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ON A METHOD OF SOLVING TORSION AND BENDING PROBLEMS OF CONTINUOUS PANEL STRUCTURES

By Shosaburo NEGORO

In this report we treat the general solution of the torsion and bending problems of continuous panel structures under transverse forces, twisting, and bending moments, distributions of the external forces acting on the members of the structures and positions of the supported bars being quite arbitrary. The chief topics of the report are as follows: — First, using the Stieltjes integrations with accumulation functions of the external forces acting on a beam, the well-known formulae of the bending are expressed in a new form. Second, we show that, using the formulae stated above, the present problems reduce to the simultaneous equations of the first degree as to the moments and deflections at the intersecting points of the members, and the number of the equations is the same as that of the unknown quantities in the preseut problems. Therefore, the solution of these equations suffices to determine one set of unknown quantities and the problems are solved. Last, we show an applicable type for solving the above equations by means of the iteration methods practically.

1. Introduction. As to the torsion and bending problems of continuous panel structures, many papers have already been published. In this report we treat the problems again by the method as stated above, that is to say, first, the well-known formulae of the bending of a beam are expressed in a new form by using the Stieltjes integrations with accumulation functions of the external forces acting on the beam. Second, considering the conditions of the continuity of the inclination, equilibrium of the twisting, and the bending moments, and equilibrium of the shearing forces at the intersecting points of the members, the present problems reduce to the simultaneous equations of the first degree as to the twisting, and the bending moments and deflections at the intersecting points of the members and the simultaneous equations have the same number of unknown quantities as that of the linear equations. Then these equations suffice to define one set of the unknown quantities and the problems are solved. It is, however, practically impossible to obtain an exact solution of the equations by the usual method of employing the determinant, as the number of the unknown quantities is too large, so that the well-known iteration methods are needed for the solution of the simultaneous equations. Last, we show an applicable type for solving the above equations by means of the iteration methods practically.

2. Formulae of bending. Using the Stieltjes integrations with the accumulation functions ($v(x)$, $M(x)$) of the external forces and the bending moments acting on a beam, the relations among the shearing force (F), the bending moment (M), the inclination (i), and the deflection (y) of the beam are expressed by the following formulae

$$\left.\begin{aligned} -\underset{x}{F_{\rightarrow}} &= \underset{0}{F_{\leftarrow}} + V_0(x), \quad \underset{x}{M_{\rightarrow}} = -\underset{0}{M_{\leftarrow}} + \underset{0}{F_{\leftarrow}} + V_1(x), \\ E\,I\,i &= E\,I\,i_0 - \underset{0}{M_{\leftarrow}}\,x + \frac{1}{2}\underset{0}{F_{\leftarrow}}\,x^2 + \frac{1}{2}V_2(x), \\ E\,I\,y &= E\,I(y_0 + i_0\,x) - \frac{1}{2}\underset{0}{M_{\leftarrow}}\,x^2 + \frac{1}{6}\underset{0}{F_{\leftarrow}}\,x^3 + \frac{1}{6}V_3(x) \end{aligned}\right\} \quad (1),$$

in which

$$V_s(x) = \alpha_s(x) - s\,\beta_{s-1}(x), \quad \alpha_s(x) = \int_0^x (x-t)^s\,dv(t),$$

$$\beta_s(x) = \int_0^x (x-t)^s\,dM(t), \quad s = (0, 1, 2, 3),$$

E = Young's Modulus, I = Moment of Inertia,

and suffix 0 denotes the value at the starting point and mark ($\rightarrow$) the direction of the outward-drawn normal of a cross-section x.

In the Eqs. (1), the functions $V_s(x)$ are known functions, if the external forces acting on the beam are given, and unknown quantities (F_0, M_0, i_0, y_0) are found from the bending conditions of the beam.

Eliminating the unknown quantities (i_0, F_0) from the 2nd, 3rd and 4th of the Eqs. (1), we have

$$E\,I\,i = \frac{E\,I}{x}(y - y_0) + \frac{x}{6}(2\underset{x}{M_{\rightarrow}} - \underset{0}{M_{\leftarrow}}) - \frac{x}{3}V_1(x) + \frac{1}{2}V_2(x) - \frac{1}{6x}V_3(x) \quad (2),$$

and from the 2nd of the Eqs. (1)

$$\left.\begin{aligned} \underset{0}{F_{\leftarrow}} &= \{\underset{1}{M_{\rightarrow}} + \underset{0}{M_{\leftarrow}} - V_1(l_1)\}/l_1 = -(\underset{0}{F_{\rightarrow}} + P_0) \\ -\underset{0}{F_{\rightarrow}} &= \{\underset{1}{M_{\rightarrow}} + \underset{0}{M_{\leftarrow}} - \overline{V}_1(l_1)\}/l_1 \end{aligned}\right\} \quad (3_1),$$

and from the 1st of Eqs. (1) and Eqs. (3_1)

$$\left.\begin{aligned} -\underset{1}{F_{\rightarrow}} &= \underset{0}{F_{\leftarrow}} + V_1(l_1) = \{\underset{1}{M_{\rightarrow}} + \underset{0}{M_{\rightarrow}} + U_1(l_1)\}/l_1 = \underset{1}{F_{\leftarrow}} + P_1 \\ \underset{1}{F_{\leftarrow}} &= \{\underset{1}{M_{\rightarrow}} + \underset{0}{M_{\leftarrow}} + \overline{U}_1(l_1)\}/l_1 \end{aligned}\right\} \quad (3_2),$$

in which

P_0, P_1 = external forces acting on a beam at the points (0, 1),

$$V_1(l_1) = \int_0^{l_1} (l_1 - t)\,dv(t), \quad \overline{V}_1(l_1) = V_1(l_1) \text{ excepting the effect of } P_0,$$

$$U_s(l_1) = \gamma_s(l_1) + s\,\delta_{s-1}(l_1), \quad \gamma_s(l_1) = \int_0^{l_1} t^s\,dv(t), \quad \delta_s(l_1) = \int_0^{l_1} t^s\,dM(t),$$

l_1 = length of the considering span, $\overline{U_1}(l_1) = U_1(l_1)$ excepting the effect of P_1.

Moreover, eliminating the unknown quantities M_0 and F_0 from the 3rd and the 4th of Eqs. (1), we have

$$\left.\begin{aligned} F_{\underset{0}{\leftarrow}} &= \frac{6EI}{l_1^2}\{(i_0 + i_1) + 2(y_0 - y_1)/l_1\} - 3V_2(l_1)/l_1^2 + 2V_3(l_1)/l_1^3 \\ M_{\underset{0}{\leftarrow}} &= \frac{2EI}{l_1}\{(2i_0 + i_1) + 3(y_0 - y_1)/l_1\} - V_2(l_1)/l_1 + V_3(l_1)/l_1^2 \end{aligned}\right\} (4_1),$$

and from Eqs. (4_1) directly,

$$\left.\begin{aligned} F_{\underset{1}{\rightarrow}} &= -\frac{6EI}{l_1^2}\{(i_0 + i_1) + 2(y_0 - y_1)/l_1\} - 3U_2(l_1)/l_1^2 + 2U_3(l^1)/l_1^3 \\ M_{\underset{1}{\rightarrow}} &= \frac{2EI}{l_1}\{(i_0 + 2i_1) + 3(y_0 - y_1)/l_1\} + U_2(l_1)/l_1 - U_3(l_1)/l_1^2 \end{aligned}\right\} (4_2),$$

in which

$$V_s(l') = U_s(l), \quad \alpha_s(l') = \gamma_s(l), \quad \beta_s(l') = -\delta_s(l)$$

l' = the span taking the opposite direction of l the positive side.

Eqs. (4) are the well-known formulea of the inclinations and the deflections at the ends of a beam.

3. Fundamental principle of the theory. Let us first consider the relations among the shearing force (F), the twisting moment (T), the bending moment (M), the inclination (i) and the deflection (y) at any intersecting point (n, r) of the members of the structures.

Now, we take the Cartesian co-ordinate axes (x, y, z) of the right hand system, the y-axis being coincident with the direction of the loads, and distinguish the quantities referring to the x- and z axes respectively by suffix s ($= 1, 2$) attached on the left side. And, having discontinuities of the shearing forces and the moments at the intersecting points of the members where the external forces and the moments are acting, we take the shearing forces, the bending moments and the twisting ones at the points apart infinitesimal small quantities (ε) from the intersecting points for the convenience of the following calculations. But, as to the inclination and the displacement, we have no discontinuity in all range of the structure. Then we take the inclinations and the displacements at the intersecting points in the following calculation.

Here, we consider the conditions of the continuity of the inclination, equilibrium of the twisting, and the bending moments and equilibrium of the shearing forces at the intersecting points of the members as stated in the introduction.

That is to say, as to inclination, from Eq. (2)

$$i_{n\cdot r} = \Delta y_{n\cdot r} + B_{n.r} + Q_{n\cdot r}, \quad i'_{n\cdot r} = \Delta y'_{n\cdot r} + B'_{n\cdot r} + Q'_{n\cdot r} \quad (5_1),$$

in which

$$_1\Delta y_{n\cdot r} = (y_{n\cdot r} - y_{n-1\cdot r})/_1l_{n\cdot r}, \quad _2\Delta y_{n\cdot r} = (y_{n\cdot r} - y_{n\cdot r-1})/_2l_{n\cdot r},$$

$$_1\Delta y'_{n\cdot r} = (y_{n\cdot r} - y_{n+1\cdot r})/_1l_{n+1\cdot r}, \quad _2\Delta y'_{n\cdot r} = (y_{n\cdot r} - y_{n\cdot r+1})/_2l_{n\cdot r+1},$$

$$_1B_{n\cdot r} = {_1\mu_{n\cdot r}}(2\,{_1M}_{\underset{n-\varepsilon\cdot r}{\rightarrow}} - {_1M}_{\underset{n-1+\varepsilon\cdot r}{\leftarrow}}), \quad _2B_{n\cdot r} = {_2\mu_{n\cdot r}}(2\,{_2M}_{\underset{n\cdot r-\varepsilon}{\rightarrow}} - {_2M}_{\underset{n\cdot r-1+\varepsilon}{\leftarrow}}),$$

$$_1B'_{n\cdot r} = {_1\mu_{n+1\cdot r}}(2\,{_1M}_{\underset{n+\varepsilon\cdot r}{\rightarrow}} - {_1M}_{\underset{n+1-\varepsilon\cdot r}{\leftarrow}}), \quad _2B'_{n\cdot r} = {_2\mu_{n\cdot r+1}}(2\,{_2M}_{\underset{n\cdot r+\varepsilon}{\rightarrow}} - {_2M}_{\underset{n\cdot r+1-\varepsilon}{\leftarrow}}),$$

$$_sQ_{n\cdot r} = {_s\Big[}\Big(\frac{1}{EI}\Big)_{n\cdot r}\Big\{-\frac{l_{n\cdot r}}{3}V_1(l_{n\cdot r}) + \frac{1}{2}V_2(l_{n\cdot r}) - \frac{1}{6l_{n\cdot r}}V_3(l_{n\cdot r})\Big\}\Big], \quad s = (1\cdot 2),$$

$$_sQ'_{n\cdot r} = {_s\Big[}\Big(\frac{1}{EI}\Big)_q\Big\{-\frac{l_q}{3}U_1(l_q) + \frac{1}{2}U_2(l_q) - \frac{1}{6l_q}U_3(l_q)\Big\}\Big],$$

$$(s = 1,\ q = n+1\cdot r),\ (s = 2,\ q = n\cdot r+1),\ \mu_{n\cdot r} = \Big(\frac{l}{6EI}\Big)_{n\cdot r}.$$

