A Note on the Laminar Flow in the Inlet Length of a Circular Pipe

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https://doi.org/10.5109/7156996

出版情報：Reports of Research Institute for Applied Mechanics. 2 (6), pp.75-79, 1953-06. 九州大学応用力学研究所
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NOTES

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By Michio OHJI

Lately the present writer has had a chance to calculate the "Anlaufsprobleme" of the Poiseuille flow according to the method similar to that was used in his previous paper on the laminar nonisothermal flow in a circular pipe (Rep. Res. Inst. App. Mech., vol. 1, No. 2, 1952, pp. 23-38). Though the calculation is only a tentative one, it will be briefly noted below with the results obtained.

As is usually done the somewhat simplified equation of motion

$$u'\frac{\partial w'}{\partial r'} + w'\frac{\partial w'}{\partial z'} = -\frac{1}{2}\frac{dp'}{dz'} + \frac{2}{R}\left(\frac{\partial^2 w'}{\partial r'^2} + \frac{1}{r'}\frac{\partial w'}{\partial r'}\right), \tag{1a}$$

and

$$\frac{\partial p'}{\partial r'} = 0. \tag{1b}$$

are considered,[1] together with the continuity equation

$$\frac{\partial}{\partial r'}(r'u') + \frac{\partial}{\partial z'}(r'w') = 0. \tag{2}$$

First, let us assume the shape of w' in the form

$$w' \doteq Z\{1 - r'^2 e^{-(1-r')/\lambda}\}, \tag{3}$$

where it is seen

$$Z = \frac{1}{1 - 2J_3(\lambda)} \tag{4}$$

from the condition of constancy of flux (i.e., the integral form of (2)). Here we write

$$J_3(\lambda) = \int_0^1 r'^3 e^{-(1-r')/\lambda} dr', \tag{5}$$

and λ is a function of $\chi = z/aR$ only. (3) always satisfies the conditions that

$$w' = 0 \text{ at } r' = 1, \text{ and } \frac{\partial w'}{\partial r'} = 0 \text{ at } r' = 0. \tag{6}$$

Further when $\lambda = 0$, J_3 becomes 0 and (3) represents the uniform distribution $w' \equiv 1$ (except at $r' = 1$). And as $\lambda \to \infty$, J_3 tends to 1/4 and (3) approaches to the Poiseuille's pattern $2(1 - r'^2)$ in an asymptotic manner.

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[1] These are written nondimensionally, that is $r' = r/a$, $z' = z/a$, where r = radial distance from that center, z = axial distance from the enrty and a = radius of the pipe.

And $u' = u/w_m$, $w' = w/w_m$, where u = radial velocity, w = axial velocity and w_m = mean velocity.

Further $p' = p/\rho\frac{w^2{}_m}{2}$ and $R = (2a)\,w_m/\nu$, where p = pressure, ρ = density and ν = kinematic viscosity of the fluid.

Now how can this form describe the actual state of the flow near the entry? The easiest way to check this is to utilize the momentum equation

$$\frac{d}{d\chi}\int_0^1 w'^2\, r'\, dr' = -\frac{1}{4}\frac{dp'}{d\chi} + 2\left(\frac{\partial w'}{\partial r'}\right)_{r'=1}, \tag{7}$$

and the w'-equation (1a) at the center of the pipe, viz.

$$\frac{1}{2}\frac{d}{d\chi}(w')^2_{r'=0} = -\frac{1}{2}\frac{dp'}{d\chi} + 4\left(\frac{\partial^2 w'}{\partial r'^2}\right)_{r'=0}. \tag{8}$$

Upon using (3), these two equations furnish an approximate solution of the problem, where χ and p' are (numerically) expressed in terms of the profile parameter λ. The results are embodied in the Tables 1, 2 and Fig's 1, 2 in comparison with the Nikuradse's measurements and Schiller's theory etc.

