresponds to the so-called forced vibration, let us put as usual

$$\psi_n = \varphi_n e^{ipt}; \tag{2}$$

then φ_n will be determined by a system of the equations

$$\left.\begin{aligned} &\varphi_{n-1}+x\,\varphi_n+\varphi_{n+1}=Q_n,\ (n=1,2,\cdots,N),\\ &\varphi_0=\varphi_{N+1}\equiv 0,\\ \text{where}\quad &x=\mu\,a\,p^2\,T^{-1}-2. \end{aligned}\right\} \tag{3}$$

If for the sake of shortness we write

$$G(N)\equiv\begin{vmatrix} x & 1 & 0 & 0 & \cdots & & & \\ 1 & x & 1 & 0 & \cdots & & & \\ 0 & 1 & x & 1 & 0 & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & \cdots & 0 & 1 & x & 1 \\ & & & & \cdots & 0 & 1 & x \end{vmatrix}, \tag{4}$$

which is a determinant of the Nth order, and further

$$H(N;n)\equiv\begin{vmatrix} x & 1 & 0 & \cdots & 0 & Q_1 & 0 & \cdots \\ 1 & x & 1 & 0 & \cdots & 0 & Q_2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & \cdots & 0 & 1 & Q_n & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & \cdots & & 0 & Q_N & 0 & \cdots \end{vmatrix}, \tag{5}$$

(the column of Q's being the nth)

which is a determinant of the Nth order as before and is formed by substitution of Q's in the nth column of $G(N)$, our solution may be written

$$\varphi_n = H(N;n)/G(N). \tag{6}$$

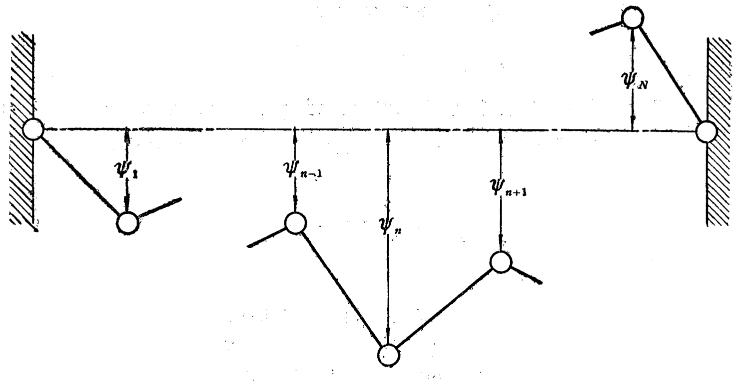

FIG. 1.

Reserving the evaluation of $G(N)$ for the later discussions,[1] we shall confine ourselves to obtaining a formal expression of $H(N; n)$ in terms of $G(N)$.

Now from the property of a determinant it is obvious that H can be expanded in the form

$$H(N;\, n) = \sum_{m=1}^{N} Q_m \cdot h(N;\, n,\, m), \tag{7}$$

where

$$h(N;\,n,m) \equiv \begin{array}{c} \overbrace{}^{n}\ \overbrace{}^{N-n+1} \\ \left|\begin{array}{cccccccccc} x & 1 & 0 & \cdots & & & 0 & \cdots & & \\ 1 & x & 1 & 0 & \cdots & & 0 & \cdots & & \\ 0 & 1 & x & 1 & 0 & \cdots & \vdots & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & & & \\ & & & & \cdots & 0 & 1 & 0 & \cdots & \\ & & & & & & 0 & & & \\ & & & & & & \vdots & & & \\ & & & & \cdots & & 0 & \cdots\ 0 & 1 & x \end{array}\right| \end{array} \tag{8}$$

(the first m rows are marked by a brace m on the right, and the last $N-m+1$ rows by a brace $N-m+1$ on the left)

which is the determinant of the Nth order formed by substitution of " $\overbrace{0,\, 0,\, \cdots,\, 0,\, 1}^{m}$, $0, \cdots, 0$ " in the nth column of $G(N)$.