Then considering the condition of the continuity of the inclination ($i_{n\cdot r} = -i'_{n\cdot r}$) and Eqs. ($5_1$), we have

$$_s[\Delta_s i'_{n\cdot r} + B_{n\cdot r} + B'_{n\cdot r} + Q_{n\cdot r} + Q'_{n\cdot r}] = 0 \quad (5_2),$$

in which

$$_1\Delta_1 i'_{n\cdot r} = {_1\Delta y_{n\cdot r}} - {_1\Delta y_{n+1\cdot r}}, \quad _2\Delta_2 i'_{n\cdot r} = {_2\Delta y_{n\cdot r}} - {_2\Delta y_{n\cdot r+1}}, \quad s = (1, 2),$$

from the condition of the equilibrium of the moments,

$$\left.\begin{aligned} &_1[M_{n\cdot r} + M_{\underset{n+\varepsilon\cdot r}{\rightarrow}} + M_{\underset{n-\varepsilon\cdot r}{\leftarrow}}] + {_2[}T_{\underset{n\cdot r+\varepsilon}{\rightarrow}} + T_{\underset{n\cdot r-\varepsilon}{\leftarrow}}] = 0,\\ &_2[M_{n\cdot r} + M_{\underset{n\cdot r+\varepsilon}{\rightarrow}} + M_{\underset{n\cdot r-\varepsilon}{\leftarrow}}] + {_1[}T_{\underset{n+\varepsilon\cdot r}{\rightarrow}} + T_{\underset{n-\varepsilon\cdot r}{\leftarrow}}] = 0 \end{aligned}\right\} \quad (6),$$

in which

$_sM_{n\cdot r}$ = external bending moment acting at the intersecting point (n, r),

from the relations between the twisting moments and the inclinations,

$$\left.\begin{aligned} &_1[i_{n\cdot r} - i_{n\cdot r-1}] = {_2[}\lambda_{n\cdot r}T_{\underset{n\cdot r-1+\varepsilon}{\longrightarrow}}] = {_2[}\lambda_{n\cdot r}T_{\underset{n\cdot r-\varepsilon}{\longrightarrow}}] = {_2[}\Delta_1 i_{n\cdot r} + \Delta_1 B_{n\cdot r} + \Delta_1 Q_{n\cdot r}]\\ &_2[i_{n\cdot r} - i_{n-1\cdot r}] = {_1[}\lambda_{n\cdot r}T_{\underset{n-1+\varepsilon\cdot r}{\longrightarrow}}] = {_1[}\lambda_{n\cdot r}T_{\underset{n-\varepsilon\cdot r}{\longrightarrow}}] = {_1[}\Delta_2 i_{n\cdot r} + \Delta_2 B_{n\cdot r} + \Delta_2 Q_{n\cdot r}] \end{aligned}\right\} \quad (7),$$

in which

$$_1\Delta_2\ i_{n\cdot r} = {_2\Delta y_{n\cdot r}} - {_2\Delta y_{n-1\cdot r}}, \quad _2\Delta_1 i_{n\cdot r} = {_1\Delta y_{n\cdot r}} - {_1\Delta y_{n\cdot r-1}},$$

$$_1\Delta_2 B_{n\cdot r} = {_2B_{n\cdot r}} - {_2B_{n-1\cdot r}}, \quad _2\Delta_1 B_{n\cdot r} = {_1B_{n\cdot r}} - {_1B_{n\cdot r-1}},$$

$$_1\Delta_2 Q_{n\cdot r} = {_2Q_{n\cdot r}} - {_2Q_{n-1\cdot r}}, \quad _2\Delta_1 Q_{n\cdot r} = {_1Q_{n\cdot r}} - {_1Q_{n\cdot r-1}}, \quad \lambda_{n\cdot r} = (l/GC)_{n\cdot r},$$

G = torsional rigidity, C = torsional constant,

and from the equilibrium condition of the shearing forces,

$$ {}_1F_{\underset{n-\varepsilon\cdot r}{\leftarrow}} + {}_1F_{\underset{n+\varepsilon\cdot r}{\rightarrow}} + {}_2F_{\underset{n\cdot r-\varepsilon}{\leftarrow}} + {}_2F_{\underset{n\cdot r+\varepsilon}{\rightarrow}} + P_{n\cdot r} + R_{n\cdot r} = 0 \tag{8$_1$} $$

in which

$P_{n\cdot r}$ = external force acting on the member at the point (n, r),

$R_{n\cdot r}$ = reactional force acting on the supported bar at the point (n, r).

Further, considering Eqs. (3), the above equation is also rewritten in the form

$$ \left.\begin{aligned} -R_{n\cdot r} &= {}_1[(M_{\underset{n-\varepsilon\cdot r}{\leftarrow}} + M_{\underset{n-1+\varepsilon\cdot r}{\leftarrow}})/l_{n\cdot r} - (M_{\underset{n+1-\varepsilon\cdot r}{\rightarrow}} + M_{\underset{n+\varepsilon\cdot r}{\leftarrow}})/l_{n+1\cdot r}] \\ &+ {}_2[(M_{\underset{n\cdot r-\varepsilon}{\rightarrow}} + M_{\underset{n\cdot r-1+\varepsilon}{\leftarrow}})/l_{n\cdot r} - (M_{\underset{n\cdot r+1-\varepsilon}{\rightarrow}} + M_{\underset{n\cdot r+\varepsilon}{\leftarrow}})/l_{n\cdot r+1}] + \Theta_{n\cdot r} \end{aligned}\right\} \tag{8$_2$} $$

in which

$$ \Theta_{n\cdot r} = {}_1[\overline{U}_1(l_{n\cdot r})/l_{n\cdot r} + \overline{V}_1(l_{n+1\cdot r})/l_{n+1\cdot r}] + {}_2[\overline{U}_1(l_{n\cdot r})/l_{n\cdot r} + \overline{V}_1(l_{n\cdot r+1})/l_{n\cdot r+1}] + P_{n\cdot r}. $$

Moreover, the supported bar being elastic and the form of the cross-section being uniform, we have

$$ y_{n\cdot r} = -[hR/EA]_{n\cdot r} \tag{9$_1$} $$

for the relation between the reaction and deflection, in which $A_{n\cdot r}$ denotes the area of the cross-section and $h_{n\cdot r}$ the length of the supported bar and, further, the bar being approximately regarded as a rigid body,

$$ E_{n\cdot r} \to \infty, \quad y_{n\cdot r} \fallingdotseq 0 \tag{9$_2$} $$

Here, having the supported bar at the intersecting point (n, r), we have one independent equation as to y by considering Eqs. (8) and (9), and having no supported bar at the point, Eqs. (8) are the independent relations as to moments.

After all, the above results tell us that, as regards the unknown quantities, say, the moments $(M, T)_{n\cdot r}$ and the deflection $(y_{n\cdot r})$ at the intersecting points (n, r), we have the simultaneous equations of the Ist degree with the 9 unknown quantities and the same number of the equations. Now, for simplicity of writing the following parts, we call these equations ((5), (6), (7), and (8)) the fundamental equations.

Next, let us consider the above relations at the ends of the members.

(a) In such cases when the intersecting points (n, r) are on the boundary members.

(i) case r = max. or min.:—

In these cases, we have

$$ \left.\begin{aligned} &{}_2[F_{\underset{n\cdot r+\varepsilon}{\rightarrow}},\ M_{\underset{n\cdot r+\varepsilon}{\rightarrow}},\ T_{\underset{n\cdot r+\varepsilon}{\rightarrow}}] = 0 \quad (r = \text{max.}), \\ &{}_2[F_{\underset{n\cdot r-\varepsilon}{\leftarrow}},\ M_{\underset{n\cdot r-\varepsilon}{\leftarrow}},\ T_{\underset{n\cdot r-\varepsilon}{\rightarrow}}] = 0 \quad (r = \text{min.}), \end{aligned}\right\} \tag{10$_1$} $$

for boundary conditions, and see easily that the 2nd of Eqs. (5_2) does not hold for the case (r = max.) and the 2nd of Eqs. (5_2) and the 1st of Eqs. (7) do not hold for the case (r = min.). Hence, excepting these equations, and putting these terms expressed by Eqs. (10_1) equal to zero, the previous fundamental equations hold also for the present cases. Then, in the case (r = max.), we have the simultaneous equations consisting of 7 equations for 8 unknown quantities, and, in the case (r = min.), consisting of 6 equations for 7 unknown quantities. The above descriptions tell us that, considering both of the cases together, we have as many simultaneous as the unknown quantities.

(ii) case n = max. or min.: —

In these cases, we have

$$\left.\begin{aligned} {}_1[F_{\underset{n+\varepsilon\cdot r}{\rightarrow}},\ M_{\underset{n+\varepsilon\cdot r}{\rightarrow}},\ T_{\underset{n+\varepsilon\cdot r}{\rightarrow}}] &= 0 \qquad (n = \text{max.}) \\ {}_1[F_{\underset{n-\varepsilon\cdot r}{\leftarrow}},\ M_{\underset{n-\varepsilon\cdot r}{\leftarrow}},\ T_{\underset{n-\varepsilon\cdot r}{\leftarrow}}] &= 0 \qquad (n = \text{min.}) \end{aligned}\right\} \tag{10_2}$$

for boundary conditions, and see easily that the 1st of Eqs. (5_2) does not hold for the case (n = max.) and the 1st of Eqs. (5_2) and the 2nd of Eqs. (7) do not hold for the case (n = min.). Hence, similarly as in case (i), excepting the above equations and putting these terms expressed by Eqs. (10_2) equal to zero, the fundamental equations hold also for the present cases. Then, in the case (n = max.), we have the simultaneous equations consisting of 8 equations for 7 unknown quantities and, in the case (n = min.), consisting of 6 equations for 7 unknown quantities. These descriptions tell us that, considering both of the cases together, we have as many simultaneous equations as the unknown quantities.

(iii) case (n, r) = max. or min.: —

In theses cases, we have

$$\left.\begin{aligned} &{}_1[F_{\underset{n+\varepsilon\cdot r}{\rightarrow}},\ M_{\underset{n+\varepsilon\cdot r}{\rightarrow}},\ T_{\underset{n+\varepsilon\cdot r}{\rightarrow}}] = 0, \quad {}_2[F_{\underset{n\cdot r+\varepsilon}{\leftarrow}},\ M_{\underset{n\cdot r+\varepsilon}{\leftarrow}},\ T_{\underset{n\cdot r+\varepsilon}{\leftarrow}}] = 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad ((n\cdot r) = \text{max.}) \\ &{}_1[F_{\underset{n-\varepsilon\cdot r}{\leftarrow}},\ M_{\underset{n-\varepsilon\cdot r}{\leftarrow}},\ T_{\underset{n-\varepsilon\cdot r}{\leftarrow}}] = 0, \quad {}_2[F_{\underset{n\cdot r-\varepsilon}{\leftarrow}},\ M_{\underset{n\cdot r-\varepsilon}{\leftarrow}},\ T_{\underset{n\cdot r-\varepsilon}{\leftarrow}}] = 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad ((n\cdot r) = \text{min.}) \end{aligned}\right\} \tag{10_3}$$

for boundary conditions and see easily that Eqs. (5_2) do not hold for the case (n, r) = max. and Eqs. (5_2) and Eqs. (7) do not hold for the case (n, r) = min. Hence, similarly as in the cases (i) and (ii), excepting the above equations and putting these terms expressed by Eqs. (10_3) equal to zero, the others of the fundamental equations hold for each case. Then, in the case (n, r) = max., we have the simultaneous equations consisting of 7 equations for 5 unknown quantities and, in the case (n, r) = min., consisting of 3 equations for 5 unknown quantities. Therefore, considering both of the cases together, we have as many simultaneous equations as the unknown quantities.

