An Approximate Calculation of the Laminar Boundary Layer of a Flat Plate Having a Parabolic Plan Form

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By Jun-ichi OKABE

1. Let us start our discussion from the same basic equation

$$n^2 U \frac{\partial u_1}{\partial x} = \nu \left(\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2} \right) \tag{1.1}$$

in terms of the same notations as in the foregoing note.[1] The origin of the coördinates is situated at the vertex of a flat plate having the parabolic plan form in the x, z-plane in the x, z-plane which is defined by the equation

$$z^2 = 4cx, \tag{1.2}$$

where c, a constant, denotes the focal distance. If we pass from the cartesian coördinates (x, y, z) to a new system (η, ζ) derived by the relations

$$y = \sqrt{4c\,x}\, \sinh \eta \cdot \sin \zeta,$$

and

$$z = \sqrt{4c\,x}\, \cosh \eta \cdot \cos \zeta, \tag{1.3}$$

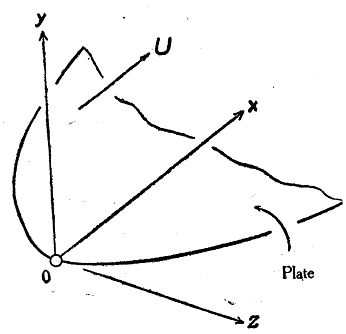

Fig. 1.

we can prove without difficulty that the equation (1.1) is transformed into

$$-\sinh 2\eta \frac{\partial u}{\partial \eta} + \sin 2\zeta \frac{\partial u}{\partial \zeta} = \frac{1}{R_0}\left(\frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 u}{\partial \zeta^2} \right), \tag{1.4}$$

where $R_0 = n^2 U c / \nu$.

Now, it would be quite natural to expect that the velocity, u, is a function of η only and independent of ζ, in other words, that the contours for equal values of u are the ellipses coincident with $\eta =$ constant. Owing to this assumption — we shall discuss about it in the next section — we can write (1.4) in the form

$$\frac{d\theta}{d\eta} + R_0\, \theta \sinh 2\eta = 0, \tag{1.5}$$

where $\theta = du/d\eta$.

The solution which satisfies the boundary conditions

$$u = 0 \quad \text{at} \quad \eta = 0$$

and

$$u = 1 \quad \text{at} \quad \eta = \infty$$

is readily found to be

1) Okabe, J. An Approximate Calculation of the Laminar Boundary Layer of a Flat Plate of a Constant Breadth. *Rep. Res. Inst. App. Mech.*, vol. 2, No. 6. (1953), *Note* 80.

$$u = \int_0^\eta \exp\left(-\frac{R_0}{2}\cosh 2t\right) dt \Big/ \int_0^\infty \exp\left(-\frac{R_0}{2}\cosh 2t\right) dt . \tag{1.6}$$

Using the formula

$$K_0(z) = \int_0^\infty e^{-z\cosh\theta}\, d\theta \,,^{1)}$$

we have

$$u = 2\int_0^\eta \exp\left(-\frac{R_0}{2}\cosh 2t\right) dt \Big/ K_0\left(\frac{R_0}{2}\right), \tag{1.7}$$

where K_0 denotes the modified Bessel function of the order zero.

2. With a view to examining the validity of the assumption ($\partial u/\partial\zeta = 0$) made in 1 on one hand, and the accuracy of the method described in the foregoing note[2] on the other, we shall substitute for u in (1.4) the same polynomial as before, viz.

$$u = 2\gamma - 2\gamma^3 + \gamma^4, \qquad \text{where} \quad \gamma = \eta/\delta, \tag{2.1}$$

