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Takabeya, Fukuhei
Research Institute for Applied Mechanics, Kyushu University

Murakami, Yoshimaru
Research Institute for Applied Mechanics, Kyushu University

Takabeya, Ryohei
Research Institute for Applied Mechanics, Kyushu University

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**STRESS INVESTIGATIONS OF RIGID FRAMES,
CONSIDERING THE EFFECTS OF BENDING MOMENTS,
NORMAL AND TANGENTIAL STRESSES**

By Fukuhei TAKABEYA, Yoshimaru MURAKAMI
and Ryohei TAKABEYA.

In the problem of rigid frames, all redundant reactions and stresses are generally due to the bending flexures and partly due to the normal and tangential forces. However, when compared with the effect of the first, that of the latter two forces is usually very small and accustomed to be neglected as a factor unworthy of notice.

The senior author would work out, for some thirty years, the stress calculations usually applying approximate solutions of the Slope Deflection Method, in which the effect of bending moments is only taken into account, and not of axial and shearing forces.

In the note, there have been firstly derived equations of the redundant reactions and moments, for a rigid frame of single storied-two bays, fixed at its column ends, considering these three forces. Secondly the authors investigated to what extent the reactions and bending moments due to the axial and shearing forces are worthy of comparison with that due to the bending moments.

Finally, the problems of the following rigid frames, carrying loads vertical and uniformly distributed on the total span, have been worked out by the junior authors and tabulated in Appendix Tables A ~ E:

- (1) Rigid frames of hinged supports, single storied and single bay,
- (2) Rigid frames of hinged supports, two storied and single bay,
- (3) Rigid frames of hinged supports, single storied and two bays,
- (4) Rigid frames of fixed supports, single storied and single bay,
- (5) Rigid frames of fixed supports, two storied and single bay.

1. Introduction. The computations of the statically indeterminate stresses in high building frames with rigid connecting joints, composed of rectangular elements are so complicated that, in general, the analysis of rigid frames, commonly treated by the Slope Deflection Method, is based upon the assumptions: (i) the connections between the columns and beams are perfectly rigid; (ii) the length of a member is not changed by direct stress; (iii) the deformation of a member due to the internal shearing stress is zero. Under these assumptions, the stress computations of rigid frames are accustomed to be worked out, disregarding the effects of shearing stress as well as direct one, because the effects of the two are

negligible small, when compared with the effect of flexure; but it is however very obscure to confirm the grade of no effects and the present paper deals with such problems as to consider both terms of shearing and direct stresses in equations of Virtual Work and to investigate the relations between height, span length and the stresses of rigid frames.

2. Development of Elastic Equations and Redundant Forces. The current method of solution used in the case of high storied building frames is based upon the well known Slope Deflection Method, in which the term of bending moments is merely taken into consideration; while, as we here consider the terms of bending moments, shears as well as direct stresses, it seems to be advantageous to apply the equation of Virtual Work to the solution of a two-span-rigid frame, as shown in Fig. 1.

The rigid frame is cut at a convenient point, as at the central joint D and the internal forces X_1 , X_2 and X_3 , acting at this section, are treated as the unknown redundancies; the following three equations contain the three redundant stresses as well as the stresses due to external loads, with all redundant members removed.

$$X_1 [t_{1.1}] + X_2 [t_{1.2}] + X_3 [t_{1.3}] = t_{1.0} \quad (1)$$

$$X_1 [t_{2.1}] + X_2 [t_{2.2}] + X_3 [t_{2.3}] = t_{2.0} \quad (2)$$

$$X_1 [t_{3.1}] + X_2 [t_{3.2}] + X_3 [t_{3.3}] = t_{3.0} \quad (3)$$

where $t_{1.1}$, $t_{1.2}$... represent the followings:

$$t_{1.1} = \int \frac{M_1 M_1}{EI} ds + \int \frac{N_1 N_1}{EA} ds + \int \alpha \frac{S_1 S_1}{GA} ds \quad (4)$$

$$t_{1.2} = \int \frac{M_1 M_2}{EI} ds + \int \frac{N_1 N_2}{EA} ds + \int \alpha \frac{S_1 S_2}{GA} ds \quad (5)$$

$$t_{r.s} = \int \frac{M_r M_s}{EI} ds + \int \frac{N_r N_s}{EA} ds + \int \alpha \frac{S_r S_s}{GA} ds \quad (6)$$

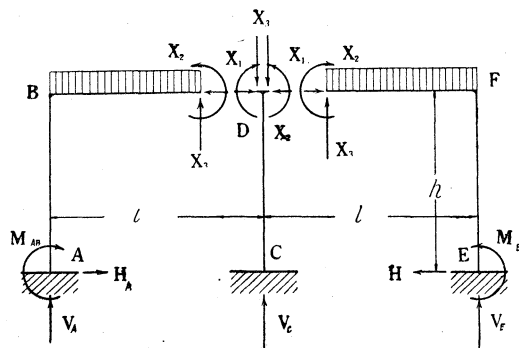


Fig. 1

In the above expressions we denote by

- M the bending moment,
- N the direct stress,
- S the shearing force,
- α a constant depending upon the shape of the cross-section of the member,
- E the modulus of elasticity of the material in flexure,
- G the modulus of elasticity of the material in shearing,
- I the moment of inertia of the section of the member about the neutral axis,
- A the cross sectional area of the member.

After tedious labour of tiresome calculation, solving these simultaneous equations, we get:

$$X_1 = \frac{n}{8 h \xi} \{1 + 16\Psi_2 - 9\Psi_3\} w l^2 \quad (7)$$

$$X_2 = -\frac{1}{24\xi} \left\{ n(3n+2) + 2(6n+1)\Psi_1 - 4n(3n+4)\Psi_2 \right. \\ \left. + 9n(n+2)\Psi_3 - 16(3n+1)\Psi_1\Psi_2 + 18(2n+1)\Psi_1\Psi_3 \right\} w l^2 \quad (8)$$

$$X_3 = \frac{1}{8\xi} \left\{ n(5n+4) + 4(5n+1)\Psi_1 + 3n(n+4)\Psi_3 \right. \\ \left. + 12(n+1)\Psi_1\Psi_3 \right\} w l \quad (9)$$

where

$$\xi = n(n+1) + (4n+1)\Psi_1 + n(n+4)\Psi_2 + 4(n+1)\Psi_1\Psi_2 \quad (10)$$

$$\Psi_1 = 3 \left(\frac{r_b}{h} \right)^2 \left[1 + \frac{\alpha n \beta}{G_0} \right] \quad (11)$$

$$\Psi_2 = 3 \left(\frac{r_b}{l} \right)^2 \left[3n\beta + \frac{\alpha}{G_0} \right] \quad (12)$$

$$\Psi_3 = 4 \left(\frac{r_b}{l} \right)^2 \left[2n\beta + \frac{\alpha}{G_0} \right] \quad (13)$$

$$\left. \begin{aligned} G_0 = G/E, \quad n = \frac{I_b h}{I_c l} \\ \text{and} \\ \beta = \left(\frac{r_c}{r_b} \right)^2 \end{aligned} \right\} \quad (14)$$

in which I_b , I_c and r_b , r_c denoting the moment of inertia and the radius of gyration of the beam and column-sections respectively.