(iv) case (n = max., r = min.) or (n = min., r = max.): —

In these cases, we have

$$\left.\begin{array}{l} {}_1[F_{\underset{n+\varepsilon\cdot r}{\longrightarrow}},\ M_{\underset{n+\varepsilon\cdot r}{\longrightarrow}},\ T_{\underset{n+\varepsilon\cdot r}{\longrightarrow}}]=0,\quad {}_2[F_{\underset{n\cdot r-\varepsilon}{\longleftarrow}},\ M_{\underset{n\cdot r-\varepsilon}{\longleftarrow}},\ T_{\underset{n\cdot r-\varepsilon}{\longleftarrow}}]=0 \\ \qquad\qquad (n=\text{max. and } r=\text{min.}) \\ {}_1[F_{\underset{n-\varepsilon\cdot r}{\longleftarrow}},\ M_{\underset{n-\varepsilon\cdot r}{\longleftarrow}},\ T_{\underset{n-\varepsilon\cdot r}{\longleftarrow}}]=0,\quad {}_2[F_{\underset{n\cdot r+\varepsilon}{\longrightarrow}},\ M_{\underset{n\cdot r+\varepsilon}{\longrightarrow}},\ T_{\underset{n\cdot r+\varepsilon}{\longrightarrow}}]=0, \\ \qquad\qquad (n=\text{min. and } r=\text{max.}) \end{array}\right\} \quad (10_4)$$

for boundary conditions and see easily that Eqs. (5_2) and the 1st of Eqs. (7) do not hold for the case (n = max. and r = min.) and Eqs. (5_2) and the 2nd of Eqs. (7) do not hold for the case (n = min. and r = max.). Hence, similarly as in the above cases, excepting the above equations and putting these terms shown by Eqs. (10_4) equal to zero, the others of the fundamental equations hold for each case. Then in both of the cases, we have the simultaneous equations consisting of 5 equations for 5 unknown quantities. These descriptions tell us that, considering both of the cases togeteher, we have as many simultaneous equations as the unknown quantities.

(b) In such cases when the ends (n, r) of the members of the structure are built.

Firstly, as well known, we have

$$ {}_1i_{n\cdot r} = {}_2i_{n\cdot r} = 0, \quad y_{n\cdot r} = 0 \quad (11)$$

for boundary conditions and, therefore, the following are satisfied.

(i) case r = max. or min.: —

In these cases, from the boundary conditions (II) and the fundamental equations, we have

$$\left.\begin{array}{ll} y_{n\cdot r-1}/{}_2l_{n\cdot r} = {}_2[B_{n\cdot r}+Q_{n\cdot r}] & (r=\text{max.}), \\ y_{n\cdot r+1}/{}_2l_{n\cdot r+1} = {}_2[B'_{n\cdot r}+Q'_{n\cdot r}] & (r=\text{min.}), \end{array}\right\} \quad (12_1)$$

$$ {}_2[T_{\underset{n\cdot r+\varepsilon}{\longrightarrow}} + T_{\underset{n\cdot r-\varepsilon}{\longleftarrow}}] = 0,\ {}_2[M_{\underset{n\cdot r+\varepsilon}{\longrightarrow}} + M_{\underset{n\cdot r-\varepsilon}{\longleftarrow}}] = 0 \qquad (r=\text{max. or min.}) \quad (13_1)$$

$$\left.\begin{array}{l} -{}_1i_{n-1\cdot r} = {}_2[\lambda_{n\cdot r}\, T_{\underset{n\cdot r-1+\varepsilon}{\longrightarrow}}] = {}_2[\lambda_{n\cdot r}\, T_{\underset{n\cdot r-\varepsilon}{\longrightarrow}}] = -{}_1[\Delta y_{n\cdot r-1} + B_{n\cdot r-1} + Q_{n\cdot r-1}] \\ \qquad\qquad (r=\text{max.}) \end{array}\right\} \quad (14_1)$$

and

$$\left.\begin{array}{ll} -R_{n\cdot r} = F_{\underset{n\cdot r-\varepsilon}{\longleftarrow}} = {}_2[\{M_{\underset{n\cdot r-\varepsilon}{\longrightarrow}} + M_{\underset{n\cdot r-1+\varepsilon}{\longleftarrow}} + \overline{U}_1(l_{n\cdot r})\}/l_{n\cdot r}] & (r=\text{max.}) \\ -R_{n\cdot r} = F_{\underset{n\cdot r+\varepsilon}{\longrightarrow}} = -{}_2[\{M_{\underset{n\cdot r+1-\varepsilon}{\longrightarrow}} + M_{\underset{n\cdot r+\varepsilon}{\longleftarrow}} - \overline{V}_1(l_{n\cdot r+1})\}/l_{n\cdot r+1}] & (n=\text{min.}) \end{array}\right\} \quad (15_2)$$

Then, in the case r = max., we have the simultaneous equation consisting of 5 equations (Eqs. (12_1), (13_1) and (14_1)) for 4 unknown quantities and, in the case r = min., consisting of 3 equations (Eqs. (12_1) and (13_1)) for 4 unknown quantities. Therefore, considering both of the cases together, we have as many equations as the unknown quantities. Moreover, Eqs. (15) give the values of the reactions and the shearing forces at the end points (n, r).

(ii) case $n = \max.$ or $\min.$: —

In these cases, similarly as case (i), from the boundary conditions (II) and the fundamental equations, we have

$$\left.\begin{aligned} y_{n-1\cdot r}/{}_1l_{n\cdot r} &= {}_1[B_{n\cdot r} + Q_{n\cdot r}] \qquad (n = \max.) \\ y_{n+1\cdot r}/{}_1l_{n+1\cdot r} &= {}_1[B'_{n\cdot r} + Q'_{n\cdot r}] \qquad (n = \min.) \end{aligned}\right\} \quad (12_2),$$

$${}_1[T_{\underset{n-\varepsilon\cdot r}{\longleftarrow}} + T_{\underset{n+\varepsilon\cdot r}{\longrightarrow}}] = 0, \quad {}_1[M_{\underset{n-\varepsilon\cdot r}{\longleftarrow}} + M_{\underset{n+\varepsilon\cdot r}{\longrightarrow}}] = 0, \quad (n = \max. \text{ or } \min.) \quad (13_2)$$

$$\left.\begin{aligned} -{}_2i_{n-1\cdot r} = {}_1[\lambda_{n\cdot r}T_{\underset{n-1+\varepsilon\cdot r}{\longrightarrow}}] = {}_1[\lambda_{n\cdot r}T_{\underset{n-\varepsilon\cdot r}{\longrightarrow}}] = -{}_2[\Delta y_{n-1\cdot r} + B_{n-1\cdot r} + Q_{n-1\cdot r}] \\ (n = \max.) \end{aligned}\right\} \quad (14_2)$$

and

$$\left.\begin{aligned} -R_{n\cdot r} &= {}_1F_{\underset{n-\varepsilon\cdot r}{\longleftarrow}} = {}_1[\{M_{\underset{n-\varepsilon\cdot r}{\longrightarrow}} + M_{\underset{n-1+\varepsilon\cdot r}{\longleftarrow}} + \bar{U}_1(l_{n\cdot r})\}/l_{n\cdot r}] \qquad (n = \max.) \\ -R_{n\cdot r} &= {}_1F_{\underset{n+\varepsilon\cdot r}{\longrightarrow}} = -{}_1[\{M_{\underset{n+1-\varepsilon\cdot r}{\longrightarrow}} + M_{\underset{n+\varepsilon\cdot r}{\longleftarrow}} - \bar{V}_1(l_{n+1\cdot r})\}/l_{n+1\cdot r}] \ (n = \min.) \end{aligned}\right\} \quad (15_2)$$

Then, in the case ($r = \max.$), we have the simultaneous equation consisting of 5 equations (Eqs. (12_2), (13_2) and (14_2)) for 4 unknown quantities and, in the case ($n = \min.$), consisting of 3 equations (Eqs. (12_2) and (13_2)) for 4 unknown quantities. Therefore, considering both of the cases together, we have as many simultaneous equations as the unknown quantities. Moreover, Eqs. (15) give the values of the reactions and the shearing forces at the end points.

In conclusion, the above descriptions tell us that we have as many equations of the form of the fundamental equations as the intersecting points. Therefore, these equations suffice to define one set of the unknown quantities and, from these results and Eqs. (1), all of the requirements are found. Then the problems are solved.

It is, however, practically impossible to obtain the solutions of the equations by usual method of employing the determinant, as the number of the unknown quantities is too large in general, so that the well-known iteration methods are needed for solving the equations practically. For convinience of the actual calculation, we rewrite the simultaneous equations with the unknown quantities $({}_1i, {}_2i, y)_{n\cdot r}$ referring to the inclinations and the deflections at the points (n, r) instead of the previous fundamental equations over again.

First, from Eqs. (4)

$$\left.\begin{aligned} {}_1M_{\underset{n+\varepsilon\cdot r}{\longleftarrow}} &= -({}_1M_{\underset{n+\varepsilon\cdot r}{\longrightarrow}} + {}_1M_{n+\varepsilon\cdot r}) = {}_1C_{n+1\cdot r}({}_1I_{n+1\cdot r} + 3\,{}_1\Delta y'_{n\cdot r}) + {}_1S_{n+1\cdot r}, \\ {}_1M_{\underset{n+\varepsilon\cdot r}{\longrightarrow}} &= -\{{}_1C_{n+1\cdot r}({}_1I_{n+1\cdot r} + 3\,{}_1\Delta y'_{n\cdot r}) + {}_1\bar{S}_{n+1\cdot r}\}, \\ {}_1M_{\underset{n+1-\varepsilon\cdot r}{\longrightarrow}} &= -({}_1M_{\underset{n+1-\varepsilon\cdot r}{\longleftarrow}} + {}_1M_{n+1-\varepsilon\cdot r}) = {}_1C_{n+1\cdot r}({}_1I'_{n+1\cdot r} + 3\,{}_1\Delta y'_{n\cdot r}) + {}_1K_{n+1\cdot r}, \\ {}_1M_{\underset{n+1-\varepsilon\cdot r}{\longleftarrow}} &= -\{{}_1C_{n+1\cdot r}({}_1I'_{n+1\cdot r} + 3\,{}_1\Delta y'_{n\cdot r}) + {}_1\bar{K}_{n+1\cdot r}\}, \\ {}_2M_{\underset{n\cdot r+\varepsilon}{\longleftarrow}} &= -({}_2M_{\underset{n\cdot r+\varepsilon}{\longrightarrow}} + {}_2M_{n\cdot r+\varepsilon}) = {}_2C_{n\cdot r+1}({}_2I_{n\cdot r+1} + 3\,{}_2\Delta y'_{n\cdot r}) + {}_2S_{n\cdot r+1}, \\ {}_2M_{\underset{n\cdot r+\varepsilon}{\longrightarrow}} &= -\{{}_2C_{n\cdot r+1}({}_2I_{n\cdot r+1} + 3\,{}_2\Delta y'_{n\cdot r}) + {}_2\bar{S}_{n\cdot r+1}\}, \end{aligned}\right\}$$

$$
\begin{aligned}
{}_2M_{\overrightarrow{n\cdot r+1-\varepsilon}} &= -({}_2M_{\overleftarrow{n\cdot r+1-\varepsilon}} + {}_2M_{n\cdot r+1-\varepsilon}) = {}_2C_{n\cdot r+1}({}_2I'_{n\cdot r+1} + 3\,{}_2\Delta y'_{n\cdot r}) + {}_2K_{n\cdot r+1}\,, \\
{}_2M_{\overleftarrow{n\cdot r+1-\varepsilon}} &= -\{{}_2C_{n\cdot r+1}({}_2I'_{n\cdot r+1} + 3\,{}_2\Delta y'_{n\cdot r}) + {}_2\overline{K}_{n\cdot r+1}\}\,,
\end{aligned} \tag{16}
$$

where

$$
{}_sC_{n\cdot r} = 2_s(EI/l)_{n\cdot r}\,, \quad {}_1I_{n\cdot r} = {}_1[i_{n\cdot r} + 2i_{n-1\cdot r}]\,, \quad {}_1I'_{n\cdot r} = {}_1[2i_{n\cdot r} + i_{n-1\cdot r}]\,,
$$

$$
{}_2I_{n\cdot r} = {}_2[i_{n\cdot r} + 2i_{n\cdot r-1}]\,, \quad {}_2I'_{n\cdot r} = {}_2[2i_{n\cdot r} + i_{n\cdot r-1}]\,,
$$

$$
{}_1S_{n+1\cdot r} = -V_2({}_1l_{n+1\cdot r})/{}_1l_{n+1\cdot r} + V_3({}_1l_{n+1\cdot r})/{}_1l^2_{n+1\cdot r}\,, \quad {}_1\overline{S}_{n+1\cdot r} = {}_1S_{n+1\cdot r} + {}_2M_{n+\varepsilon\cdot r}\,,
$$

