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ON THE CALCULATION OF FREE OSCILLATIONS WITH INTERMEDIATE NON-LINEARITY[1]

By Hikoji YAMADA

Equation of free oscillation is solved approximately in the form that the velocity is a function of the displacement. For velocity a functional form which contains a polynomial with undetermined coefficients is assumed, and these coefficients determined by the *method of moments*. The approximations is fairly accurate either when the non-linear term is small or when the number of coefficients is large. To secure good approximation in cases of intermediate non-linearity, the necessary increase of the number of coefficients is replaced by a sucessive approximation method, which consists of the calculation of correction terms for velocity and amplitude, linearizing the equation by reason of the smallness of corrections. As examples non-linear damped oscillation and the Van der Pol's self-excited oscillation are treated with good accuracies.

1. Method of approximation. We have proposed a simple method of approximate solution of differential equation, especially of the non-linear one, and some applications have also been reported.[2] Some types of the non-linear oscillation seem to be under the scope of this method, and here we take up the free oscillation of the type:

$$\Delta(v,\sigma)\equiv v\frac{dv}{d\sigma}-f(\sigma,v;\alpha)=0 \qquad (1)$$

where σ and v are the non-dimensional displacement and velocity, and α the amplitude of oscillation, which is to be determined.

We consider the velocity v as a function of the displacement σ, and then take one swing only at a time in view, apart from the other swings. This separate treatment of each swing brings with it labors in the calculation of transcient oscillations, but none in the most important case of cyclic oscillations. As the differential equation (1), with the domain of solution $\sigma(-1, +1)$, is equivalent to the *moment-equations*

$$\int_{-1}^{+1}\Delta(\sigma,v;\alpha)\,\sigma^n\,d\sigma=0, \qquad (2)$$
$$n=0,1,2,\cdots,N(N\to\infty)$$

1) Former half of this paper has been reported, in Japanese, in the Reports of the Research Institute for Fluid Engineering, Vol. VII, No. 2 (1950).

2) *c.f.* On a method of approximate solution of differential equations, Reports of the Research Institute for Fluid Engineering, Vol. VI, No. 2 (1950); A method of approximate integration of the laminar boundary layer equation, ibid.

we consider the problem in the form (2), with a finite and appropriately chosen N, *i.e.* approximately.

For the solution v of (2) we can assume any function of σ which satisfies the boundary conditions and contains N undetermined constants. In the present case, however, two ends of the swing are singular points, and introduction of the factor $\sqrt{1-\sigma^2}$ into the assumed solution will secure better approximation with fewer number of undetermined constants. This factor brings the possibility of $\Delta = 0$ at the both ends of the swing, as we can see by the equation (1). We then assume:

$$v = \sqrt{1-\sigma^2} \cdot P(\sigma; c_0, c_1, \cdots, c_{N-1}) \tag{3}$$

where P is a certain function finite (and positive) in $\sigma(-1, +1)$, with undetermined constants c's. Assumed form of P will have large influence on the results; when N tends to infinity linear conbination of functions from any complete set which satisfy a certain boundary condition will suffice, but with a rather small number N, the degree of approximation depends on the form we chose. At present, knowing not the most adequate set of functions, we employ a polynomial, *i.e.* the set of functions $1, \sigma, \sigma^2, \cdots$, thus:

$$P = c_0 + c_1\sigma + \cdots \cdots \cdots \cdots \cdots + c_{N-1}\sigma^{N-1}. \tag{3'}$$

Introducing (3), (3′) into (1) we obtain

$$\Delta(\sigma) = -\sigma P^2 + (1-\sigma^2) P \frac{dP}{d\sigma} - f(\sigma, v; \alpha), \tag{1'}$$

