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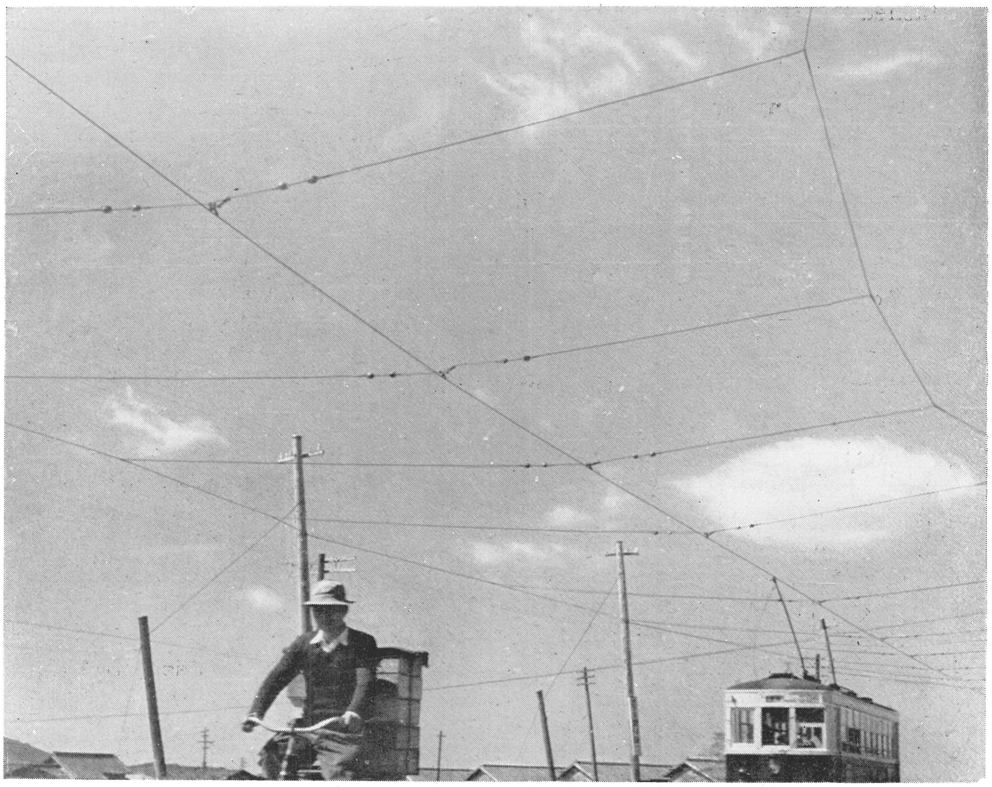
ON THE MOTION OF A TROLLEY-WIRE

By Jun-ichi OKABE

Mathematics of the motion of a large number of particles attached uniformly to a long string upon which a concentrated load is running with a constant speed is discussed. In three cases when the *Mach number*, so to speak, is equal to 1/2, 1 and 2, the motion is calculated in detail using the assumed values of the parameters.

The string is a model of the main line of a trolley-wire whose rigidity is assumed small and the particles represent auxiliary wires crossing the main line perpendicularly at equal intervals.

1. In those gloomy days after the end of the war it was our daily routine to wait for a street-car for a long, long time in a long, long line, because the number of the cars had been cut down from terrible shortage



of electricity. When the trolley-wire against the dusk evening sky was found moving at last, it was nothing but a sign of the approaching relief from both hunger and weariness. The following is a *Pride and Prejudice* of a young computer who was helpless in the vacuum of the war-torn country.

2. A trolley-line as shown in the picture consists of a long main wire upon which a trolley rolls or a pantagraph slides and a number of transverse wires crossing the main line at equal intervals. If we are interested not in the individual movements of the auxiliary (transversal) wires in detail but solely in the general aspect of the behavior of the main line, it is quite legitimate to suppose that some fraction (we cannot say *how much* exactly) of the mass of each transverse works as an additive mass attached to the main line at equal intervals while the latter is in motion. Besides, supposing we shall content ourselves with discussing the ideal case when the whole system of the wires lies in a horizontal plane, the vertical component of the tension of each transverse will resist the motion of the main to restore the equilibrium of the system, cf. **4**, (iii) and Fig. 3.

As the model of a trolley-wire let us take the following system: great numbers of particles of mass m which is equal to each other are attached at equal intervals l to a string of infinite length extending to both directions. When a particle is disturbed, it is brought back to the position of equilibrium by a spring whose one end is fixed at a constant level. Take any one of the particles and name it No. 0. Starting from No. 0, we shall name the particles No.'s 1, 2, 3, $\cdots$ one after another to the right, and accordingly No.'s $-1, -2, -3, \cdots$ to the left, Fig. 1. Before going further we must admit that the rigidity of the wire resisting the bending may not be vanishing in some cases when its diameter is not negligible compared with the span. Then some modification will become necessary to the result of the following computations in which the wire is considered as a string to simplify the mathematics. This is a problem reserved for another paper and let us go back to *our main line*.

Originally the string and the particles were resting on the straight line passing through the both extremities of the system, but now they are being disturbed incessantly out of equilibrium by a point load of a constant magnitude running since the time $t = -\infty$ with a constant speed V. We shall choose the origin of the time in such a way that the load arrives at $x = 0$ (the particle No. 0) at $t = 0$ and runs away in the direction of x increasing afterwards. We shall denote the displacement of the particle No. n ($n = 0, 1, 2, \cdots$) and that of the string at the point of the abscissa x by $Y_n(t)$ and $y(x, t)$, respectively. If we make use of the delta function of Dirac for the mathematical expression of the moving load, the idealized trolley, and further if we ignore the effect of gravity which we assume cannot have fundamental importance in the nature of the vibration in question, then the equation of motion of the string has the form under the well-known assumptions,

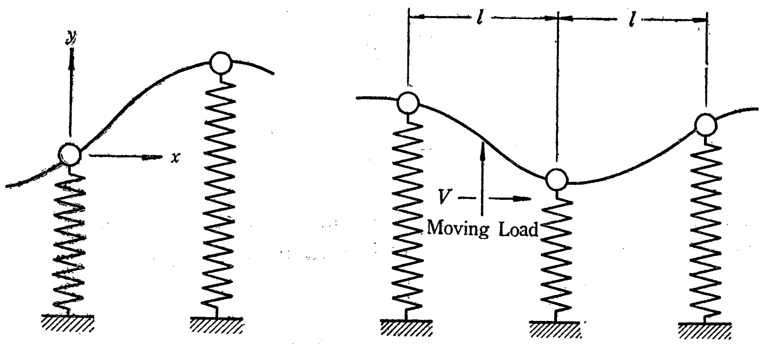


FIG. 1.

$$\rho \frac{\partial^2 y}{\partial t^2} = -\nu \frac{\partial y}{\partial t} + T \frac{\partial^2 y}{\partial x^2} + f\delta\left[\frac{c}{l}\left(t - \frac{x}{V}\right)\right], \tag{2.1}$$

where ρ is the linear density, and T the tension, of the string, both of which are assumed constant spatially as well as temporally: ν and f are the constants showing the viscosity of the surrounding medium and the intensity of the running load respectively. The argument of the delta function has been reduced dimensionless by means of the constants l and c defined as

$$c = \sqrt{T/\rho}, \tag{2.2}$$

which is the speed of propagation of lateral vibration of the string.

After multiplying the factor $\exp(-ipt)$, let us integrate the both hand sides of (2.1) with respect to t from $-\infty$ to ∞. Denoting by $\bar{y}(x, p)$ the integral

$$\int_{-\infty}^{\infty} y(x, t)\, e^{-ipt}\, dt,$$

the Fourier transform of y, we have finally

$$\frac{d^2\bar{y}}{dx^2} + \left(\frac{p^2}{c^2} - i\frac{\nu p}{T}\right)\bar{y} = -\frac{fl}{cT} e^{-ipx/V}. \tag{2.3}$$

In deriving this equation from (2.1) we must take into consideration the fact that y and $\partial y/\partial t$ become zero when t approaches to $\pm\infty$ and the property of the delta function as well, i.e.

$$\int_{-\infty}^{\infty} F(\xi)\,\delta(\xi)\,d\xi = F(0).$$ [1]

By putting

[1] Dirac, P.A.M. *Ryoshi Rikigaku* (*The principles of Quantum Mechanics*), Iwanami (1943), §20 (3), p. 98.

$$\frac{p^2}{c^2} - i\frac{\nu p}{T} = \theta^2, \tag{2.4}$$

we can rewrite (2.3) in the form

$$\frac{d^2\bar{y}}{dx^2} + \theta^2\,\bar{y} = -\frac{fl}{cT}\,e^{-ipx/V}. \tag{2.5}$$

Since two independent solutions of the homogeneous equation derived from (2.5) are $\exp(i\,\theta\,x)$ and $\exp(-i\,\theta\,x)$,[1] and a particular solution of (2.5) can be calculated from the formula

$$\bar{y} = -\frac{fl}{cT}\int^x e^{-ip\eta/V}\begin{vmatrix} e^{i\theta\eta} & e^{-i\theta\eta} \\ e^{i\theta x} & e^{-i\theta x}\end{vmatrix}\,d\eta \div \begin{vmatrix} e^{i\theta\eta} & e^{-i\theta\eta} \\ i\theta e^{i\theta\eta} & -i\theta e^{-i\theta\eta}\end{vmatrix},$$

we can obtain the integral of (2.5) as follows:

$$\bar{y}(x, p) = \varphi\,e^{i\theta x} + \psi\,e^{-i\theta x} + \frac{fl}{cT}\left(\frac{p^2}{V^2} - \frac{p^2}{c^2} + i\frac{\nu p}{T}\right)^{-1} e^{-ipx/V}; \tag{2.6}$$

φ and ψ are constants, which may, however, vary from a span to another. If we denote φ and ψ in the interval $n\,l \leqq x \leqq (n+1)\,l$ $(n = 0, \pm 1, \pm 2, \cdots)$ by φ_n and ψ_n respectively, they are determined by the boundary conditions

$$\bar{y}_{x=nl} = \bar{Y}_n, \quad \bar{y}_{x=(n+1)l} = \bar{Y}_{n+1},$$

where
$$\bar{Y}_n(p) = \int_{-\infty}^{\infty} Y_n(t)\,e^{-ipt}\,dt, \quad \text{etc.} \tag{2.7}$$

Namely we have from (2,6)

$$\begin{aligned} &\varphi_n\,e^{i\theta nl} + \psi_n\,e^{-i\theta nl} + \sigma\,e^{-ipnl/V} = \bar{Y}_n, \\ &\varphi_n\,e^{i\theta(n+1)l} + \psi_n\,e^{-i\theta(n+1)l} + \sigma\,e^{-ip(n+1)l/V} = \bar{Y}_{n+1}, \end{aligned} \tag{2.8}$$

if we write for shortness

$$\frac{fl}{cT}\left(\frac{p^2}{V^2} - \frac{p^2}{c^2} + i\frac{\nu p}{T}\right)^{-1} = \sigma. \tag{2.9}$$

By solving these equations we readily find that

$$\varphi_n = \frac{i}{2}\,e^{-i\theta nl}\,\{\bar{Y}_n\,e^{-i\theta l} - \bar{Y}_{n+1} - \sigma\,e^{-ipnl/V}(e^{-i\theta l} - e^{-ipl/V})\}\ \operatorname{cosec}\theta l,$$

and

$$\psi_n = \frac{i}{2}\,e^{i\theta nl}\,\{\bar{Y}_{n+1} - \bar{Y}_n\,e^{i\theta l} - \sigma\,e^{-ipnl/V}(e^{-ipl/V} - e^{i\theta l})\}\ \operatorname{cosec}\theta l. \tag{2.10}$$

3. Let us next consider the motion of the particles attached to the main line. The force acting on a particle comes from the three sources: i) the

[1] We shall specify θ by that branch of the function which tends to p/c as ν becomes zero.

restoring force due to the elongated spring, which is therefore proportional to the deviation of the particle from its position of equilibrium, ii) the damping force from the surrounding medium which is assumed varying as the speed of the displacement of the particle, and iii) the y-components of the tensions of the two wires, running on both sides of the particle, at the junction with the particle, cf. Fig. 2.

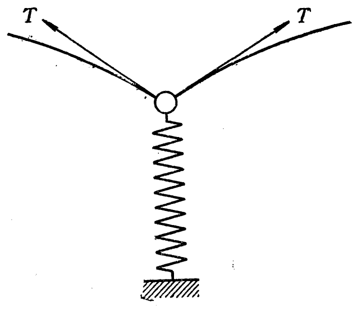


FIG. 2.

