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FLEXURAL VIBRATION OF THE SQUARE PLATE WITH A CENTRAL CIRCULAR HOLE

By Toyoji KUMAI

The experimental studies on the effect of a central circular hole on the frequencies of the flexural vibrations of the square plate with clamped and simply supported edge are carried out in the cases of a few modes of nodal vibrations. The results of the measurements are compared with those of the approximate calculations allowing small residual deflections at some portion of the outer boundary of the square plate.

1. Introduction. The effect of a lightening hole or a manhole perforated at the centre of the rectangular plate on the flexural vibrations is an important problem in view of regarding the stiffeness of the element of ships' hull or of the airplane structure. The results of experimental or theoretical studies of the problem, however do not seem to have yet been published except those treated by W. Hort and M. König[1] on the vibration of the circular ring with free-free peripheries.

The results obtained from the case of circular ring may not be applicable to the practical design, because of the form of outside boundary of the panel in the structure of light constructions is generally square or rectangular and a central hole is circular or elliptical. In the present paper, the flexural vibrations of the clamped and simply supported square plate with a central circular hole are observed and the frequencies of the plates with different diameter of a hole are measured up to the second mode of nodal vibrations by the method of Chladni figure. Moreover the frequencies of the vibration of the square plate having a central hole with given modes are computed by means of the superposition of the solutions of circular ring plate satisfying the boundary conditions at several points along the outside edges of the square plate. Such a method of approximation for satisfying the boundary conditions as above mentioned has already been applied to the vibration of a rectangular plate clamped at four edges by Porf. K. Sezawa.[2]

2. Experimental Studies. The model specimens used in the present experiments are made of the celluloid plate with 7 cm effective breadth

1) W. Hort und M. König, "Studien über Schwingungen von Kreisplatten und Ringen." Zeitschrift für tech. Physik. Nr. 10, 1928.

2) K. Sezawa, "On the Lateral Vibration of a Rectangular Plate Clamped at four Edges". Rep. of Aero. R. I. Tokyou Univ. No. 70. April, 1931.

and 0.05 cm thickness. The tests are taken place in both cases of the square plate and of circular ring with the diameter of equal length to a side of the square plate. The model plate is clamped by the aluminium frame. In the case of simply supported edge, the outer boundary is connected to the frame by the cross hinge made of the alternatively bridged paper strip as shown in fig. 1.

The frequencies of the vibration of the model specimens with given mode of the vibration are determined by the method of Chradni figure using the carborundum. The vibration of the specimen is excited by the sound oscillation through the air column as wili be seen in fig. 2.

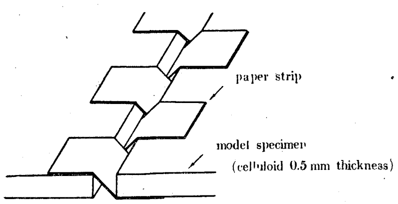

Fig. 1. Details of procedure for simply supported edge of the model plate.

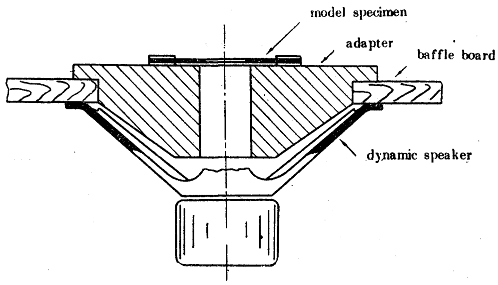

Fig. 2. Apparatus for excitation of vibration of the plate by the use of dynamic speaker.

The dynamic speaker of 10 inch diameter is oscillated by the beat oscillator with the range of the frequencies from 150 to 2000 cps. in the present observations. The relations between the frequencies of the vibration of the perforated plate and the diameter ratio λ are shown in fig. 3. a), b), c) and d).

3. Analysis.

Nomenclature

x, y	cartesian co-ordinate parallel to both edges of the square plate with the origin at the centre of the plate,
r, θ	polar co-ordinate with the origin at the centre of the plate,
w, t, φ	deflection of vibration of the plate, time and phase angle respectively,
λ	ratio of the diameter of a central hole and the breadth of the square plate or outside diameter of circular ring,
h	half thickness of the plate,
E, ν, ρ	Young's modulus, Poisson's ratio and density of material respectively,
p, f	circular frequency and number of frequency in cps. of vibration of the plate. respectively.