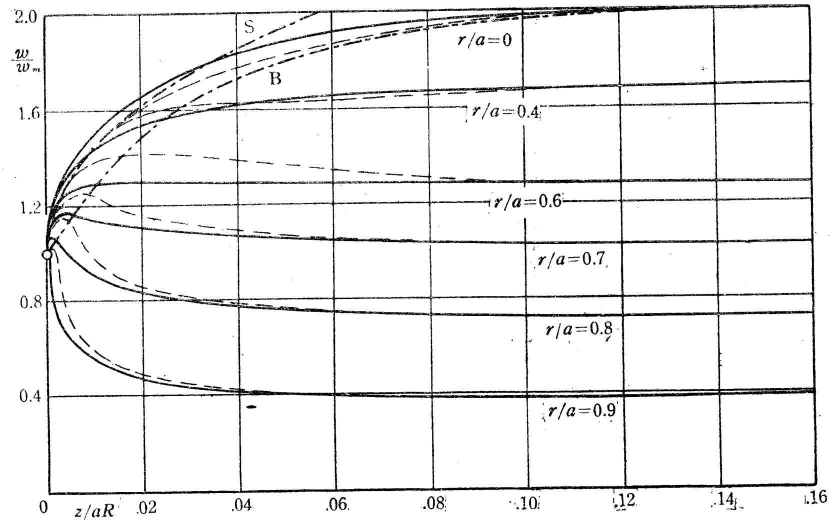

Fig. 1 Velocity Distribution in the Inlet Flow
——— Present Theory I
–·–·– Former Theory ($r/a = 0$); B : Boussinesq S : Schiller
– – – Experiment (Nikuradse)

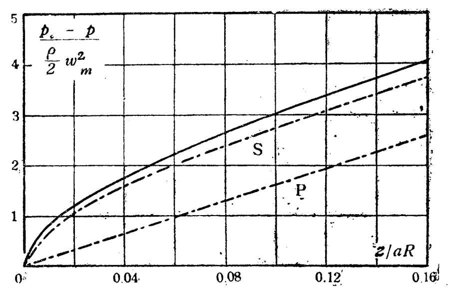

Fig. 2. Pressure Distribution in the Inlet Flow
S : Schiller, P : Poiseuille

TABLE 1. Results of Integrations

λ	0	0.04	0.10	0.20	0.30	0.50	0.70	1.0	2.0	10	100
z/aR	0	0.000200	0.001165	0.003828	0.006924	0.012889	0.018080	0.02457	0.03955	0.08336	0.16039
Z	1	1.0766	1.1776	1.3105	1.4090	1.5420	1.6268	1.7081	1.8311	1.9614	1.9960
$p_0 - p/\frac{1}{2}\rho w_m^2$	0	0.1590	0.3867	0.7175	0.9866	1.3910	1.6842	2.0083	2.6555	4.2268	6.7418

p_0: the pressure at the entry

TABLE 2.

authors	L/aR	$\Delta p/\frac{1}{2}\rho\, u^{m2}$
Boussinesq	0.13	2.24
Schiller	0.0575	2.16
Goldstein and Atkinson	0.12	2.41
present writer	0.14	2.55

L: total inlet-length,
Δp: inlet-correction for the pressure-drop

Discussions

We have now an approximate solution which reproduces the smooth transition from the uniform distribution to the parabolic one without introducing the thickness of the boundary layer explicitly. But it is noted that we have hitherto disregarded the dynamical condition at the wall of the pipe. That is, putting $r' = 1$ in the w'-equation, (1a) becomes

$$\frac{1}{2}\frac{dp'}{d\chi} = 2\left(\frac{\partial^2 w'}{\partial r'^2} + \frac{1}{r'}\frac{\partial w'}{\partial r'}\right)_{r'=1}, \tag{9}$$

which is, however, not satisfied by the present solution except for large values of χ. Conversely, when (9) is used in place of (8) to evaluate the pressure drop the result is that the total inlet length becomes considerably shorter. This will be readily understood from the fact that $dp'/d\chi \sim O(1/\lambda^2)$ from (9) while $dp'/d\chi \sim O(1/\lambda)$ from (8) as $\lambda \to 0$ so long as the profile (3) is assumed. So that in order to avoid this difficulty and obtain more legitimate solution the assumption (3) must be properly modified. Along this line consider now the form

$$w' = Z\left[1 - \left\{\left(1 + \frac{1}{2\lambda}\right) - \frac{r'}{2\lambda}\right\} r'^2\, e^{-(1-r')/\lambda}\right], \tag{10}$$

for which the condition of constancy of the flux requires that

$$Z=\left[1-\left\{\left(2+\frac{1}{\lambda}\right)J_3-\frac{1}{\lambda}J_4\right\}\right]^{-1}. \tag{11}$$[1]

In this case the profile parameter is λ alone as before, but the pressure gradient $dp'/d\chi$ near the entry becomes same order of magnitude ($O(1/\lambda)$) when either of (8) or (9) is used. Thus we can obtain two kinds of approximate solution, each of which is denoted by II and II* respectively in the following way:

II, obtained from the momentum equation and the dynamical condition at the center of pipe, (8), and

II*, obtained from the momentum equation and the dynamical condition at the wall of pipe, (9).

(The notation I is used for the above solution based on (3)). Of course the calculation is more complicated than before and has not been fully pursued. In Table 3, therefore, only the asymptotic forms of the maximum velocity w_0' near $\chi=0$ are tabulated together with the corresponding formulae of the other authors. Although not satisfactorily, it will suggest the degrees of approximation of these solutions to some extent.