$$
{}_2S_{n\cdot r+1} = -V_2({}_2l_{n\cdot r+1})/{}_2l_{n\cdot r+1} + V_3({}_2l_{n\cdot r+1})/{}_2l^2_{n\cdot r+1}\,,
$$

$$
{}_2\overline{S}_{n\cdot r+1} = {}_2S_{n\cdot r+1} + {}_2M_{n\cdot r+\varepsilon}\,, \quad {}_2K_{n\cdot r+1} = -U_2({}_2l_{n\cdot r+1})/{}_2l_{n\cdot r+1} + U_3({}_2l_{n\cdot r+1})/{}_2l^2_{n\cdot r+1}\,,
$$

$$
{}_2\overline{K}_{n\cdot r+1} = {}_2K_{n\cdot r+1} + {}_2M_{n\cdot r+1-\varepsilon}\,,
$$

and substituting Eqs. (5_2) and (7) into the 2nd brackets of Eqs. (6),

$$
\begin{aligned}
{}_2[T_{\overrightarrow{n\cdot r+\varepsilon}} + T_{\overleftarrow{n\cdot r-\varepsilon}}] &= {}_2[\Delta_1^2\, i_{n\cdot r+1} + \Delta_1^2\, Q_{n\cdot r+1}] + \frac{{}_1B_{n\cdot r+1}}{{}_2\lambda_{n\cdot r+1}} + \frac{{}_1B_{n\cdot r-1}}{{}_2\lambda_{n\cdot r}} \\
&\quad + {}_2\Gamma_{n\cdot r+1}\,{}_1[\Delta_1 i' + B'_{n\cdot r} + Q_{n\cdot r} + Q'_{n\cdot r}]\,,
\end{aligned}
$$

$$
\begin{aligned}
{}_1[T_{\overrightarrow{n+\varepsilon\cdot r}} + T_{\overleftarrow{n-\varepsilon\cdot r}}] &= {}_1[\Delta_2^2\, i_{n+1\cdot r} + \Delta_2^2\, Q_{n+1\cdot r}] + \frac{{}_2B_{n+1\cdot r}}{{}_1\lambda_{n+1\cdot r}} + \frac{{}_2B_{n-1\cdot r}}{{}_1\lambda_{n\cdot r}} \\
&\quad + {}_1\Gamma_{n+1\cdot r}\,{}_2[\Delta_2 i' + B'_{n\cdot r} + Q_{n\cdot r} + Q'_{n\cdot r}]\,,
\end{aligned}
$$

where

$$
{}_1\Gamma_{n+1\cdot r} = \frac{1}{{}_1\lambda_{n+1\cdot r}} + \frac{1}{{}_1\lambda_{n\cdot r}}\,, \qquad {}_2\Gamma_{n\cdot r+1} = \frac{1}{{}_2\lambda_{n\cdot r+1}} + \frac{1}{{}_2\lambda_{n\cdot r}}\,,
$$

$$
\begin{aligned}
{}_2\Delta_1^2\, i_{n\cdot r+1} &= {}_2\!\left[\frac{\Delta_1 i_{n\cdot r+1}}{\lambda_{n\cdot r+1}} - \frac{\Delta_1 i_{n\cdot r}}{\lambda_{n\cdot r}}\right] = ({}_1\Delta y_{n\cdot r+1} - {}_1\Delta y_{n\cdot r})/{}_2\lambda_{n\cdot r+1} \\
&\quad - ({}_1\Delta y_{n\cdot r} - {}_1\Delta y_{n\cdot r-1})/{}_2\lambda_{n\cdot r} = \frac{{}_1\Delta y_{n\cdot r+1}}{{}_2\lambda_{n\cdot r+1}} + \frac{{}_1\Delta y_{n\cdot r-1}}{{}_2\lambda_{n\cdot r}} - {}_2\Gamma_{n\cdot r+1}\,{}_1\Delta y_{n\cdot r}\,,
\end{aligned}
$$

$$
\begin{aligned}
{}_1\Delta_2^2\, i_{n+1\cdot r} &= {}_1\!\left[\frac{\Delta_2 i_{n+1\cdot r}}{\lambda_{n+1\cdot r}} - \frac{\Delta_2 i_{n\cdot r}}{\lambda_{n\cdot r}}\right] = ({}_2\Delta y_{n+1\cdot r} - {}_2\Delta y_{n\cdot r})/{}_1\lambda_{n+1\cdot r} \\
&\quad - ({}_2\Delta y_{n\cdot r} - {}_2\Delta y_{n-1\cdot r})/{}_1\lambda_{n\cdot r} = \frac{{}_2\Delta y_{n+1\cdot r}}{{}_1\lambda_{n+1\cdot r}} + \frac{{}_2\Delta y_{n-1\cdot r}}{{}_1\lambda_{n\cdot r}} - {}_1\Gamma_{n+1\cdot r}\,{}_2\Delta y_{n\cdot r}\,,
\end{aligned}
$$

$$
\begin{aligned}
{}_2\Delta_1^2\, Q_{n\cdot r+1} &= {}_2\!\left[\frac{\Delta_1 Q_{n\cdot r+1}}{\lambda_{n\cdot r+1}} - \frac{\Delta_1 Q_{n\cdot r}}{\lambda_{n\cdot r}}\right] = ({}_1Q_{n\cdot r+1} - {}_1Q_{n\cdot r})/{}_2\lambda_{n\cdot r+1} \\
&\quad - ({}_1Q_{n\cdot r} - {}_1Q_{n\cdot r-1})/{}_2\lambda_{n\cdot r} = \frac{{}_1Q_{n\cdot r+1}}{{}_2\lambda_{n\cdot r+1}} + \frac{{}_1Q_{n\cdot r-1}}{{}_2\lambda_{n\cdot r}} - {}_2\Gamma_{n\cdot r+1}\,{}_1Q_{n\cdot r}\,,
\end{aligned}
$$

$$
\begin{aligned}
{}_1\Delta_2^2\, Q_{n+1\cdot r} &= {}_1\!\left[\frac{\Delta_2 Q_{n+1\cdot r}}{\lambda_{n+1\cdot r}} - \frac{\Delta_2 Q_{n\cdot r}}{\lambda_{n\cdot r}}\right] = ({}_2Q_{n+1\cdot r} - {}_2Q_{n\cdot r})/{}_1\lambda_{n+1\cdot r} \\
&\quad - ({}_2Q_{n\cdot r} - {}_2Q_{n-1\cdot r})/{}_1\lambda_{n\cdot r} = \frac{{}_2Q_{n+1\cdot r}}{{}_1\lambda_{n+1\cdot r}} + \frac{{}_2Q_{n-1\cdot r}}{{}_1\lambda_{n\cdot r}} - {}_1\Gamma_{n+1\cdot r}\,{}_2Q_{n\cdot r}\,.
\end{aligned}
$$

Then, substituting the above equations and Eqs. (16) into Eqs. (6) and (8), the fundamental equations are rewritten in the following forms:—

$$\left.\begin{aligned}
&{}_1[\alpha_1 i_{n\cdot r-1} + \alpha_2 i_{n-1\cdot r} + \alpha_1' i_{n\cdot r+1} + \alpha_2' i_{n+1\cdot r} + (2\alpha_2 + \alpha_3') i_{n\cdot r}] \\
&\qquad - {}_1[\beta\, y_{n-1\cdot r} + (\beta' - \beta)\, y_{n\cdot r} - \beta'\, y_{n+1\cdot r}] = [{}_1H_{n\cdot r}]_0, \\
&{}_2[\alpha_1 i_{n-1\cdot r} + \alpha_2 i_{n\cdot r-1} + \alpha_1' i_{n+1\cdot r} + \alpha_2' i_{n\cdot r+1} + (2\alpha_2 + \alpha_3') i_{n\cdot r}] \\
&\qquad - {}_2[\beta\, y_{n\cdot r-1} + (\beta' - \beta)\, y_{n\cdot r} - \beta'\, y_{n\cdot r+1}] = [{}_2H_{n\cdot r}]_0, \\
&{}_1[\beta\, i_{n-1\cdot r} + (\beta - \beta')\, i_{n\cdot r} - \beta'\, i_{n+1\cdot r}] + {}_2[\beta\, i_{n\cdot r-1} + (\beta - \beta')\, i_{n\cdot r} - \beta'\, i_{n\cdot r+1}] \\
&\qquad + {}_1[\delta_1\, y_{n+1\cdot r} + \delta_2\, y_{n-1\cdot r}] + {}_2[\delta_1\, y_{n\cdot r+1} + \delta_2\, y_{n\cdot r-1}] \\
&\qquad\qquad + \delta\, y_{n\cdot r} = -(R_{n\cdot r} + \Theta_{n\cdot r}),
\end{aligned}\right\} \quad (17_0),$$

in which

$${}_1\alpha_1 = {}_2(GC/l)_{n\cdot r},\ {}_1\alpha_1' = {}_2(GC/l)_{n\cdot r+1},\ {}_1\alpha_2 = -2\,{}_1(EI/l)_{n\cdot r},$$
$${}_1\alpha_2' = -2\,{}_1(EI/l)_{n+1\cdot r},\ {}_1\alpha_3 = -\{4\,{}_1(EI/l)_{n\cdot r} + {}_2(GC/l)_{n\cdot r+1} + {}_2(GC/l)_{n\cdot r}\},$$
$${}_1\alpha_3' = -\{4\,{}_1(EI/l)_{n+1\cdot r} + {}_2(GC/l)_{n\cdot r+1} + {}_2(GC/l)_{n\cdot r}\},\ {}_1\beta = 6\,{}_1(EI/l^2)_{n\cdot r},$$
$${}_1\beta' = 6\,{}_1(EI/l^2)_{n+1\cdot r},\ {}_1\delta_1 = 12\,{}_1(EI/l^3)_{n+1\cdot r},\ {}_1\delta_2 = 12\,{}_1(EI/l^3)_{n\cdot r},$$
$${}_2\alpha_1 = {}_1(GC/l)_{n\cdot r},\ {}_2\alpha_1' = {}_1(GC/l)_{n+1\cdot r},\ {}_2\alpha_2 = -2\,{}_2(EI/l)_{n\cdot r},$$
$${}_2\alpha_2' = -2\,{}_2(EI/l)_{n\cdot r+1},$$
$${}_2\alpha_3 = -\{4\,{}_2(EI/l)_{n\cdot r} + {}_1(GC/l)_{n+1\cdot r} + {}_1(GC/l)_{n\cdot r}\},$$
$${}_2\alpha_3' = -\{4\,{}_2(EI/l)_{n\cdot r+1} + {}_1(GC/l)_{n+1\cdot r} + {}_1(GC/l)_{n\cdot r}\},$$
$${}_2\beta = 6\,{}_2(EI/l^2)_{n\cdot r},\ {}_2\beta' = 6\,{}_2(EI/l^2)_{n\cdot r+1},\ {}_2\delta_1 = 12\,{}_2(EI/l^3)_{n\cdot r+1},\ {}_2\delta_2 = 12\,{}_2(EI/l^3)_{n\cdot r},$$
$$\delta = -({}_1\delta_1 + {}_1\delta_2 + {}_2\delta_1 + {}_2\delta_2),$$ E = Young's Modulus, G = Torsional Rigidity, C = Torsional Constant,