On this understanding, we shall first work out the case when $m \geqq n$. Then by expanding h with respect to the first column, we have

$$h(N;\, n,\, m) = x\, h(N-1;\, n-1,\, m-1) - h(N-2;\, n-2,\, m-2)\,. \tag{9}$$

If we define $\vartheta(\sigma)$, the generating function of h, by the infinite series

$$\vartheta(\sigma) = \sum_{\nu=1}^{\infty} h(N-n+\nu;\, \nu,\, m-n+\nu)\, \sigma^{\nu} \equiv \sum_{\nu=1}^{\infty} h_{\nu}\, \sigma^{\nu}, \tag{10}$$

we can prove easily that

$$\vartheta(\sigma) = \frac{h_1\, \sigma - (x\, h_1 - h_2)\, \sigma^2}{1 - \sigma(x-\sigma)} \tag{11}$$

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[1] Okabe, J. On the Roots of the Equation$\cdots$(*Note*), in the present issue. Abbreviation " II " will be used for future quotations.

by making use of the relation (9). But $h_2 \equiv h(N-n+2\,;2, m-n+2)$ is found to be equal to $x\,h(N-n+1\,;1, m-n+1) \equiv x\,h_1$ when we expand h_2 in terms of the first row, and further by the same procedure it becomes clear that

$$\begin{aligned} h_1 \equiv h(N-n+1\,;\,1, m-n+1) &= -h(N-n\,;\,1, m-n) \\ = \cdots &= (-)^{m-n}\,h(N-m+1\,;\,1, 1) = (-)^{m-n}\,G(N-m)\,. \end{aligned}$$

Therefore,

$$\vartheta(\sigma) = \frac{(-)^{m-n}\,G(N-m)\,\sigma}{1-\sigma(x-\sigma)} = (-)^{m-n}\,G(N-m)\,\sigma\Gamma(\sigma)\,, \tag{12}$$

where $\Gamma(\sigma)$ is the generating function of $G(N)$; see (II, 5). We can readily write down accordingly

$$h(N;\,n, m) = (-)^{m-n}\,G(N-m)\,G(n-1)\,, \tag{13}$$

since $h(N;\,n, m)$ is the coefficient of σ^n in the series (10).

In order to arrive at the similar expression for the case when $n \geqq m+1$ we shall write

$$h(N;\,n, m) \equiv h'(N;\,N-n+1, N-m+1)\,; \tag{14}$$

see (8). From the expansion of h' in terms of the Nth column we have as before

$$\begin{aligned} h'(N\,;\,N-n+1, N-m+1) &= x\,h'(N-1\,;\,N-n, N-m) \\ &\quad - h'(N-2\,;\,N-n-1, N-m-1)\,. \end{aligned} \tag{15}$$

The generating function $\vartheta'(\sigma)$, defined by

$$\vartheta'(\sigma) = \sum_{\nu=1}^{\infty} h'(n+\nu-1\,;\,\nu, n+\nu-m)\,\sigma^{\nu} \equiv \sum_{\nu=1}^{\infty} h_{\nu}'\,\sigma^{\nu}\,,$$

will be easily verified to be equal to

$$(-)^{n-m}\,G(m-1)\,\sigma\Gamma(\sigma)\,,$$

accordingly we have by comparing the coefficients of σ^{N-n+1} of these expressions

$$h'(N;\,N-n+1, N-m+1) \equiv h(N;\,n, m) = (-)^{n-m}\,G(m-1)\,G(N-n) \tag{16}$$

for the case when $n \geqq m+1$.

In conclusion, combining (7), (13) and (16), the results may be summarized as follows:

$$\begin{aligned} H(N\,;\,n) = &\sum_{m=1}^{n-1} (-)^{n-m}\,Q_m\,G(N-n)\,G(m-1) \\ &+ \sum_{m=n}^{N} (-)^{m-n}\,Q_m\,G(N-m)\,G(n-1)\,, \end{aligned} \tag{17}$$

$G(j)$ being a polynomial of x of the jth order, whose explicit form will be shown in (II, 6). Owing to this formulation φ_n can be calculated from (6).

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