To proceed to higher approximation, in which the dynamical conditions at both of the boundaries ($r'=0$ and 1) are taken into consideration, a polynomial with suitable coefficients can be added. For instance, we take

$$w'=Z\left[1-\left\{\left(1+\frac{1}{2\lambda}\right)-\frac{r'}{2\lambda}\right\}r'^2e^{-(1-r')/\lambda}\right]$$
$$+2\left[1-Z\left\{\left(2+\frac{1}{\lambda}\right)J_3-\frac{1}{\lambda}J_4\right\}\right](1-r'^2), \tag{12}$$

where Z and λ are profile parameters which should be determined independently. Now solving (7), (8) and (9), the simultaneous equations of Z, λ and p, we have the desired solutionI III. See Table 3.

Finally, what is the accuracy of these solutions? Is not there any other form of w' which gives better approximation than our present one? And further to what extent do the terms neglected in the complete equations of motion influence the actual state of flow? Unfortunately these questions are so complicated that no definite answer has been given so far.

TABLE 3.

authors	w_0' near $\chi=0$
Schiller	$1+5.164\sqrt{\chi}$
Atkinson and Goldstein	$1+4.867\sqrt{\chi}$
Mochizuki	$1+5.657\sqrt{\chi}$
present I	$1+5.656\sqrt{\chi}$
〃 II	$1+5.117\sqrt{\chi}$
〃 II*	$1+5.765\sqrt{\chi}$
〃 III	$1+5.591\sqrt{\chi}$

Supplementary note

Recently at least two papers published in this country concerning the same subject. The one is that by S. Mochizuki (Memoirs of Faculty of Technology, The Tokyo

1) $J_4(\lambda)=\int_0^1 r'^4e^{-(1-r')/\lambda}dr'$

Metropolitan University, No. 2, 1952, pp. 41-47) and the other is by T. Tatsumi (Journal of the Physical Society of Japan, vol. 7, No.5, 1952, pp. 489-495).

In Mochizuki's paper the velocity profile is assumed to be

$$w' = (1+f)(1-r'^{2/f}), \qquad f = f(\chi), \tag{13}$$

where f is a profile parameter, and the momentum equation is considered as well as the w'-equation both at $r'=0$ and $r'=1$. Under the particular assumption on the pressure distribution, he obtained the result which agrees very well with the experiments.

In Tatsumi's paper, on the other hand, the equation is solved by iteration method under the assumption of "almost similarity" of the velocity profile across the boundary layer along the wall. Although this method is applicable for relatively narrow range compared with the total inlet length ($\chi \lesssim 0.01$), it is shown in subsequent paper that the solution is quite adequate for his primary purpose, i.e., for the discussion of the stability of the inlet flow (T. Tatsumi, *ibid.* pp. 495-502).

References

1) Goldstein, S., Modern Development in Fluid Dynamics, vol. 1, (1938), Chap. VII, § 139.
2) Prandtl, L. and Tietjens, O., Hydro-und Aerodynamik, Bd. 2, (1932), Kapitel III, §§ 14-20.
3) Schiller, L., Handbuch der Experimentalphysik, IV, Teil 4, (1932), Kapitel III, § 2.

(Received February 18, 1953)

ERRATA

An Approximate Calculation of the Laminar Nonisothermal Flow in a Circular Pipe (I)

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Page 24, line 13, for "viz; ⋯" read "viz, ⋯."
Page 28, line 5, for "simaltaneous" read "simultaneous."
Page 28, line 1 of the footnote, for "computation" read "computations."
Page 29, Table 1, top of the seventh column, for "$-2\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0}$" read "$-2\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1}$."
Page 30, line 2 from the bottom, for "chosing" read "choosing."
Page 32, line 8, for "(4.11)" read "(4.9)."
Page 35, formula (I.2), for "$\frac{1}{\eta}\left(\frac{\partial^2\theta^*}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial\theta^*}{\partial\eta}\right)_{\eta=1} \equiv \left(\frac{F}{\lambda}+\frac{\partial F}{\partial\eta}\right)_{\eta=1}+\cdots$"
read "$\left(\frac{\partial^2\theta^*}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial\theta^*}{\partial\eta}\right)_{\eta=1} = \frac{1}{\eta}\left(\frac{F}{\lambda}+\frac{\partial F}{\partial\eta}\right)_{\eta=1}+\cdots$."
Page 36, formula (I.3), for "$F_0 = 1+\frac{1}{\lambda}$" read "$F_0 = 1+\frac{1}{4\lambda}$."
Page 37, line 7 from the bottom, for "Acoording" read "According."