and by (2) $N+1$ algebraic equations with regard to the N constants c's and the amplitude α, the required approximate solution resulting from the values of constants thus determined. The sets of roots of the algebraic equations which have physical meanings are, in general, not unique, but the selection of the appropriate one is usually easy, and mathematically we are guided by the fact that the value of Δ for the selected solution must not depart largely from zero along the interval $\sigma(-1, +1)$. The present method of solution is quite parallel to the usual one of employing Fourier series of phase angle θ of finite terms. $v = c_0\sqrt{1-\sigma^2}$ is a simple harmornic oscillation and corresponds to the Fourier's first term $b_1 \cos\theta$. Introduction of terms $b_n \cos n\theta$, in non-linear cases, in the latter is responded by the generalization of c_0 to $c_0 + c_1\sigma + \cdots$ in the former, and the methods of determination of the constants are also parallel. This parallelism would have been more direct had we employed any set of orthogonal polynomials instead of the set $1, \sigma, \sigma^2, \cdots$, but owing to the occurrence of the factor $\sqrt{1-\sigma^2}$ such orthogonalization brings no gain. This consideration reveals also the fact that much is not gained by our method except in the cases where $f(\sigma, v; \alpha)$ is simple polynomial of σ and v, but these exceptional cases are rather usual in practice and our method may fit in this respect. More decisive benifit, however, will be brought when we intend the improvement of approximation, as will be seen in § 4.

As the determination of the Fourier coefficients above alluded is laborious when the number of them is large, so also in our case the solution of the algebraic equations (2) requires labor and cannot increase the number of coefficients sufficiently. In such a case the fulfilment of the equation $\Delta = 0$ at a certain particular points of the displacement range brings some simplification. For example, at the centre of oscillation $\sigma = 0$:

$$c_0 c_1 = f(0,\ c_0;\ \alpha) \tag{4}$$

and at the ends $\sigma = -1$ and $+1$:

$$P(-1) = \sqrt{f(-1,\ 0;\ \alpha)}, \qquad P(+1) = \sqrt{-f(+1,\ 0;\alpha)}. \tag{4'}$$

By the introduction of some of these relations the number of the *moment-equations* (2) must, of course, be reduced by the number of those relations.

The next integral *i.e.* the relation between σ and θ, θ being an angle variable proportional to time, is obtained by a single quadrature which can surely be accomplished analytically:

$$\theta = \int_{-1}^{\sigma} \frac{d\sigma}{\sqrt{1-\sigma^2}\cdot P(\sigma)}. \tag{5}$$

2. Damped oscillation. As the first example we take the simple harmonic system under the hydrodynamical resistance. The governing equations of motion are:

$$v = dx/d\theta, \qquad v\,(dv/dx) + Av + Bv^2 + x = 0, \tag{6}$$

which are to be solved under the initial conditions:

$$(x)_{\theta=0} = -1, \qquad (v)_{\theta=0} = 0. \tag{6'}$$

We denote the other end of the swing by $x = \lambda$, and the transformation

$$1 + \lambda = \mu, \qquad 1 + x = \mu s \tag{7}$$

brings (6) into the form:

$$\Delta(s) = (v/\mu)\,(dv/ds) + Av + Bv^2 + \mu s - 1 = 0, \tag{8}$$

here the domain of solution being $s\,(0,\ 1)$.[3]

For v we assume a very simple form:

$$v = \sqrt{2\mu\, s\,(1-s)}\ (a_0 + a_1 s + a_2 s^2), \tag{9}$$

and the requirement $\Delta = 0$ at $s = 0$ (*i.e.* $x = -1$) results in $a_0 = 1$; the remaining three constants a_1, a_2 and μ are to be determined by the relations:

$$\int_0^1 \Delta(s)\, s^m\, ds = 0, \qquad m = 0,\ 1,\ 2, \tag{10}$$

which can easily be written down as follow:

3) We used the variable s accidentally, instead of the preceding section, and as the process is all the same we keep to it here.