δ being the thickness of the boundary layer in elliptic coördinates and in general a function of ζ. Integrating the both hand sides of the equation thus obtained with respect to γ from zero to unity, we are led to the equation

$$R_0(15\delta\cosh 2\delta - 15\sinh 2\delta - 10\delta^3 + 15\delta + 3\delta^4\delta'\sin 2\zeta) = 20\delta^3 + 3\delta^4\delta'', \tag{2.2}$$

where we write for simplicity δ' and δ'' in place of $d\delta/d\zeta$ and $d^2\delta/d\zeta^2$, respectively. This is the differential equation which defines δ as a function of $\zeta (0 \leqq \zeta \leqq \pi/2)$. However, since the boundary conditions imposed upon δ are

$$\delta' = 0 \quad \text{at} \quad \zeta = 0 \text{ and } \pi/2, \tag{2.3}$$

the ordinary numerical integration of a differential equation is not easily applicable to our problem.

Because we may presume that the general behavior of δ defined by (2.2) as a function of ζ will not possibly be influenced by a particular value of R_0, and, further, because we have to resort to a numerical method in what follows, we shall make a scrutiny into the case $R_0 = 1$ as an example, assuming that the general conclusion obtained in this case will hold for all values of R_0.

When $R_0 = 1$, (2.2) reduces to

$$\delta^4\delta'' - \delta^4\delta'\sin 2\zeta - 5\delta\cosh 2\delta + 5\sinh 2\delta + 10\delta^3 - 5\delta = 0 . \tag{2.4}$$

The simplest assumption compatible with the boundary conditions (2.3) is obviously

$$\delta = \text{constant} \equiv \varepsilon, \tag{2.5}$$

and on substituting in (2.4) we find that the value of ε is determined as a positive root of the transcendental equation

$$\varepsilon\cosh 2\varepsilon - \sinh 2\varepsilon - 2\varepsilon^3 + \varepsilon = 0,$$

i.e.

$$\varepsilon = 1.521. \tag{2.6}$$

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1) Watson, G. *Theory of Bessel Functions* (Cambridge, 1922), 6·15, (5), p. 172.

2) *Op. cit.* on page 86.

Making another step forward, let us put

$$\delta = \varepsilon + \lambda \cos 2\zeta \tag{2.7}$$

as the second approximation satisfying (2.3), ε and λ being assumed constant. After substituting (2.7) for δ in (2.4) we integrate the both hand sides with regard to ζ from zero to $\pi/2$, then if we make use of the formulas

$$\int_0^{\pi/2} \cos^2 2\zeta \cdot \sin^2 2\zeta \, d\zeta = \frac{\pi}{16},$$

$$\int_0^{\pi/2} \cos^4 2\zeta \cdot \sin^2 2\zeta \, d\zeta = \frac{\pi}{32},$$

$$\int_0^{\pi/2} \exp(2\lambda \cos 2\zeta) \, d\zeta = \frac{\pi}{2} I_0(2\lambda),$$

$$\int_0^{\pi/2} \exp(2\lambda \cos 2\zeta) \cos 2\zeta \, d\zeta = \frac{\pi}{2} I_1(2\lambda), \quad \text{etc.,}$$

where I_0 and I_1 are the modified Bessel functions, we have finally

$$\begin{aligned} &40 I_0(2\lambda)(\sinh 2\varepsilon - \varepsilon \cosh 2\varepsilon) - 40\lambda I_1(2\lambda) \sinh 2\varepsilon + \lambda^5 - 48\varepsilon\lambda^4 + 12\varepsilon^2\lambda^3 \\ &\quad - (64\varepsilon^3 - 120\varepsilon)\lambda^2 + 8\varepsilon^4\lambda + 80\varepsilon^3 - 40\varepsilon = 0. \end{aligned} \tag{2.8}$$

In order to find another equation between ε and λ, we shall use the relation resulting from the equality

$$\int_0^{\pi/2} \Delta \cdot \cos 2\zeta \, d\zeta = 0,$$

where Δ stands for the expression on the left-hand side of (2.4). $\cos 2\zeta$ has been chosen out of the complete system $\sin n\zeta$, $\cos n\zeta$ $(n = 0, 1, 2 \cdots)$ considering the convenience of the integration. In other words we are going to approximate the equality

$$\Delta = 0$$

by the two kinds of the average

$$\int_0^{\pi/2} \Delta \cdot d\zeta = 0, \tag{2.9}$$

and

$$\int_0^{\pi/2} \Delta \cdot \cos 2\zeta \, d\zeta = 0. \tag{2.10}$$ [1]