Considering that the terms $\left(\frac{r_b}{h}\right)^2$ and $\left(\frac{r_b}{l}\right)^2$ are very small in Equations (11) and (12); and further disregarding therefore the terms $\left(\frac{r_b}{h}\right)^2 \times \left(\frac{r_b}{l}\right)^2$, we obtain from equations (7), (8) and (9),

$$X_1 = \frac{nw l^2}{8h \xi'} \{1 + 16\Psi_2 - 9\Psi_3\} \quad (15)$$

$$X_2 = \frac{-w l^2}{24\xi'} \left\{ \begin{array}{l} n(3n+2) + 2(6n+1)\Psi_1 \\ - 4n(3n+4)\Psi_2 + 9n(n+2)\Psi_3 \end{array} \right\} \quad (16)$$

$$X_3 = \frac{w l}{8\xi'} \{n(5n+4) + 4(5n+1)\Psi_1 + 3n(n+4)\Psi_3\} \quad (17)$$

where

$$\xi' = n(n+1) + (4n+1)\Psi_1 + n(n+4)\Psi_2 \quad (18)$$

With these three redundant stresses, we obtain:

Horizontal reactions at A and E :

$$H_A = H_E = X_1 = \frac{n}{8h \xi'} \{1 + 16\Psi_2 - 9\Psi_3\} w l^2 \quad (19)$$

Vertical reactions at A , E and C :

$$\begin{aligned} V_A = V_E = w l - X_3 \\ = w l - \frac{1}{8\xi'} \{n(5n+4) + 4(5n+1)\Psi_1 + 3n(n+4)\Psi_3\} w l \\ = \frac{w l}{8\xi'} \left\{ \begin{array}{l} n(3n+4) + 4(1+3n)\Psi_1 \\ + 8n(4+n)\Psi_2 - 3n(4+n)\Psi_3 \end{array} \right\} \end{aligned} \quad (20)$$

$$\begin{aligned} V_c = 2w l - 2V_A = 2X_3 \\ = \frac{1}{4\xi'} \{n(5n+4) + 4(5n+1)\Psi_1 + 3n(n+4)\Psi_3\} w l \end{aligned} \quad (21)$$

Taking the moment at A or E , we get:

$$M_A - X_2 - X_3 l - X_1 h + \frac{w l^2}{2} = 0$$

From this equation we have:

$$M_A = M_E = \frac{w l^2}{24\xi'} \{n - 2\Psi_1 + 16n\Psi_2 - 9n\Psi_3\} \quad (22)$$

And at the central joint D :

$$\begin{aligned} M_{DB} = -M_{DF} = -X_2 \\ = \frac{w l^2}{24\xi'} \left\{ \begin{array}{l} n(3n+2) + 2(6n+1)\Psi_1 \\ - 4n(3n+4)\Psi_2 + 9n(n+2)\Psi_3 \end{array} \right\} \end{aligned} \quad (23)$$

At the left joint B :

$$M_{BD} + \frac{w l^2}{2} - X_3 l - X_2 = 0$$

Namely:

$$M_{BD} = -\frac{w l^2}{12\xi'} \{n + \Psi_1 + 16n\Psi_2 - 9n\Psi_3\} \quad (24)$$

If we neglect the effect of shearing forces, we put $\alpha = 0$ in equations (11) ~ (13) and obtain :

$$\Psi_1' = 3 \left(\frac{r_b}{h} \right)^2, \Psi_2' = 9n \beta \left(\frac{r_b}{l} \right)^2, \Psi_3' = 8n \beta \left(\frac{r_b}{l} \right)^2 \quad (25)$$

We use Ψ' instead of Ψ for the previous equations obtained. Furthermore, if we disregard both effects of shearing and direct stresses, we put $\Psi = 0$ in the expressions of the redundancies as well as all the expressions of forces.

3. Assumptions Adopted. The assumptions adopted in the paper are as follows :

- (a). Loads applied are uniformly and vertically distributed on the total span of the rigid frame, as shown already in the previous articles.
- (b). The stiffness of the beam or column is constant along the whole length.
- (c). The sectional area of the beam or column is constant.
- (d). The cross section of the column is expressed by $d \times d$ and that of the beam by $d \times \mu d$, in which μd shows its length of the depth.
- (e). The values of the elastic modulus are the same for beams and columns.
- (f). The constant depending upon the shape of the cross section with respect to shearing forces is taken $\alpha = 1.2$, because of the rectangular or square section.
- (g). The value of Poisson's number $m = 6$ is considered, principally for concrete beams and columns.
- (h). The effect of reinforcing steel bars and also of the haunches of the concrete beams is not considered for the problems in question.
- (i). The effects of shearing forces as well as direct stresses are expressed, as found afterwards, as a function of $\left(\frac{r_b}{h} \right)^2$, in which $r_b^2 = \mu^2 d^2 / 12$. The value r_b is determined by the depth of beam μd .

4. Effects of Shearing and Direct Stresses. Common designers of building frames are accustomed to work out, as we mentioned previously, the redundant stresses by considering merely bending moments; but we have calculated the expressions of redundant forces, after tedious computation labour, considering the bending moments, direct stresses as well as shearing forces.

In practices, it is said that the effects of direct stress and shearing forces are very small, but it is the chief object, in the note, to investigate how small is the effect actually for the case of concrete structures.

As numerical examples, taking the column section $d \times d$ and the beam section $d \times \mu d$ and $\alpha = 1.2$, $m = 6$, we have then $\beta = 1/\mu^2$, $n = \mu^3 (h/l)$ and $G_0 = G/E = m/2 (m + 1) = 3/7$.

Further, let $h/l = \eta$ and $r_b/h = x$, then $r_b/l = \eta x$. From equations (11) ~ (13), we obtain:

$$\left. \begin{aligned} \Psi_1 &= 3 \{1 + 2.8\mu\eta\} x^2, & \Psi_2 &= 3 \{2.8 + 3\mu\eta\} \eta^2 x^2, \\ \Psi_3 &= 8 \{1.4 + \mu\eta\} \eta^2 x^2, \\ \Psi_1 \Psi_2 &= 9 \{1 + 2.8\mu\eta\} \{2.8 + 3\mu\eta\} \eta^2 x^4, \\ \Psi_1 \Psi_3 &= 24 \{1 + 2.8\mu\eta\} \{1.4 + \mu\eta\} \eta^2 x^4. \end{aligned} \right\} \quad (26)$$

(A). General Formulae. From equation (19) we obtain:
Horizontal reactions at A and E :

$$H_A = H_E = \frac{n}{8h\xi'} [1 + 24(1.4 + 3\mu\eta)\eta^2 x^2] wl^2 \quad (27)$$

where

$$\xi' = n(n+1) + 3[(4n+1)(1+2.8\mu\eta) + n(n+4)(2.8+3\mu\eta)\eta^2] x^2 \quad (28)$$

Horizontal reaction at C :

$$H_C = 0 \quad (29)$$

Vertical reactions at A and E :

$$V_A = V_E = \frac{wl}{8\xi'} \left\{ \begin{aligned} n(3n+4) + 12x^2[(1+3n)(1+2.8\mu\eta)] \\ + 4n\eta^2(4+n)(0.7+\mu\eta) \end{aligned} \right\} \quad (30)$$