The fundamental equation of the flexural vibration of the plate with uniform thickness is shown as the following form

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 w_1 + \rho\frac{\partial^2 w_1}{\partial t^2} = 0 \ . \tag{1}$$

Put

$$w_1 = w(r, \theta)\begin{matrix}\cos pt \\ \sin pt\end{matrix}, \tag{2}$$

$w(r, \theta)$ yield as belows

$$w = \sum_{n=0}^{\infty}\left\{A_n J_n(z) + B_n I_n(z) + C_n Y_n(z) + D_n K_n(z)\right\}\cos(n\theta + \varphi), \tag{3}$$

where A_n, B_n, C_n and D_n are integration constants and $J_n(z)$, $Y_n(z)$ are Bessel functions of the first kind and of the second kind respectively and $I_n(z)$, $K_n(z)$ are modified Bessel functions of $J_n(z)$, $Y_n(z)$ respectively, and z is the eigenvalue of the solutions and is presented by

$$z^2 = \frac{p r^2}{h}\sqrt{\frac{3\rho(1-\nu^2)}{E}} \ . \tag{4}$$

Giving diameter ratio, the argument z is easily obtained in the case of circular ring plate with any type of edge conditions. In the case of the square plate, however, it is very difficult to make the solution satisfy the conditions of outside boundary and that of the periphery of a central hole at the same time. In the present paper, eigenvalues are computed by superposing the solutions more than two groups of n terms selected in accordance with the mode of vibrations of the square plate. Thus, the boundary conditions of the outside edges are satisfied at the several points, for example, at sixteen points, in the fundamental mode of the square plate with clamped or supported edges when two groupes of n are taken into consideration.

Some examples of the values of n and φ are shown as belows.

mode of vibration	n	φ
fundamental	0, 4, 8, ...	0
$x = 0$ node	1, 3, 5, ...	0
$x = y$ node	1, 3, 5, ...	$\pi/4$

The boundary conditions and the equations which determine the eigenvalues in both cases of the edge conditions are presented as belows.

i. clamped edge,

$$w_i = 0; \quad r = r_i \quad \theta = \theta_i, \tag{5}$$

$$\left(\frac{\partial w}{\partial x}\right)_i = 0; \ \text{for} \ x = \pm 1, \quad \left(\frac{\partial w}{\partial y}\right)_i = 0; \ \text{for} \ y = \pm 1, \tag{6}$$

where

$$\left.\begin{aligned}\frac{\partial w}{\partial x} &= \frac{\partial w}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial w}{\partial \theta}\sin\theta\,,\\ \frac{\partial w}{\partial y} &= \frac{\partial w}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial w}{\partial \theta}\cos\theta\,.\end{aligned}\right\} \qquad (6)_c'$$

ii. simply supported edge,

$$w_i = 0\,; \qquad r = r_i \quad \theta = \theta_i\,, \qquad (5)$$

$$\left.\begin{aligned}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)_i &= 0; \quad \text{for} \quad x = \pm 1,\\ \left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) &= 0; \quad \text{for} \quad y = \pm 1.\end{aligned}\right\} \qquad (6)_s$$

where

$$\left.\begin{aligned}\frac{\partial^2 w}{\partial x^2} &= \frac{\partial^2 w}{\partial r^2}\cos^2\theta + \left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\sin^2\theta\\ &\quad -2\left(\frac{1}{r}\frac{\partial^2 w}{\partial r\,\partial\theta} - \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\sin\theta\cos\theta\,,\\ \frac{\partial^2 w}{\partial y^2} &= \frac{\partial^2 w}{\partial r^2}\sin^2\theta + \left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\cos^2\theta\\ &\quad -2\left(\frac{1}{r}\frac{\partial^2 w}{\partial r\,\partial\theta} - \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\sin\theta\cos\theta\,.\end{aligned}\right\} \qquad (6)_s'$$

along the periphery of a hole, in both cases,

$$\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\left(\frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial^2 w}{\partial \theta^2}\right) = 0; \qquad r = r_0\,, \qquad (7)$$

$$\begin{aligned}&\frac{\partial}{\partial r}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\\ &\qquad + (1-\nu)\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial^2 w}{\partial r\,\partial\theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right) = 0\,;\\ &\qquad\qquad r = r_0\,, \end{aligned} \qquad (8)$$

