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AN APPROXIMATE CALCULATION OF THE LAMINAR NONISOTHERMAL FLOW IN A CIRCULAR PIPE (I)

THE CASE OF INVARIABLE VISCOSITY (POISEUILLE FLOW)

By Michio OHJI

In the present report the heat transfer of a laminar nonisothermal flow in a circular pipe is calculated according to the method analogous to the well-known Kármán-Pohlhausen's approximation in the boundary layer theory, under the assumption of Poiseuille's velocity distribution across the pipe. The results are compared with those of the existing theories and it is concluded that our solution can be regarded as an acceptable one of the problem. The application of this method to the case of variable viscosity is now being planned and will be published in future report.

1. Introductory Remarks. The problem of heat transfer of a steady laminar flow in a circular pipe has been already studied since the last century. For the case when the wall temperature is kept constant, in particular, Graetz[1] first obtained a theoretical solution, assuming that the physical properties of the fluid, i.e., the density, viscosity and thermal conductivity, etc. do not change with temperature and that accordingly the Poiseuille's velocity distribution is maintained throughout the motion. Although his solution was improved numerically or modified by later authors[2], their results still appear to show somewhat considerable and systematic discrepancies with those of the experiments.

In reality, however, the Graetz's assumption is not always correct; that is, among the physical properties of fluid, the viscosity, especially that of liquid, is in general rather sensitive to the changes of temperature, so it should be considered as a variable quantity from point to point in a field of nonisothermal flow. Since this fact might influence the state of flow as well as the exchange of heat to some extent the discrepancies stated above is likely to be ascribed, at least partly, to this effect. Thus various attempts have been made to introduce the dependency of viscosity on tem-

1) Graetz, L., Ann. d. Phys., 18 (1883), 79; 25 (1885), 337.

2) Among them the followings must be mentioned: Nusselt, W., V.D.I., (1910), 1154; Drew, T.B., Trans. Am. Inst. Chem. Engrs., 26 (1931), 26; Lévêque, M.A., Ann. Mines, 13 (1928), 283; Yamagata, K., Memoirs of the Faculty of Engineering, Kyushu Univ., Japan, vol. 8, No. 6 (1940), 365, in the last of which the further calculation for the case of variable viscosity is also shown with the proposed formulae convenient to represent the experimental data.

perature. But up to the present, the problem being so complicated, no one seems to have succeded to obtain a definite solution beyond that of more or less an empirical nature.[1]

Now the present author takes another way of attack on this problem, basing on the similar idea as that of the momentum equation frequently used in hydrodynamical theories, e.g., the Kármán-Pohlhausen's method of approximate calculation of the laminar boundary layer. Recently this idea has been extended and reinterpreted mathematically by Professor Yamada of this Institute and found to be very effective for a number of problems.[2]

At first, however, the case of invariable viscosity or Poiseuille flow is treated in order to examine whether such an approximating procedure has a possibility to serve our primary purpose, viz; the evaluation of the effect of variable viscosity. The first report is thus devoted to this rather preliminary work, apart from the direct comparison with the experiments.

2. Statement of the Problem. Let us consider a steady laminar flow of fluid filling a straight circular pipe of uniform radius $a\,(=d/2)$ and suppose that its wall temperature changes from T_i to T_w discontinuously, each of which is kept constant all the time. Then the flow will be heated when $T_i < T_w$ or cooled when $T_i > T_w$, approaching to the final equilibrium, where its temperature becomes entirely equal to T_w. The problem is now to find the temperature distribution in the flow and the heat transfer coefficient in the course of the fluid motion.

Here the following assumptions are adopted:

(i) The fluid is incompressible and the free convection, radiation and dissipation are of no importance.

(ii) The velocity distribution is governed everywhere by the law of Poiseuille.

(iii) The molecular heat conduction in the direction of the flow can be neglected in comparison with that across the flow.

Then taking cylindrical coordinates (r, ϕ, z), the origin of which is at the center of the cross-section where the wall temperature jumps from T_i to T_w, the equation of heat transfer for steady state becomes,[3]

$$2w_m\left[1-\left(\frac{r}{a}\right)^2\right]\frac{\partial T}{\partial z} = \kappa\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\,\frac{\partial T}{\partial r}\right), \tag{2.1}$$

1) *See*, for instance, McAdams, W., "Heat Transmission", pp. 181 ~ 192, 2nd Edition, McGraw-Hill (1942), where an exhaustive bibliography is found.

2) Yamada, H., Reports of the Research Institute for Fluid Engineering, Kyushu Univ., Japan, vol. 6, No. 2 (1950), 42 and 87; Kurihara, M., *ibid.* vol. 6, No. 2 (1950), 62; Yamada, H., p. 11 of the present volume of this journal.