Vertical reaction at C :

$$V_C = \frac{wl}{4\xi'} \left\{ \begin{aligned} n(5n+4) + 12x^2[(5n+1)(1+2.8\mu\eta)] \\ + 2n(n+4)(1.4+\mu\eta)\eta^2 \end{aligned} \right\} \quad (31)$$

Bending moments at A and E :

$$\begin{aligned} M_A = X_1 h + X_2 + X_3 l - \frac{wl^2}{2} &= M_E \\ &= \frac{wl^2}{24\xi'} \left\{ n - 6x^2[(1+2.8\mu\eta) - 4n\eta^2(1.4+3\mu\eta)] \right\} \end{aligned} \quad (32)$$

Bending moments at D and B :

$$M_{DB} = \frac{wl^2}{24\xi'} \left\{ \begin{aligned} n(3n+2) + 6(6n+1)(1+2.8\mu\eta)x^2 \\ - 12n(3n\mu\eta - 5.6)\eta^2 x^2 \end{aligned} \right\} \quad (33)$$

$$M_{BD} = -\frac{wl^2}{12\xi'} \left\{ n + 3(1+2.8\mu\eta)x^2 + 24n\eta^2 x^2(1.4+3\mu\eta) \right\} \quad (34)$$

and

$$M_{DC} = M_{CD} = 0 \quad (35)$$

(B). Expressions of Redundant Forces, when Shears Neglected. Considering only the bending moments and the direct stresses, disregarding the shearing forces, we obtain from equations (25) and (18):

$$\left. \begin{aligned} \Psi_1' &= 3x^2, & \Psi_2' &= 9\mu\eta^3 x^2, & \Psi_3' &= 8\mu\eta^3 x^2 \\ \xi' &= n(n+1) + (4n+1)\Psi_1' + n(n+4)\Psi_2' \end{aligned} \right\} \quad (36)$$

With these Ψ_1' , Ψ_2' , Ψ_3' and ξ' , we get horizontal and vertical reactions as follows:

Horizontal reaction:

$$H_A = \frac{n(1+72\mu\eta^3 x^2)wl^2}{8h\xi'} \quad (37)$$

where

$$\xi' = n(n+1) + 3(4n+1)x^2 + 9n(n+4)\mu\eta^3 x^2 \quad (38)$$

Vertical reactions:

$$V_A = \frac{wl}{8\xi'} \left\{ n(3n+4) + 12x^2 [(1+3n) + 4n(4+n)\mu\eta^3] \right\} \quad (39)$$

$$V_C = \frac{wl}{4\xi'} \left\{ n(5n+4) + 12x^2 [(5n+1) + 2n(n+4)\mu\eta^3] \right\} \quad (40)$$

In a similar way we have:

Bending moments:

$$M_A = \frac{wl^2}{24\xi'} \left\{ n - 6x^2 [1 - 12n\mu\eta^3] \right\} \quad (41)$$

$$M_{DB} = \frac{wl^2}{24\xi'} \left\{ n(3n+2) + 6x^2 [(6n+1) - 6n^2\mu\eta^3] \right\} \quad (42)$$

$$M_{BD} = -\frac{wl^2}{12\xi'} \left\{ n + 3x^2 (1 + 24n\mu\eta^3) \right\} \quad (43)$$

TABLE I
Table of Vertical and Horizontal Reactions,
and Reaction Moments

Reactions	(A)	(B)
	Frames of Different Stiffness Coeff.: $\frac{wl}{24h(n+1)}$	Frames of Equal Stiffness Coeff.: $\frac{wl}{48h}$
$H_A = H_F$	$3l$	$3l$
$V_A = V_F$	$3h(3n+4)$	$21h$
V_C	$6h(5n+4)$	$54h$
$M_A = M_F$	hl	hl
M_{DB}	$hl(3n+2)$	$5hl$
M_{DF}	$-hl(3n+2)$	$-5hl$
M_{BD}	$-2hl$	$-2hl$

(C). Expressions of Redundant Reactions, when Shears and Direct Stresses Neglected. If we disregard shears and direct stresses in work equations, we substitute $\Psi = 0$ in equations (19) ~ (24) and obtain Table I(A).

For rigid frames of equal stiffness $n = 1$, we get further Table I(B).

5. Comparisons of Reactions and Moments. Calculation results for reactions and moments have been compared with each other among the following three cases:

Case (i): Bending Moments only considered (Direct stresses N and Shearing stresses S disregarded).

Case (ii): Bending Moments M and Direct Stresses N considered (Shearing stresses S disregarded).

Case (iii): Bending Moments M , Direct Stresses N and Shearing Stresses S all considered).

All reactions and end moments for case (i) are obtained by Table I; and equations (37) ~ (43) and equations (27) ~ (34) give reactions and moments for case (ii) and case (iii) respectively. As numerical examples we have calculated the reactions and end moments for a single storied-two bays-fixed frame for these three different cases, under the two given conditions as follows: (1) $n = 1$, $h/l = \eta = 1$, $\mu = 1$ and $\mu d = l/10$; (2) $n = 2$, $h/l = \eta = 0.4$, $\mu = \left(\frac{n}{\eta}\right)^{1/3} = 1.71$ and $\mu d = l/10$.

Table II shows the calculation results for the first example and Table III shows the reactions and moments for the second example.

As Table II and Table III show, the effects of bending moments M are very large and, on the other hand, those of direct stresses N and shears S are very small.

TABLE II
Table of Reactions and Moments
Rigid Frames of Fixed Supports, Single Storied-Two Bays.
Stiffness Ratio $n = 1$, Height of Beams $\mu d = l/10$,
 $h/l = \eta = 1$, $\mu = 1$

Reactions and Moments	Case (i) (M considered)	Case (ii) (M and N considered)	Case (iii) (M , N and S considered)
H_A	0.0625 wl	0.0646 wl	0.0642 wl
V_A	0.4375 wl	0.4415 wl	0.4417 wl
V_C	1.1250 wl	1.1172 wl	1.1167 wl
M_A	0.0208 wl^2	0.0214 wl^2	0.0210 wl^2
M_{DB}	0.1042 wl^2	0.1017 wl^2	0.1020 wl^2
M_{BD}	-0.0417 wl^2	-0.0432 wl^2	-0.0431 wl^2

TABLE III

Table of Reactions and Moments
Rigid Frames of Fixed Supports, Single Storied Two Bays.
Stiffness Ratio $n = 2$, Height of Beams $\mu d = l/10$,

$$h/l = \eta = 0.4, \quad \mu = \left(\frac{n}{\eta}\right)^{1/3} = 1.71$$

Reactions and Moments	Case (i) (M considered)	Case (ii) (M and N considered)	Case (iii) (M , N and S considered)
H_A	0.1042 wl	0.1049 wl	0.1019 wl
V_A	0.4167 wl	0.4185 wl	0.4183 wl
V_C	1.1667 wl	1.1630 wl	1.1633 wl
M_A	0.0139 wl^2	0.0138 wl^2	0.0130 wl^2
M_{DB}	0.1111 wl^2	0.1097 wl^2	0.1094 wl^2
M_{BD}	-0.0278 wl^2	-0.0282 wl^2	-0.0278 wl^2

We find in Table II that the reaction H shows the largest value for Case (ii), in which M and N are taken into consideration; while the value H which has been computed by considering M , N and S , is, however, smaller than that computed under consideration of M and N , neglecting the effects of S .