$$\begin{aligned}
[{}_1H_{n\cdot r}]_0 = &-{}_1\Big[M_{n\cdot r} + {}_2\Gamma_{n\cdot r+1}(Q_{n\cdot r} + Q'_{n\cdot r}) + \frac{{}_2\Delta_1\, Q_{n\cdot r+1}}{{}_2\lambda_{n\cdot r+1}} - \frac{{}_2\Delta_1\, Q_{n\cdot r}}{{}_2\lambda_{n\cdot r}} \\
&+ \frac{{}_1\mu_{n\cdot r+1}}{{}_2\lambda_{n\cdot r+1}}(2\,K_{n\cdot r+1} - S_{n\cdot r+1}) + \frac{{}_1\mu_{n\cdot r-1}}{{}_2\lambda_{n\cdot r}}(2\,K_{n\cdot r-1} - S_{n\cdot r-1}) \\
&- (1 + 2\,{}_2\Gamma_{n\cdot r+1}\,{}_1\mu_{n+1\cdot r})\,\overline{S}_{n+1\cdot r} - {}_1\overline{K}_{n\cdot r} + {}_2\Gamma_{n\cdot r+1}\,{}_1\overline{K}_{n+1\cdot r}\Big],
\end{aligned}$$

$$\begin{aligned}
[{}_2H_{n\cdot r}]_0 = &-{}_2\Big[M_{n\cdot r} + {}_1\Gamma_{n+1\cdot r}(Q_{n\cdot r} + Q'_{n\cdot r}) + \frac{{}_1\Delta_2\, Q_{n+1\cdot r}}{{}_1\lambda_{n+1\cdot r}} - \frac{{}_1\Delta_2\, Q_{n\cdot r}}{{}_1\lambda_{n\cdot r}} \\
&+ \frac{{}_2\mu_{n+1\cdot r}}{{}_1\lambda_{n+1\cdot r}}(2\,K_{n+1\cdot r} - S_{n+1\cdot r}) + \frac{{}_2\mu_{n-1\cdot r}}{{}_1\lambda_{n\cdot r}}(2\,K_{n-1\cdot r} - S_{n-1\cdot r}) \\
&- (1 + 2\,{}_1\Gamma_{n+1\cdot r}\,{}_2\mu_{n\cdot r+1})\,\overline{S}_{n\cdot r+1} - \overline{K}_{n\cdot r} + {}_1\Gamma_{n+1\cdot r}\,{}_2\mu_{n\cdot r+1}\,\overline{K}_{n\cdot r+1}\Big],
\end{aligned}$$

$${}_1\Gamma_{n+1\cdot r} = \frac{1}{{}_1\lambda_{n+1\cdot r}} + \frac{1}{{}_1\lambda_{n\cdot r}}, \quad {}_2\Gamma_{n\cdot r+1} = \frac{1}{{}_2\lambda_{n\cdot r+1}} + \frac{1}{{}_2\lambda_{n\cdot r}},$$

$$\lambda_{n\cdot r} = (l/GC)_{n\cdot r}, \quad \mu_{n\cdot r} = (l/6EI)_{n\cdot r},$$

$${}_sS_{n\cdot r} = -V_2({}_sl_{n\cdot r})/{}_sl_{n\cdot r} + V_3({}_sl_{n\cdot r})/{}_sl^2_{n\cdot r},$$
$${}_sK_{n\cdot r} = -U_2({}_sl_{n\cdot r})/{}_sl_{n\cdot r} + U_3({}_sl_{n\cdot r})/{}_sl^2_{n\cdot r}, \quad {}_2\overline{K}_{n\cdot r} = {}_2K_{n\cdot r} + {}_2M_{n\cdot r-\varepsilon},$$

$$_2\overline{S}_{n\cdot r} = {}_2S_{n\cdot r} + {}_2M_{n\cdot r-1+\varepsilon}\,,\quad {}_1\overline{K}_{n\cdot r} = {}_1K_{n\cdot r} + {}_1M_{n-\varepsilon\cdot r}\,,\quad {}_1\overline{S}_{n\cdot r} = {}_1S_{n\cdot r} + {}_1M_{n-1+\varepsilon\cdot r}\,,$$

$$_2\Delta_1\, Q_{n\cdot r+1} = {}_1Q_{n\cdot r+1} - {}_1Q_{n\cdot r}\,,\quad {}_1\Delta_2\, Q_{n+1\cdot r} = {}_2Q_{n+1\cdot r} - {}_2Q_{n\cdot r}\,,$$

$$_sQ_{n\cdot r} = {}_s\left[-\frac{l_{n\cdot r}}{3}V_1(l_{n\cdot r}) + \frac{1}{2}V_2(l_{n\cdot r}) - \frac{1}{6l_{n\cdot r}}V_3(l_{n\cdot r})\right],$$

$$\Theta_{n\cdot r} = P_{n\cdot r} + {}_1\left[\frac{\overline{U}_1(l_{n\cdot r})}{l_{n\cdot r}} + \frac{\overline{V}_1(l_{n+1\cdot r})}{l_{n+1\cdot r}}\right] + {}_2\left[\frac{\overline{U}_1(l_{n\cdot r})}{l_{n\cdot r}} + \frac{\overline{V}_1(l_{n\cdot r+1})}{l_{n\cdot r+1}}\right]$$
$$+ \frac{1}{{}_1l_{n\cdot r}}({}_1K_{n\cdot r} + {}_1S_{n\cdot r}) - \frac{1}{{}_1l_{n+1\cdot r}}({}_1K_{n+1\cdot r} + {}_1S_{n+1\cdot r})$$
$$+ \frac{1}{{}_2l_{n\cdot r}}({}_2K_{n\cdot r} + {}_2S_{n\cdot r}) - \frac{1}{{}_2l_{n\cdot r+1}}({}_2K_{n\cdot r+1} + {}_2S_{n\cdot r+1})\,.$$

Next, at the points (n, r) on the boundary, we have forms slightly different from Eqs. (17_0), because some of Eqs. (5) and (7) do not hold as the previous discriptions. Then, let us show these equations at the points (n, r)on the boundary.

(i) case $r =$ max. or min.: —

In these cases, the boundary conditions are expressed by Eqs. (10_1) and the 2nd of Eqs. (5_2) does not hold, as the previous descriptions. Moreover, in the case ($r =$ min.), we take the following equations for the 2nd of Eqs. (7)

$$-_2[i'_{n\cdot r} - i'_{n-1\cdot r}] = {}_1[\lambda_{n\cdot r}\, T_{\underset{n-1+\varepsilon\cdot r}{\longrightarrow}}] = {}_1[\lambda_{n\cdot r}\, T_{\underset{n-\varepsilon\cdot r}{\longrightarrow}}]$$
$$= -_1[\Delta_2\, i''_{n\cdot r} + \Delta_2\, B''_{n\cdot r} + \Delta_2\, Q''_{n\cdot r}]$$

in which

$$_1\Delta_2\, i''_{n\cdot r} = {}_2\Delta y'_{n\cdot r} - {}_2\Delta y'_{n-1\cdot r} = (y_{n\cdot r} - y_{n\cdot r+1})/{}_2l_{n\cdot r+1}$$
$$- (y_{n-1\cdot r} - y_{n-1\cdot r+1})/{}_2l_{n-1\cdot r+1}$$

$$_1\Delta_2\, B''_{n\cdot r} = {}_2B'_{n\cdot r} - {}_2B'_{n-1\cdot r} = {}_2\mu_{n\cdot r+1}\, {}_2[2\, M_{\underset{n\cdot r+\varepsilon}{\longrightarrow}} - M_{\underset{n\cdot r+1-\varepsilon}{\longleftarrow}}]$$
$$- {}_2\mu_{n-1\cdot r+1}\, {}_2[2\, M_{\underset{n-1\cdot r+\varepsilon}{\longrightarrow}} - M_{\underset{n-1\cdot r+1-\varepsilon}{\longleftarrow}}]$$

$$_1\Delta_2\, Q''_{n\cdot r} = {}_2[Q'_{n\cdot r} - Q'_{n-1\cdot r}]$$

Then, considering the suffix of the unknown quantities in the fundamental equations and putting the quantities about the fictious members equal to zero, we have the following equations by similar ways as the case (i): —

$$\left.\begin{aligned}
&{}_1\left[\alpha_2'\, i_{n+1\cdot r} + (2\,\alpha_2 + \alpha_3')\, i_{n\cdot r} + \alpha_2\, i_{n-1\cdot r} + \begin{Bmatrix}\alpha_1\, i_{n\cdot r-1}\\ \alpha_1'\, i_{n\cdot r+1}\end{Bmatrix}\right]\\
&\quad + {}_1\left[\beta'(y_{n+1\cdot r} - y_{n\cdot r}) + \beta(y_{n\cdot r} - y_{n-1\cdot r})\right] = \begin{Bmatrix}[{}_1H_{n\cdot r}]_1' \\ [{}_1H_{n\cdot r}]_1''\end{Bmatrix}\begin{matrix}(r = \text{max.})\\ (r = \text{min.})\end{matrix}\,,\\
&{}_2\left[\alpha_1'\, i_{n+1\cdot r} + \alpha_1\, i_{n-1\cdot r} + \begin{Bmatrix}\alpha_3\, i_{n\cdot r} + \alpha_2\, i_{n\cdot r-1}\\ \alpha_3'\, i_{n\cdot r} + \alpha_2'\, i_{n\cdot r+1}\end{Bmatrix}\right]\\
&\quad + {}_2\left[\begin{Bmatrix}\beta(y_{n\cdot r} - y_{n\cdot r-1})\\ \beta'(y_{n\cdot r+1} - y_{n\cdot r})\end{Bmatrix}\right] = \begin{Bmatrix}[{}_2H_{n\cdot r}]_1' \\ [{}_2H_{n\cdot r}]_1''\end{Bmatrix}\begin{matrix}(r = \text{max.})\\ (r = \text{min.})\end{matrix}\,,
\end{aligned}\right\}$$

$$\left.\begin{aligned}&{}_1\!\left[\beta\, i_{n-1\cdot r}+(\beta-\beta')\, i_{n\cdot r}-\beta'\, i_{n-1\cdot r}\right]+{}_2\!\left[(\beta-\beta')\, i_{n\cdot r}+\left\{\begin{matrix}\beta\, i_{n\cdot r-1}\\ -\beta'\, i_{n\cdot r+1}\end{matrix}\right\}\right]\\ &\qquad+\delta y_{n\cdot r}+{}_1[\delta_1 y_{n+1\cdot r}+\delta_2 y_{n-1\cdot r}]+{}_2\!\left[\left\{\begin{matrix}\delta_2 y_{n\cdot r-1}\\ \delta_1 y_{n\cdot r+1}\end{matrix}\right\}\right]\\ &\qquad\qquad=-\left\{\begin{matrix}[R_{n\cdot r}+\Theta_{n\cdot r}]_1{}'\\ [R_{n\cdot r}+\Theta_{n\cdot r}]_1{}'\end{matrix}\right\}\quad\begin{matrix}(r=\max.)\\ (r=\min.)\end{matrix}\,,\end{aligned}\right\} \tag{17$_1$}$$

in which

$$ {}_1\alpha'_3=-\left\{4\,{}_1\!\left(\frac{EI}{l}\right)_{n+1\cdot r}+{}_2\!\left(\frac{GC}{l}\right)_{n\cdot r}\right\}\ (r=\max.)\,,$$

$$ {}_1\alpha_3'=-\left\{4\,{}_1\!\left(\frac{EI}{l}\right)_{n+1\cdot r}+{}_2\!\left(\frac{GC}{l}\right)_{n\cdot r+1}\right\}\ (r=\min.)\,,$$

$$\delta=-12\left\{{}_1\!\left(\frac{EI}{l^3}\right)_{n+1\cdot r}+{}_1\!\left(\frac{EI}{l^3}\right)_{n\cdot r}+{}_2\!\left(\frac{EI}{l^3}\right)_{n\cdot r}\right\}\ (r=\max.)\,,$$

$$\delta=-12\left\{{}_1\!\left(\frac{EI}{l^3}\right)_{n+1\cdot r}+{}_1\!\left(\frac{EI}{l^3}\right)_{n\cdot r}+{}_2\!\left(\frac{EI}{l^3}\right)_{n\cdot r+1}\right\}\ (r=\min.)\,,$$