3) Goldstein, S., "Modern Developments in Fluid Dynamics", pp. 616 ~ 622, 1st Edition, Oxford (1938).

where T is the temperature at a point (r, ϕ, z), κ the thermometric conductivity and w_m the mean velocity over the cross-section. Or for convenience (2.1) is rewritten in nondimensional form

$$(1-\eta^2)\frac{\partial\theta}{\partial\zeta} = \frac{1}{\sigma R}\left(\frac{\partial^2\theta}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial\theta}{\partial\eta}\right), \tag{2.1a}$$

introducing dimensionless quantities

$$\left.\begin{aligned} &\eta = r/a\,, \quad \zeta = z/a\,, && \theta = (T - T_w)\,/\,(T_i - T_w)\,, \\ &\sigma = \nu/\kappa\,, && \text{(Prandtl number)}, \\ \text{and}\quad &R = w_m\, d/\nu\,, && \text{(Reynolds number)}, \\ \text{namely}\quad &\sigma R = w_m\, d/\kappa\,, && \text{(Péclet number)}, \end{aligned}\right\} \tag{2.2}$$

where ν is the kinematic viscosity of the fluid. Further putting

$$\chi = \zeta/2\sigma R = \kappa\, z/w_m\, d^2\,, \tag{2.3}$$

(2.1a) reduces to

$$(1-\eta^2)\frac{\partial\theta}{\partial\chi} = 2\left(\frac{\partial^2\theta}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial\theta}{\partial\eta}\right), \tag{2.1b}$$

which is to be solved under the boundary conditions

$$\left.\begin{aligned} &\theta(0,\eta) \equiv 1 && (T \equiv T_i\,), \\ \text{besides}\quad &\theta(\chi, 1) = 0 && (T = T_w), \\ \text{and}\quad &\frac{\partial\theta}{\partial\eta}(\chi, 0) = 0 && \left(\frac{\partial T}{\partial r} = 0\right), \end{aligned}\right\} \tag{2.4}$$

for $\chi > 0$.

Next, the heat transfer coefficient or Nusselt number Nu concerning the pipe length l is related to the temperature distribution $\theta(\chi, \eta)$ in the following way:

$$Nu = \frac{w_m\, d^2}{2\kappa\, l}\left(\frac{1-\theta_M}{1+\theta_M}\right), \tag{2.5}$$

where

$$\theta_M = \frac{T_M - T_w}{T_i - T_w} = \int_0^a \theta \cdot w \cdot 2\pi r\, dr \Big/ \int_0^a w \cdot 2\pi r\, dr\,, \tag{2.6}$$

i.e., the nondimensional mean temperature over the cross-section at $z = l$ or $\chi = \kappa\, l\,/\,w_m\, d^2$, weighted with respect to the axial velocity w.[1] Substituting $w = 2w_m(1-\eta^2)$ into (2.6), we have

$$\theta_M = 4\int_0^1 \theta\,(1-\eta^2)\,\eta\, d\eta = 4\,(I_1 - I_3)\,, \tag{2.6a}$$

[1] Goldstein, S., *loc. cit.*

for Poiseuille flow, where I_n stands for the nth moment of the temperature distribution, viz, $I_n = \int_0^1 \theta \cdot \eta^n \, d\eta$. Thus Nu can be calculated after (2.6a) and (2.5), if the termperature distribution is found from (2.1b).

3. Method of Approximation. Consider now the form

$$\left.\begin{aligned} \theta^*(\chi,\eta) &= 1 - e^{-(1-\eta)/\lambda}\left(1+\frac{1}{4\lambda}-\frac{\eta^2}{4\lambda}\right) \\ &+ e^{-(1/\lambda)}\left(\frac{1}{\lambda}+\frac{1}{4\lambda^2}\right)\left(\eta-\frac{8}{5}\eta^2+\frac{3}{5}\eta^3\right), \end{aligned}\right\} \quad (3.1)^{1)}$$

in which $\lambda = \lambda(\chi)$ is an arbitrary function of $\chi\,(\geqq 0)$, but has an initial value $\lambda(0) = 0$. Then it is easily verified that (3.1) completely satisfies the boundary conditions (2.4) as well as the equation (2.1b) at $\eta = 1$, i.e., $(\partial^2\theta^* / \partial\eta^2 + \partial\theta^* / \eta\partial\eta)_{\eta=1} = 0$, automatically. This property is still conserved, if we add polynomials of any order so far as their coefficients are properly chosen; for instance, the form

$$\theta(\chi, \eta) = \theta^*(\chi, \eta) + \vartheta(\chi)\left(1 - \frac{4}{3}\eta^2 + \frac{1}{3}\eta^4\right) \tag{3.2}$$

satisfies these conditions provided that $\vartheta(0) = 0$.

The two unknown functions $\lambda(\chi)$ and $\vartheta(\chi)$ are left to be determined from the requirement that the equation (2.1b) should be satisfied by (3.2) at $\eta = 0$ and also in its integral form, that is, its first moment with respect to η which corresponds to the momentum equation in the case of the boundary layer theory. This is, of course, the same procedure as that of Kármán and Pohlhausen in principle.[2]

So that putting $\eta = 0$ in (2.1b) we have

$$\frac{d\theta_0}{d\chi} = 4\left(\frac{\partial^2\theta}{\partial\eta^2}\right)_{\eta=0}, \tag{3.3}$$

where $\theta_0 = (\theta)_{\eta=0}$, and multiplying the both hand sides of (2.1b) by η and then integrating from $\eta = 0$ to 1, we have

$$\frac{d(I_1 - I_3)}{d\chi} = 2\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1}, \tag{3.4}$$

for the first moment. If we write these equations explicitly in terms of λ, ϑ and χ after (3.2) and solve them with respect to λ and ϑ we will have an approximate solution of the original equation definitely in the form of (3.2).