If we assume for simplicity that the running load, whenever it arrives at any particle, jumps immediately into the next segment of wire without giving a blow to that particle, then the equation of motion of a particle has the form

$$M \frac{d^2 Y_n}{dt^2} = -k Y_n - \mu \frac{d Y_n}{dt} + T \left\{ \left(\frac{\partial y}{\partial x}\right)_{x=nl+0} - \left(\frac{\partial y}{\partial x}\right)_{x=nl-0} \right\}. \quad (3.1)$$

Denoting by $\Delta(\partial y/\partial x)_{nl}$ the quantity within bracket on the right-hand side, and then taking the Fourier transform of the equation, we are led to the result

$$(-M p^2 + i \mu p + k) \overline{Y}_n = T \Delta(d\overline{y}/dx)_{nl} \quad (3.2)$$

under the assumption that $Y_n = dY_n/dt = 0$ when $t = \pm \infty$. On the other hand, however, from the definition of $\Delta(d\overline{y}/dx)$ it can readily be found that

$$\Delta(d\overline{y}/dx)_{nl} = i\,\theta\,\{(\varphi_n - \varphi_{n-1})\, e^{i\theta nl} - (\psi_n - \psi_{n-1})\, e^{-i\theta nl}\},$$

and by making use of the relations (2.10) we can prove that

$$\begin{aligned}\Delta(d\overline{y}/dx)_{nl} = \theta\,\{&\overline{Y}_{n+1} - 2\overline{Y}_n \cos\theta l + \overline{Y}_{n-1} \\ &+ 2\sigma\, e^{-ipnl/V} (\cos\theta l - \cos pl/V)\} \operatorname{cosec}\theta l. \end{aligned} \quad (3.3)$$

Accordingly (3.2) becomes after slight modifications

$$\begin{aligned}\overline{Y}_{n+1} - \{2 \cos\theta l + \frac{\sin\theta l}{T\theta}(-M p^2 + i \mu p + k)\}\, \overline{Y}_n + \overline{Y}_{n-1} \\ = 2\sigma\, e^{-ipnl/V}(\cos\frac{pl}{V} - \cos\theta l). \end{aligned} \quad (3.4)$$

If we write for brevity,

$$\left.\begin{aligned} -2 \cos\theta l - \frac{\sin\theta l}{T\theta}(-M p^2 + i \mu p + k) &\equiv P, \\ \text{and} \qquad 2\sigma(\cos pl/V - \cos\theta l) &\equiv Q, \end{aligned}\right\} \quad (3.5)$$

$\overline{Y}_n$ will be determined by an infinite number of the equations of the type

$$\overline{Y}_{n+1} + P\overline{Y}_n + \overline{Y}_{n-1} = Q\, e^{-ipnl/V}, \quad (3.6)$$

where $n = 0, \pm 1, \pm 2, \cdots$.

Now that, however, the load has been running uniformly since $t=-\infty$ and besides the wire extends to an infinite distance on both sides, the following recurrence formula necessarily holds among any of the neighboring members:

$$Y_{n-1}\left(t-\frac{l}{V}\right)=Y_n(t)=Y_{n+1}\left(t+\frac{l}{V}\right).$$

Or, in terms of Fourier transforms,

$$\left.\begin{aligned}\overline{Y}_{n-1}&=\overline{Y}_n \exp(i\,pl/V),\\ \text{and}\qquad \overline{Y}_{n+1}&=\overline{Y}_n \exp(-i\,pl/V).\end{aligned}\right\}\tag{3.7}$$

By virtue of these relations, (3.6) can be readily solved, and we have

$$\overline{Y}_n=Q\ \exp(-i\,pnl/V)(2\cos pl/V+P)^{-1}. \tag{3.8}$$

Or more explicitly

$$\overline{Y}_n=\frac{2fl}{cT}\frac{\left(\cos\frac{pl}{V}-\cos\theta l\right)\exp\left(-i\frac{pnl}{V}\right)}{\left(\frac{p^2}{V^2}-\frac{p^2}{c^2}+i\frac{\nu p}{T}\right)\left\{2\cos\frac{pl}{V}-2\cos\theta l-\frac{\sin\theta l}{T\theta}(-Mp^2+i\mu p+k)\right\}}, \tag{3.9}$$

$n=0,\ \pm1,\ \pm2,\ \cdots$.

The inversion from $\overline{Y}_n(p)$ to $Y_n(t)$ is enabled by the formula

$$Y_n(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{Y}_n(p)\,e^{ipt}\,dp, \tag{3.10}$$

and on introducing the non-dimensional quantities τ, ζ, α, β, γ, ε and λ defined respectively by

$$\begin{aligned} &t-\frac{nl}{V}\equiv\frac{l\tau}{c}, && ip\equiv\frac{c\zeta}{l},\\ &\frac{Mc^2}{Tl}=\frac{M}{\rho l}\equiv\alpha,\quad \frac{\mu c}{T}\equiv\beta,\quad \frac{kl}{T}\equiv\gamma,\\ &\frac{\nu lc}{T}\equiv\varepsilon, \qquad \text{and} && \frac{c}{V}\equiv\lambda,\end{aligned}\tag{3.11}$$

we can rewrite (3.9) in the form

$$\begin{aligned}Y_n(\tau)=\frac{2fl^2}{T}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}&(\cosh\lambda\zeta-\cosh\sqrt{\zeta^2+\varepsilon\zeta})\exp(\tau\zeta)(\zeta^2-\lambda^2\zeta^2+\varepsilon\zeta)^{-1}\\ &\times\{2\cosh\lambda\zeta-2\cosh\sqrt{\zeta^2+\varepsilon\zeta}-(\alpha\zeta^2+\beta\zeta+\gamma)(\zeta^2+\varepsilon\zeta)^{-1/2}\\ &\times\sinh\sqrt{\zeta^2+\varepsilon\zeta}\}^{-1}d\zeta;\end{aligned}\tag{3.12}$$

the integration should be carried out along the imaginary axis of the ζ-plane from $-i\infty$ to $i\infty$.

Incidentally, we have to notice first that the integrand is one-valued, and second that the right-hand side is independent of the number n. The latter is a natural consequence, for from our assumption the motion of each particle should be equivalent with regard to the non-dimensional time τ defined in (3.11). Accordingly we shall omit the suffix n of Y_n, if no confusion may arise. Before computing the integration, however, we feel it convenient to discuss briefly about the expected magnitudes of the constants involved in the integral.[1]

4. A sketch of a small portion of the vibrating trolley-wire is shown in the figure. Suppose the main line running through the joints A, B, C, ··· has been deformed into an elevated position A, B', C, ··· . An auxiliary wire crossing the main line perpendicularly at a joint is denoted by D B' E. Bearing in mind that the integral (3.12) depends upon the 6 parameters α, β, γ, ε, λ and τ defined by (3.11),——

(i) $\alpha = M/\rho l$: ρl is obviously the mass of the wire whose span is AB or BC. M, on the other hand, which represented in the preceding computation the mass of a particle attached to the string, is not in reality necessarily the mass of a joint, e.g. B'. On the contrary, the mass of a single joint itself may be comparatively small, and a considerable portion of the mass of the auxiliary wire moving with the joint *en bloc* must be involved in M. The magnitude of the additive mass due to the auxiliary wire, which presumably constitutes the greater part of M as a matter of fact, changes every moment in the course of vibration. Although it is impossible therefore to mention the precise value of the additive mass, it would be

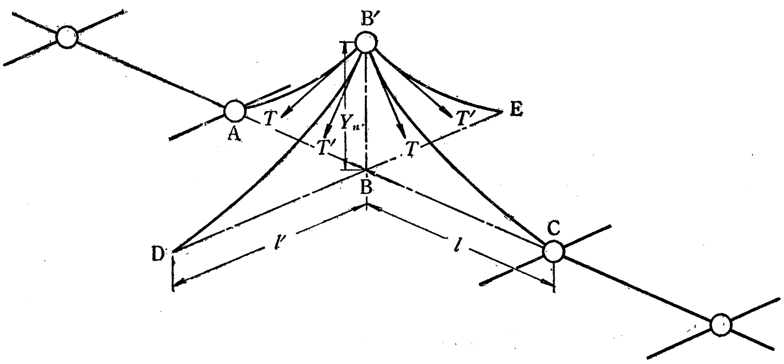


FIG. 3.

[1] For the convenience of the *Note* An Illustrative Example of Solving a Sum Equation ··· (in the present issue), it is added that in terms of the new variables P is expressed by

$$P = -2\cosh\sqrt{\zeta^2+\varepsilon\zeta} - (\alpha\zeta^2+\beta\zeta+\gamma)(\zeta^2+\varepsilon\zeta)^{-1/2}\sinh\sqrt{\zeta^2+\varepsilon\zeta},$$

and especially if $\varepsilon = \gamma = 0$, then

$$P = -2\cosh\zeta - (\alpha\zeta+\beta)\sinh\zeta.$$

no great error to say if l' is very roughly as long as l that M is of the same order of magnitude as ρl and this we write for brevity

$$M \approx \rho l\,,$$

or

$$\alpha \approx 1\,. \tag{4.1}$$

(ii) $\beta = \mu c/T$: At the beginning (cf. 2) it was assumed that the value of T and $c = (T/\rho)^{1/2}$ were not affected by the motion of a wire and that a wire while undisturbed lay on a straight line passing through joints at an infinite distance, the effect of gravity being neglected. By the first of these assumptions we are enabled to evaluate T or c from the state of rest of the wire; by the second, however, T would not be determined from the condition of equilibrium. In other words, therefore, in spite of the fact that the nature of the vibration of the wire may well be revealed under the second assumption, this is not exactly valid so long as the wire has its own weight, and we have to take account of the deformation of the wire owing to its own weight in order to know the reasonable value of the tension of the wire.

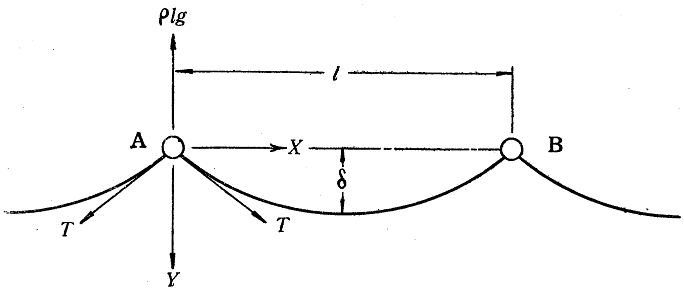


FIG. 4.

Let the arc AB be a small portion of the main wire stretched through the joints A, B, C, ⋯, cf. Fig. 4. Now we can approximate the form of a resting wire by the parabola whose equation in terms of the coördinates X and Y shown in the figure is

$$Y = 4\,\delta l^{-2} X(l - X)\,, \tag{4.2}$$

where δ is the deflection of the middle point. From the condition of balance of the forces acting on A, we obtain

$$\rho l g = 2T(dY/dX)_{X=0} = 8T\,\delta/l\,.$$

Hence

$$T = \rho l^2 g/8\delta\,,$$

$$c = \sqrt{\frac{T}{\rho}} = l\sqrt{\frac{g}{8\delta}}\,,$$

and

$$\beta = \frac{\mu c}{T} = \frac{\mu}{\rho l}\sqrt{\frac{8\delta}{g}}\,. \tag{4.3}$$

The following example will be useful in giving us an idea about a reasonable value of β: $l = 10\,\text{m}$, $\delta = 20\,\text{cm}$, $\rho = 8.93 \times \pi \times 0.5^2 = 7.01\,\text{gr/cm}$ (linear density of copper wire, 1 cm in diameter), $g = 980\,\text{cm/sec}^2$. From these data it is found that

$$c \approx 25\,\text{m/sec},$$

and

$$\beta \approx \mu/17300. \qquad (\mu \text{ in C.G.S.}) \tag{4.4}$$

A number of sources of dissipation may be involved in μ, the factor of damping of an attached particle: the resistance of air, the inner friction of an auxiliary wire, etc. All of these dissipations, linear or non-linear, were combined altogether into an equivalent single factor, somewhat fictitious, proportional to the velocity; see (3.1). It is very difficult, therefore, to specify a definite value of μ in general circumstances. So in order to simplify the following calculations, let us confine ourselves to discussion of the case when the value of μ in C.G.S. is not very large, being at the most of the order of 1000, and accordingly β is at the most several per cent (we shall call β infinitesimal of the first order).

