An Approximate Calculation of the Laminar Boundary Layer of a Flat Plate of a Constant Breadth

Okabe, Jun-ichi
Research Institute for Applied Mechanics, Kyushu University

https://doi.org/10.5109/7156998

出版情報：Reports of Research Institute for Applied Mechanics. 2 (6), pp.80-85, 1953-06. 九州大学応用力学研究所

Reports of Research Institute for Applied Mechanics
Vol. II, No. 6 June, 1953

An Approximate Calculation of the Laminar Boundary Layer of a Flat Plate of a Constant Breadth

By Jun-ichi OKABE

1. Subsequent to the paper by Howarth concerning Rayleigh's problem for a semi-infinite plate,[1] Hashimoto published an elaborate computation of the similar problem for a plate of finite breadth.[2] Namely he showed thatthe vorticity of the fluid can be found as a solution of a certain integral equation, and by connecting smoothly two kinds of its asymptotic solutions which correspond to $t \to 0$ and $t \to \infty$ (t = time), he could tabulate the total skin friction of the plate for various values of the parameter.[3] Independently, the present writer who has been interested in the problem of the title formulated another type of the solution of the extended Rayleigh's problem. This solution is simpler than that of Hashimoto but is valid only in a very limited region. The idea underlying the method was explained in the previous note.[4]

2. The basic equation is

$$n^2 U \frac{\partial u_1}{\partial x} = \nu \left(\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2} \right). \tag{2.1}$$

The origin of the coördinates is placed at the center of the leading edge of the plate, whose breadth is denoted by $2a$. The x-, and the z-axes are taken in the plane of the plate and the remaining y-axis perpendicularly to both of them. The general flow, U, coincides with the x-direction; u_1 is the x-component of the velocity of the fluid, and n^2 is a constant less than unity.[5] Transforming these quantities by the relations

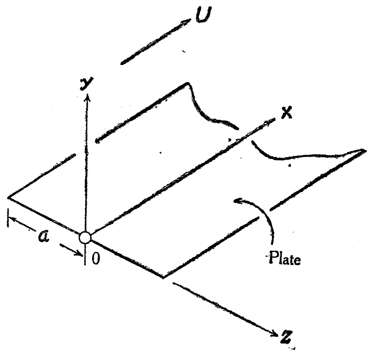

Fig. 1.

$$\begin{aligned} u_1 &= U u, \quad R = U a / \nu, \\ x &= \frac{n^2 U a^2}{2\nu} \xi = \frac{n^2 a}{2} R \xi, \\ y &= a \sinh \eta \cdot \sin \zeta, \\ z &= a \cosh \eta \cdot \cos \zeta, \end{aligned} \tag{2.2}$$

1) Howarth, L. Rayleigh's Problem for a Semi-infinite Plate. *Proc. Camb. Phil. Soc.* 46 (1950), 127.

2) Hashimoto, H. Rayleigh's Problem for a Plate of Finite Breadth. *Proc. 1st Japan National Congress App. Mech.* (1951), 447.

3) Table 1, *loc. cit.*

4) Okabe, J. Modified Oseen's Equation for a Flow Past a Semi-infinite Plate. *Rep. Res. Inst. App. Mech.*, vol. 1, No. 4 (1952), *Note*, 151.

5) Yamada, H. and Okabe, J. An Approximate Calculation of the Laminar Wake Behind a Flat Plate. *Rep. Res. Inst. Fluid Engg*, vol. 4, No. 1 (1947), 1. Okabe, J. *op. cit.*

we obtain from (2.1) the following equation of the non-dimensional quantities u, ξ, η and ζ:

$$(\cosh 2\eta - \cos 2\zeta)\frac{\partial u}{\partial \xi} = \frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 u}{\partial \zeta^2}. \tag{2.3}$$

Since the formal solution of this equation will contain the Mathieu functions which are not convenient for numerical calculations, let us take a different way by putting

$$u = 2\gamma - 2\gamma^3 + \gamma^4,$$

where

$$\gamma = \eta/\delta(\xi, \zeta). \tag{2.4}$$

δ is the thickness of the boundary layer in elliptic coördinates and is therefore a function of ξ and ζ. (2.4) has the same form as the approximate expression for the velocity profile in the two-dimensional boundary layer of a flat plate,[1] and satisfies the following conditions:

$$u = \frac{d^2u}{d\gamma^2} = 0 \quad \text{at} \quad \gamma = 0 \quad (\text{i.e.} \quad \eta = 0),$$

and

$$u = 1, \quad \frac{du}{d\gamma} = \frac{d^2u}{d\gamma^2} = 0 \quad \text{at} \quad \gamma = 1 \quad (\text{i.e} \quad \eta = \delta).$$