Unfortunately, however, there exists an undesirable singularity, that is, $d\vartheta / d\chi \big/ d\lambda / d\chi$ diverges to infinity at about $\chi = 0.04$ and the approximation becomes utterly invalid. Thus the following expedient is taken to avoid this difficulty and obtain a consistent solution for all values of $\chi \geqq 0$. Namely, let us introduce an another parameter c into (3.2) and write

1) APPENDIX I, pp. 35, 36.

2) Pohlhausen, K., Z.A.M.M. vol. 1 (1921), 252.

$$\theta = \theta^* + \vartheta\left[\left(1 - \frac{4}{3}\eta^2 + \frac{1}{3}\eta^4\right) + c\left(\frac{5}{3}\eta^2 - \frac{8}{3}\eta^4 + \eta^6\right)\right], \tag{3.2a}$$

which has the same property at the boundaries as (3.1) or (3.2). Then (3.3) and (3.4) can be written

$$A(\lambda)\frac{d\lambda}{d\chi} - \frac{d\vartheta}{d\chi} = \Phi(\lambda) + \frac{8}{3}\vartheta(4 - 5c), \tag{3.3a}$$

and

$$B(\lambda)\frac{d\lambda}{d\chi} - \left(\frac{11}{72} + \frac{19}{360}c\right)\frac{d\vartheta}{d\chi} = \Psi(\lambda) + \frac{8}{3}\vartheta(1 + c), \tag{3.4a}$$

respectively, where A, B, Φ and Ψ are known functions of λ (*see* next page (4.5)).

It is found that λ becomes in general larger and larger with χ, whereas $A(d\lambda/d\chi)$, $B(d\lambda/d\chi)$, Φ and Ψ decrease rapidly, leaving the equations concerning ϑ only which are incompatible simaltaneously when $c \equiv 0$; this corresponds to the singularity stated above. Such being the case, let us determine the value of c so as to make equations (3.3a) and (3.4a) reduce to a single equation when λ-terms vanish, regarding most simply c as a constant. This is realized if

$$c = -0.2621241, \tag{3.5}$$ [1]

which will be used for the later calculation. Thus the final form of the assumed profile becomes

$$\left.\begin{aligned} \theta(\chi, \eta) = 1 - e^{-(1-\eta)/\lambda}\left[1 + \frac{0.25}{\lambda}(1 - \eta^2)\right] + e^{-(1/\lambda)}\left(\frac{1}{\lambda} + \frac{0.25}{\lambda^2}\right) \\ \times (\eta - 1.6\eta^2 + 0.6\eta^3) + \vartheta(1 - 1.770207\eta^2 + 1.032331\eta^4 - 0.2621241\eta^6), \end{aligned}\right\} \tag{3.6}$$

corresponding to which we have

$$\theta_0 = 1 - e^{-(1/\lambda)}\left(1 + \frac{0.25}{\lambda}\right) + \vartheta, \tag{3.7}$$

and

$$\left.\begin{aligned} \theta_M &= 4\int_0^1 \theta(\eta - \eta^3)\,d\eta \\ &= 1 - 16\lambda^2 + 72\lambda^3 - 144\lambda^4 + 120\lambda^5 - e^{-(1/\lambda)}\Big[\lambda + 4\lambda^2 - 12\lambda^3 \\ &\quad - 24\lambda^4 + 120\lambda^5 - 0.0342857\left(\frac{4}{\lambda} + \frac{1}{\lambda^2}\right)\Big] + 0.555774\vartheta. \end{aligned}\right\} \tag{3.8}$$

This expedient is essentialy quite similar to that proposed by Dryden[2] in his modification of Pohlhausen's method and seems to be inevitable in this order of approximation. Its mathematical argument has been also given by Professor Yamada in some detail.[3]

1) One more value of c is possible to fulfil this requirement, but we can't adopt it, because it gives a wavy temperature profile which is physically unreasonable.

2) Dryden, H., N.A.C.A. Tech. Rep. No. 497 (1934), 435.

3) Yamada, H., *op. cit.*

In addition it is noticed here that generally the assumed shape of the profile in this method is not determined uniquely but rather arbitrarily; it must be chosen in view of the convenience for calculation and physical features of the problem (*see* APPENDIX I).