(iii) $\gamma = k\,l/T$: As was mentioned in **2**, the restoring force acting on the elevated joint B′ owing to a spring in the model (Fig. 3) results in reality from the vertical component of the tension T' of the auxiliary wire crossing perpendicularly the main line at that joint. Remembering that k is the constant of restoration and using the notation $\approx$ in the same meaning as in the preceding section,

$$k\,Y_n(\text{the restoring force}) \approx 2T'\,Y_n/l' \begin{pmatrix}\text{the vertical component}\\ \text{of } 2T',\text{ approximately}\end{pmatrix},$$

where l' is a half-length of the auxiliary wire. Suppose furthermore that

$$T' \approx T, \quad l' \approx l,$$

then from the above estimation we have

$$k\,Y_n \approx 2T\,Y_n/l,$$

that is to say

$$k\,l/T \approx 2.$$

2 on the right-hand side, however, has very little meaning; our conclusion should be rather

$$\gamma \approx 1. \tag{4.5}$$

(iv) $\varepsilon = \nu lc/T$: The parameter ε relates through ν to the resistance of the surrounding air against the lateral vibration of the wire. According to (2.1) $\nu\,\partial y/\partial t$ is the force from air per unit length of the wire. In order to estimate roughly the order of its magnitude, let us use the following expedient. Let d and U be the diameter, and the mean, ***spatial and temporal***, speed of vibration, of the wire, respectively. Suppose $d = 1\,\text{cm}$, $U = 3\,\text{cm/sec}$. Then the Reynolds number for this motion is roughly

$$R = \frac{U \cdot d}{\text{kinematic viscosity of air at } 20\,^\circ\text{C}} = \frac{3}{0.15} = 20.$$

Now if we assume that we are able to know the order of magnitude of ν by the resistance of air to which the unit length of a circular cylinder infinitely long is subjected when placed in a steady flow of air perpendicular to its axis, then the drag coefficient defined by

$$C_D = \frac{D \text{ (drag)}}{(1/2)\, \rho_{air}\, U^2\, d}$$

has the value about 2.0 for this Reynolds number,[1] ρ_{air} being the density of air, namely 0.0012 gr/cm³. Therefore

$$\nu \approx \frac{D}{U} = C_D \frac{1}{2} \rho_{air}\, Ud = 2.0 \times \frac{1}{2} \times 0.0012 \times 3 \times 1$$
$$= 0.0036 \quad \text{(in C.G.S.)}.$$

Further, by the previous example l and c/T being 1000 and 1/17300 in C.G.S., respectively,

$$\varepsilon = \nu\, l\, c/T \approx 0.00021. \tag{4.6}$$

Next, when U is ten times as large, i.e. 30 cm/sec, by the same procedure we find $R = 200$, $C_D = 1.3$, $\nu \approx 0.023$ and $\varepsilon \approx 0.0014$. From these examples therefore, we may say, ε is an infinitesimal of the second order according to our nomenclature.

Summarizing the above discussions we can arrive at the following conclusions: α and γ are the numbers of the order of unity and β and ε are the infinitesimals of the first and the second order, respectively. However, two more parameters are still left: λ and τ. λ is the ratio of the speed of propagation of lateral vibration to the speed of a street-car (i.e. the reciprocal of the Mach number used in aerodynamics), and is generally greater than unity. But it would be of interest to examine the case $\lambda \leqq 1$ to establish an analogy with supersonic aerodynamics. τ, on the other hand, which is the non-dimensional time, measured from the instant when the trolley arrives at the particle whose motion we are going to discuss, is subjected essentially to no restrictions and varies from $-\infty$ to $+\infty$.

To return to our integral (3.12), as it has been clarified that ε is an infinitesimal of higher order than α, β or γ, we may safely neglect the terms including ε so long as we are interested in the property of the solution in the neighborhood of $\tau = 0$. In spite of this neglect of ε, however, it is assumed as before that the premises underlying our Fourier transforms are still valid, viz.

$$y = \partial y/\partial t = 0, \quad \text{when} \quad t = \pm\infty.$$

In conclusion under this approximation, (3.12) can be written

[1] Goldstein, S. *Modern Developments in Fluid Dynamics*, Oxford (1938), vol. I, I, 5. FIG. 1, p. 15.

$$Y(\tau)=\frac{2fl^2}{T}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{(\cosh\lambda\zeta-\cosh\zeta)\exp(\tau\zeta)\,d\zeta}{(1-\lambda^2)\zeta^2\{2\cosh\lambda\zeta-2\cosh\zeta-(\alpha\zeta+\beta+\gamma/\zeta)\sinh\zeta\}}; \tag{4.7}$$

this is the integral which we are going to evaluate in the following pages.

5. As is easily verified, $\zeta=0$ is not a singular point of the integrand; the only singularities are an infinite number of the simple poles arising from the zeros of the denominator, defined by

$$2\cosh\lambda\zeta-2\cosh\zeta-(\alpha\zeta+\beta+\gamma/\zeta)\sinh\zeta=0. \tag{5.1}$$

The path of integration, on the other hand, is the imaginary axis of the ζ-plane from $-i\infty$ to $i\infty$. But if we can prove that the same integral vanishes on a semicircle of infinite radius on either side of the imaginary axis with center at the origin, then by computing the integral along a closed curve consisting of the imaginary axis and one of those semicircles, we shall be enabled to reduce the integration to the calculations of the residues corresponding to the simple poles above-mentioned. It becomes necessary, therefore, to know the asymptotic behavior of the integrand for $|\zeta|$ tending to infinity.

Writing for brevity

$$\frac{\cosh\lambda\zeta-\cosh\zeta}{2\cosh\lambda\zeta-2\cosh\zeta-(\alpha\zeta+\beta+\gamma/\zeta)\sinh\zeta}\equiv\frac{N}{D},$$

and $\zeta\equiv R\exp(i\vartheta)$,

we shall examine the properties of N and D when $R\to\infty$ in the range of ϑ such that

$$-\frac{\pi}{2}+\varpi\leq\vartheta\leq\frac{\pi}{2}-\varpi$$

for a sufficiently small value of ϖ chosen in such a way that $R\varpi\to\infty$ as $R\to\infty$. The following relations are easily obtained:

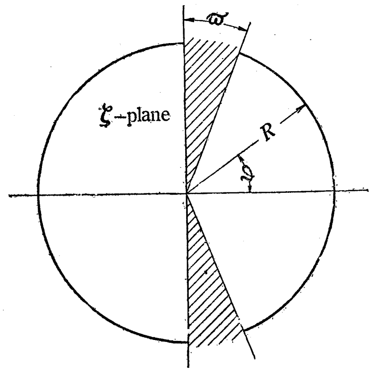


FIG. 5.

$$|\cosh\lambda\zeta|=\{\cos^2(\lambda R\sin\vartheta)+\sinh^2(\lambda R\cos\vartheta)\}^{1/2},$$

$$|\cosh\zeta|=\{\cos^2(R\sin\vartheta)+\sinh^2(R\cos\vartheta)\}^{1/2},$$

and $$|\sinh\zeta|=\{\sin^2(R\sin\vartheta)+\sinh^2(R\cos\vartheta)\}^{1/2}.$$

For various values of λ, three cases have to be discussed separately.

(i) When $\lambda<1$:—Since

$$|N|\approx|\cosh\zeta|,$$

and $$|D|\approx|\alpha\zeta\sinh\zeta|$$

for very large value of $|\zeta|$, it follows that

$$\left|\frac{N}{D}\right| \approx \frac{|\cosh \zeta|}{\alpha|\zeta||\sinh \zeta|} \approx \frac{1}{\alpha|\zeta|} \to 0 .$$

So Q, the integrand of (4.7), tends to zero as $|\zeta|$ becomes very large.

(ii) When $\lambda = 1$:—At the beginning of 7 by taking the limiting form for λ tending to unity, we shall prove directly that $Q \to 0$; see page 125.

(iii) When $\lambda > 1$:—Because

$$|\cosh \lambda \zeta| \gg |\cosh \zeta| ,$$

it follows that

$$|N| \approx |\cosh \lambda \zeta| ,$$

and

$$|D| \approx |2\cosh \lambda \zeta| .$$

Therefore obviously

$$|Q| = \frac{1}{|1-\lambda^2||\zeta^2|}\left|\frac{N}{D}\right| \to 0 .$$

Completely analogous relations hold in the range $\pi/2 + \tilde{\omega} \leqq \vartheta \leqq (3\pi/2) - \tilde{\omega}$.

Finally our scrutiny will be completed by evaluating Q at a point lying on the imaginary axis. For this purpose, put $\zeta = i\eta$, where η is a real number, positive or negative. Then we have

$$N = \cos \lambda \eta - \cos \eta ,$$

and

$$D = 2\cos \lambda \eta - 2\cos \eta + (\alpha \eta - i\beta - \gamma/\eta) \sin \eta .$$

So except the points where $\sin \eta = 0$ (this restriction is not important), $|N/D| \to 0$ as $\eta \to \pm\infty$.

To sum up, we can conclude that the integrand of (4.7) vanishes uniformly on the circle of infinite radius with center at the origin. According to Jordan's lemma,[1] therefore, we can deform the path of integration into a closed curve consisting of the imaginary axis and a large semi-circle: viz. when $\tau > 0$,

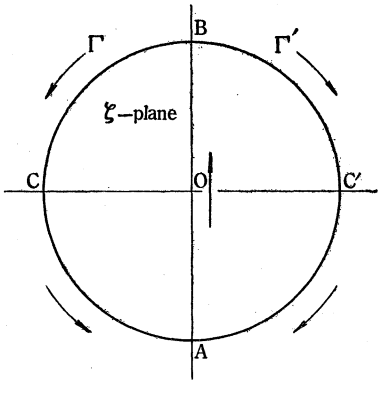


Fig. 6.

$$\left.\begin{aligned} &\int_{-i\infty}^{i\infty} d\zeta = \int_{AB} d\zeta + \int_{BCA} d\zeta = \int_{\Gamma} d\zeta , \\ &\text{and when } \tau < 0, \\ &\int_{-i\infty}^{i\infty} d\zeta = \int_{AB} d\zeta + \int_{BC'A} d\zeta = \int_{\Gamma'} d\zeta . \end{aligned}\right\} \quad (5.2)$$

Γ and Γ' are the contours shown in the figure.

[1] Whittaker, E. T. and Watson, G. N. *A Course of Modern Analysis*, 4th ed., Cambridge (1928). 6·222. p. 115.

6. In the preceding section we were enabled to reduce the evaluation of the integral (4.7) to the calculation of the residues relative to an infinite number of simple poles. But still the computation is very laborious, as will be fully shown in the later sections. To treat a similar problem preliminarily by means of a model as simplified as possible before entering into the detailed considerations will help us, therefore, if the model is properly chosen, in obtaining beforehand a general view of the property of the solution which we are striving for.

We have been dealing so far with the vibrating system consisting of a number of particles attached at equal intervals to the string, each particle being connected elastically with a fixed level. One of the most natural simplifications will be provided by a fictitious model produced by diffusing the effects of the particles and of the springs uniformly over the string. So a few pages will be devoted to the calculation of this simplified motion in order to examine later how much of the real motion can be reproduced by this model. The equation written in terms of the same notations as before is

$$\rho' \frac{\partial^2 y}{\partial t^2} = -\kappa\, y - \nu' \frac{\partial y}{\partial t} + T \frac{\partial^2 y}{\partial x^2} + f \delta \left[\frac{c'}{l} \left(t - \frac{x}{V} \right) \right], \tag{6.1}$$

where $\rho' = \rho + M/l$, $\nu' = \nu + \mu/l$, $\kappa = k/l$ and $c' = (T/\rho')^{1/2}$. The Fourier transform of this equation is readily found to be

$$\frac{d^2 \bar{y}}{dx^2} + \theta'^2 \bar{y} = -\frac{f\, l}{c'\, T} e^{-ipx/V} \tag{6.2}$$

by writing

$$\frac{p^2}{c'^2} - i \frac{\nu' p}{T} - \frac{\kappa}{T} = \theta'^2. \tag{6.3}$$

Since the solution of this equation which remains finite where $x \to \pm\infty$ is

$$\bar{y}(x, p) = \frac{f\, l}{c'\, T} \frac{\exp(-i\, p\, x/V)}{\dfrac{p^2}{V^2} - \dfrac{p^2}{c'^2} + i \dfrac{\nu' p}{T} + \dfrac{\kappa}{T}}, \tag{6.4}$$

$y(x, t)$ is given by the formula

$$y(x, t) = \frac{f\, l}{c'\, T} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(-i\, p\, x/V + i\, p\, t)}{\dfrac{p^2}{V^2} - \dfrac{p^2}{c'^2} + i \dfrac{\nu' p}{T} + \dfrac{\kappa}{T}} dp. \tag{6.5}$$

If we put similarly as in (3.11)

$$\left.\begin{aligned} t - \frac{x}{V} &= \frac{l\, \tau'}{c'}, \qquad i\, p = \frac{c'\, \zeta'}{l}, \\ \frac{k\, l}{T} = \frac{\kappa\, l^2}{T} &= \gamma, \qquad \frac{\nu'\, l\, c'}{T} = \varepsilon' \quad \text{and} \quad \frac{c'}{V} = \lambda', \end{aligned}\right\} \tag{6.6}$$

we can transform (6.5) into the non-dimensional form

$$y(\tau') = \frac{f l^2}{T} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\exp(\tau' \zeta')}{(1-\lambda'^2)\zeta'^2 + \varepsilon' \zeta' + \gamma} d\zeta'. \tag{6.7}$$

Thus the problem will be classified into three cases according to the value of λ'. In each case the denominator tends to ∞ uniformly as $|\zeta'|$ increases indefinitely, so by Jordan's lemma the path of integration can be replaced by the contours mentioned in (5.2). The result of the computation will be summarized in what follows.