If we substitute (2.4) for u in (2.3) and integrate the both hand sides of the equation with regard to γ from zero to unity, we can arrive at the equation

$$\{6\delta^5 \cos 2\zeta - 30(\delta^2 + 2)\sinh 2\delta + 75\delta \cosh 2\delta - 10\delta^3 + 45\delta\}\partial\delta/\partial\xi$$
$$+ 40\delta^4 + 6\delta^5\, \partial^2\delta/\partial\zeta^2 = 0. \tag{2.5}$$

The variable ξ, however, derived from x by the relation (2.2) is usually a very small quantity, since R, the Reynolds number, is quite large in practical problems. δ for small values of ξ must be likewise small, and if we are interested in only such a value of ξ as is so small that the higher powers of δ can be safely neglected, after expanding each term and reärranging, (2.5) can be found to reduce into the form

$$\left(12\delta \sin^2\zeta + \frac{20}{7}\delta^3\right)\frac{\partial\delta}{\partial\xi} = 40. \tag{2.6}$$

Integrating with the initial condition that $\delta = 0$ at $\xi = 0$, we have

$$\delta = \left\{\left(\frac{441}{25}\sin^4\zeta + 56\xi\right)^{1/2} - \frac{21}{5}\sin^2\zeta\right\}^{1/2}. \tag{2.7}$$

It must be remembered that (2.7) is valid only in the range of ξ where δ^2 and $\delta \cdot \partial^2\delta/\partial\zeta^2$ are small compared with unity.

The skin friction $\tau = \rho\nu(\partial u_1/\partial y)_{y=0}$ can be proved to be identical to

$$\tau = \rho\nu\frac{U}{a \sin\zeta}\left(\frac{\partial u}{\partial\eta}\right)_{\eta=0} = \rho U^2 \frac{2}{R\delta \sin\zeta} \tag{2.8}$$

from the relation (2.2). In the following table, δ, $y/a = \sinh\delta \cdot \sin\zeta$, $z/a = \cosh\delta \cdot \cos\zeta$ (the contour of the outer edge of the boundary layer) and $R\tau/\rho U^2$ are shown for several small values of ξ.

---

1) Goldstein, S. *Modern Developments in Fluid Dynamics* (1938), vol. 1, **60**, p. 156.

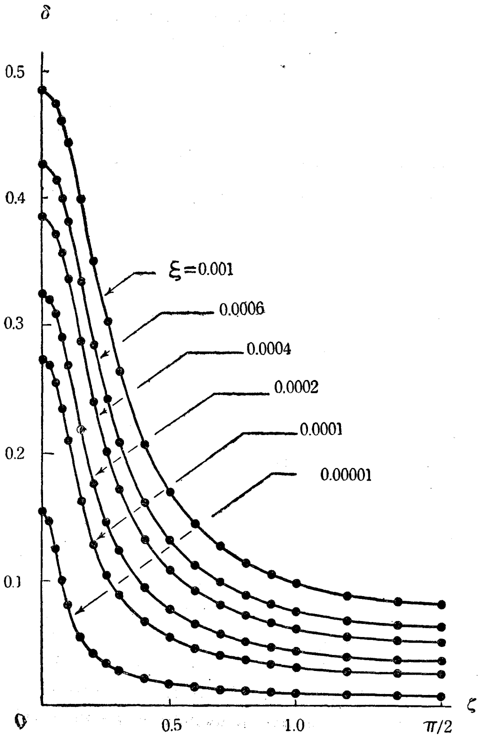

Fig. 2. (2.7)

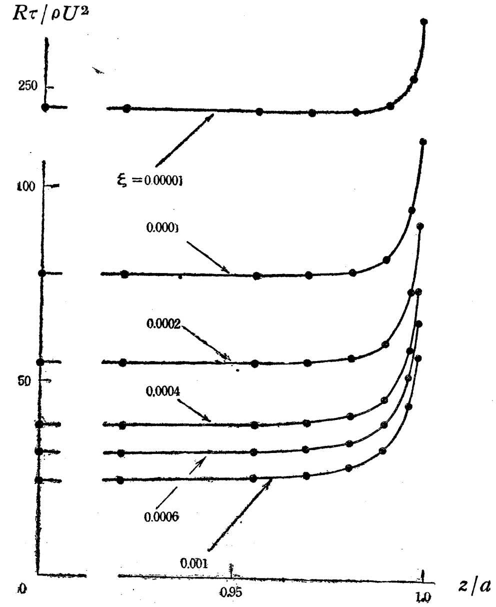

Fig. 3. (2.8)