4. Results of Calculation. The simaltaneous differential equations (3.3a) and (3.4a) can be integrated numerically, when it is convenient to change the independent variable from χ to λ. Noting then $d\vartheta / d\chi = (d\vartheta / d\lambda) \times (d\lambda / d\chi)$ and (3.5), they are rewritten

$$\left[A(\lambda) - \frac{d\vartheta}{d\lambda}\right]\frac{d\lambda}{d\chi} = \Phi(\lambda) + 14.16165\vartheta, \tag{4.1}$$

and $$\left[B(\lambda) - 0.1389435\frac{d\vartheta}{d\lambda}\right]\frac{d\lambda}{d\chi} = \Psi(\lambda) + 1.967669\vartheta, \tag{4.2}$$

which give readily

$$\frac{d\vartheta}{d\lambda} = \frac{A(\Psi + 1.967669\vartheta) - B(\Phi + 14.16165\vartheta)}{\Psi - 0.1389435\Phi}, \tag{4.3}$$

and $$\frac{d\chi}{d\lambda} = \frac{B - 0.1389435A}{\Psi - 0.1389435\Phi}. \tag{4.4}$$

In these expressions A, B, Φ and Ψ are evaluated as follows, substituting (3.6) into (3.3) and (3.4):

$$\left.\begin{aligned}
A(\lambda) &= e^{-(1/\lambda)}\left(\frac{0.75}{\lambda^2} + \frac{0.25}{\lambda^3}\right),\\
B(\lambda) &= 8\lambda - 54\lambda^2 + 144\lambda^3 - 150\lambda^4 + e^{-(1/\lambda)}\Big[\frac{0.25}{\lambda} + 1.25 - \lambda\\
&\quad - 15\lambda^2 + 6\lambda^3 + 150\lambda^4 + 0.0342857\left(\frac{1}{\lambda^2} - \frac{0.5}{\lambda^3} - \frac{0.25}{\lambda^4}\right)\Big],\\
\Phi(\lambda) &= e^{-(1/\lambda)}\left(\frac{10.8}{\lambda} + \frac{7.2}{\lambda^2} + \frac{1}{\lambda^3}\right),\\
\text{and}\quad \Psi(\lambda) &= \frac{1}{\lambda} + e^{-(1/\lambda)}\left(\frac{0.8}{\lambda} + \frac{0.2}{\lambda^2}\right).
\end{aligned}\right\} \tag{4.5}$$

Then (4.3) was integrated after the Runge-Kutta's procedure and (4.4) after the Simpson rule.

The results of integration and some related data are tabulated in Tables 1 to 3, while the nature of the present solution is also visualized in Figures 1 to 4 for ready comparison with the previous theories or experiments.[1]

[1] In these Figures the numerical computation of the previous solutions are carried out after Professor Yamagata (*op. cit.*) and the experimental points are taken from Goldstin's book (*loc. cit.*). *See* also APPENDIX II, pp. 36, 37.

TABLE 1. Numerical Table of the Functions in the Equations.

λ	A	B	Φ	Ψ	$-4\left(\frac{\partial^2\theta}{\partial\eta^2}\right)_{\eta=0}$	$-2\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0}$
0	0	0	0	∞	0	∞
0.04	6×10^{-8}	0.2424320	2.8×10^{-7}	25.00000	2.8×10^{-7}	25.00000
0.1	0.01475498	0.3846459	0.08299108	10.00127	0.08481164	10.00152
0.2	0.3368974	0.3206358	2.418923	5.060642	2.550837	5.078970
0.5	0.6766764	0.1383712	7.903580	2.324805	9.377380	2.529580
1	0.3678794	0.0580941	6.989709	1.367878	10.30439	1.828430
2	0.1326786	0.01970862	4.442837	0.7729388	9.270556	1.443719
10	0.00701249	0.00102257	1.043278	0.1741967	6.308742	0.9057985
20	0.00181328	0.000264269	0.5309049	0.0885248	5.351267	0.7582826
10^2	0.0000745013	0.0000108560	0.1076392	0.0179402	3.558390	0.4973994
10^3	7.5×10^{-7}	1.09×10^{-7}	0.0107964	0.0017994	1.757308	0.2444657
10^4	7.5×10^{-9}	1.09×10^{-9}	0.0010800	0.0001800	0.8708484	0.1210286
10^5	7.5×10^{-11}	1.09×10^{-11}	0.0001080	0.0000180	0.4315633	0.0599662
Reference Formula	(4.5)				I	II

$$\text{I}: \quad -4\left(\frac{\partial^2\theta}{\partial\eta^2}\right)_{\eta=0} = \Phi(\lambda) + 14.16165\,\vartheta \qquad (\textit{see } (4.1))$$

$$\text{II}: \quad -2\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} = \Psi(\lambda) + 1.967669\,\vartheta \qquad (\textit{see } (4.2))$$

TABLE 2. Final Results of the Calculation.