(i) When $\lambda' > 1$ (i.e. $c' > V$):— Assuming ε' is so small that ε'^2 may be safely neglected, we can prove without difficulty that

$$y(\tau') = \frac{f l^2}{T} \frac{1}{2\sqrt{\gamma(\lambda'^2-1)}} \exp\left\{\frac{\frac{1}{2}\varepsilon' - \sqrt{\gamma(\lambda'^2-1)}}{\lambda'^2-1}\tau'\right\} \quad \text{for } \tau' > 0,$$

and

$$y(\tau') = \frac{f l^2}{T} \frac{1}{2\sqrt{\gamma(\lambda'^2-1)}} \exp\left\{\frac{\frac{1}{2}\varepsilon' + \sqrt{\gamma(\lambda'^2-1)}}{\lambda'^2-1}\tau'\right\} \quad \text{for } \tau' < 0. \tag{6.8}$$

In Fig. 7 the general aspect of the motion is shown as a function of τ', the non-dimensional time.

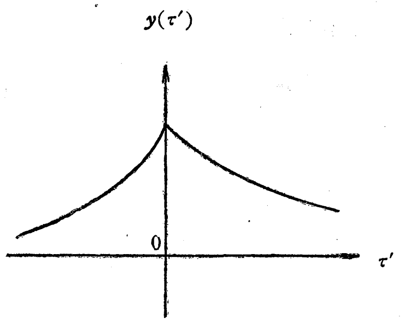


Fig. 7.

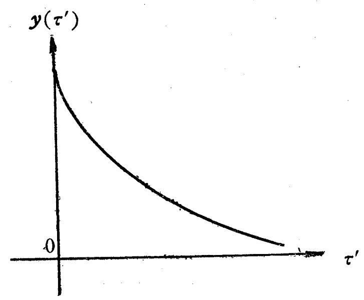


Fig. 8.

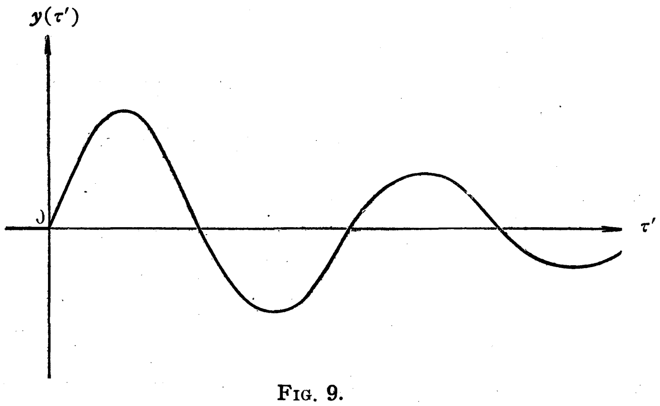


Fig. 9.

(ii) When $\lambda' = 1$ (i.e. $c' = V$):—We have

$$\left.\begin{aligned} y(\tau') &= \frac{f l^2}{T}\frac{1}{\varepsilon'}\exp\left(-\frac{\gamma}{\varepsilon'}\tau'\right) \qquad &&\text{for} \quad \tau' > 0, \\ \text{and} \qquad y(\tau') &= 0 &&\text{for} \quad \tau' < 0. \end{aligned}\right\} \tag{6.9}$$

The wave-pattern is illustrated by Fig. 8. Incidentally it is worth noticing that as ε' decreases the deflection of the string approaches the figure of the delta function itself.

(iii) And finally when $\lambda' < 1$ (i.e. $c' < V$):—For $\tau' > 0$

$$\left.\begin{aligned} y(\tau') &= \frac{f l^2}{T}\frac{1}{\sqrt{\gamma(1-\lambda'^2)}}\exp\left\{-\frac{\varepsilon'}{2(1-\lambda'^2)}\tau'\right\}\sin\left(\sqrt{\frac{\gamma}{1-\lambda'^2}}\tau'\right), \\ \text{and } y(\tau') &= 0 \qquad\qquad \text{for } \tau' < 0; \text{ see Fig. 9.} \end{aligned}\right\} \tag{6.10}$$

Conclusion: first, when $\lambda' \leqq 1$, or when the Mach number $\geqq 1$ if aerodynamically spoken, no disturbance is transferred ahead of the trolley. This is of course in accordance with the common sense of physics. Second, as λ' decreases the type of motion of a point on the string changes from non-oscillatory to oscillatory character.

7. Let us go back to our integral (4.7), viz.

$$Y(\tau) = \frac{2f l^2}{T}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{(\cosh\lambda\zeta - \cosh\zeta)\exp(\tau\zeta)\,d\zeta}{(1-\lambda^2)\zeta^2\{2\cosh\lambda\zeta - 2\cosh\zeta - (\alpha\zeta + \beta + \gamma/\zeta)\sinh\zeta\}}. \tag{7.1}$$

In evaluating this integral it is convenient to begin with the simplest case in which $\lambda = 1$, i.e. $c = V$.

Making $\lambda \to 1$, we can prove without difficulty that the integrand has the value

$$\exp(\tau\zeta)\{2(\alpha\zeta^2 + \beta\zeta + \gamma)\}^{-1}$$

as the limit. Evidently the denominator tends to infinity as $|\zeta|$ increases indefinitely, so owing to Jordan's lamma we are enabled to replace the path of integration by the contours Γ and Γ' for $\tau > 0$ and $\tau < 0$, respectively, cf. (5.2). The poles of the integrand are situated at the points

$$\zeta = \frac{-\beta \pm \sqrt{\beta^2 - 4\gamma\alpha}}{2\alpha},$$

and neglecting β^2 which we assumed very small, we can approximate them by

$$\zeta = -\frac{\beta}{2\alpha} \pm i\sqrt{\frac{\gamma}{\alpha}}. \tag{7.2}$$

So by the well-known procedure of contour integration we can obtain the following:

$$\left.\begin{aligned} Y(\tau) &= \frac{2fl^2}{T}\frac{1}{2\sqrt{\gamma\alpha}}\exp\left(-\frac{\beta}{2\alpha}\tau\right)\sin\left(\sqrt{\frac{\gamma}{\alpha}}\tau\right) && \text{for} \quad \tau > 0 \\ \text{and} \qquad Y(\tau) &= 0 && \text{for} \quad \tau < 0. \end{aligned}\right\} \tag{7.3}$$

Obviously $Y(0) = 0$ consistently from these two different expressions.
If we assume as a numerical example

$$\alpha = \gamma = 2, \quad \beta = 0.01, \tag{7.4}$$

then from (7.3) we have

$$\left.\begin{aligned} KY(\tau) &= \frac{1}{4}\exp\left(-\frac{\tau}{400}\right)\sin\tau && \text{for} \quad \tau > 0, \\ \text{and} \qquad KY(\tau) &= 0 && \text{for} \quad \tau < 0, \end{aligned}\right\} \tag{7.5}$$

where we write for convenience

$$K \equiv \frac{T}{2fl_n}. \tag{7.6}$$

The curve of (7.5) was worked out in the range of τ from 0 to 10 at intervals of 0.1 and is reproduced in Fig. 10 at the end of **10.** Note that contrary to our expectation (7.3) has some analogy with (6.10) instead of (6.9).

8. Our next problem is to evaluate (7.1) in the case when $\lambda \neq 1$ (i.e. $c \neq V$). Since the origin $\zeta = 0$ is not a singular point of the integrand, the only singularities are situated at those points which satisfy the equation

$$2\cosh\lambda\zeta - 2\cosh\zeta - (\alpha\zeta + \beta + \gamma/\zeta)\sinh\zeta = 0. \tag{8.1}$$

This equation may or may not have real roots and has certainly a number of conjugate complex roots of the type

$$\zeta = \xi + i\eta, \quad \tilde{\zeta} = \xi - i\eta. \tag{8.2}$$

To compute the real roots, if any, is not so difficult anyhow. In order to calculate the complex roots, substituting for ζ in (8.1) $\xi + i\eta$ and putting the real and imaginary parts equal to zero separately, we obtain

$$\text{Real Part: } 2\cosh\lambda\xi\cos\lambda\eta - 2\cosh\xi\cos\eta - \left(\alpha\xi + \beta + \frac{\gamma\xi}{\xi^2+\eta^2}\right)\sinh\xi\cos\eta + \left(\alpha\eta - \frac{\gamma\eta}{\xi^2+\eta^2}\right)\cosh\xi\sin\eta = 0, \tag{8.3}$$

$$\text{Imaginary Part: } 2\sinh\lambda\xi\sin\lambda\eta - 2\sinh\xi\sin\eta - \left(\alpha\eta - \frac{\gamma\eta}{\xi^2+\eta^2}\right)\sinh\xi\cos\eta - \left(\alpha\xi + \beta + \frac{\gamma\xi}{\xi^2+\eta^2}\right)\cosh\xi\sin\eta = 0. \tag{8.4}$$

Strictly speaking, we have to solve the simultaneous equations (8.3) and (8.4) for a set of given values of α, β, γ and λ. However, this is too laborious as a matter of fact.

If we put $\beta=0$ in these equations, both of them will be reduced to the forms symmetrical with respect to ξ and η, and we have only to compute in detail their common roots lying in the first quadrant of the ζ-plane. Generally the common roots corresponding to $(\alpha, \beta, \gamma, \lambda)$ will not be different so much from those for $(\alpha, 0, \gamma, \lambda)$, since β is assumed quite small. It is far simpler to solve the equations (8.3) and (8.4) putting first $\beta=0$ and then to find out the necessary corrections than to treat the original equations containing β. After simplifying the problem so much, we must still work numerically. So we shall take up the following example:

$$\alpha=\gamma=2, \quad \beta=0,$$

cf. (7.4). To visualize the effect of speed of the trolley upon the mode of vibration we shall assume two kinds of values for λ, viz.

$$\lambda=2\ (c=2V) \quad \text{and} \quad \lambda=1/2\ (2c=V).$$

When $\lambda=2$, from (8.3) and (8.4) we have

$$\text{Real Part:} \quad \cosh 2\xi \cos 2\eta-\cosh \xi \cos \eta-\left(\xi+\frac{\xi}{\xi^2+\eta^2}\right) \sinh \xi \cos \eta + \left(\eta-\frac{\eta}{\xi^2+\eta^2}\right) \cosh \xi \sin \eta=0, \tag{8.5}$$

$$\text{Imaginary Part:} \quad \sinh 2\xi \sin 2\eta-\sinh \xi \sin \eta-\left(\eta-\frac{\eta}{\xi^2+\eta^2}\right) \sinh \xi \cos \eta - \left(\xi+\frac{\xi}{\xi^2+\eta^2}\right) \cosh \xi \sin \eta=0; \tag{8.6}$$

when $\lambda=1/2$, on the other hand,

$$\text{Real Part:} \quad \cosh \frac{\xi}{2} \cos \frac{\eta}{2}-\cosh \xi \cos \eta-\left(\xi+\frac{\xi}{\xi^2+\eta^2}\right) \sinh \xi \cos \eta + \left(\eta-\frac{\eta}{\xi^2+\eta^2}\right) \cosh \xi \sin \eta=0, \tag{8.7}$$

$$\text{Imaginary Part:} \quad \sinh \frac{\xi}{2} \sin \frac{\eta}{2}-\sinh \xi \sin \eta-\left(\eta-\frac{\eta}{\xi^2+\eta^2}\right) \sinh \xi \cos \eta - \left(\xi+\frac{\xi}{\xi^2+\eta^2}\right) \cosh \xi \sin \eta=0. \tag{8.8}$$

Evidently (8.6) and (8.8) are satisfied by

$$\xi=0 \quad \text{or} \quad \eta=0. \tag{8.9}$$

So much for the complex roots.

When $\lambda = 2$, (8.5) and (8.6) are satisfied by a pair of the real roots, positive and negative, $(\xi, 0)$, ξ being determined as the root of the equation

$$\cosh 2\xi - \cosh \xi - (\xi + \xi^{-1}) \sinh \xi = 0. \qquad (8.10)$$

When $\lambda = 1/2$, we can readily prove on the contrary that (8.7) and (8.8) cannot have any real roots.