From (2.10) after the similar calculations as before, we can arrive at the result

$$\begin{aligned} &20\lambda I_0(2\lambda) \cosh 2\varepsilon + I_1(2\lambda)(20\varepsilon \sinh 2\varepsilon - 30 \cosh 2\varepsilon) \\ &+ 5\lambda^5 - 2\varepsilon\lambda^4 + (36\varepsilon^2 - 15)\lambda^3 - 4\varepsilon^3\lambda^2 + (8\varepsilon^4 - 60\varepsilon^2 + 10)\lambda = 0. \end{aligned} \tag{2.11}$$

ε and λ can be found as a pair (or pairs if possible) of the common roots of (2.8) and (2.11).

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[1] Yamada, H. On a Method of Approximate Solution of Differential Equations (English Abstract). *Rep. Res. Inst. Fluid Engg.*, vol. 6, No. 2 (1950), 42.

In order to remove some ambiguity arising from arbitrary selection of $\cos 2\zeta$ out of the system of the functions, let us construct another relation between ε and λ by a different procedure. To this end, we shall integrate the both hand sides of (1.4), multiplied by the weight δ^4, first with respect to η from zero to $\delta(\zeta)$ and then with respect to ζ from zero to $\pi/2$. As is easily verified, we have in conclusion assuming $R_0 = 1$,

$$\int_0^{\pi/2} (\delta^4\delta'' - \delta^4\delta' \sin 2\zeta - 5\delta\cosh 2\delta + 5\sinh 2\delta + 10\delta^3 - 5\delta)\, d\zeta = 0,$$

which is nothing but the relation expressed by (2.9). If we adopt another weight function instead of δ^4, we shall obtain necessarily another equation; in particular when we use δ^5 as the weight we are led to the third relation between ε and λ, viz.

$$I_0(2\lambda)\{40\varepsilon\sinh 2\varepsilon - 40(\varepsilon^2+\lambda^2)\cosh 2\varepsilon\} + I_1(2\lambda)(60\lambda\cosh 2\varepsilon - 80\varepsilon\lambda\sinh 2\varepsilon)$$

$$-10\lambda^6 + 5\varepsilon\lambda^5 + (-120\varepsilon^2+30)\lambda^4 + 20\varepsilon^3\lambda^3 + (-80\varepsilon^4+240\varepsilon^2-20)\lambda^2 + 8\varepsilon^5\lambda + 80\varepsilon^4 - 40\varepsilon^2 = 0. \quad (2.12)$$

The loci of the roots of these equations, i.e. (2.8), (2.11) and (2.12), are shown in the table and also in the figure.[1] It is concluded from both of them that the only one common root of these three equations does exist (at least in the range of λ permissible for our problem, since for a too large value of λ there will happen a case when $\delta < 0$, which is nonsense) and that it is situated at the point $\varepsilon = 1.521$, $\lambda = 0$, the same point as given by (2.6) already.

Although, of course, the above discussion is by no means a mathematical proof for the validity of the assumption that u is independent of ζ for all values of R_0, we may infer that this is highly plausible.