$$\begin{aligned}[{}_1H_{n\cdot r}]_1{}'=-{}_1\!\Big[&M_{n\cdot r}+\frac{Q'_{n\cdot r}+Q_{n\cdot r-1}}{{}_2\lambda_{n\cdot r}}+\frac{{}_1\mu_{n\cdot r-1}}{{}_2\lambda_{n\cdot r}}(2\,K_{n\cdot r-1}-S_{n\cdot r-1})\\ &-\left(1+2\,\frac{{}_1\mu_{n+1\cdot r}}{{}_2\lambda_{n\cdot r}}\right)\overline{S}_{n+1\cdot r}-\overline{K}_{n\cdot r}+\frac{{}_1\mu_{n+1\cdot r}}{{}_2\lambda_{n\cdot r}}\overline{K}_{n+1\cdot r}\Big]\,,\end{aligned}$$

$$\begin{aligned}[{}_1H_{n\cdot r}]_1{}''=-{}_1\!\Big[&M_{n\cdot r}+\frac{Q_{n\cdot r+1}+Q'_{n\cdot r}}{{}_2\lambda_{n\cdot r+1}}+\frac{{}_1\mu_{n\cdot r+1}}{{}_2\lambda_{n\cdot r+1}}(2\,K_{n\cdot r+1}-S_{n\cdot r+1})\\ &-\left(1+2\,\frac{{}_1\mu_{n+1\cdot r}}{{}_2\lambda_{n\cdot r+1}}\right)\overline{S}_{n+1\cdot r}-\overline{K}_{n\cdot r}+\frac{{}_1\mu_{n+1\cdot r}}{{}_2\lambda_{n\cdot r+1}}\overline{K}_{n+1\cdot r}\Big]\,,\end{aligned}$$

$$\begin{aligned}[{}_2H_{n\cdot r}]_1{}'=-{}_2\!\Big[&M_{n\cdot r}+{}_1\Delta_2^2\,Q_{n+1\cdot r}-\overline{K}_{n\cdot r}+\frac{{}_2\mu_{n+1\cdot r}}{{}_1\lambda_{n+1\cdot r}}(2\,K_{n+1\cdot r}-S_{n+1\cdot r})\\ &+\frac{{}_2\mu_{n-1\cdot r}}{{}_1\lambda_{n\cdot r}}(2\,K_{n-1\cdot r}-S_{n-1\cdot r})-{}_1\Gamma_{n+1\cdot r}\,{}_2\mu_{n\cdot r}(2\,K_{n\cdot r}-S_{n\cdot r})\Big]\,,\end{aligned}$$

$$\begin{aligned}[{}_2H_{n\cdot r}]_1{}''=-{}_2\!\Big[&M_{n\cdot r}-{}_1\Delta_2^2\,Q''_{n+1\cdot r}-\overline{S}_{n\cdot r+1}+\frac{{}_2\mu_{n+1\cdot r+1}}{{}_1\lambda_{n+1\cdot r}}(2\,\overline{S}_{n+1\cdot r+1}-\overline{K}_{n+1\cdot r+1})\\ &-{}_1\Gamma_{n+1\cdot r}\,{}_2\mu_{n\cdot r+1}(2\,\overline{S}_{n\cdot r+1}-\overline{K}_{n\cdot r+1})+\frac{{}_2\mu_{n-1\cdot r+1}}{{}_1\lambda_{n\cdot r}}(2\,\overline{S}_{n-1\cdot r+1}-\overline{K}_{n-1\cdot r+1})\Big]\,,\end{aligned}$$

$$\begin{aligned}[Q_{n\cdot r}]_1{}'=P_{n\cdot r}&+{}_1\!\left[\overline{U}_1(l_{n\cdot r})/l_{n\cdot r}+\overline{V}_1(l_{n+1\cdot r})/l_{n+1\cdot r}\right]+\overline{U}_1({}_2l_{n\cdot r})/{}_2l_{n\cdot r}\\ &+\frac{1}{{}_1l_{n\cdot r}}({}_1K_{n\cdot r}+{}_1S_{n\cdot r})-\frac{1}{{}_1l_{n+1\cdot r}}({}_1K_{n+1\cdot r}+{}_1S_{n+1\cdot r})\\ &+\frac{1}{{}_2l_{n\cdot r}}({}_2K_{n\cdot r}+{}_2S_{n\cdot r})\,,\end{aligned}$$

$$[Q_{n\cdot r}]_1'' = P_{n\cdot r} + {}_1\Big[\bar{U}_1(l_{n\cdot r})/l_{n\cdot r} + \bar{V}_1(l_{n+1\cdot r})/l_{n+1\cdot r}\Big] + \bar{V}_1({}_2l_{n\cdot r+1})/{}_2l_{n\cdot r+1}$$

$$+ \frac{1}{{}_1l_{n\cdot r}}({}_1K_{n\cdot r} + {}_1S_{n\cdot r}) - \frac{1}{{}_1l_{n+1\cdot r}}({}_1K_{n+1\cdot r} + {}_1S_{n+1\cdot r})$$

$$- \frac{1}{{}_2l_{n\cdot r+1}}({}_2K_{n\cdot r+1} + {}_2S_{n\cdot r+1}),$$

$${}_1\Delta_2^2 Q''_{n+1\cdot r} = {}_1\left[\frac{\Delta_2 Q''_{n+1\cdot r}}{\lambda_{n+1\cdot r}} - \frac{\Delta_2 Q''_{n\cdot r}}{\lambda_{n\cdot r}}\right] = \frac{{}_2Q'_{n+1\cdot r} - {}_2Q'_{n\cdot r}}{{}_1\lambda_{n+1\cdot r}} - \frac{{}_2Q'_{n\cdot r} - {}_2Q'_{n-1\cdot r}}{{}_1\lambda_{n\cdot r}}$$

$$= \frac{{}_2Q'_{n+1\cdot r}}{{}_1\lambda_{n+1\cdot r}} + \frac{{}_2Q'_{n-1\cdot r}}{{}_1\lambda_{n\cdot r}} - {}_1\Gamma_{n+1\cdot r}\,{}_2Q'_{n\cdot r},$$

$${}_1\Delta_2^2 Q_{n+1\cdot r} = \frac{{}_2Q_{n+1\cdot r}}{{}_1\lambda_{n+1\cdot r}} + \frac{{}_2Q_{n-1\cdot r}}{{}_1\lambda_{n\cdot r}} - {}_1\Gamma_{n+1\cdot r}\,{}_2Q_{n\cdot r},$$

and the other coefficients coincide with that of Eqs. (17_0).

(ii) case $n = \text{max.}$ or min.: —

In these cases, the boundary conditions are expressed by Eqs. (10_2) and the 1st of Eqs. (5_2) does not hold as the previous descriptions. Moreover, in the case ($n = \text{min.}$), we take the following equations for the 1st of Eqs. (7)

$$-{}_1[i'_{n\cdot r} - i'_{n\cdot r-1}] = {}_2[\lambda_{n\cdot r} T_{\xrightarrow[n\cdot r-1+\varepsilon]{}}] = {}_2[\lambda_{n\cdot r} T_{\xrightarrow[n\cdot r-\varepsilon]{}}]$$

$$= -{}_2[\Delta_1 i''_{n\cdot r} + \Delta_1 B''_{n\cdot r} + \Delta_1 Q''_{n\cdot r}]$$

in which

$${}_2\Delta_1 i''_{n\cdot r} = {}_1\Delta y'_{n\cdot r} - {}_1\Delta y'_{n\cdot r-1} = (y_{n\cdot r} - y_{n+1\cdot r})/{}_1l_{n+1\cdot r}$$

$$- (y_{n\cdot r-1} - y_{n+1\cdot r-1})/{}_1l_{n+1\cdot r-1},$$

$${}_2\Delta_1 B''_{n\cdot r} = {}_1B'_{n\cdot r} - {}_1B'_{n\cdot r-1} = {}_1\mu_{n+1\cdot r}(2\,{}_1M_{\xrightarrow[n+\varepsilon\cdot r]{}} - {}_1M_{\xleftarrow[n+1-\varepsilon\cdot r]{}})$$

$$- {}_1\mu_{n+1\cdot r-1}(2\,{}_1M_{\xrightarrow[n+\varepsilon\cdot r-1]{}} - {}_1M_{\xleftarrow[n+1-\varepsilon\cdot r]{}}).$$

Then, considering the suffix of the unknown quantities in the fundamental equations, and putting the quantities about the fictious members equal to zero, we have the following equations by a similar method as in the treatments for case (i) and (ii): —

$$\left.\begin{aligned}
&{}_1\left[\alpha_1 i_{n\cdot r-1} + \alpha_1' i_{n\cdot r+1} + \begin{Bmatrix}\alpha_3\, i_{n\cdot r} + \alpha_2\, i_{n-1\cdot r}\\ \alpha_3'\, i_{n\cdot r} + \alpha_2'\, i_{n+1\cdot r}\end{Bmatrix}\right]\\
&\quad + \begin{Bmatrix}{}_1\beta\,(y_{n\cdot r} - y_{n-1\cdot r})\\ {}_1\beta'(y_{n+1\cdot r} - y_{n\cdot r})\end{Bmatrix} = \begin{Bmatrix}[{}_1H_{n\cdot r}]_2' & (n = \text{max.})\\ [{}_1H_{n\cdot r}]_2'' & (n = \text{min.})\end{Bmatrix},\\
&{}_2\left[\alpha_2 i_{n\cdot r-1} + \alpha_2' i_{n\cdot r+1} + (2\alpha_2 + \alpha_3')\, i_{n\cdot r} + \begin{Bmatrix}\alpha_1\, i_{n-1\cdot r}\\ \alpha_1'\, i_{n+1\cdot r}\end{Bmatrix}\right]\\
&\quad + {}_2[\beta(y_{n\cdot r} - y_{n\cdot r-1}) + \beta'(y_{n\cdot r+1} - y_{n\cdot r})]\\
&\quad = \begin{Bmatrix}[{}_2H_{n\cdot r}]_2' & (n = \text{max.})\\ [{}_2H_{n\cdot r}]_2'' & (n = \text{min.})\end{Bmatrix},
\end{aligned}\right\} \quad (17_2),$$

$$
{}_{2}\!\left[(\beta-\beta')\,i_{n\cdot r}+\begin{Bmatrix}\beta\ i_{n-1\cdot r}\\ -\beta'\,i_{n+1\cdot r}\end{Bmatrix}\right]-{}_{2}[\beta'\,i_{n\cdot r+1}+(\beta'-\beta)\,i_{n\cdot r}-\beta\,i_{n\cdot r-1}]
$$

$$
+\,\delta y_{n\cdot r}+{}_{2}[\delta_1\,y_{n\cdot r+1}+\delta_2 y_{n\cdot r-1}]+\begin{Bmatrix}{}_1\delta_2\,y_{n-1\cdot r}\\ {}_1\delta_1\,y_{n+1\cdot r}\end{Bmatrix}
$$

$$
=-\begin{Bmatrix}[R_{n\cdot r}+\Theta_{n\cdot r}]_2{}' & (n=\text{max.})\\ [R_{n\cdot r}+\Theta_{n\cdot r}]_2{}'' & (n=\text{min.})\end{Bmatrix},
$$

in which

$$
{}_1\alpha_3{}'=-\left\{4\,{}_1\!\left(\frac{EI}{l}\right)_{n+1\cdot r}+{}_2\!\left(\frac{GC}{l}\right)_{n\cdot r+1}\right\}\quad(n=\text{min.}),
$$

$$
{}_2\alpha_3{}'=-\left\{4\,{}_2\!\left(\frac{EI}{l}\right)_{n\cdot r+1}+{}_1\!\left(\frac{GC}{l}\right)_{n\cdot r}\right\}\quad(n=\text{max.}),
$$

$$
{}_2\alpha_3{}'=-\left\{4\,{}_2\!\left(\frac{EI}{l}\right)_{n\cdot r+1}+{}_1\!\left(\frac{GC}{l}\right)_{n+1\cdot r}\right\}\quad(n=\text{min.}),
$$

$$
[{}_1H_{n\cdot r}]_2{}'=-{}_1\!\Big[M_{n\cdot r}+{}_2\Delta_1{}^2\,Q_{n\cdot r+1}-{}_1\overline{K}_{n\cdot r}+\frac{{}_1\mu_{n\cdot r+1}}{{}_2\lambda_{n\cdot r+1}}(2K_{n\cdot r+1}-S_{n\cdot r+1})
$$