λ	χ	ϑ	θ_0	θ_M	$w_m d^2/\kappa l$	Nu
0	0	0	1	1	∞	∞
0.04	0.00014	—	1.00000	0.97865	7196	38.82
0.1	0.00156	0.00013	0.99997	0.89908	640.6	17.019
0.2	0.00675	0.00931	0.99416	0.75779	148.07	10.202
0.5	0.02102	0.10407	0.90107	0.55902	47.584	6.730
1	0.03356	0.23406	0.77421	0.45188	29.801	5.625
2	0.04529	0.34090	0.65855	0.37568	22.078	5.010
10	0.07248	0.37181	0.44435	0.24894	13.796	4.148
20	0.08402	0.34038	0.37726	0.21067	11.902	3.880
10^2	0.11245	0.24367	0.25114	0.13978	8.8930	3.356
10^3	0.15888	0.12333	0.12408	0.06898	6.2940	2.741
10^4	0.20536	0.06142	0.06149	0.03418	4.8694	2.274
10^5	0.25185	0.03047	0.03047	0.01694	3.9706	1.9192
*1.423×10^8	0.375	0.00533	0.00533	0.00296	2.6667	1,3255
*2.26×10^{11}	0.5	0.00091	0.00091	0.00050	2	0.9990
*5.69×10^{17}	0.75	0.00003	0.00003	0.00001	1.3333	0.6665
*1.437×10^{24}	1	—	—	—	1	0.5000
Reference Formula	(4.4)	(4.3)	(3.7)	(3.8)	(2.3)	(2.5)

The values of λ with an asterisk were evaluated, chosing χ as the independent variable.

TABLE 3. Temperature Distributions across the Pipe at Three Typical Sections. (*see* FIG. 2)

λ	0.1	10	10^4
η \ χ	0.00156	0.07248	0.20536
0	0.99997	0.44435	0.061492
0.1	0.99983	0.43656	0.060410
0.2	0.99920	0.41388	0.057239
0.3	0.99740	0.37798	0.052199
0.4	0.99270	0.33130	0.045636
0.5	0.98065	0.27690	0.037997
0.6	0.95269	0.21815	0.029783
0.7	0.88448	0.15847	0.021502
0.8	0.74303	0.10094	0.013605
0.9	0.45746	0.04781	0.006405
1.0	0	0	0

Thus we have now an approximate solution of the problem represented by a single function in unified manner for all χ. When χ is very small (or $w_m d^2 / \kappa l$ is very large), in particular, λ is also small and we see that

$$\frac{d\chi}{d\lambda} = 8\lambda^2 + O(\lambda^3), \qquad \text{that is,} \qquad \chi \sim \frac{8}{3}\lambda^3, \tag{4.6}$$

and ϑ must be vanishingly small with the factor $e^{-(1/\lambda)}$. We have, therefore, asymptotically

$$\left.\begin{aligned} &\theta_M \sim 1 - 8.320\chi^{2/3} && (see\ (3.8)), \\ \text{and}\quad &Nu \sim 2.080\,(w_m d^2 / \kappa l)^{1/3} && (see\ (2.5)), \end{aligned}\right\} \tag{4.7}$$

for $\chi \to 0$. These are to be compared with the Lévêque's solution, giving

$$\left.\begin{aligned} &\theta_M \sim 1 - 6.460\chi^{2/3}, \\ \text{and}\quad &Nu \sim 1.615\,(w_m d^2 / \kappa l)^{1/3}, \end{aligned}\right\} \tag{4.8}$$

which are a little different from (4.7) in numerical factors (*see* APPENDIX II).

As is stated in the last section λ increases monotonously with χ, reducing the equations (3.3a) and (3.4a) at last to a single relation

$$\left.\begin{aligned} &\frac{d\vartheta}{d\chi} \sim -14.16165\,\vartheta\,, \\ &\text{from which we have} \\ &\vartheta \sim \theta_0 \sim 1.078e^{-14.16165\chi} \quad \text{and} \quad \theta_M \sim 0.5993e^{-14.16165\chi}\,, \end{aligned}\right\} \quad (4.9)$$

for large χ. (4.11) corresponds to the first partial solution of Graetz, which gives

$$\theta_0 \sim 1.4764e^{-14.627\chi} \quad \text{and} \quad \theta_M \sim 0.81906e^{-14.627\chi} \qquad (4.10)$$

(*see* APPENDIX II).

In conclusion, it may be permissible to suppose that the present results are fairly reasonable as a whole, implying the possibilities of this method for future applications.

5. Acknowledgements for the First Report. The author wishes to express his hearty thanks to Professor Yamada for his cordial guidances and encouragements in the course of this study. The acknowledgements are also due to Professors Kurihara and Okabé of this Institute, who were interested in this work and kind enough to give valuable advices at every chance.

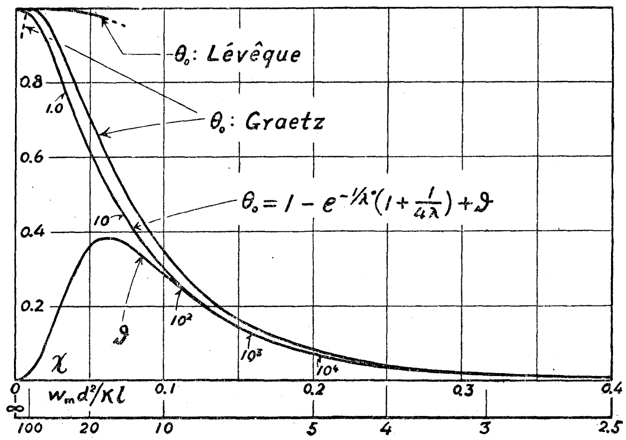


FIG. 1 Temperature Distribution along the Pipe.