Simpler is the case in which $\lambda = 1/2$. Then the values of η satisfying the equations (8.7) and (8.8) for various values of ξ are tabulated in Tables I and II and are shown by Fig. 11. We can observe that the roots of (8.7) and (8.8) tend to $(n+1/2)\pi$ and $(n+1)\pi$ $(n = 0, 1, 2, \cdots)$, respectively as ξ increases indefinitely. And as mentioned above, by inversion with regard to the ξ-, and the η-axes, we can find at once those

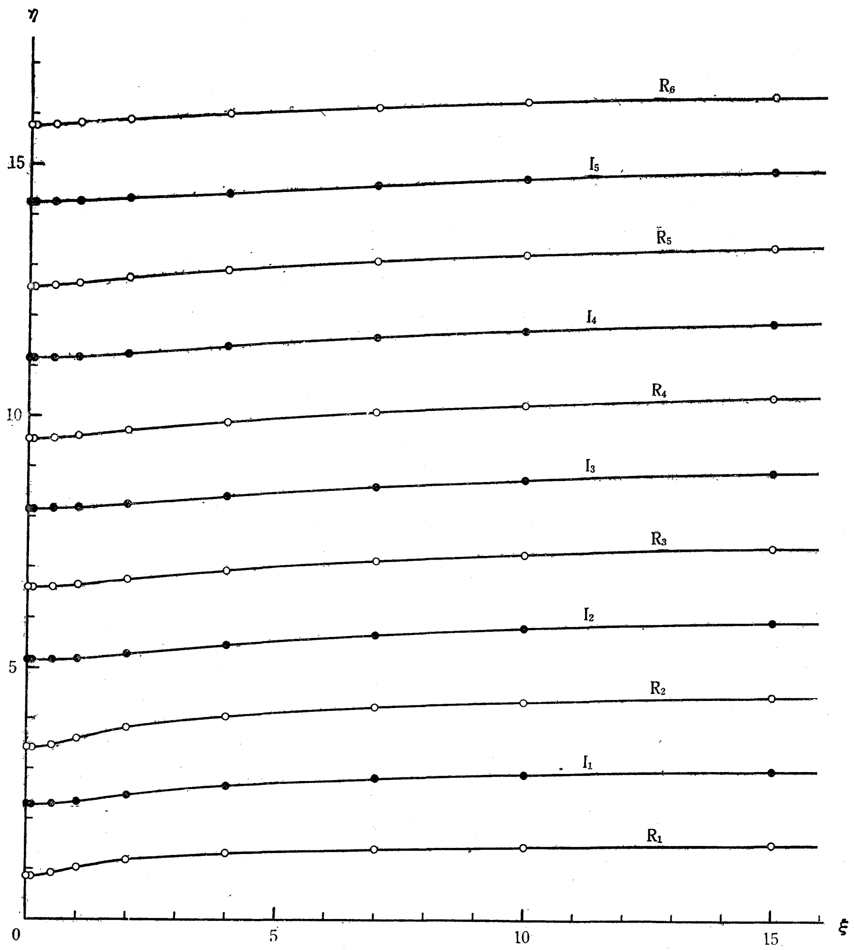


FIG. 11. $\lambda = 1/2$.

roots which are situated in another quadrant. The common roots of the simultaneous equations (8.7) and (8.8) are thus found to be

$$\pm i \times (0.84727,\ 3.41095,\ 6.58957,\ 9.53629,\ 12.56637,\ 15.76959,\ \cdots).$$

TABLE I. Roots of (8.7).

$\xi \backslash \eta$	R_1	R_2	R_3	R_4	R_5	R_6
0	0.84727	3.41095	6.58957	9.53629	12.56637	15.76959
0.1	0.84946	3.41363	6.59046	9.53737	12.56748	15.77021
0.5	0.91560	3.47128	6.60989	9.56102	12.59174	15.78397
1.0	1.03222	3.59648	6.65643	9.61592	12.64844	15.81700
2.0	1.18526	3.81111	6.75929	9.72798	12.76270	15.88905
4.0	1.31692	4.04916	6.93626	9.90151	12.92870	16.01113
7.0	1.40081	4.23527	7.13738	10.10143	13.11596	16.16942
10.0	1.44235	4.34159	7.27579	10.25027	13.26186	16.30382
15.0	1.47936	4.44349	7.42228	10 42093	13.44081	16.48051
∞	1.57080	4.71239	7.85393	10.99557	14.13717	17.27876

TABLE II. Roots of (8.8).

$\xi \backslash \eta$	I_1	I_2	I_3	I_4	I_5
0	2.28268	5.15242	8.14844	11.14622	14.25155
0.1	2.28341	5.15291	8.14876	11.14658	14.25182
0.5	2.29997	5.16348	8.15652	11.15440	14.25797
1.0	2.34556	5.19422	8.17868	11.17712	14.27576
2.0	2.46852	5.28798	8.24944	11.24855	14.33298
4.0	2.66199	5.47897	8.41165	11.40755	14.46552
7.0	2.81049	5.67775	8.61252	11.60299	14.63872
10.0	2.88721	5.80369	8.75808	11.75176	14.77954
15.0	2.96010	5.93064	8.91897	11.92827	14.95834
∞	3.14159	6.28319	9.42478	12.56637	15.70796

The situation is not so favorable in the case when $\lambda = 2$. Now the roots of (8.5) and (8.6) are shown for various values of ξ in Tables III, IV and Fig. 12. It is easily proved as before that each branch of R and I tends to $(2n+1)\pi/4$ and $(n+1)\pi/2$, respectively as $\xi \to \infty$. The common roots of these equations are therefore

(i) $\pm i \times (1.59778,\ 3.60636,\ 6.28319,\ 9.62569,\ 12.56637,\ 15.83266,\ \cdots)$, yielded as the intersections of (8.5) and the η-axis,

(ii) ± 1.04128, from (8.5) and the ξ-axis,[1]

(iii) $\pm 2.08378 \pm i\ 7.46426,\ \cdots$, from (8.5) and (8.6).[1]

[1] It is worth noticing that in spite of the fact that the system, since β is assumed zero, has no damping element, these components of the vibration exhibit some decreasing of the amplitude with time, the dissipation of energy.

TABLE III. Roots of (8.5).
(An asterisk indicates the absence of the

ξ \ η	R_1	R_2	R_3	R_4	R_5	R_6	R_7
0	*	1.59778	3.60636	6.28319	*	*	9·62569
0.1	*	1.60266	3.60969	6.28239	*	*	9.62794
0.5	*	1.70193	3.67539	6.26181	*	*	9.67729
1.0	*	1.88067	3.78605	6.18082	*	*	9.79063
1.04128	0	—	—	—	*	*	—
1.05	0.07729	—	—	—	*	*	—
1.1	0.19696	—	—	—	*	*	—
1.25	0.35216	—	—	—	*	*	—
1.5	0.48203	—	—	—	*	*	—
2.0	0.61023	2.12391	3.90237	5.87738	*	*	10.00196
2.05	—	—	—	—	7.51946	7.82960	—
2.1	—	—	—	—	7.44365	7.91819	—
2.5	—	—	—	—	7.22197	8.23004	—
3.0	—	—	—	—	7.13396	8.40148	—
4.0	0.75487	2.30925	3.93504	5.56590	7.08206	8.54986	10.18029
7.0	0.78301	2.35284	3.92834	5.50215	7.06823	8.63400	10.20950
∞	0.78540	2.35619	3.92699	5.49779	7.06858	8.63938	10.21018

TABLE IV. Roots of (8.6).

ξ \ η	I_1	I_2	I_3	I_4	I_5	I_6	I_7
0	0.64304	2.75284	4.94452	*	*	8.27469	11.12864
0.1	0.64952	2.75398	4.94436	*	*	8.27644	11.12864
0.5	0.77777	2.78040	4.94109	*	*	8.31995	11.12954
1.0	1.00987	2.84733	4.92869	*	*	8.46800	11.13016
1.5	—	—	—	*	*	8.70169	—
1.95	—	—	—	7.02896	7.31395	—	—
2.0	1.32142	2.99043	4.87200	6.94319	7.38781	8.92970	11.11403
4.0	1.52236	3.11753	4.75727	6.34305	7.80418	9.34370	11.03743
7.0	1.56709	3.14019	4.71606	6.28603	7.85029	9.42052	10.99922
∞	1.57080	3.14159	4.71239	6.28319	7.85398	9.42478	10.99557

The procedure to work out the last root is far from being simple. Being difficult to calculate it accurately by graphical method, the following alternative was adopted:—Our problem is to find out the essentially complex roots of the equation, retaining β and λ in a general form for the future convenience,

$$\cosh \lambda \zeta - \cosh \zeta - (\zeta + \beta' + \zeta^{-1}) \sinh \zeta = 0, \tag{8.13}$$

where

$$\beta' = \beta/2. \tag{8.14}$$

root.)

R_8	R_9	R_{10}	R_{11}
12.56637	*	*	15.83266
12.56598	*	*	15.83415
12.55550	*	*	15.86780
12.51324	*	*	15.95344
—	*	*	—
—	*	*	—
—	*	*	—
—	*	*	—
—	*	*	—
12.28910	?	?	16.16352
—	—	—	—
—	—	—	—
—	—	—	—
—	—	—	—
11.88563	—	—	—
11.78738	—	—	—
11.78097	13.35177	14.92257	16.49336

I_8	I_9	I_{10}
*	*	14.32883
*	*	14.32947
*	*	14.34571
*	*	14.40690
*	*	—
?	?	—
?	?	14.85930
12.68776	14.08441	15.57184
12.57210	14.13347	15.70085
12.56637	14.13717	15.70796

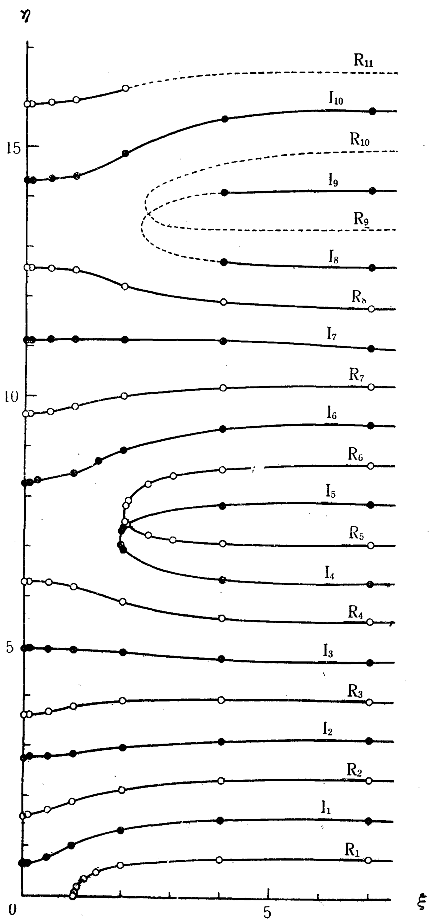


FIG. 12. $\lambda = 2$.

The dotted lines indicate those parts of the curves which were interpolated on the figure.

Let an approximate value of the root, found out graphically or otherwise, be $\zeta_0 = \xi_0 + i\eta_0$; we can further the approximation by computing a correcting quantity for ζ_0. Namely making use of Newton's method for a complex variable, a more accurate value of the root is found to be

$$\zeta_0 + \zeta_1 = (\xi_0 + \xi_1) + i(\eta_0 + \eta_1), \tag{8.15}$$

where ζ_1, which is assumed small, is given by the relation

$$\zeta_1 = \frac{\cosh \lambda \zeta_0 - \cosh \zeta_0 - (\zeta_0 + \beta' + \zeta_0^{-1}) \sinh \zeta_0}{-\lambda \sinh \lambda \zeta_0 + (2 - \zeta_0^{-2}) \sinh \zeta_0 + (\zeta_0 + \beta' + \zeta_0^{-1}) \cosh \zeta_0}. \tag{8.16}$$

We can now return to our particular case by putting

$$\beta' = 0 \quad \text{and} \quad \lambda = 2;$$

then we have

$$\zeta_1 = \frac{\cosh 2\zeta_0 - \cosh \zeta_0 - (\zeta_0 + \zeta_0^{-1}) \sinh \zeta_0}{-2\sinh 2\zeta_0 + (2 - \zeta_0^{-2}) \sinh \zeta_0 + (\zeta_0 + \zeta_0^{-1}) \cosh \zeta_0}, \tag{8.17}$$

or more explicitly,

$$\begin{aligned}
\xi_1 + i\,\eta_1 = \Bigg[& \Bigg\{ \cosh 2\xi_0 \cos 2\eta_0 - \cosh \xi_0 \cos \eta_0 - \left(\xi_0 + \frac{\xi_0}{\xi_0^2 + \eta_0^2}\right) \sinh \xi_0 \cos \eta_0 \\
& + \left(\eta_0 - \frac{\eta_0}{\xi_0^2 + \eta_0^2}\right) \cosh \xi_0 \sin \eta_0 \Bigg\} + i \Bigg\{ \sinh 2\xi_0 \sin 2\eta_0 - \sinh \xi_0 \sin \eta_0 \\
& - \left(\eta_0 - \frac{\eta_0}{\xi_0^2 + \eta_0^2}\right) \sinh \xi_0 \cos \eta_0 - \left(\xi_0 + \frac{\xi_0}{\xi_0^2 + \eta_0^2}\right) \cosh \xi_0 \sin \eta_0 \Bigg\} \Bigg] \\
& \div \Bigg[\Bigg\{ -2 \sinh 2\xi_0 \cos 2\eta_0 + \left(2 - \frac{\xi_0^2 - \eta_0^2}{(\xi_0^2 + \eta_0^2)^2}\right) \sinh \xi_0 \cos \eta_0 \\
& - \frac{2\xi_0 \eta_0}{(\xi_0^2 + \eta_0^2)^2} \cosh \xi_0 \sin \eta_0 + \left(\xi_0 + \frac{\xi_0}{\xi_0^2 + \eta_0^2}\right) \cosh \xi_0 \cos \eta_0 \\
& - \left(\eta - \frac{\eta_0}{\xi_0^2 + \eta_0^2}\right) \sinh \xi_0 \sin \eta_0 \Bigg\} \\
& + i \Bigg\{ -2 \cosh 2\xi_0 \sin 2\eta_0 + \frac{2\xi_0 \eta_0}{(\xi_0^2 + \eta_0^2)^2} \sinh \xi_0 \cos \eta_0 \\
& + \left(2 - \frac{\xi_0^2 - \eta_0^2}{(\xi_0^2 + \eta_0^2)^2}\right) \cosh \xi_0 \sin \eta_0 + \left(\eta_0 - \frac{\eta_0}{\xi_0^2 + \eta_0^2}\right) \cosh \xi_0 \cos \eta_0 \\
& + \left(\xi_0 + \frac{\xi_0}{\xi_0^2 + \eta_0^2}\right) \sinh \xi_0 \sin \eta_0 \Bigg\} \Bigg].
\end{aligned} \tag{8.18}$$

This procedure can be repeated until at last we have $\xi_1 = \eta_1 = 0$ practically.