On the contrary, the value V_C shows in Table II that the computed value for case (i), in which M is only considered, takes the largest and therefrom, we know, the calculation value by current method gives only errors of safe side; more precisely to say, in the given example, the vertical reaction V_C becomes smaller when computed by taking M , N and S into considerations or when M and N considered.

The end moment M_{DB} shows also the similar characteristics as shown in Table II. Likewise, in Table III, for the second example, the reaction V_C and end moment M_{DB} show the same characteristics as in Table II; while the end moment M_A shows the largest value for case (i) in the second example, but the same moment shows the largest for case (ii) in the first example.

6. Error Percentages. Let the horizontal reaction H_A for the case in which M and N are regarded be H_A' ; and H_A for the case in which M only regarded be H_A'' ; then the difference $\Delta H_A = H_A' - H_A''$ is expressed as follows:

$$\Delta H_A = \frac{\{CT - AD\} x^2}{8hC\{C + Dx^2\}} w l^2 \quad (44)$$

in which

$$\left. \begin{aligned} A &= n, & C &= n(n+1) \\ D &= 3\{(4n+1) + 3n(n+4)\mu\eta^3\} \\ T &= 72n\mu\eta^3 \end{aligned} \right\} \quad (45)$$

In a similar way, we obtain for the vertical reaction V_A :

$$\Delta V_A = V_A' - V_A'' = \frac{\{CW - GD\}x^2}{8C\{C + Dx^2\}} w l \quad (46)$$

in which

$$\left. \begin{aligned} W &= 12\{(1+3n) + 4n(4+n)\mu\eta^3\} \\ G &= n(3n+4) \\ C \text{ and } D, & \text{ see equ. (45)} \end{aligned} \right\} \quad (47)$$

and for the vertical reaction V_C :

$$\Delta V_C = V_C' - V_C'' = \frac{(LC - KD)x^2}{4C(C + Dx^2)} w l \quad (48)$$

in which

$$\left. \begin{aligned} K &= n(5n+4) \\ L &= 12\{(5n+1) + 2n(n+4)\mu\eta^3\} \end{aligned} \right\} \quad (49)$$

For the bending moment M_A :

$$\Delta M_A = \frac{(QC - AD)x^2}{24C(C + Dx^2)} w l^2 \quad (50)$$

For the moments M_{DB} and M_{BD} :

$$\Delta M_{DB} = \frac{(CS - FD)x^2}{24C(C + Dx^2)} w l^2 \quad (51)$$

$$\Delta M_{BD} = \frac{(AD - UC)x^2}{12C(C + Dx^2)} w l^2 \quad (52)$$

where

$$\left. \begin{aligned} Q &= -6(1 - 12n\mu\eta^3) \\ F &= n(3n+2) \\ S &= 6\{(6n+1) - 6n^2\mu\eta^3\} \\ U &= 3\{1 + 24n\mu\eta^3\} \end{aligned} \right\} \quad (53)$$

In order to investigate the error percentages that result from the case where the reaction H_A'' is used instead of H_A' , we work out further as follows:

Let the ratio $\Delta H_A/H_A''$ be expressed by $r_{(H_A)}$, then we have:

$$r_{(H_A)} = \frac{\{CT - AD\}x^2}{A\{C + Dx^2\}} \quad (54)$$

and its percentage is given by

$$100r_{(H_A)} = R_{(H_A)} = \left\{ C \left(\frac{T}{A} \right) - D \right\} \frac{100x^2}{C + Dx^2} \quad (55)$$

Similarly, we obtain error percentages: $R_{(V_A)}$, $R_{(V_C)}$, $R_{(M_A)}$ etc. as shown in Table IV.

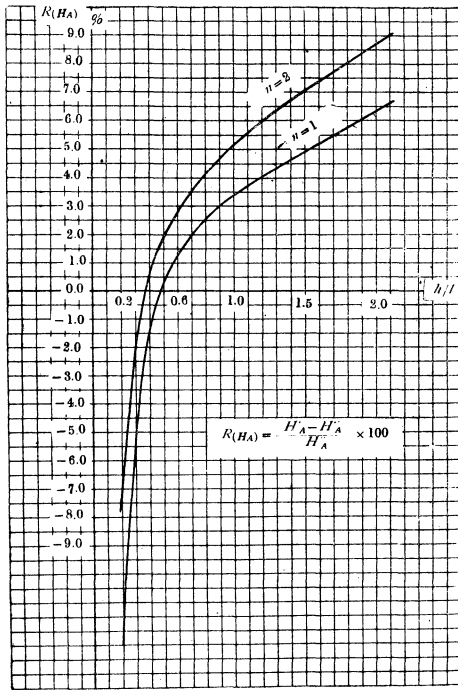


Fig. 2.
Relation between Error Percentage $R(H_A)$ and Ratio h/l for $n = 1$ and $n = 2$

TABLE IV
Equations of Error Percentages for Reactions and Moments Rigid Frames of Fixed Supports, Single Storied-Two Bays.

Reactions and Moments	Equations of R Coeff. = $\frac{100x^2}{C + Dx^2}$	Reference Number
$R_{(H_A)}$	$C \left(\frac{T}{A} \right) - D$	(I)*
$R_{(V_A)}$	$C \left(\frac{W}{G} \right) - D$	(II)
$R_{(V_C)}$	$C \left(\frac{L}{K} \right) - D$	(III)
$R_{(M_A)}$	$C \left(\frac{Q}{A} \right) - D$	(IV)
$R_{(M_{DB})}$	$C \left(\frac{S}{F} \right) - D$	(V)
$R_{(M_{BD})}$	$C \left(\frac{U}{A} \right) - D$	(VI)

* see equ. (55).

Further, giving $n = 1$ and $\mu d = l/10$, there are shown in Table V the relations between error percentages R and η ; and Table VI gives the similar relations when stiffness ratio n is equal to 2.

These tables show the error percentages for $\eta = 0.2, 0.4, 0.6, \dots, 2.0$; while Figs. 2 ~ 7 show the error percentages for the other intermediate values of η . The characteristics which $R_{(H_A)}$, $R_{(V_A)}$, $R_{(V_C)}$ etc. show for relatively smaller values of η , are of importance; namely for a rigid frame of very short height as well as very long spans the effects of disregarding N and S are remarkable as found in. Figs 2 ~ 7, Tables V and VI.

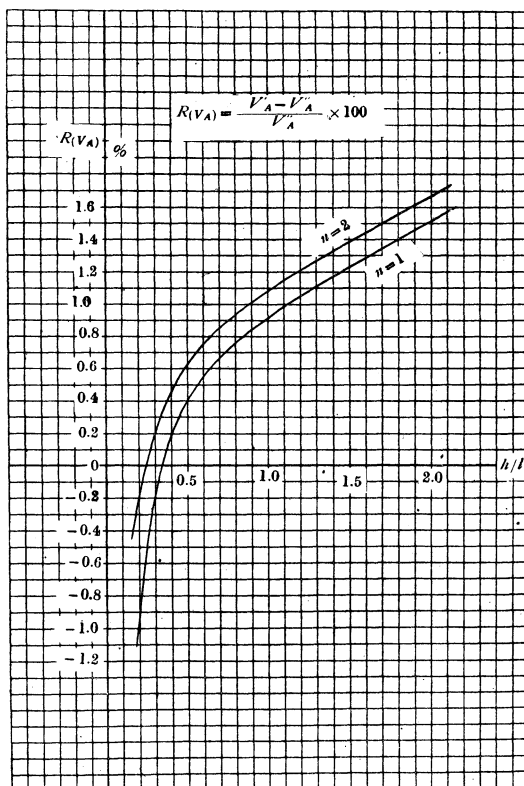


Fig 3

Relation between Error Percentage
 $R(V_A)$ and Ratio h/l for $n = 1$ and $n = 2$

TABLE V

Table of Error Percentages.