(a) θ_0, ϑ vs. χ.

Figures attached to the θ_0-curve indicate the corresponding values of λ.

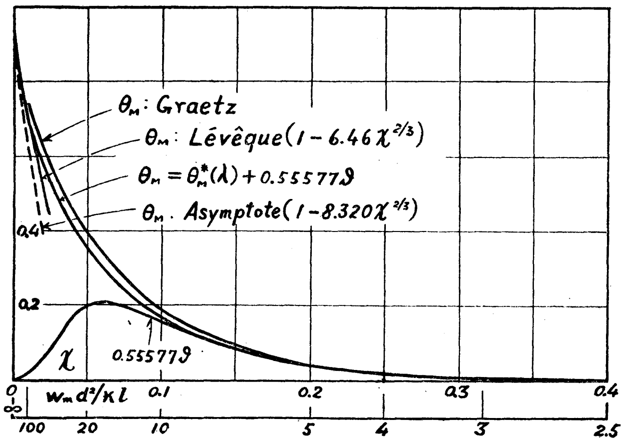
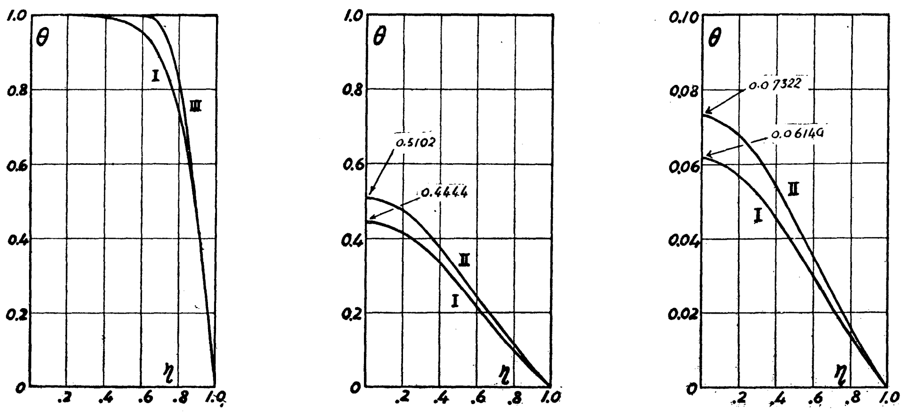


FIG. 1.

(b) θ_M vs. χ.

$$\theta_M^*(\lambda) = 1 - 16\lambda^2 + 72\lambda^3 - 144\lambda^4 + 120\lambda^5$$
$$- e^{-(1/\lambda)}\left[\lambda + 4\lambda^2 - 12\lambda^3 - 24\lambda^4 + 120\lambda^5 - 0.0342857\left(\frac{4}{\lambda} + \frac{1}{\lambda^2}\right)\right]. \quad (cf\ (3.8)).$$



(a)
$\chi = 0.00156$,
$\lambda = 0.1$,
$\vartheta = 0.00013$.

(b)
$\chi = 0.07248$,
$\lambda = 10$,
$\vartheta = 0.37181$.

(c)
$\chi = 0.20536$,
$\lambda = 10^4$,
$\vartheta = 0.06142$.

FIG. 2 Temperature Distribution across the Pipe.

I : Present Calculation
II : Graetz
III : Lévêque

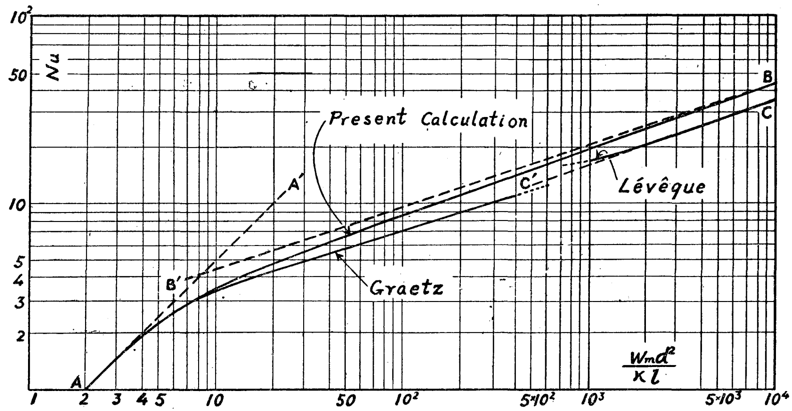


FIG. 3 Theoretical Plot of Nusselt Numbers.

AA': Asymptote, $Nu = 0.5\ (w_m\, d^2/\kappa l)$.
BB': Asymptote, $Nu = 2.080\ (w_m\, d^2/\kappa l)^{1/3}$.
CC': Asymptote, $Nu = 1.615\ (w_m\, d^2/\kappa l)^{1/3}$.