To sum up, we can solve the simultaneous equations with sufficient accuracy for the particular case in which

$$\alpha = \gamma = 2, \quad \beta = 0 \quad \text{and} \quad \lambda = 2 \quad \text{or} \quad 1/2.$$

Our next problem is this: — How much will these roots vary, if we should give a small but positive value to β, retaining α, γ and λ unchanged?

9. Before answering this question, however, we should have some preliminary knowledge about the behavior of the roots. As we have shown in **7** and **8**, the aspect of the figures of the loci of the roots of the equations

(8.3) and (8.4) with respect to various values of ξ are completely different from each other among the three cases calculated: $\lambda = 1$, $1/2$ and 2. Namely if we regard a common root of (8.3) and (8.4), the intersection of two loci R and I from them in other words, as the function of our parameters α, β, γ and λ, that function is thus proved discontinuous at the point $\lambda = 1$. Then, is it continuous with regard to β in the neighborhood of $\beta = 0$? If not, it is nonsense to try to estimate the approximate value of a root for a small and positive value of β from the case $\beta = 0$. It seems, however, very difficult to give a mathematical proof to this problem. So as a partial suggestion, we shall take up the branch of the locus of the root from the case $\lambda = 2$, denoted by R_1 in Table III and also in Fig. 12 to solve the original equations (8.3) for three values of β namely 0.01, 0.1 and 1. Comparing them with each other and with the case $\beta = 0$, we shall be able to obtain some visual conception about the aspect of the variation of this root for small values of β. If we should have plenty of time to repeat this calculation for all other branches of the roots, we can certainly arrive at far more definite conclusion in this problem.

Inferring from the result of the computation reproduced in Table V and Fig. 13, it is highly plausible that the roots will vary continuously near the point $\beta = 0$, and that the effect of β upon the roots will be so small, provided β is small, that we may safely take the value corresponding to $\beta = 0$ as the first approximation when we are going to work out the root for *a small but positive value of* β (let us denote it by $\beta \gtrsim 0$ for brevity). Although not impossible to describe all cases in a unified manner, it is more convenient to classify the common roots of (8.3) and (8.4) into two groups: those which are situated near the imaginary axis and those which are not.

TABLE V.

ξ \ β	0	0.01	0.1	1.0
1.04128	0	*	*	*
1.04312	—	0	*	*
1.05	0.07729	0.06862	*	*
1.05959	—	—	0	*
1.1	0.19696	0.19389	0.16369	*
1.21726	—	—	—	0
1.25	0.35216	0.35068	0.33720	0.14357
1.5	0.48203	0.48121	0.47383	0.39167
2.0	0.61023	0.60984	0.60651	0.57247
4.0	0.75487	0.75484	0.75454	0.75134
7.0	0.78301	0.78297	0.78297	0.78280
∞	0.78540	0.78540	0.78540	0.78540

(i) The roots near the imaginary axis: — In both of the cases when $\lambda = 1/2$ and 2, we have found a number of the common roots (we shall refer simply as the "roots" in what follows) on the imaginary axis. These roots will still remain near the axis when $\beta \gtrsim 0$. Now writing

$$\zeta = \xi + i\eta\,,$$

we shall neglect the terms containing ξ^2, $\beta\xi$ and higher orders. In the degree of our approximation we have therefore

$$\cosh\xi \fallingdotseq 1 \quad \text{and} \quad \sinh\xi \fallingdotseq \xi\,. \tag{9.1}$$

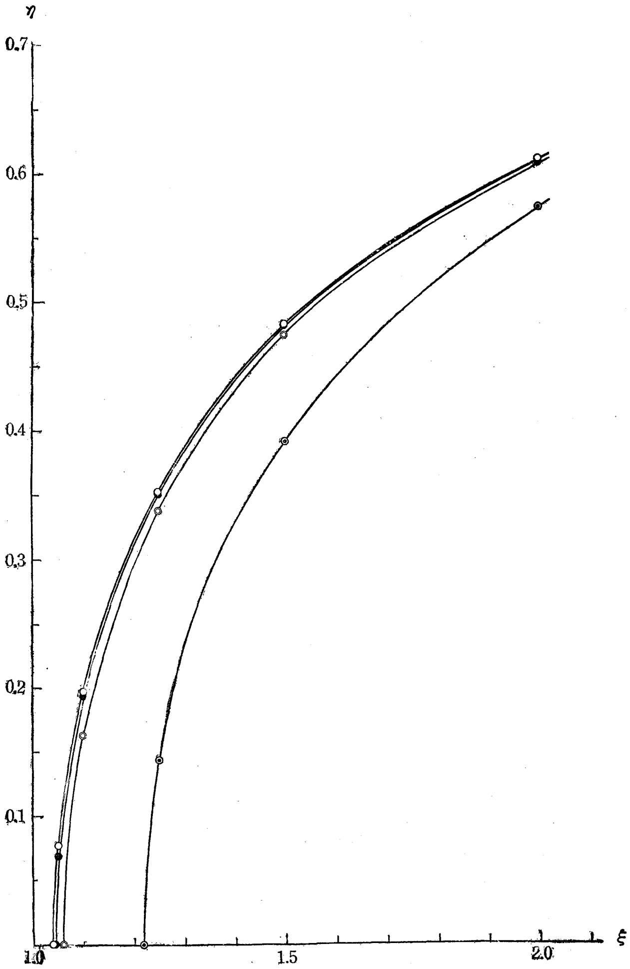


FIG. 13.

○ $\beta=0$ ◎ $\beta=0.1$

● $\beta=0.01$ ⊙ $\beta=1.0$

Supposing λ is of the order of unity, we shall assume the expansion

$$\cosh \lambda \xi \fallingdotseq 1 \quad \text{and} \quad \sinh \lambda \xi \fallingdotseq \lambda \xi . \tag{9.2}$$

Then (8.3) and (8.4) reduce to

$$2 \cos \lambda \eta - 2 \cos \eta + (\alpha \eta - \gamma/\eta) \sin \eta = 0 , \tag{9.3}$$

and

$$2\lambda \xi \sin \lambda \eta - 2\xi \sin \eta - (\alpha \eta - \gamma/\eta) \xi \cos \eta - (\alpha \xi + \beta + \gamma \xi/\eta^2) \sin \eta = 0 , \tag{9.4}$$

respectively. From (9.3) we can readily find that the imaginary parts of the roots for $\beta \gtrsim 0$ are, in our approximation, unchanged from their values corresponding to $\beta = 0$. Substituting these values for η in (9.4), we can solve this equation and we have

$$\xi = -\frac{\beta}{2+\alpha+\gamma/\eta^2+(\alpha\eta-\gamma/\eta)\cot\eta-2\lambda\sin\lambda\eta\operatorname{cosec}\eta}. \quad (9.5)$$

Namely when $\alpha = \gamma = 2$ and $\lambda = 1/2$,

$$\xi = -\frac{\beta'}{2+\eta^{-2}+(\eta-\eta^{-1})\cot\eta-(1/4)\sec(\eta/2)}, \quad (9.6)$$

and on the other hand when $\alpha = \gamma = 2$ and $\lambda = 2$,

$$\xi = -\frac{\beta'}{2+\eta^{-2}+(\eta-\eta^{-1})\cot\eta-4\cos\eta}, \quad (9.7)$$

where $\beta' = \beta/2$, cf. (8.14). It is worth noticing that (9.5), accordingly (9.6) and (9.7), are the even functions of η.

Incidentally, the physical common sense that when $\lambda < 1$ (i.e. $c < V$), no disturbance can travel ahead of the trolley, the source of disturbance in other words, is equivalent in mathematical language to the fact that when $\lambda < 1$ no singular points can exist on the right of the imaginary axis, that is to say all possible values of ξ given by (9.5) in a case when $\lambda < 1$ are negative. That they are actually so can be proved very easily. In order to do this, from (9.3) we have

$$\alpha\eta - \frac{\gamma}{\eta} = \frac{2\cos\eta - 2\cos\lambda\eta}{\sin\eta}.$$

Substituting this in the denominator of (9.5), it becomes

$$D \equiv \alpha + \frac{\gamma}{\eta^2} + \frac{2-2\cos\lambda\eta\cos\eta-2\lambda\sin\lambda\eta\sin\eta}{\sin^2\eta}.$$

But since $0 < \lambda < 1$,

$$-2|\sin\lambda\eta\sin\eta| < 2\lambda\sin\lambda\eta\sin\eta < 2|\sin\lambda\eta\sin\eta|.$$

D lies therefore between

$$\alpha + \frac{\gamma}{\eta^2} + \frac{2-2\cos(1\mp\lambda)\eta}{\sin^2\eta},$$

so that $D > 0$, which means $\xi < 0$. *Q.E.D.*[1)]

To return, when $\alpha = \gamma = 2$, $\beta = 0.01$ and $\lambda = 1/2$, from (9.6) we obtain

1) Notice that when $\lambda = 1/2$ all the roots are found near the imaginary axis. This will be a general conclusion when $0 < \lambda < 1$.

$$\begin{aligned}\xi = &-0.001770 \text{ (corresponding to } \eta = 0.84727), \\ &-0.000328 \ (\eta = 3.41095), \quad -0{,}000221 \ (\eta = 6.58957), \\ &-0.000061 \ (\eta = 9.53629), \quad 0 \qquad (\eta = 12.56637),^* \\ &-0.000019 \ (\eta = 15.76959), \ \cdots .\end{aligned}$$

The fifth one with an asterisk is clearly extraordinary, but if we remember that $12.56637\cdots = 4\pi$, we find that this is not the pole of the integrand of (7.1) and must be removed from the list, because this makes the numerator vanish at the same time. Thus the poles of the integrand are shown for $\lambda = 1/2$ in Table VI. As there are no roots of (8.1) which are not near the imaginary axis, this series will exhaust the poles when $\lambda = 1/2$.

When, on the other hand, $\alpha = \gamma = 2$, $\beta = 0.01$ and $\lambda = 2$, removing by the same reason $6.28319 = 2\pi$ and $12.56637 = 4\pi$ from the list the following poles have been found; see the first columm of Table VII. However, since there are still other roots which do not belong to this kind when $\lambda = 2$, they will be discussed separately in what follows.

(ii) The real roots: — We can easily estimate the real roots for $\beta \gtrsim 0$ from their values corresponding to $\beta = 0$. Or it does not take so much labor to solve directly the equation

$$2\cosh \lambda\xi - 2\cosh \xi - (\alpha\xi + \beta + \gamma/\xi)\sinh \xi = 0, \tag{9.8}$$

derived from (8.3) by putting $\eta = 0$. By the latter procedure we obtain

$$\xi = 1.04312 \quad \text{and} \quad -1.03944$$

when $\alpha = \gamma = \lambda = 2$ and $\beta = 0.01$. Since they were situated originally at $\xi = \pm 1.04128$ when $\beta = 0$, we note the symmetry has been slightly broken by introduction of small but finite magnitude of friction.

(iii) Other roots: — When $\lambda = 2$, in addition to two kinds of the roots above-mentioned, there still remain others which were calculated by successive approximation in (8.17) or (8.18), namely those roots which are

TABLE VI.

$\alpha = \gamma = 2$, $\beta = 0.01$, $\lambda = 1/2$.

$-0.001770 \pm i\ 0.84727$
$-0.000328 \pm i\ 3.41095$
$-0.000221 \pm i\ 6.58957$
$-0.000061 \pm i\ 9.53629$
$-0.000019 \pm i\ 15.76959$
$\cdots$

TABLE VII.

$\alpha = \gamma = 2$, $\beta = 0.01$, $\lambda = 2$.