Rigid Frames of Fixed Supports, Single Storiad-Two Bays.

Stiffness Ratio $n = 1$, Height of Beams $\mu d = l/10$, $\mu = \left(\frac{n}{\eta}\right)^{1/3}$.

$\eta = h/l$	$R_{(H_A)}$	$R_{(V_A)}$	$R_{(V_C)}$	$R_{(M_A)}$	$R_{(M_{DB})}$	$R_{(M_{BD})}$
0.2	-12.22	-0.87	+0.67	-22.97	+0.94	-6.85
0.4	- 1.59	+0.19	-0.15	- 4.57	-0.83	-0.10
0.6	+ 1.16	+0.53	-0.41	- 0.18	-1.50	+1.83
0.8	+ 2.51	+0.74	-0.58	+ 1.75	-1.96	+2.89
1.0	+ 3.40	+0.91	-0.70	+ 2.92	-2.33	+3.64
1.5	+ 4.98	+1.23	-0.97	+ 4.76	-3.12	+5.09
2.0	+ 6.25	+1.51	-1.17	+ 6.12	-3.82	+6.31

TABLE VI
 Table of Error Percentages.
 Rigid Frames of Fixed Supports, Single Storied-Two Bays.
 Stiffness Ratio $n = 2$, Height of Beams $\mu d = l/10$, $\mu = \left(\frac{n}{\eta}\right)^{1/3}$.

$\eta = h/l$	$R_{(H_A)}$	$R_{(H_B)}$	$R_{(V_C)}$	$R_{(M_A)}$	$R_{(M_{DB})}$	$R_{(M_{BD})}$
0.2	-6.76	-0.22	+0.15	-7.71	-0.17	-3.92
0.4	+0.71	+0.44	-0.32	+0.46	-1.30	+1.47
0.6	+2.69	+0.72	-0.52	+2.80	-1.88	+3.59
0.8	+4.20	+0.92	-0.66	+4.14	-2.34	+4.58
1.0	+5.16	+1.08	-0.77	+5.12	-2.73	+5.40
1.5	+7.07	+1.44	-1.03	+7.05	-3.60	+7.18
2.0	+8.70	+1.66	-1.26	+8.69	-4.39	+8.77

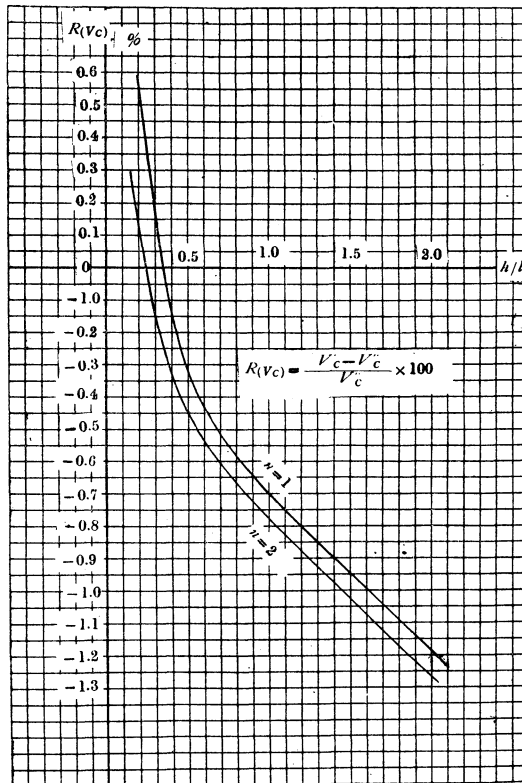


Fig. 4
 Relation between Error Percentage
 $R(V_c)$ and Ratio h/l for $n = 1$ and $n = 2$

7. Sectional Summary and Conclusions. The general conclusions to be drawn from the investigations described in this note are included in the resumé below :

(1) Ordinary approximate calculations of rigid frames in which the effect of bending distortions only is considered will probably be quite accurate for practical purposes; and horizontal reaction H_A is then dependent on the values n and h/l ; while vertical reactions V_A, V_C, V_E and end moments M_A, M_E, M_{DB}, M_{BD} are independent of h/l , but dependent on n .

(2) After close investigations worked out by more accurate computations of rigid frames, in which the effect of deformations owing to the normal forces is taken into consideration, it became obvious that the effect of the distortions are not always negligible in comparison with that induced by the distortion due to bending; besides the percentage of error

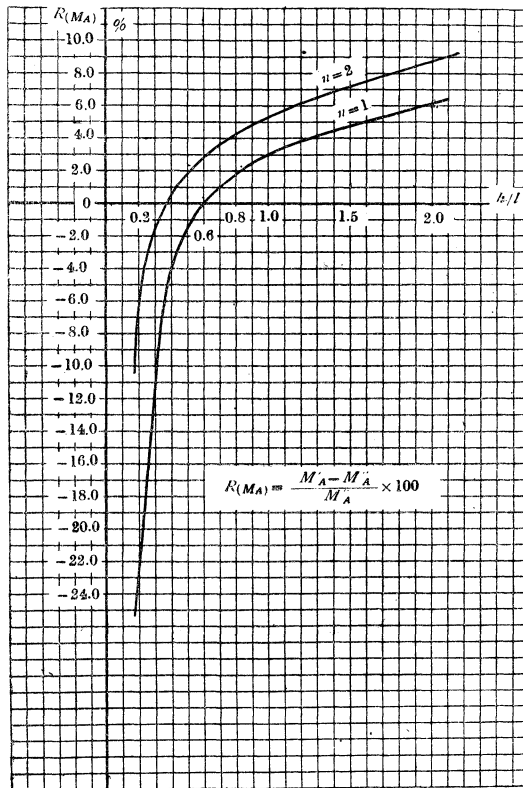


Fig. 5

Relation between Error Percentage
 $R(M_A)$ and Ratio h/l
 for $n = 1$ and $n = 2$

caused by current approximation is not always on the safe side. All the reactions and bending moments are, in this time, dependent on the values of n , h/l and μ .

(3) Furthermore, the effect of the deformations due to bending moments, normal stresses as well as tangential forces is not always negligible when compared with that computed under consideration of the effect of the bending moments, as in the case of smaller values η .

(4) In this note, a single storied-two bays-fixed frame has been treated and there have been compared reactions and moments for the following three cases:

- Case (i) M considered; Case (ii) M and N considered;
- Case (iii) M , N and S considered.