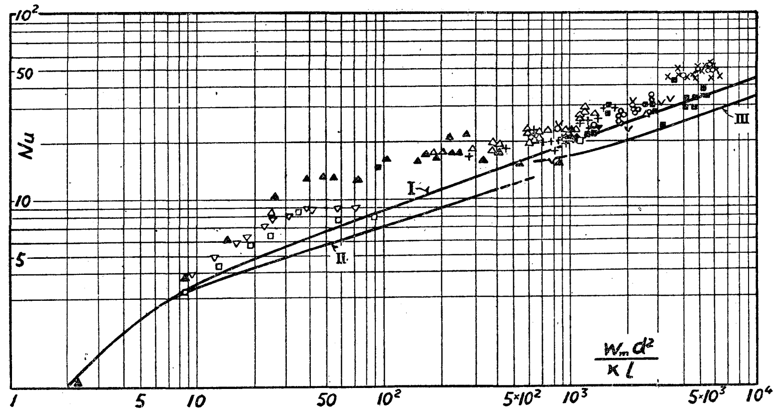


FIG. 4 Comparison with the Experimental Data.

I : Present Calculation
II . Graetz
III: Lévêque

(a) Heating of Oil.

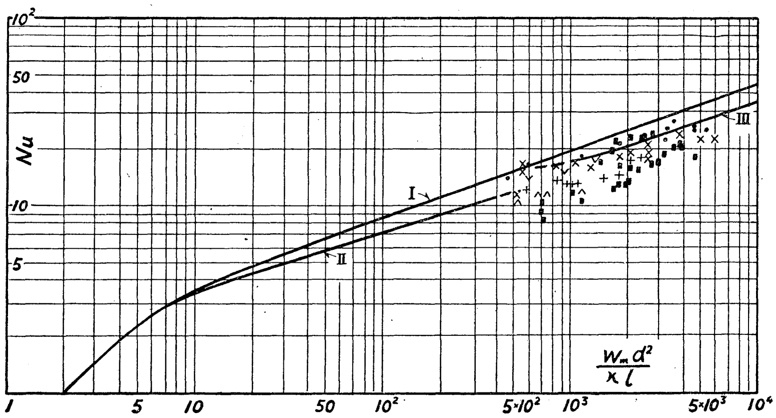


FIG. 4

(b) Cooling of Oil.

APPENDIX

I. The Derivation of θ^*. At the biginning of the study Professor Yamada suggested to the author that the discontinuous change of the boundary value $(\theta)_{\eta=1}$ at $\chi = 0$ (from 1 to 0) will be represented in a simple manner by a form containing the discontinuous factor $e^{-(1-\eta)/\lambda}$, which has been successfully applied to the calculation of the laminar jet[1] or the laminar wake of a plate.[2] Since, however, the final form of the assumed profile should be determined according to the nature of the problem, a little consideration was necessary to arrive at (3.1) in the present case. It is shown briefly in the following: put at first

$$\theta^* = 1 - e^{-(1-\eta)/\lambda} F(\chi, \eta) \tag{I.1}$$

and make it satisfy at $\eta = 1$ the equation and boundary condition. Since these conditions are written

$$\left.\begin{aligned} &\frac{1}{\eta}\left(\frac{\partial^2\theta^*}{\partial\eta^2} + \frac{1}{\eta}\,\frac{\partial\theta^*}{\partial\eta}\right)_{\eta=1} \\ &\equiv \left(\frac{F}{\lambda} + \frac{\partial F}{\partial\eta}\right)_{\eta=1} + \left(\frac{F}{\lambda^2} + \frac{2}{\lambda}\,\frac{\partial F}{\partial\eta} + \frac{\partial^2 F}{\partial\eta^2}\right)_{\eta=1} = 0, \\ \text{and} \qquad &(\theta^*)_{\eta=1} \equiv 1 - (F)_{\eta=1} = 0, \end{aligned}\right\} \tag{I.2}$$

we have

[1] Okabé, J., Reports of the Research Institute for Fluid Engineering, Kyushu Univ. Japan, vol. 5, No. 1 (1948) 15.

[2] By Yamada, H., unpublished.

$$\left.\begin{array}{l} F_0(1+\lambda)+F_1(1+3\lambda+\lambda^2)+F_2(1+5\lambda+4\lambda^2)+\cdots=0, \\ \text{and} \quad F_0+F_1+F_2+\cdots=1, \end{array}\right\} \quad \text{(I. 2a)}$$

assuming a polynomial of η for F, that is, putting

$$F = F_0(\lambda) + F_1(\lambda)\eta + F_2(\lambda)\eta^2 + \cdots,$$

where λ should be considered as a function of χ only. Naturally many different sets of F_n satisfying these conditions are possible, of which the simplest one is

$$F_0 = 1 + \frac{1}{\lambda}, \qquad F_1 = 0, \qquad F_2 = -\frac{1}{4\lambda}. \qquad \text{(I. 3)}$$

The other sets were given up because they would lead to more complicated calculation. Then we have

$$\theta^* = 1 - e^{-(1-\eta)/\lambda}\left(1 + \frac{1}{4\lambda} - \frac{\eta^2}{4\lambda}\right), \qquad \text{(I. 1a)}$$

which, however, does not satisfy itself the condition $(\partial\theta^* / \partial\eta)_{\eta=0} = 0$. So the third order polynomial

$$\Delta(\chi)\left(\eta - \frac{8}{5}\eta^2 + \frac{3}{5}\eta^3\right)$$

is added, where

$$\Delta = e^{-(1/\lambda)}\left(\frac{1}{\lambda} + \frac{1}{4\lambda^2}\right) \qquad \text{(I. 4)}$$

from the condition at $\eta = 0$. The form (3.1) was thus derived.