$-0.002022 \pm i\ 1.59778$	1.04312	$2.08394 \pm i\,7.46364$
$-0.000407 \pm i\ 3.60636$	-1.03944	$-2.08361 \pm i\,7.46489$
$-0.000095 \pm i\ 9.62569$		$\cdots$
$-0.000038 \pm i\ 15.83266$		
$\cdots$		

represented by $\pm 2.08378 \pm i\,7.46426$. How much change will they undergo when we give a positive value to β? In order to know it, we shall return to (8.16). Supposing now $\zeta_0 = \xi_0 + i\,\eta_0$ is the pole when $\beta = 0$, we write ζ_2 as a necessary correction of ζ_0 owing to $\beta \gtrsim 0$. Then, since ζ_0 satisfies the equation

$$\cosh 2\,\zeta_0 - \cosh \zeta_0 - (\zeta_0 + \zeta_0^{-1}) \sinh \zeta_0 = 0$$

(put $\beta' = 0$ in (8.13)), from (8.16) ζ_2 is given by the equation

$$\zeta_2 = \frac{-\beta' \sinh \zeta_0}{-2\sinh 2\zeta_0 + (2 - \zeta_0^{-2}) \sinh \zeta_0 + (\zeta_0 + \beta' + \zeta_0^{-1}) \cosh \zeta_0}.$$

However, β' being assumed small, we have approximately

$$\zeta = \frac{-\beta' \sinh \zeta_0}{-2\sinh 2\zeta_0 + (2 - \zeta_0^{-2}) \sinh \zeta_0 + (\zeta_0 + \zeta_0^{-1}) \cosh \zeta_0}. \quad (9.7)$$

After some tedious computations we find that

$$2.08378 \pm i\,7.46426 \text{ move to } 2.08394 \pm i\,7.46364,$$

and

$$-2.08378 \pm i\,7.46426 \text{ move to } -2.08361 \pm i\,7.46489,$$

respectively. Note that the new roots are symmetrical with regard to the real axis, but slightly unsymmetrical with regard to the imaginary axis. Table VII was constructed in this way. It is observed that there are only two real roots, while both of the series of the complex roots are infinite.

10. We are now in a position to evaluate the integral (7.1). By putting

$$\alpha = \gamma = 2, \quad \beta = 2\beta' = 0.01, \quad \lambda = 2 \text{ or } 1/2, \quad \frac{T}{2 f l^2} = K, \quad (10.1)$$

we can rewrite (7.1) in the form

$$KY(\tau) = \frac{1}{2\pi i}\frac{1}{2(1-\lambda^2)}\int_{-i\infty}^{i\infty} \frac{(\cosh \lambda\,\zeta - \cosh \zeta)\exp(\tau\,\zeta)\,d\zeta}{\zeta^2\{\cosh \lambda\,\zeta - \cosh \zeta - (\zeta + \beta' + \zeta^{-1}) \sinh \zeta\}}. \quad (10.2)$$

And as we showed in **5**, owing to Jordan's lemma we can evaluate the integral in the following manner: for $\tau > 0$, $KY(\tau)$ = the sum of the residues arising from the poles existing on the left of the imaginary axis, and for $\tau < 0$, $-KY(\tau)$ = the sum of the residues arising from the poles existing on the right of the imaginary axis. To write in a unified manner, let us introduce another notation $\pm\sum$ to denote the summation of two kinds above-mentioned. Since $\zeta = 0$ is an ordinary point and the poles are all simple, we have by differentiating the terms within brace of (10.2)

$$KY(\tau) = \pm\frac{1}{2(\lambda^2 - 1)} \sum [(\cosh \lambda\,\zeta - \cosh \zeta) \exp(\tau\,\zeta)\,\zeta^{-2}$$
$$\{-\lambda \sinh \lambda\,\zeta + (2 - \zeta^{-2}) \sinh \zeta + (\zeta + \beta' + \zeta^{-1}) \cosh \zeta\}^{-1}]_{\zeta=\text{pole}}. \quad (10.3)$$

Taking into consideration the fact that in (10.3) the pole represented by $\xi + i\eta$ is always accompanied by its conjugate $\xi - i\eta$, we can rewrite the above formula in the form

$$KY(\tau) = \pm\frac{1}{\lambda^2-1}\Sigma'\ \Re\,[(\cosh\lambda\zeta - \cosh\zeta)\exp(\tau\zeta)\,\zeta^{-2}$$
$$\{-\lambda\sinh\lambda\zeta + (2-\zeta^{-2})\sinh\zeta + (\zeta+\beta'+\zeta^{-1})\cosh\zeta\}^{-1}]_{\zeta=\text{pole}}, \quad (10.4)$$

where Σ' means the summation with regard to the poles lying above the real axis and $\Re[\ \ ]$ is the real part of the function within bracket. The expression (10.4) ceases to be valid for the real roots, then we have to go back to (10.3).

(i) When $\lambda = 1/2$: — Using the approximate expansions (9.1) and (9.2), we can derive from (10.4) the following result:

$$KY(\tau) = -\frac{4}{3}\Sigma' e^{\xi\tau}\Big[\cos\eta\tau\,[\xi\Big\{(\cos\frac{\eta}{2} - \cos\eta)\{\frac{\eta^2}{4}\cos\frac{\eta}{2} + \eta\sin\frac{\eta}{2}$$
$$- 5\eta^2\cos\eta + (\eta^3 - 5\eta)\sin\eta\} + (\frac{1}{2}\sin\frac{\eta}{2} - \sin\eta)\{\frac{\eta^2}{2}\sin\frac{\eta}{2}$$
$$- (\eta^3-\eta)\cos\eta - (2\eta^2+1)\sin\eta\}\Big\} - \beta'(\cos\frac{\eta}{2} - \cos\eta)\,\eta^2\cos\eta]$$
$$+ \sin\eta\tau\,[(\cos\frac{\eta}{2} - \cos\eta)\{\frac{\eta^2}{2}\sin\frac{\eta}{2} - (\eta^3-\eta)\cos\eta$$
$$- (2\eta^2+1)\sin\eta\}]\Big]$$
$$\div\{\frac{\eta^2}{2}\sin\frac{\eta}{2} - (\eta^3-\eta)\cos\eta - (2\eta^2+1)\sin\eta\}^2 \text{ for } \tau > 0,$$

and $\quad KY(\tau) = 0$ for $\tau < 0$. (10.5)

The necessary values of ξ and η being shown in Table VI, we have

$$\begin{aligned} KY(\tau) = \ & 0.21896\exp(-0.001770\tau)\sin(0.84727\tau) \\ & +0.0^{4}1813\exp(-0.001770\tau)\cos(0.84727\tau)^{1)} \\ & -0.02344\exp(-0.000328\tau)\sin(3.41095\tau) \\ & -0.0^{4}1215\exp(-0.000328\tau)\cos(3.41095\tau) \\ & -0.00874\exp(-0.000221\tau)\sin(6.58957\tau) \\ & -0.0^{5}480\exp(-0.000221\tau)\cos(6.58957\tau) \\ & -0.00169\exp(-0.000061\tau)\sin(9.53629\tau) \\ & -0.0^{6}92\exp(-0.000061\tau)\cos(9.53629\tau) \\ & -0.00032\exp(-0.000019\tau)\sin(15.76959\tau) \\ & -0.0^{7}9\exp(-0.000019\tau)\cos(15.76959\tau) \\ & -\cdots \quad \text{for } \tau > 0, \end{aligned}$$

1) E.g. $0.0^{3}1 = 0.0001$.

and $KY(\tau) = 0$ for $\tau < 0$. (10.6)

In order that these different solutions may connect with each other smoothly at $\tau = 0$, the algebraic sum of the cosine terms must vanish at $\tau = 0$. In reality, however, we have, by putting $\tau = 0$,

$$KY(0) = 0.0^1\,1813 - 0.0^4\,1215 - 0.0^5\,480 - 0.0^6\,92 - 0.0^7\,9 = 0.0^6\,17,$$

which does not satisfy our requirement completely. But we can expect with certainty that if we are so untiring as to take into account of higher harmonics on one hand and of smaller quantities ξ^2, $\beta\xi$, $\cdots$ on the other, this remainder will be found to vanish finally.

(ii) When $\lambda = 2$:—By the similar procedure, only a little more tedious, let us omit the general expression similar to (10.5) which is too lengthy, we can arrive at the final result

$$\begin{aligned} KY(\tau) = \; & 0.113192 \exp(-1.039440\tau)\dagger \\ & -0.002715 \exp(-2.083614\tau) \sin(7.46489\tau)^* \\ & -0.004754 \exp(-2.083614\tau) \cos(7.46489\tau)^* \\ & +0.051307 \exp(-0.002022\tau) \sin(1.59778\tau) \\ & -0.000353 \exp(-0.002022\tau) \cos(1.59778\tau) \\ & +0.006941 \exp(-0.000407\tau) \sin(3.60636\tau) \\ & -0.0^5\,17 \exp(-0.000407\tau) \cos(3.60636\tau) \\ & +0.000650 \exp(-0.000095\tau) \sin(9.62569\tau) \\ & +0.0^6\,2 \exp(-0.000095\tau) \cos(9.62569\tau) \\ & +0.000159 \exp(-0.000038\tau) \sin(15.83266\tau) \\ & +0.0^7\,4 \exp(-0.000038\tau) \cos(15.83266\tau) \\ & \cdots \quad \text{for} \quad \tau > 0, \end{aligned}$$

$$\begin{aligned} \text{and} \quad KY(\tau) = \; & 0.112839 \exp(1.043117\tau)\dagger \\ & +0.002717 \exp(2.083940\tau) \sin(7.46364\tau)^* \\ & -0.004755 \exp(2.083940\tau) \cos(7.46364\tau)^* \\ & \quad \text{for} \quad \tau < 0. \end{aligned} \qquad (10.7)$$

Substituting $\tau = 0$, we have from these expressions $KY(0) = 0.108083$ and 0.108084, respectively; this discrepancy will be removed provided we should further the approximation as before.

The curves of vibration embodied by (10.6) and (10.7) were calculated varying τ at intervals of 0.1. They are shown in Figs. 14 and 15, respectively. The terms with an asterisk in (10.7) are the waves travelling ahead of the trolley running at *subsonic* speed. Because they are almost masked by other larger vibrations, they are reproduced in Fig. 16 in particular. The terms with a cross resemble the deflection mentioned in **6**, cf. (6.8). To compare the order of magnitude, we put $\tau' = 0$, $\gamma = 2$ in (6.8), then we have

$$Ky(0) = \frac{1}{4\sqrt{2(\lambda'^2 - 1)}}.$$

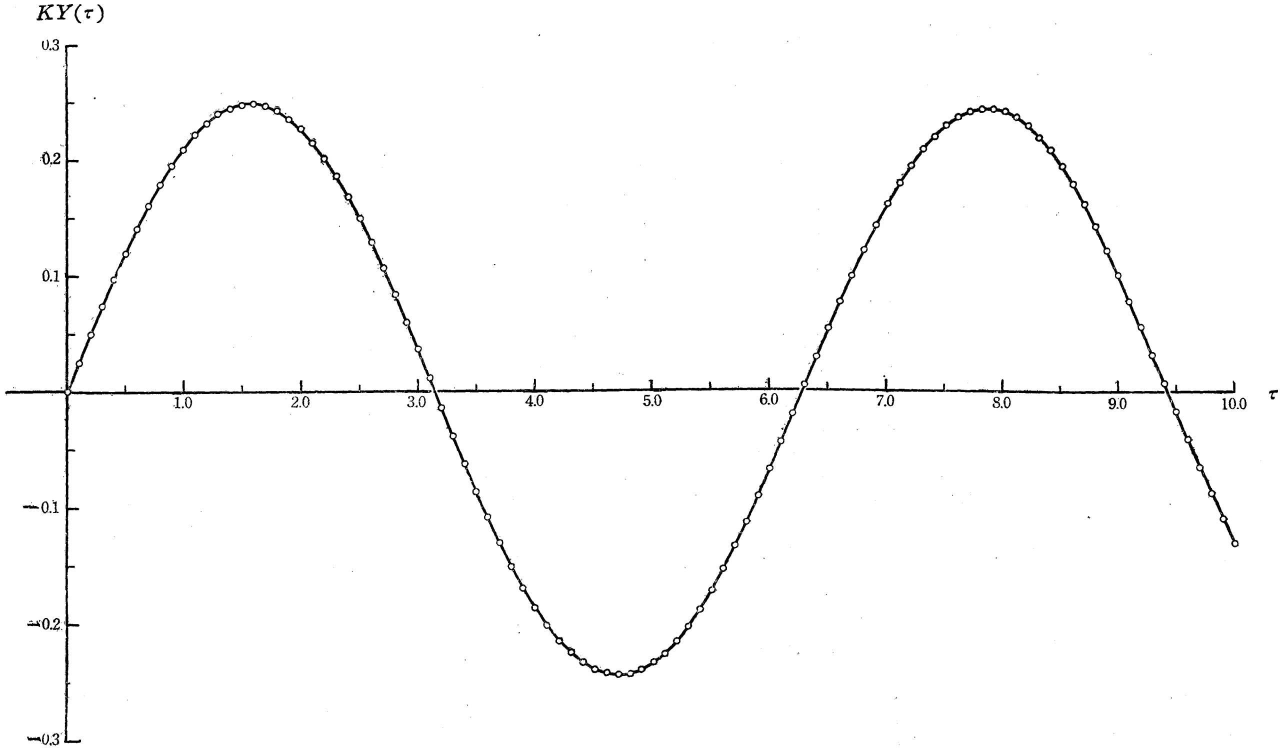


FIG. 10. $\lambda = 1$.