(5) First Example ($n = 1$, $\eta = 1$, $\mu = 1$)

Ranking orders of magnitudes for each reaction and moment calculated

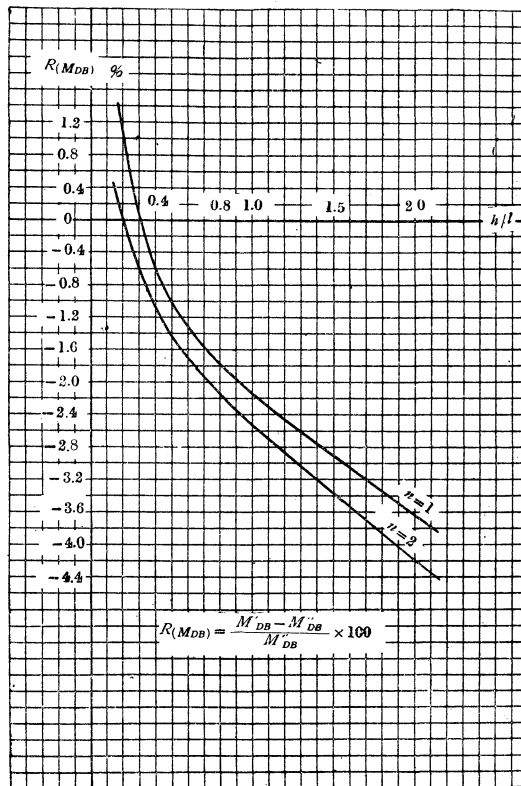


Fig 6

Relation between Error Percentage
 $R(M_{DB})$ and Ratio h/l
 for $n = 1$ and $n = 2$

for these three different cases are as follows:

H_A : Case (ii) →
Case (iii) → Case (i)

V_A : Case (iii) →
Case (ii) → Case (i)

V_C : Case (i) →
Case (ii) → Case (iii)

M_A : Case (ii) →
Case (iii) → Case (i)

M_{DB} : Case (i) →
Case (ii) → Case (iii)

M_{BD} : Case (ii) →
Case (iii) → Case (i)

(6) Second Example ($n = 2$,
 $\eta = 0.4$, $\mu = 1.71$)

In a similar way, we obtain:

H_A : Case (ii) →
Case (i) → Case (iii)

V_A : Case (ii) →
Case (iii) → Case (i)

V_C : Case (i) →
Case (iii) → Case (ii)

M_A : Case (i) →
Case (ii) → Case (iii)

M_{DB} : Case (i) →
Case (ii) → Case (iii)

M_{BD} : Case (ii) →
Case (i) → Case (iii)

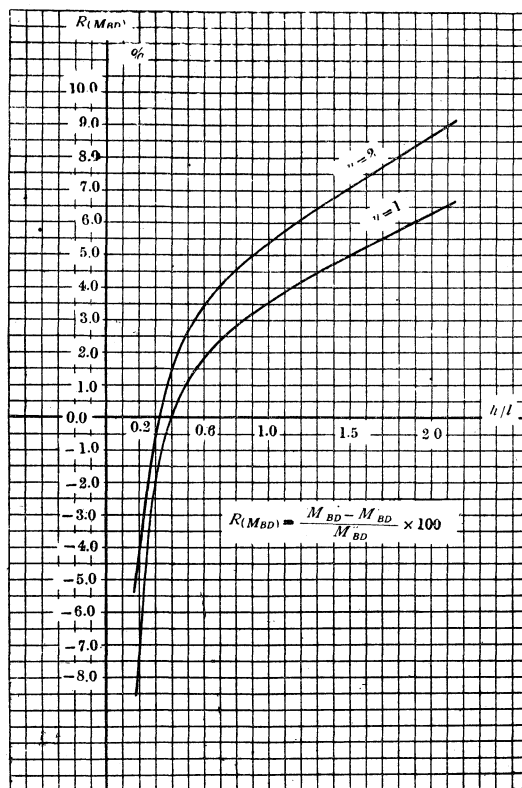


Fig. 7.
Relation between Error Percentage
 $R(M_{BD})$ and Ratio h/l
for $n = 1$ and $n = 2$

In this note, there are investigated Error Percentages for different reactions and moments for the examples above stated. For relatively smaller values of η , Error Percentages $R(H_A)$, $R(V_A)$, $R(V_C)$, $R(M_A)$, $R(M_{DB})$, $R(M_{BD})$ are not negligible and the effects of disregarding N , S are noticeable for rigid frames of shorter columns as well as very long bays.

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APPENDIX

The following Tables A ~ E are important in some degree to explain that the effect of the normal and tangential forces is usually of little significance and might be disregarded as an inconsiderable magnitude in comparison with the effect of bending moments and further to show that the relative effects of the two forces would not be always insignificant, depending upon the shape of the rigid frame, for example, in the case the height of the column is very short and the span length is very long.

TABLE A (1)
 Rigid Frames of Hinged Supports,
 Single Storied-Single Bay.
 $n = 1, h/l = \eta = 1, \mu = 1, \mu d = l/10$
 $x^2 = 0.000833$

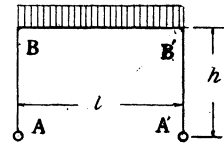


Fig. A.

Reactions and Moments	Case (i) (M considered)	Case (ii) (M and N considered)	Case (iii) (M, N and S considered)
H_A	$+0.05 wl$	$+0.04998 wl$	$+0.04984 wl$
$M_{BB'}$	$-0.05 wl^2$	$-0.04998 wl^2$	$-0.04984 wl^2$

TABLE A (2) Rigid Frames of Hinged Supports, Single Storied-Single Bay.

$n = 2, h/l = \eta = 0.4, \mu = \left(\frac{n}{\eta}\right)^{1/3} = 1.71, \mu d = l/10, x^2 = 0.00521$

Reactions and Moments	Case (i) (M considered)	Case (ii) (M and N considered)	Case (iii) (M, N and S considered)
H_A	$+0.0893 wl$	$+0.0891 wl$	$+0.0883 wl$
$M_{BB'}$	$-0.0357 wl^2$	$-0.0356 wl^2$	$-0.0353 wl^2$

TABLE A (3) Rigid Frames of Hinged Supports, Single Storied-Single Bay.

$n = 1, \eta = 0.2, \mu = \left(\frac{n}{\eta}\right)^{1/3} = 1.71, x^2 = 0.020833$

Reactions and Moments	Case (i) (M considered)	Case (ii) (M and N considered)	Case (iii) (M, N and S considered)
H_A	$+0.25 wl$	$+0.2469 wl$	$+0.2412 wl$
$M_{BB'}$	$-0.05 wl^2$	$-0.0494 wl^2$	$-0.0482 wl^2$

TABLE A (4) Rigid Frames of Hinged Supports, Single Storied-Single Bay.

$$n = 2, \quad \eta = 0.2, \quad \mu = \left(\frac{n}{\eta}\right)^{1/3} = 2.154, \quad x^2 = 0.020833$$

Reactions and Moments	Case (i) (M considered)	Case (ii) (M and N considered)	Case (iii) (M , N and S considered)
H_A	+0.1786 wl	+0.1770 wl	+0.1733 wl
$M_{BR'}$	-0.0357 wl^2	-0.0354 wl^2	-0.0347 wl^2

TABLE B (1)

Rigid Frames of Hinged Supports,
Two storied-Single Bay.

$$n = 1, \quad h/l = \eta = 1, \quad \mu = 1, \quad \mu d = l/10 \\ x^2 = 0.000833$$

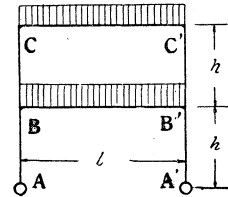


Fig. B.