II. A Brief Account of the Previous Theories.[1] The equation (2.1b), that is,

$$(1-\eta^2)\frac{\partial\theta}{\partial\chi} = 2\left(\frac{\partial^2\theta}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial\theta}{\partial\eta}\right)$$

is a linear partial differential equation of the separable variables and its solution can be written in a form of an infinite series

$$\theta = \sum_{j=1}^{\infty} A_j e^{-2\beta_j^2\chi} Y_j(\eta;\beta_j), \qquad \text{(II. 1)}$$

where $Y_j(\eta;\beta_j)$'s are the particular solutions of the equations

$$\frac{d^2Y_j}{d\eta^2} + \frac{1}{\eta}\frac{dY_j}{d\eta} + \beta_j^2(1-\eta^2)Y_j = 0, \qquad j = 1, 2, 3, \cdots \qquad \text{(II. 2)}$$

β_j's being the eigenvalues of the equations which corresponds to the conditions $(\theta)_{\eta=1} = 0$, and A_j's the coefficients of the series expansion. Y_j's,

[1] Goldstein, S., *loc. cit.*; Yamagata, K., *op. cit.* *See* also the footnote 2) p. 23.

which are free from singularities at $\eta = 0$, are called Graetz functions and can be evaluated in the form of infinite series, too. This is the Graetz's theory. The calculation has been carried out so far up to the fourth term; it is shown by Professor Yamagata that

$$\left.\begin{aligned} &\beta_1 = 2.704365, \quad \beta_2 = 6.67903, \quad \beta_3 = 10.67340, \quad \beta_4 = 14.6712, \\ &A_1 = 1.4764, \quad A_2 = -0.80612, \quad A_3 = 0.5888, \quad A_4 = -0.4758, \\ \text{and} \quad &\theta_M = 0.81906\, e^{-14.627\chi} + 0.09753\, e^{-89.22\chi} \\ &\qquad + 0.03250\, e^{-227.8\chi} + 0.0155\, e^{-430\chi}. \end{aligned}\right\} \quad \text{(II. 3)}$$

When χ becomes small, however, this solution ceases to be an appropriate one because of its insufficient convergency and hence is replaced by that proposed by Lévêque. Considering the fact that when χ is very small the temperature gradient must be concentrated in the narrower layer adjacent to the pipe wall, and assuming the plane boundary and linear velocity distribution instead of the actual ones, he solved the simplified equation

$$\frac{\partial^2 \theta}{\partial \eta'^2} = \eta' \frac{\partial \theta}{\partial \chi} \qquad \text{(II. 4)}$$

under the boundary conditions

$$(\theta)_{\eta'=0} = 0, \qquad (\theta)_{\eta'=\infty} = (\theta)_{\chi=0} = 1$$

where $\eta' = 1 - \eta$. As the solution he had

$$\theta = \frac{1}{\Gamma\left(\frac{4}{3}\right)} \int_0^{\eta(9\chi)^{-1/3}} e^{-t^3}\, dt, \qquad \text{(II. 5)}$$

which is called the Lévêque integral, giving

$$\left.\begin{aligned} &\theta_M = 1 - 6.460\chi^{2/3}, \\ \text{and} \qquad &Nu = \frac{1.615\chi^{-1/3}}{1 - 3.23\chi^{2/3}}. \end{aligned}\right\} \quad \text{(II. 6)}$$

The Graetz function and Lévêque integral are tabulted in, for instance, the paper of Yamagata (*op. cit.*) or that of Lee, Nelson and Others.[1] Acoording to Professer Yamagata the Graetz's solution can be used for $w_m d^2 / \kappa l < 400\, (\chi > 0.0025)$, while the Lévêque's solution should be applied for $w_m d^2 / \kappa l > 1000\, (\chi < 0.0010)$; for intermediate values of $w_m d^2 / \kappa l$, neither of these solutions are available.

III. A Note on the Notation. In many references a non-dimensional variable $W c_p / kl$ are used in place of $w_m d^2 / \kappa l$ of this paper, where W means the rate of flow, that is, the weight of the fluid passing through the

1) Lee, A., Nelson, O., Cherry, V.H. and Boelter, L.M.K., Proc. Fifth Intern. Congr. App. Mech. (1939) 571.

pipe-section in unit time, and c_p stands for the specific heat at the constant pressure, k the thermal conductivity. Numerically they are connected by the relation

$$\frac{Wc_p}{kl} = \frac{\pi}{4}\left(\frac{w_m d^2}{\kappa l}\right), \qquad \text{(III. 1)}$$

which must be noticed when we compare the results.

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