$$\left.\begin{aligned}
&\frac{B}{10}\mu a_1^2+\frac{2B}{15}\mu a_1 a_2+\frac{B}{21}\mu a_2^2\\
&+\left(\frac{B}{3}\mu+A'\sqrt{\mu}\right)a_1+\left(\frac{B}{5}\mu+\frac{5A'}{8}\sqrt{\mu}\right)a_2\\
&+\left\{\left(\frac{B}{3}+\frac{1}{2}\right)\mu+2A'\sqrt{\mu}-1\right\}=0\\
&\left(\frac{B}{15}\mu-\frac{1}{20}\right)a_1^2+\left(\frac{2B}{21}\mu-\frac{1}{15}\right)a_1 a_2+\left(\frac{B}{28}\mu-\frac{1}{42}\right)a_2^2\\
&+\left(\frac{B}{5}\mu+\frac{5A'}{8}\sqrt{\mu}-\frac{1}{6}\right)a_1+\left(\frac{2B}{15}\mu+\frac{7A'}{16}\sqrt{\mu}-\frac{1}{10}\right)a_2\\
&+\left\{\left(\frac{B}{6}+\frac{1}{3}\right)\mu+A'\sqrt{\mu}-\frac{2}{3}\right\}=0\\
&\left(\frac{B}{21}\mu-\frac{1}{15}\right)a_1^2+\left(\frac{B}{14}\mu-\frac{2}{21}\right)a_1 a_2+\left(\frac{B}{36}\mu-\frac{1}{28}\right)a_2^2\\
&+\left(\frac{2B}{15}\mu+\frac{7A'}{16}\sqrt{\mu}-\frac{1}{5}\right)a_1+\left(\frac{2B}{21}\mu+\frac{21A'}{64}\sqrt{\mu}-\frac{2}{15}\right)a_2\\
&+\left\{\left(\frac{B}{10}+\frac{1}{4}\right)\mu+\frac{5A'}{8}\sqrt{\mu}-\frac{1}{2}\right\}=0,
\end{aligned}\right\}\qquad (10')$$

where $A'=\pi A/8\sqrt{2}$.

General solution of (10′) is difficult, and we quote a numerical example: $A=1/4$ and $B=1/2$. This is the case of medium damping and of nearly equal contributions of the linear and quadratic terms of resistance. For these values (10′) gives:

$$\mu=1.4196\ (i.e.\ \lambda=0.4196);\qquad a_1=-0.5347,\qquad a_2=0.2553,$$

and then we obtain the solutions

$$\begin{aligned}
v&=1.6850\sqrt{s(1-s)}\,(1-0.5347\,s+0.2553\,s^2)\\
&=\sqrt{(1+x)(0.4196-x)}\,(0.8902-0.1463\,x+0.1504\,x^2),
\end{aligned}\qquad (11)$$

and

$$\begin{aligned}
\theta&=1.0317\tan^{-1}\left(-\frac{0.9598\sqrt{(1+x)(0.4196-x)}}{x+0.2317}\right)\\
&\quad-0.1555\log\left(\frac{1-0.1191\,x+0.3801\sqrt{(1+x)(0.4196-x)}}{1-0.1191\,x-0.3801\sqrt{(1+x)(0.4196-x)}}\right).
\end{aligned}\qquad (21)$$

The methods of error estimation and correction of the results obtained are postponed to § 4, and here we compare the results with the standard ones. We have previously calculated this case very minutely by the Runge-Kutta's method of numerical integration and the Simpson's rule; the steps of integration were so minite that the results can be regarded as the exact ones. Our present results are compared with these at a few points of interval in the following table 1 and in the figure 1. In that table *exact* means the numerical integrations and *approx.* the present results; in the figure full lines are the *exact* ones and circles the *approx.*; the agreement is sufficiently good.

TABLE 1

σ	σ exact	σ approx.	θ exact	θ approx.
−1.0	0.0000	0.0000	0.0000	0.0000
−0.9	0.4089	0.4155	0.4640	0.4584
−0.8	0.5358	0.5450	0.6736	0.6650
−0.7	0.6143	0.6180	0.8464	0.8363
−0.6	0.6593	0.6591	1.0030	0.9924
−0.5	0.6825	0.6788	1.1517	1.1412
−0.4	0.6887	0.6822	1.2973	1.288
−0.3	0.6804	0.6725	1.4431	1.436
−0.2	0.6591	0.6516	1.5922	1.586
−0.1	0.6252	0.6197	1.7477	1.743
0.0	0.5783	0.5767	1.9137	1.910
0.1	0.5165	0.5200	2.0961	2.092
0.2	0.4359	0.4450	2.3058	2.300
0.3	0.3255	0.3390	2.5680	2.553
0.4	0.1326	0.1417	3.0042	2.966
λ	0.0000	0.0000	3.3100	3.241

λ exact = 0.4202 λ approx. = 0.4196.