$$
+\frac{{}_1\mu_{n\cdot r-1}}{{}_2\lambda_{n\cdot r}}(2\,{}_1K_{n\cdot r-1}-{}_1S_{n\cdot r-1})-{}_2\Gamma_{n\cdot r+1}\,{}_1\mu_{n\cdot r}(2\,{}_1K_{n\cdot r}-{}_1S_{n\cdot r})\Big],
$$

$$
[{}_1H_{n\cdot r}]_2{}''=-{}_1\!\Big[M_{n\cdot r}-{}_2\Delta_1{}^2\,Q''_{n\cdot r+1}-{}_1\overline{S}_{n+1\cdot r}
$$

$$
+\frac{{}_1\mu_{n+1\cdot r+1}}{{}_2\lambda_{n\cdot r+1}}(2\,\overline{S}_{n+1\cdot r+1}-\overline{K}_{n+1\cdot r+1})+\frac{{}_1\mu_{n+1\cdot r-1}}{{}_2\lambda_{n\cdot r}}(2\,\overline{S}_{n+1\cdot r-1}-\overline{K}_{n+1\cdot r-1})
$$

$$
-\,{}_2\Gamma_{n\cdot r+1}\,{}_1\mu_{n+1\cdot r}(2\,\overline{S}_{n+1\cdot r}-\overline{K}_{n+1\cdot r})\Big],
$$

$$
[{}_2H_{n\cdot r}]_2{}'=-{}_2\!\Big[M_{n\cdot r}+\frac{Q'_{n\cdot r}+Q_{n-1\cdot r}}{{}_1\lambda_{n\cdot r}}+\frac{{}_2\mu_{n-1\cdot r}}{{}_1\lambda_{n\cdot r}}(2\,K_{n-1\cdot r}-S_{n-1\cdot r})
$$

$$
-\left(1+2\frac{{}_2\mu_{n\cdot r+1}}{{}_1\lambda_{n\cdot r}}\right)\overline{S}_{n\cdot r+1}-\overline{K}_{n\cdot r}+\frac{{}_2\mu_{n\cdot r+1}}{{}_1\lambda_{n\cdot r}}\,{}_2\overline{K}_{n\cdot r+1}\Big],
$$

$$
[{}_2H_{n\cdot r}]_2{}''=-{}_2\!\Big[M_{n\cdot r}+\frac{Q'_{n\cdot r}+Q_{n+1\cdot r}}{{}_1\lambda_{n+1\cdot r}}+\frac{{}_2\mu_{n+1\cdot r}}{{}_1\lambda_{n+1\cdot r}}(2\,K_{n+1\cdot r}-S_{n+1\cdot r})
$$

$$
-\left(1+2\frac{{}_2\mu_{n\cdot r+1}}{{}_1\lambda_{n+1\cdot r}}\right)\overline{S}_{n\cdot r+1}-\overline{K}_{n\cdot r}+\frac{{}_2\mu_{n\cdot r+1}}{{}_1\lambda_{n+1\cdot r}}\,{}_2\overline{K}_{n\cdot r+1}\Big],
$$

$$
[\Theta_{n\cdot r}]_2{}'=P_{n\cdot r}+\overline{U}_1({}_1l_{n\cdot r})/{}_1l_{n\cdot r}+{}_2[\overline{U}_1(l_{n\cdot r})/l_{n\cdot r}+\overline{V}_1(l_{n\cdot r+1})/l_{n\cdot r+1}]
$$

$$
+\frac{1}{{}_1l_{n\cdot r}}({}_1K_{n\cdot r}+{}_1S_{n\cdot r})+{}_2\!\Big[\frac{1}{l_{n\cdot r}}(K_{n\cdot r}+S_{n\cdot r})
$$

$$
-\frac{1}{l_{n\cdot r+1}}(K_{n\cdot r+1}+S_{n\cdot r+1})\Big],
$$

$$[\Theta_{n\cdot r}]_2'' = P_{n\cdot r} + \overline{V}_1({}_1l_{n+1\cdot r})/{}_1l_{n+1\cdot r} + {}_2[\overline{U}_1(l_{n\cdot r})/l_{n\cdot r} + \overline{V}_1(l_{n\cdot r+1})/l_{n\cdot r+1}]$$
$$- \frac{1}{{}_1l_{n+1\cdot r}}({}_1K_{n+1\cdot r} + {}_1S_{n+1\cdot r}) + {}_2\!\left[\frac{1}{l_{n\cdot r}}(K_{n\cdot r} + S_{n\cdot r}) - \frac{1}{l_{n\cdot r+1}}(K_{n\cdot r+1} + S_{n\cdot r+1})\right],$$

$$_2\Delta_1^2\, Q''_{n+1\cdot r} = {}_2\!\left[\frac{\Delta_1 Q''_{n\cdot r+1}}{\lambda_{n\cdot r+1}} - \frac{\Delta_1 Q''_{n\cdot r}}{\lambda_{n\cdot r}}\right] = \frac{{}_1Q'_{n\cdot r+1}}{{}_2\lambda_{n\cdot r+1}} + \frac{{}_1Q'_{n\cdot r-1}}{{}_2\lambda_{n\cdot r}} - {}_2\Gamma_{n\cdot r+1}\, {}_1Q'_{n\cdot r},$$

$$_2\Delta_1^2\, Q_{n+1\cdot r} = \frac{{}_1Q_{n\cdot r+1}}{{}_2\lambda_{n\cdot r+1}} + \frac{{}_1Q_{n\cdot r-1}}{{}_2\lambda_{n\cdot r}} - {}_2\Gamma_{n\cdot r+1}\, {}_1Q_{n\cdot r},$$

and the other coefficients coincide with that of Eqs. (17_0).

(iii) case $(n, r) = \text{max.}$ or min.:—

In these cases, the boundary conditions are expressed by Eqs. (10_3) and Eqs. (5_2) do not hold for the case $(n, r) = \text{max.}$, and Eqs. (5_2) and Eqs. (7) do not hold for the case $(n, r) = \text{min.}$ as the previous descriptions. Then, considering the suffix of the unknown quantities in the fundamental equations and putting the quantities about the fictious members equal to zero, we have the following equations by a similar method as the above cases:—

$$\left.\begin{aligned}
&{}_1[\alpha_3 i_{n\cdot r} + \alpha_2 i_{n-1\cdot r} + \alpha_1 i_{n\cdot r-1}] + {}_1\beta(y_{n\cdot r} - y_{n-1\cdot r}) = [{}_1H_{n\cdot r}]_3' \quad ((n\cdot r) = \text{max.}),\\
&{}_1[\alpha_3' i_{n\cdot r} + \alpha_2' i_{n+1\cdot r} + \alpha_1' i_{n\cdot r+1}] + {}_1\beta'(y_{n+1\cdot r} - y_{n\cdot r}) = [{}_1H_{n\cdot r}]_3'' \quad ((n\cdot r) = \text{min.}),\\
&{}_2[\alpha_3 i_{n\cdot r} + \alpha_2 i_{n\cdot r-1} + \alpha_1 i_{n-1\cdot r}] + {}_2\beta(y_{n\cdot r} - y_{n\cdot r-1}) = [{}_2H_{n\cdot r}]_3' \quad ((n\cdot r) = \text{max.}),\\
&{}_2[\alpha_3' i_{n\cdot r} + \alpha_2' i_{n\cdot r+1} + \alpha_1' i_{n+1\cdot r}] + {}_2\beta'(y_{n\cdot r+1} - y_{n\cdot r}) = [{}_2H_{n\cdot r}]_3'' \quad ((n\cdot r) = \text{min.}),\\
&{}_1[(\beta - \beta') i_{n\cdot r} + \beta i_{n-1\cdot r}] + {}_2[(\beta - \beta') i_{n\cdot r} + \beta i_{n\cdot r-1}]\\
&\qquad + \delta y_{n\cdot r} + {}_1\delta_2\, y_{n-1\cdot r} + {}_2\delta_2\, y_{n\cdot r-1} = -[R_{n\cdot r} + \Theta_{n\cdot r}]_3' \quad ((n\cdot r) = \text{max.}),\\
&{}_1[(\beta - \beta') i_{n\cdot r} - \beta' i_{n+1\cdot r}] + {}_2[(\beta - \beta') i_{n\cdot r} - \beta' i_{n\cdot r+1}]\\
&\qquad + \delta y_{n\cdot r+1} + {}_1\delta_1\, y_{n+1\cdot r} + {}_2\delta_1\, y_{n\cdot r+1} = -[R_{n\cdot r} + \Theta_{n\cdot r}]_3'' \quad ((n\cdot r) = \text{min.}),
\end{aligned}\right\} \quad (17_3),$$

in which

$$_1\alpha_3 = -\left\{4\,{}_1\!\left(\frac{EI}{l}\right)_{n\cdot r} + {}_2\!\left(\frac{GC}{l}\right)_{n\cdot r}\right\} \quad ((n\cdot r) = \text{max.}),$$

$$_1\alpha_3' = -\left\{4\,{}_1\!\left(\frac{EI}{l}\right)_{n+1\cdot r} + {}_2\!\left(\frac{GC}{l}\right)_{n\cdot r+1}\right\} \quad ((n\cdot r) = \text{min.}),$$

$$_2\alpha_3 = -\left\{4\,{}_2\!\left(\frac{EI}{l}\right)_{n\cdot r} + {}_1\!\left(\frac{GC}{l}\right)_{n\cdot r}\right\} \quad ((n\cdot r) = \text{max.}),$$

$$ {}_2\alpha_3' = -\left\{4\,{}_2\left(\frac{EI}{l}\right)_{n\cdot r+1} + {}_1\left(\frac{GC}{l}\right)_{n+1\cdot r}\right\} \quad ((n\cdot r) = \text{min.}), $$

$$ \delta = -12\left\{{}_1\left(\frac{EI}{l^3}\right)_{n\cdot r} + {}_2\left(\frac{EI}{l^3}\right)_{n\cdot r}\right\} \quad ((n\cdot r) = \text{max.}), $$

$$ \delta = -12\left\{{}_1\left(\frac{EI}{l^3}\right)_{n+1\cdot r} + {}_2\left(\frac{EI}{l^3}\right)_{n\cdot r+1}\right\} \quad ((n\cdot r) = \text{min.}), $$

$$ [{}_1H_{n\cdot r}]_3' = -{}_1\Big[M_{n\cdot r} + \frac{Q_{n\cdot r-1} - Q_{n\cdot r}}{{}_2\lambda_{n\cdot r}} - \overline{K}_{n\cdot r} + \frac{{}_1\mu_{n\cdot r-1}}{{}_2\lambda_{n\cdot r}}(2\,K_{n\cdot r-1} - S_{n\cdot r-1}) - \frac{{}_1\mu_{n\cdot r}}{{}_2\lambda_{n\cdot r}}(2\,K_{n\cdot r} - S_{n\cdot r})\Big], $$

$$ [{}_1H_{n\cdot r}]_3'' = -{}_1\Big[M_{n\cdot r} - \frac{Q'_{n\cdot r+1} - Q'_{n\cdot r}}{{}_2\lambda_{n\cdot r+1}} - \overline{S}_{n+1\cdot r} + \frac{{}_1\mu_{n+1\cdot r+1}}{{}_2\lambda_{n\cdot r+1}}(2\,\overline{S}_{n+1\cdot r+1} - \overline{K}_{n+1\cdot r+1}) - \frac{{}_1\mu_{n+1\cdot r}}{{}_2\lambda_{n\cdot r+1}}(2\,\overline{S}_{n+1\cdot r} - \overline{K}_{n+1\cdot r})\Big], $$

$$ [{}_2H_{n\cdot r}]_3' = -{}_2\Big[M_{n\cdot r} + \frac{Q_{n-1\cdot r}}{{}_1\lambda_{n\cdot r}} + \frac{{}_2\mu_{n-1\cdot r}}{{}_1\lambda_{n\cdot r}}(2\,K_{n-1\cdot r} - S_{n-1\cdot r}) - \overline{K}_{n\cdot r}\Big], $$