Reactions and Moments	Case (i) (M considered)	Case (ii) (M and N considered)	Case (iii) (M , N and S considered)
H_A	+0.02 wl	+0.02028 wl	+0.02055 wl
M_{BA}	+0.02 wl^2	+0.02028 wl^2	+0.02055 wl^2
$M_{BB'}$	-0.07 wl^2	-0.07001 wl^2	-0.06973 wl^2
M_{BC}	+0.05 wl^2	+0.04973 wl^2	+0.04919 wl^2
$M_{CC'}$	-0.06 wl^2	-0.05937 wl^2	-0.05843 wl^2

TABLE B (2) Rigid Frames of Hinged Supports, Two Storied-Single Bay.

$$n = 2, \quad h/l = \eta = 0.4, \quad \mu = \left(\frac{n}{\eta}\right)^{1/3} = 1.71, \quad \mu d = l/10, \quad x^2 = 0.00521$$

Reactions and Moments	Case (i) (M considered)	Case (ii) (M and N considered)	Case (iii) (M , N and S considered)
H_A	+0.04464 wl	+0.04635 wl	+0.04717 wl
M_{BA}	+0.01786 wl^2	+0.01854 wl^2	+0.01887 wl^2
$M_{BB'}$	-0.05952 wl^2	-0.05951 wl^2	-0.05871 wl^2
M_{BC}	+0.04167 wl^2	+0.04097 wl^2	+0.03984 wl^2
$M_{CC'}$	-0.04762 wl^2	-0.04536 wl^2	-0.04308 wl^2

TABLE B (3) Rigid Frames of Hinged Supports, Two Storied-Single Bay.
 $n = 1, \eta = 0.2, \mu = 1.71, x^2 = 0.020833$

Reactions and Moments	Case (i) (M considered)	Case (ii) (M and N considered)	Case (iii) (M, N and S considered)
H_A	$+0.10 wl$	$+0.1275 wl$	$+0.1337 wl$
M_{BA}	$+0.02 wl^2$	$+0.0255 wl^2$	$+0.0267 wl^2$
M_{DE}	$-0.07 wl^2$	$-0.0701 wl^2$	$-0.0684 wl^2$
M_{DC}	$+0.05 wl^2$	$+0.0446 wl^2$	$+0.0416 wl^2$
$M_{CC'}$	$-0.06 wl^2$	$-0.0476 wl^2$	$-0.0426 wl^2$

TABLE B (4) Rigid Frames of Hinged Supports, Two Storied-Single Bay.
 $n = 2, \eta = 0.2, \mu = 2.154, x^2 = 0.020833$

Reactions and Moments	Case (i) (M considered)	Case (ii) (M and N considered)	Case (iii) (M, N and S considered)
H_A	$+0.0893 wl$	$+0.1016 wl$	$+0.1044 wl$
M_{BA}	$+0.0179 wl^2$	$+0.0203 wl^2$	$+0.0209 wl^2$
M_{DE}	$-0.0595 wl^2$	$-0.0595 wl^2$	$-0.0577 wl^2$
M_{DC}	$+0.0417 wl^2$	$+0.0392 wl^2$	$+0.0369 wl^2$
$M_{CC'}$	$-0.0476 wl^2$	$-0.0395 wl^2$	$-0.0350 wl^2$

TABLE C (1)

Rigid Frames of Hinged Supports,
 Single Storied-Two Bays.

$n = 1, h/l = \eta = 1, \mu = 1, \mu d = l/10$
 $x^2 = 0.000833$

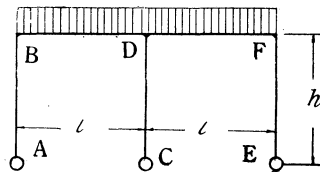


Fig. C.

Reactions and Moments	Case (i) (M considered)	Case (ii) (M and N considered)	Case (iii) (M, N and S considered)
H_A	$+0.03571 wl$	$+0.03475 wl$	$+0.03505 wl$
V_A	$+0.42857 wl$	$+0.42606 wl$	$+0.42702 wl$
V_A	$+1.14286 wl$	$+1.14787 wl$	$+1.14598 wl$
M_{BD}	$-0.03571 wl^2$	$-0.03475 wl^2$	$-0.03505 wl^2$
M_{DB}	$+0.10714 wl^2$	$+0.10869 wl^2$	$+0.10902 wl^2$

TABLE C (2)
Rigid Frames of Hinged Supports,
Single Storied-Two Bays.

$$n = 2, \quad h/l = \eta = 0.4, \quad \mu = \left(\frac{n}{\eta}\right)^{1/3} = 1.71, \quad \mu d = l/10, \quad x^2 = 0.00521$$

Reactions and Moments	Case (i) (M considered)	Case (ii) (M and N considered)	Case (iii) (M , N and S considered)
H_A	+0.05682 wl	+0.05555 wl	+0.05582 wl
V_A	+0.40909 wl	+0.40764 wl	+0.40843 wl
V_C	+1.18182 wl	+1.18473 wl	+1.18315 wl
M_{BD}	-0.02273 wl^2	-0.02222 wl^2	-0.02233 wl^2
M_{DB}	+0.11364 wl^2	+0.11458 wl^2	+0.11451 wl^2

TABLE C (3)
Rigid Frames of Hinged Supports,
Single Storied-Two Bays.

$$n = 1, \quad \eta = 0.2, \quad \mu = 1.71, \quad \mu d = l/10, \quad x^2 = 0.020833$$

Reactions and Moments	Case (i) (M considered)	Case (ii) (M and N considered)	Case (iii) (M , N and S considered)
H_A	+0.1786 wl	+0.1704 wl	+0.1665 wl
V_A	+0.4286 wl	+0.4246 wl	+0.4225 wl
V_C	+0.1429 wl	+1.1447 wl	+1.1414 wl
M_{BD}	-0.0357 wl^2	-0.0341 wl^2	-0.0333 wl^2
M_{DB}	+0.1071 wl^2	+0.1079 wl^2	+0.1081 wl^2

TABLE C (4)
Rigid Frames of Hinged Supports,
Single Storied-Two Bays.

$$n = 2, \quad \eta = 0.2, \quad \mu = 2.154, \quad \mu d = l/10, \quad x^2 = 0.020833$$

Reactions and Moments	Case (i) (M considered)	Case (ii) (M and N considered)	Case (iii) (M , N and S considered)
H_A	+0.1137 wl	+0.1097 wl	+0.1082 wl
V_A	+0.4091 wl	+0.4067 wl	+0.4058 wl
V_C	+1.1818 wl	+1.1825 wl	+1.1784 wl
M_{BD}	-0.0227 wl^2	-0.0220 wl^2	-0.0217 wl^2
M_{DB}	+0.1136 wl^2	+0.1142 wl^2	+0.1139 wl^2

TABLE D (1)

Rigid Frames of Fixed Supports,
Single Storied-Single Bay.

$$n=1, h/l = \eta = 1, \mu = 1, \mu d = l/10$$

$$x^2 = 0.000833$$

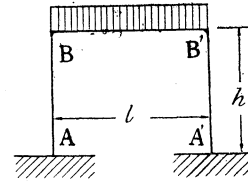


Fig. D.