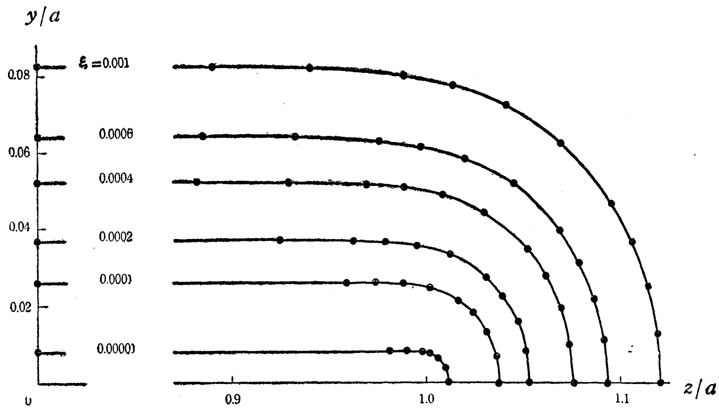

Fig. 4. Contours of Boundary Layer Thickness

3. Finally, it will be of some interest to derive the expressions corresponding to the limiting case $a \to \infty$, in other words the two-dimensional flow along a flat plate, and to compare the result with the well-known theory of Blasius or of Pohlhausen.

In the neighborhood of $\zeta = \pi/2$, from (2.7) δ is given approximately by

$$\delta = \sqrt{\frac{20}{3}\xi}\ \mathrm{cosec}\ \zeta\ ,$$

when a increases indefinitely accordingly ξ tends to zero. From (2.2), therefore, the thickness of the boundary layer in the cartesian coördinates (x, y, z) is obviously

$$y = a\sqrt{\frac{20}{3}\xi}\ .$$

But making use of the relation between ξ and x, from the above equality we are led to

$$y\sqrt{\frac{U}{\nu x}} = \frac{1}{n}\sqrt{\frac{40}{3}}\ . \tag{3.1}$$

Next, taking the limiting form of (2.8) we can prove the following:

$$\frac{\tau}{\rho U^2}\sqrt{\frac{U x}{\nu}} = n\sqrt{\frac{3}{10}}\ . \tag{3.2}$$

Now, in the joint paper mentioned before where the two-dimensional flow is calculated, it is concluded that the constant n must have the numerical value $(\sqrt{2}-1)^{1/2} = 0.64359\cdots$ when the agreement of the velocity profiles as a whole is aimed at between the approximate theory and the exact result by Blasius, while on the other hand if the agreement of the intensity of the skin friction with Blasius is preferable we have to choose $n = 0.58856\cdots$ alternatively.[1] Since we are provided with no sufficient ground to assure that these values of n may be preserved when we pass to a three-dimensional flow nor reliable data to modify the values of n, let us assume tentatively $n = 0.64359\cdots$ invariably as in a two-dimensional problem. The numerical values of $y(U/\nu x)^{1/2}$ and $\tau/\rho U^2(U x/\nu)^{1/2}$ thus calculated from (3.1) and (3.2) are shown together with the corresponding ones of Blasius and of Pohlhausen.[2]

		(3.1)	Blasius	Pohlhausen
$y\sqrt{\frac{U}{\nu x}}$	:	5.67	· ·	5.83
		(3.2)	Blasius	Pohlhausen
$\frac{\tau}{\rho U^2}\sqrt{\frac{U x}{\nu}}$	:	0.353[3]	0.332	0.343

It is concluded therefore that the value of n need not be modified sensibly when we pass from a two-dimensional problem to a three-dimensional one.

(Received March 24, 1953)

1) Yamada, H. and Okabe, J. *loc. cit.* on page 80.

2) Goldstein, S. *ibid.*, vol. 1, **60**, p. 157.

3) 0.322 when $n = 0.58856\cdots$.