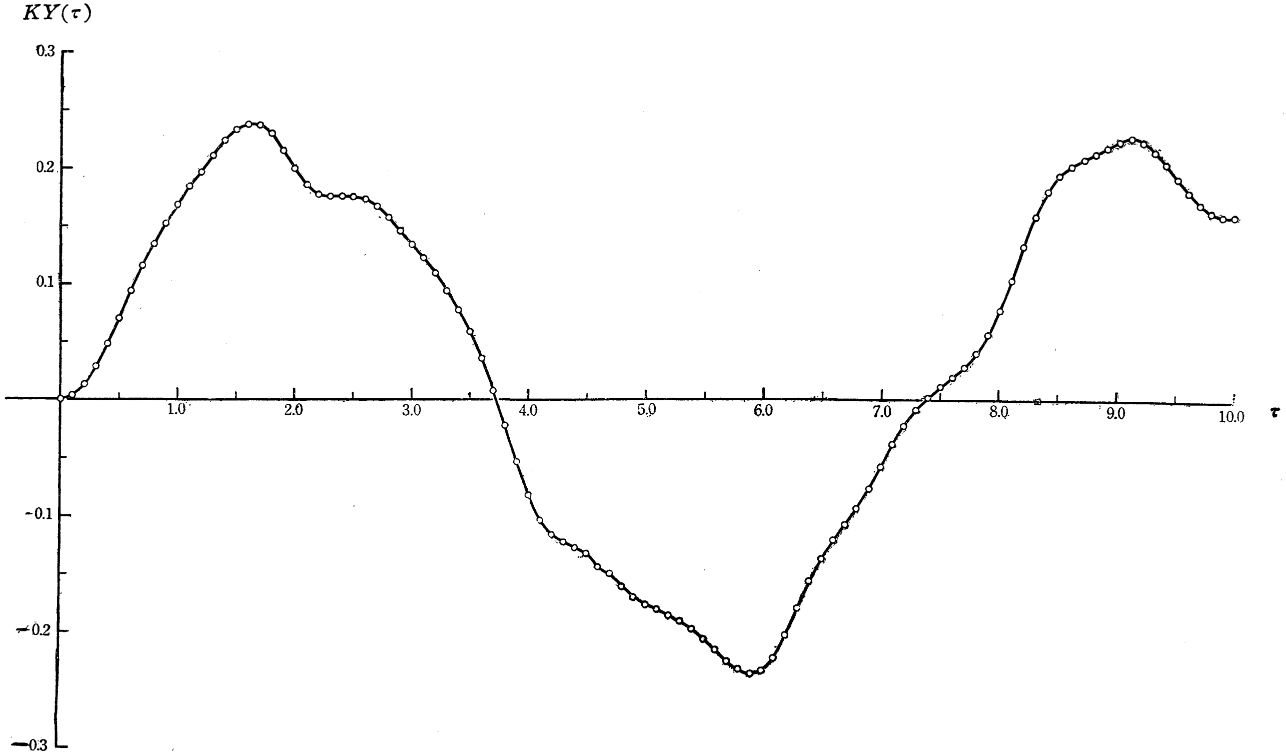


Fig. 14. $\lambda = 1/2$.

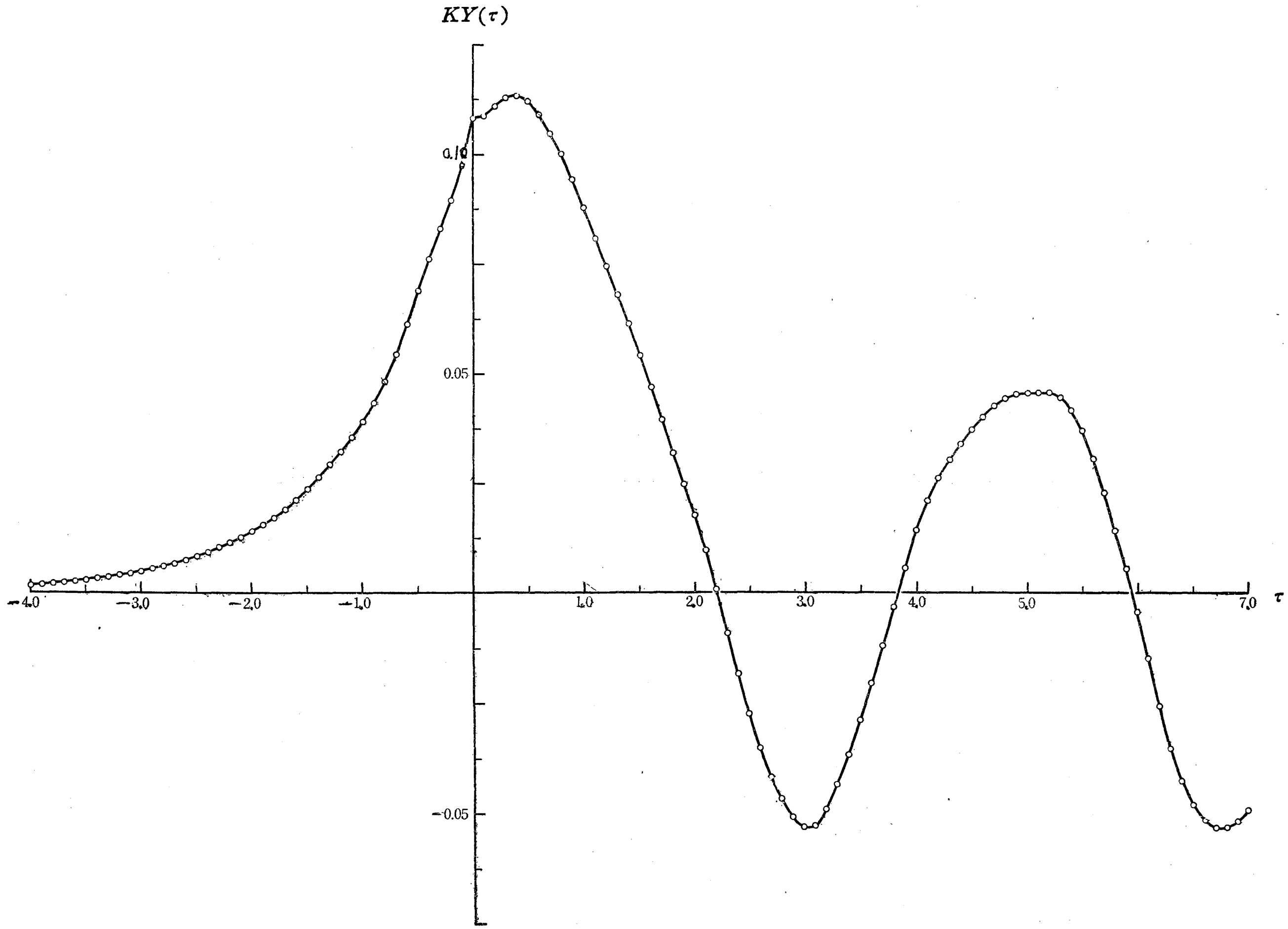


FIG. 15. $\lambda = 2$.

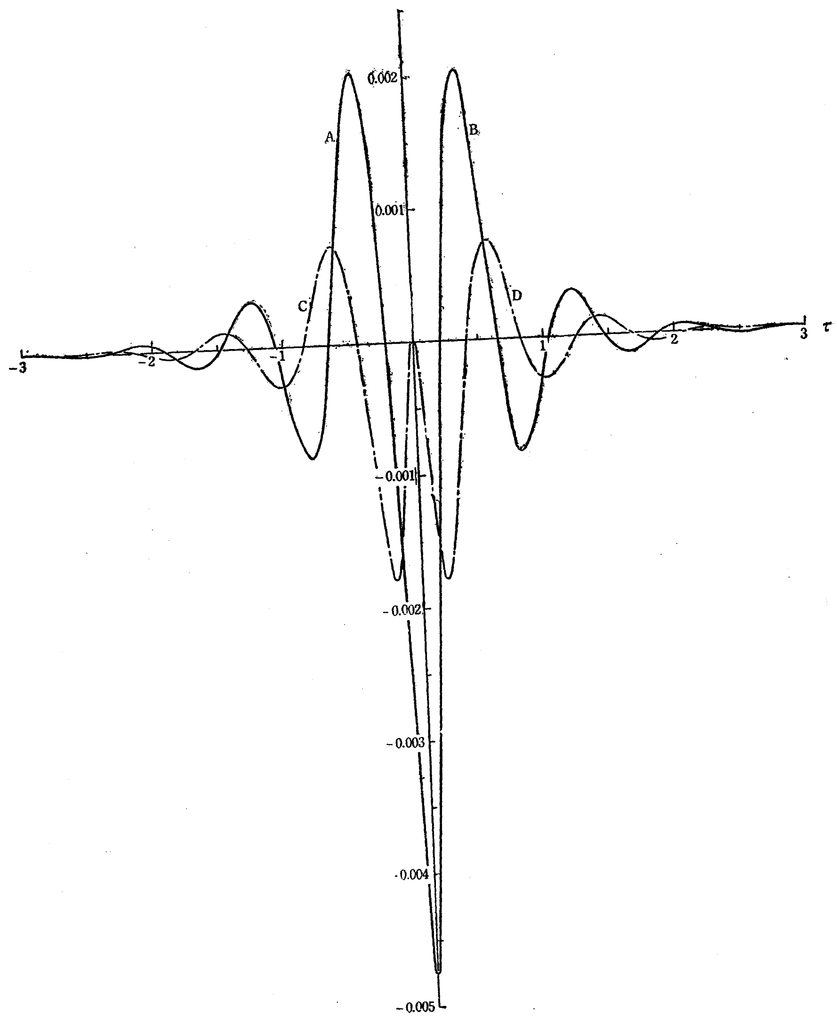


FIG. 16.

A: $-0.004755 \exp(2.083940\tau) \cos(7.46364\tau)$

B: $-0.004754 \exp(-2.083614\tau) \cos(7.46489\tau)$

C: $0.002717 \exp(2.083940\tau) \sin(7.46364\tau)$

D: $-0.002715 \exp(-2.083614\tau) \sin(7.46489\tau)$

Since

$$\lambda' = c'/V = \lambda\, c'/c = \lambda\sqrt{\rho/\rho'} = \lambda/\sqrt{3} = 2/\sqrt{3},$$

we find

$$Ky(0) \fallingdotseq 0.31,$$

i.e. three times as large as the exact value.

11. Unfortunately we have no appropriate data just at present to compare with the results calculated in the preceding section. So they must remain only *theoretical* for the time being. The writer is well aware that the model used is too much simplified so that the mathematical treatment may be possible and that accordingly a few more steps are necessary to connect the mathematics with the phenomena actually observed. He hopes to have another chance in future to discuss this problem from more practical view-point. Such being the case the conclusion of this paper will be reserved.

Supplementary notes: — i) We have calculated only the vibration of any of the particles attached to the string at equal intervals. If we should like to discuss the motion of the string itself, first we must obtain φ_n and ψ_n explicitly by substituting (3.9) or its equivalent for $\overline{Y}_n$ in (2.10). Then introducing φ_n and ψ_n in (2.6), we shall be led to the expression of $\bar{y}(x, p)$ in terms of the known quantities. The answer will be completed by inversing $\bar{y}(x, p)$ to $y(x, t)$. The solution thus obtained will contain obviously the frequencies due to Y_n together with those of the natural vibrations of the string.

ii) Suppose a trolley is approaching an end where the string is fixed and all the incident waves reflect; the solution can be easily constructed from ours by superposing at the point of the image of the load another fictitious force having the intensity of $-f$ and running with the same speed in the direction of x *decreasing*.

iii) It may appear somewhat peculiar that in (10.6) and (10.7) the higher the frequency goes up, the smaller becomes the damping, for usually in the linear vibrations higher harmonics are accompanied by larger dampings. The reason why this general rule does not hold good in our problem cannot be definitely clarified at present. That the numerical calculations were carried out carefully enough may be the only explanation now available. However, the writer's opinion is that our model is so more complicated than the ordinary vibration of a particle attached to an elastic string that there is some possibility that the rule ceases to be valid.

iv) We must admit finally that the general aspect of the motion embodied by (10.7), for c is larger than V without exceptions practically, is different more or less from what is observed in our daily life. We are able to mention a number of plausible causes responsible for this discrepancy: the model might be oversimplified, the assumed values of the paramaters might not be appropriate, etc. But above all we need to remember that we neglected the up-and-down motion of a trolley itself when a street-car is running. If we had treated the problem as the combined vibration of a trolley-wire and a trolley-pole (or a pantagraph) running in contact with it, then the solution would have become very realistic, although the discussion far more complicated would have been inevitable.

12. The theoretical part of the foregoing is the result of the discussions between Professor Yamada and the writer continued from time to time for years. The calculation was frequently interrupted by other more *serious* works and was stopped longer because of the writer's travelling abroad. Meanwhile the increasing supply of electricity in this country has made us all indifferent to the motion of a trolley-wire. The writer must apologize to the professor for publishing this paper too late. In numerical calculations and drawings the friends in the institute were so kind as to render help to the writer.

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