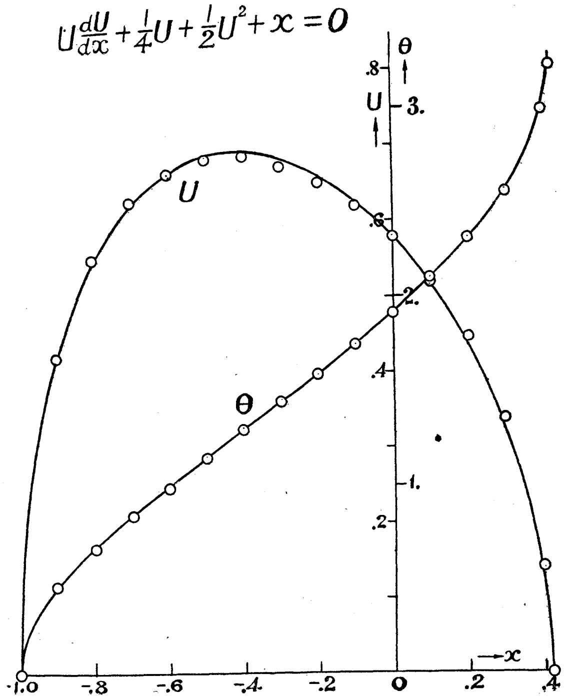

Fig. 1

simplified process along this line of thought. In our present problem, however, the differential equation is nothing but the presentation of direction field on the v-x plane (x : displacement), and we can see the deviation of the direction of integral curve from the field directly; this deviation may serve as a sort of the estimation of error, which is probably much severer than the estimation about the integral itself. This comparison of directions was done, by the solution (21), at several points of the curve, which is under the column v_0' in the table 3, numbers in parentheses being the field directions; we see deviations in spite of the accord of the general aspects.

TABLE 2

σ	v_0	η_1	v_1	η_2	v_2
−1.00	0.0000	0.0000	0.0000	0.0000	0.0000
−0.97	0.2149	−0.0231	0.1918	−0.0015	0.1903
−0.93	0.2789	−0.0238	0.2551	−0.0013	0.2538
−0.90	0.2987	−0.0138	0.2849	−0.0007	0.2842
−0.80	0.3245	+0.0345	0.3590	−0.0018	0.3572
−0.70	0.3643	+0.0614	0.4257	−0.0032	0.4225
−0.50	0.5443	+0.0296	0.5739	+0.0006	0.5745
−0.40	0.6639	+0.0002	0.6641	+0.0011	0.6652
−0.20	0.9022	−0.0329	0.8693	+0.0003	0.8696
0.00	1.1008	−0.0213	1.0795	+0.0007	1.0802
0.10	1.1822	−0.0087	1.1735	+0.0010	1.1745
0.30	1.3063	+0.0004	1.3067	+0.0010	1.3077
0.40	1.3394	−0.0090	1.3304	+0.0009	1.3313
0.50	1.3386	−0.0241	1.3145	+0.0005	1.3150
0.70	1.1593	−0.0356	1.1237	−0.0005	1.1232
0.80	0.9360	−0.0136	0.9224	−0.0005	0.9219
0.90	0.5980	+0.0293	0.6273	−0.0016	0.6257
0.93	0.4715	+0.0359	0.5074	−0.0021	0.5053
0.97	0.2759	+0.0300	0.3059	−0.0019	0.3040
0.985	0.1847	+0.0193	0.2040	−0.0011	0.2029
1.00	0.0000	0.0000	0.0000	0.0000	0.0000

TABLE 3

σ	v_0'	v_1'	v_2'
−1.0	∞ (∞)	∞ (∞)	∞ (∞)
−0.9	0.4607 (0.6255)	0.8718 (0.886)	0.8876 (0.888)
−0.8	0.2453 (0.7962)	0.6759 (0.641)	0.6488 (0.649)
−0.7	0.5764 (0.8863)	0.6700 (0.663)	0.6734 (0.673)
−0.5	1.1440 (0.8977)	0.8449 (0.8608)	0.8581 (0.8583)
−0.4	1.2271 (0.9620)	0.9556 (0.9557)	0.9535 (0.9537)
−0.2	1.1109 (1.0883)	1.0685 (1.0683)	1.0676 (1.0681)
0.0	0.8724 (1.0356)	0.9967 (1.0000)	0.9999 (1.0000)
0.1	0.7558 (0.9090)	0.8732 (0.8744)	0.8743 (0.8744)
0.3	0.4550 (0.4256)	0.4068 (0.4067)	0.4063 (0.4063)
0.5	−0.2332 (−0.3943)	−0.3876 (−0.3909)	−0.3923 (−0.3922)
0.7	−1.7140 (−1.6888)	−1.6044 (−1.6033)	−1.6066 (−1.6066)
0.8	−2.7827 (−2.5235)	−2.4568 (−2.4540)	−2.4585 (−2.4584)
0.9	−4.0051 (−3.8922)	−3.6986 (−3.709)	−3.7172 (−3.717)
1.0	−∞ (−∞)	−∞ (−∞)	−∞ (−∞)