Reactions and Moments	Case (i) (<i>M</i> considered)	Case (ii) (<i>M</i> and <i>N</i> considered)	Case (iii) (<i>M</i> , <i>N</i> and <i>S</i> considered)
H_A	+0.08333 wl	+0.08313 wl	+0.08198 wl
M_{AB}	+0.02778 wl^2	+0.02764 wl^2	+0.02688 wl^2
$M_{BB'}$	-0.05556 wl^2	-0.05549 wl^2	-0.05510 wl^2

TABLE D (2)

Rigid Frames of Fixed Supports, Single Storied-Single Bay.

$$n = 2, h/l = \eta = 0.4, \mu = \left(\frac{n}{\eta}\right)^{1/3} = 1.71, \mu d = l/10$$

$$x^2 = 0.00521$$

Reactions and Moments	Case (i) (<i>M</i> considered)	Case (ii) (<i>M</i> and <i>N</i> considered)	Case (iii) (<i>M</i> , <i>N</i> and <i>S</i> considered)
H_A	+0.15625 wl	+0.15474 wl	+0.14921 wl
M_{AB}	+0.02083 wl^2	+0.02047 wl^2	+0.01915 wl^2
$M_{BB'}$	-0.04167 wl^2	-0.04142 wl^2	-0.04054 wl^2

TABLE D (3)

Rigid Frames of Fixed Supports, Single Storied-Single Bay.

$$n = 1, \eta = 0.2, \mu = 1.71, \mu d = l/10 \quad x^2 = 0.020833$$

Reactions and Moments	Case (i) (<i>M</i> considered)	Case (ii) (<i>M</i> and <i>N</i> considered)	Case (iii) (<i>M</i> , <i>N</i> and <i>S</i> considered)
H_A	+0.41667 wl	+0.3922 wl	+0.3525 wl
M_{AB}	+0.02778 wl^2	+0.0245 wl^2	+0.0192 wl^2
$M_{BB'}$	-0.05556 wl^2	-0.0539 wl^2	-0.0513 wl^2

TABLE D (4) Rigid Frames of Fixed Supports, Single Storied-Single Bay.
 $n = 2, \eta = 0.2, \mu = 2.154, \mu d = l/10, x^2 = 0.020833$

Reactions and Moments	Case (i) (M considered)	Case (ii) (M and N considered)	Case (iii) (M, N and S considered)
H_A	$+0.3125 wl$	$+0.3008 wl$	$+0.2758 wl$
M_{AB}	$+0.02083 wl^2$	$+0.0194 wl^2$	$+0.0164 wl^2$
$M_{BB'}$	$-0.04167 wl^2$	$-0.0407 wl^2$	$-0.0387 wl^2$

TABLE E (1)

Rigid Frames of Fixed Supports,
 Fixed Frames of Two Storied-Single Bay.

$$n = 1, h/l = \eta = 1, 1/\eta = 1, \mu = 1, \mu d = l/10$$

$$x^2 = 0.0008333$$

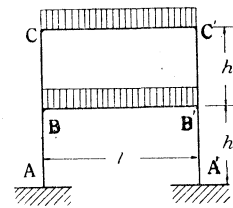


Fig. E.

Reactions and Moments	Case (i) (M considered)	Case (ii) (M and N considered)	Case (iii) (M, N and S considered)
H_A	$+0.03571 wl$	$+0.03614 wl$	$+0.03610 wl$
M_{AB}	$+0.01190 wl^2$	$+0.01211 wl^2$	$+0.01194 wl^2$
$M_{CC'}$	$-0.05952 wl^2$	$-0.05938 wl^2$	$-0.05901 wl^2$
$M_{BB'}$	$-0.07143 wl^2$	$-0.07140 wl^2$	$-0.07109 wl^2$
M_{BC}	$+0.04762 wl^2$	$+0.04737 wl^2$	$+0.04693 wl^2$
M_{BA}	$+0.02381 wl^2$	$+0.02403 wl^2$	$+0.02416 wl^2$

TABLE E (2) Rigid Frames of Fixed Supports, Two Storied-Single Bay.

$$n = 2, h/l = \eta = 0.4, 1/\eta = 2.5, \mu = \left(\frac{n}{\eta}\right)^{1/3} = 1.71, \mu d = l/10, x^2 = 0.00521$$

Reactions and Moments	Case (i) (M considered)	Case (ii) (M and N considered)	Case (iii) (M, N and S considered)
H_A	$+0.08152 wl$	$+0.08408 wl$	$+0.08295 wl$
M_{AB}	$+0.01088 wl^2$	$+0.01134 wl^2$	$+0.01088 wl^2$
$M_{CC'}$	$-0.04710 wl^2$	$-0.04660 wl^2$	$-0.04569 wl^2$
$M_{BB'}$	$-0.06159 wl^2$	$-0.06146 wl^2$	$-0.06055 wl^2$
M_{BC}	$+0.03986 wl^2$	$+0.03917 wl^2$	$+0.03825 wl^2$
M_{BA}	$+0.02173 wl^2$	$+0.02229 wl^2$	$+0.02230 wl^2$

TABLE E (3)
Rigid Frames of Fixed Supports,
Two Storied-Single Bay.

$$n = 1, \quad \eta = 0.2, \quad \mu = 1.71, \quad x^2 = 0.020833$$

Reactions and Moments	Case (i) (M considered)	Case (ii) (M and N) considered)	Case (iii) (M , N and S) considered)
H_A	+0.17855 wl	+0.22270 wl	+0.21430 wl
M_{AR}	+0.01190 wl^2	+0.01605 wl^2	+0.01434 wl^2
$M_{CC'}$	-0.05952 wl^2	-0.05667 wl^2	-0.05487 wl^2
$M_{RR'}$	-0.07143 wl^2	-0.07091 wl^2	-0.06913 wl^2
M_{BC}	+0.04762 wl^2	+0.04242 wl^2	+0.04061 wl^2
M_{BA}	+0.02381 wl^2	+0.02849 wl^2	+0.02852 wl^2

TABLE E (4)
Rigid Frames of Fixed Supports,
Two Storied-Single Bay.

$$n = 2, \quad \eta = 0.2, \quad \mu = 2.154, \quad \mu d = l/10, \quad x^2 = 0.020833$$

Reactions and Moments	Case (i) (M considered)	Case (ii) (M and N) considered)	Case (iii) (M , N and S) considered)
H_A	+0.16305 wl	+0.18180 wl	+0.17510 wl
M_{AR}	+0.01088 wl^2	+0.01262 wl^2	+0.01149 wl^2
$M_{CC'}$	-0.04710 wl^2	-0.04527 wl^2	-0.04352 wl^2
$M_{RR'}$	-0.06159 wl^2	-0.06110 wl^2	-0.05921 wl^2
M_{BC}	+0.03986 wl^2	+0.03736 wl^2	+0.03568 wl^2
M_{BA}	+0.02173 wl^2	+0.02374 wl^2	+0.02353 wl^2

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