$$ [{}_2H_{n\cdot r}]_3'' = -{}_2\Big[M_{n\cdot r} + \frac{Q'_{n\cdot r}}{{}_1\lambda_{n+1\cdot r}} - \left(1 + 2\,\frac{{}_2\mu_{n\cdot r+1}}{{}_1\lambda_{n+1\cdot r}}\right)\overline{S}_{n\cdot r+1} + \frac{{}_2\mu_{n\cdot r+1}}{{}_1\lambda_{n+1\cdot r}}\overline{K}_{n\cdot r+1}\Big], $$

$$ [\Theta_{n\cdot r}]_3' = P_{n\cdot r} + \overline{U}_1({}_1l_{n\cdot r})/{}_1l_{n\cdot r} + \overline{U}_1({}_2l_{n\cdot r})/{}_2l_{n\cdot r} + \frac{1}{{}_1l_{n\cdot r}}({}_1K_{n\cdot r} + {}_1S_{n\cdot r}) + \frac{1}{{}_2l_{n\cdot r}}({}_2K_{n\cdot r} + {}_2S_{n\cdot r}), $$

$$ [\Theta_{n\cdot r}]_3'' = P_{n\cdot r} + \overline{V}_1({}_1l_{n+1\cdot r})/{}_1l_{n+1\cdot r} + \overline{V}_1({}_2l_{n\cdot r+1})/{}_2l_{n\cdot r+1} - \frac{1}{{}_1l_{n+1\cdot r}}({}_1K_{n+1\cdot r} + {}_1S_{n+1\cdot r}) - \frac{1}{{}_2l_{n\cdot r+1}}(K_{n\cdot r+1} + S_{n\cdot r+1}), $$

and the other coefficients coincide with that of Eqs. (17_0).

(iv) case (n = max. and r = min.) or (n = min. and r = max.): —
In these cases, the boundary conditions are expressed by Eqs. (10_4) and Eqs. (5_2) and the 1st of Eqs. (7) do not hold for the case (n = max. and r = min.) and Eqs. (5_2) and the 2nd of Eqs. (7) do not hold for the case (n = min. and r = max.) as the previous descriptions. Then, considering the suffix of the unknown quantities in the fundamental equations and putting the quantities about the fictious members equal to zero, we have the following equations by similar method as the above cases: —

$$ {}_1[\alpha_1'\,i_{n\cdot r+1} + \alpha_2\,i_{n-1\cdot r} + \alpha_3\,i_{n\cdot r}] + {}_1\beta(y_{n\cdot r} - y_{n-1\cdot r}) = [{}_1H_{n\cdot r}]_4' \quad (n = \text{max. and } r = \text{min.}) $$

$$ {}_1[\alpha_1\,i_{n\cdot r-1} + \alpha_2'\,i_{n+1\cdot r} + \alpha_3'\,i_{n\cdot r}] + {}_1\beta'(y_{n+1\cdot r} - y_{n\cdot r}) = [{}_1H_{n\cdot r}]_4'' \quad (n = \text{min. and } r = \text{max.}) $$

$$\left.\begin{aligned}
&{}_2[\alpha_1 i_{n-1\cdot r} + \alpha_2' i_{n\cdot r+1} + \alpha_3' i_{n\cdot r}] + {}_2[\beta'(y_{n\cdot r+1} - y_{n\cdot r})] = [{}_2H_{n\cdot r}]_4' \\
&\qquad\qquad (n = \text{max. and } r = \text{min.}) \\
&{}_2[\alpha_1' i_{n+1\cdot r} + \alpha_2 i_{n\cdot r-1} + \alpha_3 i_{n\cdot r}] + {}_2[\beta(y_{n\cdot r} - y_{n\cdot r-1})] = [{}_2H_{n\cdot r}]_4'' \\
&\qquad\qquad (n = \text{min. and } r = \text{max.}) \\
&{}_1[(\beta - \beta') i_{n\cdot r} + \beta i_{n-1\cdot r}] + {}_2[(\beta - \beta') i_{n\cdot r} - \beta' i_{n\cdot r+1}] \\
&\qquad + \delta y_{n\cdot r} + {}_1\delta_2 y_{n-1\cdot r} + {}_2\delta_1 y_{n\cdot r+1} = -[R_{n\cdot r} + \Theta_{n\cdot r}]_4' \\
&\qquad\qquad (n = \text{max. and } r = \text{min.}) \\
&{}_1[(\beta - \beta') i_{n\cdot r} - \beta' i_{n+1\cdot r}] + {}_2[(\beta - \beta') i_{n\cdot r} + \beta i_{n\cdot r-1}] \\
&\qquad + \delta y_{n\cdot r} + {}_1\delta_1 y_{n+1\cdot r} + {}_2\delta_2 y_{n\cdot r-1} = -[R_{n\cdot r} + \Theta_{n\cdot r}]_4'' \\
&\qquad\qquad (n = \text{min. and } r = \text{max.})
\end{aligned}\right\} \quad (17_4),$$

in which

$${}_1\alpha_3 = -\left\{4\,{}_1\left(\frac{EI}{l}\right)_{n\cdot r} + {}_2\left(\frac{GC}{l}\right)_{n\cdot r+1}\right\} \quad (n = \text{max. and } r = \text{min.}),$$

$${}_1\alpha_3' = -\left\{4\,{}_1\left(\frac{EI}{l}\right)_{n+1\cdot r} + {}_2\left(\frac{GC}{l}\right)_{n\cdot r}\right\} \quad (n = \text{min. and } r = \text{max.}),$$

$${}_2\alpha_3 = -\left\{4\,{}_2\left(\frac{EI}{l}\right)_{n\cdot r} + {}_1\left(\frac{GC}{l}\right)_{n+1\cdot r}\right\} \quad (n = \text{min. and } r = \text{max.}),$$

$${}_2\alpha_3' = -\left\{4\,{}_2\left(\frac{EI}{l}\right)_{n\cdot r+1} + {}_1\left(\frac{GC}{l}\right)_{n\cdot r}\right\} \quad (n = \text{max. and } r = \text{min.}),$$

$$\delta = -12\left\{{}_1\left(\frac{EI}{l^3}\right)_{n+1\cdot r} + {}_2\left(\frac{EI}{l^3}\right)_{n\cdot r}\right\} \quad (n = \text{min. and } r = \text{max.}),$$

$$\delta = -12\left\{{}_1\left(\frac{EI}{l^3}\right)_{n\cdot r} + {}_2\left(\frac{EI}{l^3}\right)_{n\cdot r+1}\right\} \quad (n = \text{max. and } r = \text{min.}),$$

$$[{}_1H_{n\cdot r}]_4' = -{}_1\left[M_{n\cdot r} + \frac{Q_{n\cdot r+1} - Q_{n\cdot r}}{{}_2\lambda_{n\cdot r+1}} - \overline{K}_{n\cdot r} + \frac{{}_1\mu_{n\cdot r+1}}{{}_2\lambda_{n\cdot r+1}}(2K_{n\cdot r+1} - S_{n\cdot r+1}) - \frac{{}_1\mu_{n\cdot r}}{{}_2\lambda_{n\cdot r+1}}(2K_{n\cdot r} - S_{n\cdot r})\right],$$

$$[{}_1H_{n\cdot r}]_4'' = -{}_1\left[M_{n\cdot r} - \frac{Q'_{n\cdot r-1} - Q'_{n\cdot r}}{{}_2\lambda_{n\cdot r}} - {}_1\overline{S}_{n+1\cdot r} + \frac{{}_1\mu_{n+1\cdot r-1}}{{}_2\lambda_{n\cdot r}}(2\overline{S}_{n+1\cdot r-1} - \overline{K}_{n+1\cdot r-1}) - \frac{{}_1\mu_{n+1\cdot r}}{{}_2\lambda_{n\cdot r}}(2\overline{S}_{n+1\cdot r} - \overline{K}_{n+1\cdot r})\right],$$

$$[{}_2H_{n\cdot r}]_4' = -{}_2\left[M_{n\cdot r} + \frac{Q'_{n\cdot r} - Q'_{n-1\cdot r}}{{}_1\lambda_{n\cdot r}} - {}_2\overline{S}_{n\cdot r+1} - \frac{{}_2\mu_{n\cdot r+1}}{{}_1\lambda_{n\cdot r}}(2\overline{S}_{n\cdot r+1} - \overline{K}_{n\cdot r+1}) + \frac{{}_2\mu_{n-1\cdot r+1}}{{}_1\lambda_{n\cdot r}}(2\overline{S}_{n-1\cdot r+1} - \overline{K}_{n-1\cdot r+1})\right],$$

$$[{}_2H_{n\cdot r}]_4'' = -\underset{2}{\Big[}M_{n\cdot r} + \frac{Q_{n+1\cdot r} - Q_{n\cdot r}}{{}_1\lambda_{n+1\cdot r}} - {}_2\overline{K}_{n\cdot r} + \frac{{}_2\mu_{n+1\cdot r}}{{}_1\lambda_{n+1\cdot r}}(2\,K_{n+1\cdot r} - S_{n+1\cdot r})$$
$$- \frac{{}_2\mu_{n\cdot r}}{\lambda_{n+1\cdot r}}(2\,K_{n\cdot r} - S_{n\cdot r})\Big],$$

$$[\Theta_{n\cdot r}]_4' = P_{n\cdot r} + \overline{U}_1({}_1l_{n\cdot r})/{}_1l_{n\cdot r} + \overline{V}_1({}_2l_{n\cdot r+1})/{}_2l_{n\cdot r+1}$$
$$+ \frac{1}{{}_1l_{n\cdot r}}({}_1K_{n\cdot r} + {}_1S_{n\cdot r}) - \frac{1}{{}_2l_{n\cdot r+1}}({}_2K_{n\cdot r+1} + {}_2S_{n\cdot r+1}),$$

$$[\Theta_{n\cdot r}]_4'' = P_{n\cdot r} + \overline{V}_1({}_1l_{n+1\cdot r})/{}_1l_{n+1\cdot r} + \overline{U}_1({}_2l_{n\cdot r})/{}_2l_{n\cdot r}$$
$$- \frac{1}{{}_1l_{n+1\cdot r}}({}_1K_{n+1\cdot r} + {}_1S_{n+1\cdot r}) + \frac{1}{{}_2l_{n\cdot r}}({}_2K_{n\cdot r} + {}_2S_{n\cdot r}),$$

and the other coefficients coincide with that of Eqs. (17_0).

Moreover, in such case when the end of the member is built, as well known, we have the following equations instead of Eqs. (17)

$$ {}_si_{n\cdot r} = 0, \qquad y_{n\cdot r} = 0 \qquad (17').$$

In conclusion, the above results tell us that the simultaneous equations (17) can always be solved by the well-known iteration methods and the solution suffices to determine one set of the quantities (${}_si_{n\cdot r}$, $y_{n\cdot r}$) at the intersecting points in the structure. Then it is clear that substituting these values into Eqs. (1) and (4), all of the requirements are directly found and the present problems are solved.

4. Conclusion. By the present method, the torsion and bending problems of continuous panel structures can always be solved when the deflections of the members of the structure are all small. But the method, as it is, cannot be used for the large deflections, beacuse both of the restriction conditions at the supported points and the boundary conditions become far more complex than stated in the report. Moreover, when the size of the cross-sections of the members are large, the restrictions received by the deformations occured in the cross-sections of the members under the external forces are sensible quantities, so that we need consider these points fully in the practical designs of the structures.

Further, it is also obvious that vector analysis can be applied for the moments, as well known, when deflections of the members are small, so that the theory and the method presented in this report are also applicable to such cases when the members of the structures cross each other with angles slightly different from rectangle by modifying the coefficients of the fundamental equations stated in the report.

Last, in this report we neglected the effect which must appear when an external twisting moment acts on a member of the structure at any point. Thus, we are still in the course of research of the theory in which the effect of fhe twisting moment stated above is considered. And, moreover, it is intended to calculate some actual examples by the method mentioned in this report.

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