Substituting the solutions (3) in the four conditions (5), $(6)_c$, $\{(6)_s\}$, (7) and (8) and using $(6)_c'$, $\{(6)_s'\}$, the general expressions of the equations which determine the eigenvalues are shown as follows,

$$\sum_{n=0}^{\infty}\{A_n J_n(z_i) + B_n I_n(z_i) + C_n Y_n(z_i) + D_n K_n(z_i)\}\cos n\,\theta_i = 0\,, \qquad (9)$$

$$\begin{aligned}&\sum_{n=0}^{\infty}[A_n\{\alpha_{ni}J_n(z_i) - \beta_{ni}J_{n+1}(z_i)\} + B_n\{\alpha_{ni}' I_n(z_i) + \beta_{ni}I_{n+1}(z_i)\}\\ &+ C_n\{\alpha_{ni}Y_n(z_i) - \beta_{ni}Y_{n+1}(z_i)\} + AD_n\{\alpha_{ni}' K_n(z_i)\\ &- \beta_{ni}K_{n+1}(z_i)\}] = 0\,;\end{aligned} \qquad (10)$$

$$A_n\{\gamma_n J_n(z_0)+J_{n+1}(z_0)\}+B_n\{\gamma_n' I_n(z_0)-I_{n+1}(z_0)\}$$
$$+C_n\{\gamma_n Y_n(z_0)+Y_{n+1}(z_0)\}+D_n\{\gamma_n' K_n(z_0)+K_{n+1}(z_0)\}=0, \qquad (11)$$

$$A_n\{\delta_n I_n(z_0)-\zeta_n J_{n+1}(z_0)\}+B_n\{\delta_n' I_n(z_0)+\zeta_n' I_{n+1}(z_0)\}$$
$$+C_n\{\delta_n Y_n(z_0)-\zeta_n Y_{n+1}(z_0)\}+D_n\{\delta_n' K_n(z_0)-\zeta_n' K_{n+1}(z_0)\}=0, \quad (12)$$

where the coefficient α_{ni}, α_{ni}', β_{ni}, γ_n and etc. are presented by the following form,

i. clamped edge,

$$\alpha_{ni}=\alpha_{ni}'=\frac{n}{z_i}\cos(n-1)\theta_i,\quad \beta_{ni}=\cos n\theta_i\cos\theta_i;\quad (x=\pm1),$$

$$\alpha_{ni}=\alpha_{ni}'=\frac{-n}{z_i}\sin(n-1)\theta_i,\quad \beta_{ni}=\cos n\theta_i\sin\theta_i.\quad (y=\pm1).$$

ii. simply supported edge,

$$\begin{matrix}\alpha_{ni}\\ \alpha_{ni}'\end{matrix}=\frac{n(n-1)}{z_i}\cos(2-n)\theta_i\mp\frac{z_i}{1-\nu}(\cos^2\theta_i+\nu\sin^2\theta_i)\cos n\theta_i,$$

$$\beta_{ni}=-\cos 2\theta_i\cos n\theta_i+n\sin 2\theta_i\sin n\theta_i;\quad (x=\pm1),$$

$$\begin{matrix}\alpha_{ni}\\ \alpha_{ni}'\end{matrix}=-\frac{n(n-1)}{z_i}\cos(2-n)\theta_i\mp\frac{z_i}{1-\nu}(\sin^2\theta+\nu\cos^2\theta)\cos n\theta_i,$$

$$\beta_{ni}=\cos 2\theta_i\cos n\theta_i-n\sin 2\theta_i\sin n\theta_i;\quad (y=\pm1).$$

and in both cases

$$\begin{matrix}\gamma_n\\ \gamma_n'\end{matrix}=\frac{n(n-1)}{z_0}\mp\frac{z_0}{1-\nu},\qquad \begin{matrix}\delta_n\\ \delta_n'\end{matrix}=n\left\{\frac{n(n-1)}{z_0}\pm\frac{z_0}{1-\nu}\right\},$$

$$\begin{matrix}\zeta_n\\ \zeta_n'\end{matrix}=n^2\pm\frac{z^2}{1-\nu}.$$

where z_i z_0 are the values of z when $r=r_i$, $r=r_0$ respectively. If we take $i=1, 2, \cdots j$, eigenvalues are obtained by solving $4j$ equations which are deduced from (9), (10), (11) and (12) for given mode and given diameter ratio of a hole. Since the eigenvalue of the vibration of the circular ring plate with simply supported edge and with clamped edge are computed without approximation as above mentioned, the results of calculations of the square plate with a central hole are compared with that of the circular ring plate with the outside diameter same as the breadth of the square plate and with the same diameter of the hole. In the present paper, only two terms of n are taken into consideration in all cases. Also, Young's modulus and Poisson's ratio of the material of the specimens are assumed $250\,\mathrm{kg/mm^2}$ and 0.3 respectively. The results of calculations are shown in fig. 3 a) $\sim$ d). The frequency ratios are shown in the following table.