	$\xi = 0.00001$				$\xi = 0.0001$				$\xi = 0.0002$			
ζ	δ	y/a	z/a	$R\tau/\rho U^2$	δ	y/a	z/a	$R\tau/\rho U^2$	δ	y/a	z/a	$R\tau/\rho U^2$
0	0.1538	0	1.0119	∞	0.2736	0	1.0378	∞	0.3253	0	1.0533	∞
0.025	0.1455	0.0037	1.0104	550	0.2688	0.0068	1.0361	297.6	0.3213	0.0082	1.0516	249.0
0.05	0.1241	0.0062	1.0064	323	0.2551	0.0129	1.0314	156.9	0.3096	0.0157	1.0471	129.2
0.075	0.0991	0.0074	1.0021	269	0.2343	0.0177	1.0246	113.9	0.2913	0.0221	1.0397	91.6
0.1	0.0789	0.0079	0.9981	254	0.2095	0.0210	1.0168	95.6	0.2682	0.0271	1.0310	74.7
0.15	0.0542	0.0081	0.9902	247	0.1618	0.0243	1.0018	82.7	0.2182	0.0328	1.0124	61.3
0.2	0.0410	0.0082	0.9809	246	0.1269	0.0253	0.9880	79.3	0.1758	0.0351	0.9953	57.3
0.25	0.0330	0.0082	0.9694	245	0.1033	0.0256	0.9741	78.3	0.1447	0.0359	0.9791	55.9
0.3	0.0276	0.0082	0.9557	245	0.0869	0.0257	0.9590	77.9	0.1223	0.0362	0.9625	55.3
0.4	0.0210	0.0082	0.9213	245	0.0662	0.0258	0.9231	77.6	0.0934	0.0365	0.9251	55.0
0.5	0.0170	0.0082	0.8777	245	0.0538	0.0258	0.8789	77.5	0.0761	0.0365	0.8801	54.9
0.6	0.0145	0.0082	0.8254	245	0.0457	0.0258	0.8262	77.5	0.0646	0.0365	0.8271	54.8
0.7	0.0127	0.0082	0.7649	245	0.0401	0.0258	0.7655	77.5	0.0567	0.0365	0.7661	54.8
0.8	0.0114	0.0082	0.6968	245	0.0360	0.0258	0.6972	77.5	0.0509	0.0365	0.6976	54.8
0.9	0.0104	0.0082	0.6216	245	0.0330	0.0258	0.6220	77.5	0.0466	0.0365	0.6223	54.8
1.0	0.0097	0.0082	0.5403	245	0.0307	0.0258	0.5406	77.5	0.0434	0.0365	0.5408	54.8
1.2	0.0088	0.0082	0.3624	245	0.0277	0.0258	0.3625	77.5	0.0392	0.0365	0.3626	54.8
1.4	0.0083	0.0082	0.1700	245	0.0262	0.0258	0.1700	77.5	0.0371	0.0365	0.1701	54.8
$\pi/2$	0.0082	0.0082	0	245	0.0258	0.0258	0	77.5	0.0365	0.0365	0	54.8

ζ	$\xi = 0.0004$ δ	y/a	z/a	$R\tau/\rho U^2$	$\xi = 0.0006$ δ	y/a	z/a	$R\tau/\rho U^2$	$\xi = 0.001$ δ	y/a	z/a	$R\tau/\rho U^2$
0	0.3869	0	1.0758	∞	0.4281	0	1.0930	∞	0.4865	0	1.1204	∞
0.025	0.3835	0.0098	1.0739	208.6	0.4251	0.0110	1.0913	188.2	0.4838	0.0126	1.1191	165.4
0.05	0.3736	0.0191	1.0694	107.1	0.4161	0.0214	1.0864	96.2	0.4758	0.0247	1.1141	84.1
0.075	0.3577	0.0274	1.0618	74.6	0.4015	0.0309	1.0789	66.5	0.4629	0.0360	1.1060	57.7
0.1	0.3370	0.0343	1.0520	59.5	0.3823	0.0391	1.0685	52.4	0.4455	0.0459	1.0952	45.0
0.15	0.2878	0.0436	1.0301	46.5	0.3348	0.0510	1.0448	40.0	0.4009	0.0615	1.0693	33.4
0.2	0.2399	0.0481	1.0084	42.0	0.2853	0.0574	1.0201	35.3	0.3509	0.0712	1.0411	28.7
0.25	0.2010	0.0501	0.9886	40.2	0.2422	0.0605	0.9974	33.4	0.3039	0.0764	1.0140	26.6
0.3	0.1713	0.0509	0.9693	39.5	0.2080	0.0619	0.9761	32.5	0.2640	0.0789	0.9888	25.6
0.4	0.1317	0.0514	0.9291	39.0	0.1608	0.0629	0.9330	31.9	0.2063	0.0808	0.9407	24.9
0.5	0.1074	0.0516	0.8826	38.8	0.1313	0.0632	0.8851	31.8	0.1691	0.0814	0.8901	24.7
0.6	0.0913	0.0516	0.8288	38.8	0.1117	0.0632	0.8305	31.7	0.1440	0.0816	0.8339	24.6
0.7	0.0801	0.0517	0.7673	38.8	0.0980	0.0633	0.7685	31.7	0.1265	0.0817	0.7709	24.6
0.8	0.0719	0.0517	0.6985	38.8	0.0881	0.0633	0.6994	31.7	0.1137	0.0817	0.7012	24.5
0.9	0.0659	0.0517	0.6230	38.7	0.0807	0.0633	0.6236	31.6	0.1041	0.0817	0.6250	24.5
1.0	0.0613	0.0517	0.5413	38.7	0.0751	0.0633	0.5418	31.6	0.0970	0.0817	0.5429	24.5
1.2	0.0554	0.0517	0.3629	38.7	0.0678	0.0633	0.3632	31.6	0.0876	0.0817	0.3638	24.5
1.4	0.0524	0.0517	0.1702	38.7	0.0642	0.0633	0.1703	31.6	0.0828	0.0817	0.1706	24.5
$\pi/2$	0.0516	0.0517	0	38.7	0.0632	0.0633	0	31.6	0.0816	0.0817	0	24.5