TABLE

λ	$\varepsilon \sim$ Roots of (2.8)	(2.11)	(2.12)
−0.75	0.5490	0.4951	0.5176
−0.50	0.9745	0.8159	0.9225
−0.40	1.1501	0.9119	1.0978
−0.30	1.3004	0.9858	1.2630
−0.20	1.4146	1.0386	1.3955
−0.10	1.4879	1.0719	1.4831
0	1.5213	1.0863	1.5213
0.10	1.5135	1.0818	1.5085
0.20	1.4657	1.0586	1.4467
0.30	1.3782	1.0160	1.3388
0.40	1.2527	0.9531	1.1926
0.50	1.0956	0.8684	1.0259
0.75	0.6691	0.5675	0.6139

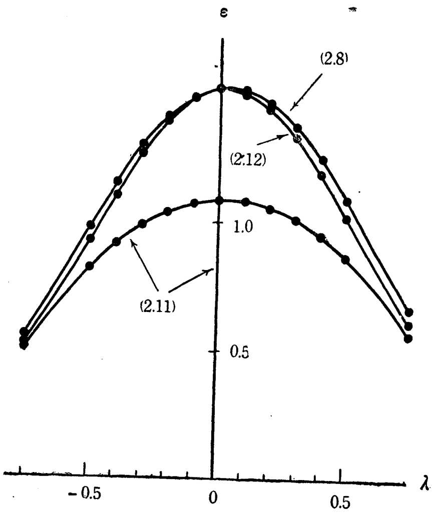

Fig. 2.

[1] It is readily proved that (2.11) is satisfied by $\lambda = 0$. Accordingly the locus of the root of (2.11) consists of two branches indicated in the figure.

3. In the last place we shall examine the accuracy of that representation of u by a polynomial of the fourth degree in terms of the elliptic coördinates which is employed in the foregoing note, cf. (2.4), and also in the present note, cf. (2.1).

We showed that u is given by the formula (1.7) so long as we assume that u is independent of ζ, what has been partially proved. If we put $R_0=1$ in (1.7), it reduces to

$$u=2\int_0^{\eta}\exp\left(-\frac{1}{2}\cosh 2t\right)\frac{dt}{K_0(0.5)}\,. \tag{3.1}$$

On expanding the integrand of the right-hand side in the power-series of t and integrating term by term, we have

$$u=\frac{2}{K_0(0.5)\sqrt{e}}\left(\eta-\frac{\eta^3}{3}+\frac{\eta^5}{30}+\frac{11}{630}\eta^7-\frac{71}{20160}\eta^9+\cdots\right)$$

$$=1.31224\left(\eta-\frac{\eta^3}{3}+\cdots\right). \tag{3.2}$$

On the other hand, by the method of polynomial representation u is given when $R_0=1$ in the form

$$u=2\left(\frac{\eta}{1.521}\right)-2\left(\frac{\eta}{1.521}\right)^3+\left(\frac{\eta}{1.521}\right)^4, \tag{3.3}$$

in the range $0\leqq\eta\leqq1.521$. Both of these expressions (3.2) and (3.3) are reproduced for ready comparison in Fig. 3. It is thus concluded that the polynomial representation has the accuracy quite satisfactory for practical purposes.

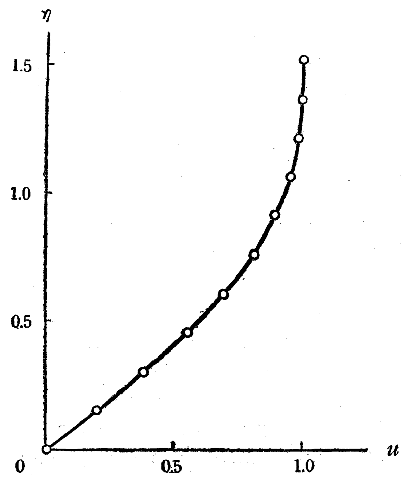

Fig. 3.

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Measurement of the Wave-making Resistance of a Ship by Means of an Electric Analogy

By Jun-ichi OKABE and Sadatoshi TANEDA

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Page 145, footnote, for "*Zôsengaku*" read "*Zosengaku.*"
Page 146, line 3 from the bottom (footnote), for "ship in" read "ship."
Page 147, line 4, for "as" read "that."
Page 147, line 1 (footnote), for "useful." read "useful:."
Page 148, line 5, for "neglection" read "neglect."