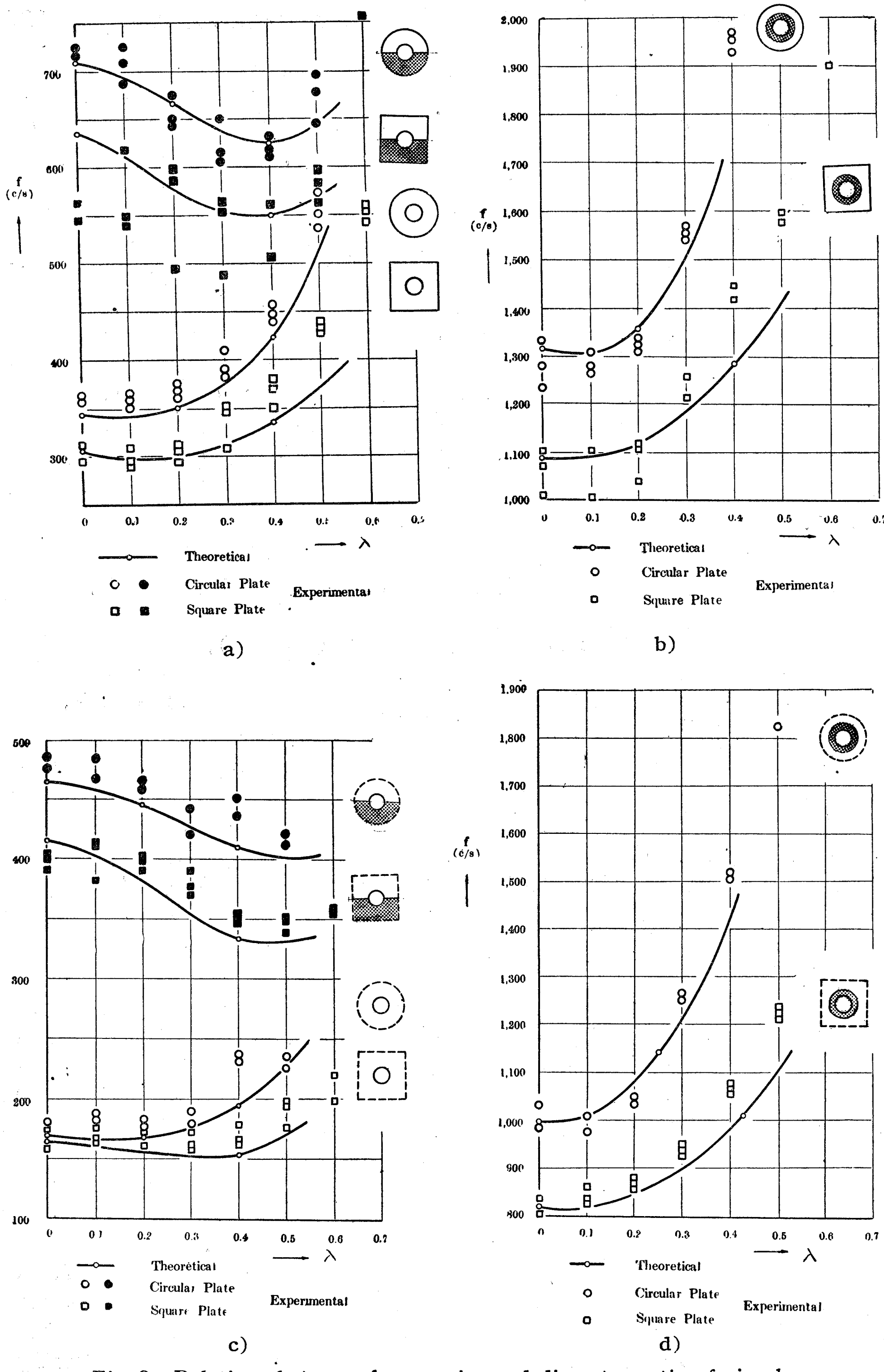

Fig. 3. Relations between frequencies and diameter ratio of circular ring plate and perforated square plate,
a), b); in the case of clamped edge,
c), d); in the case of simply supported edge.

Frequency ratios of vibration of the square plate with a central hole.

mode of vibration	λ	f/f_0	
		clamped edge	supported edge
fundamental	0	1.000	1.000
	0.2	0.986	0.985
	0.4	1.118	0.965
one nodal line $y=0$	0	1.000	1.000
	0.2	0.916	0.913
	0.4	0.876	0.804
one nodal circle	0	1.000	1.000
	0.2	1.040	1.024
	0.4	1.195	1.228

4. Conclusions and Acknowledgment. The relations between the frequencies of few mode of vibration of the perforated square plate and the diameter ratio of the central circular hole were obtained by the experiments and the approximate calculations as will be seen in fig. 3 a) $\sim$ d) in both cases of edge conditions. The tendency of the curves obtained by the present calculations are found in good agreement with that of the model experiments as shown in fig. 3.

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