For the oscillation problems there are ingeneous methods of graphical integration which give good results. Our results in the preceding sections are, then, useless as far as the numericals are concerned, and in this respect need is the method of correction. To increase the degree of polynomial will certainly improve the results and the solution of the algebraic equations there will be accomplished by some iterration processes, but probably tedious. Here we prefer rather to treat the differential equation directly by the iterration process, employing as the zeroth approximation the results given by the preceding method.

For comvenience sake, we introduce a parameter ϵ into the differential equation (1):

$$\Delta = v\frac{dv}{d\sigma} - f(\sigma, v; \epsilon, \alpha) = 0, \tag{1''}$$

and denote its solution by

$$\epsilon = \epsilon_0 + \delta, \qquad \alpha = \alpha_0 + \beta, \qquad v = v_0 + \eta, \tag{22}$$

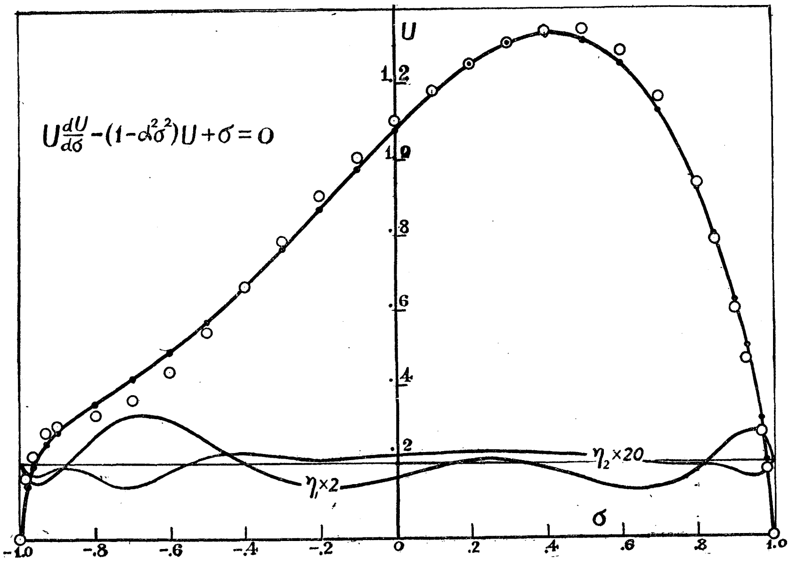

FIG. 2

where (ϵ_0, α_0, v_0) is the approximate solution somehow given. ϵ is usually a given constant, but the introduction of its variation enables us to pass to another value of ϵ. With (22) into (1″) and neglecting terms higher than first order with regard to (δ, β, η), it results the linear equation:

$$\left.\begin{aligned} \frac{d}{d\sigma}(v_0\eta) - \left(\frac{1}{v}\frac{\partial f}{\partial v}\right)_0 (v_0\eta) &= \left(\frac{\partial f}{\partial \epsilon}\right)_0 \delta + \left(\frac{\partial f}{\partial \alpha}\right)_0 \beta - \Delta_0(\sigma), \\ \Delta_0(\sigma) &= v_0\frac{dv_0}{d\sigma} - f(\sigma, v_0; \epsilon_0, \alpha_0), \end{aligned}\right\} \tag{23}$$

whose solution can be written down at once:

$$\eta = \frac{1}{v_0} e^{E(\sigma)} \left\{ \delta \int_{-1}^{\sigma} e^{-E(\sigma)} \left(\frac{\partial f}{\partial \epsilon}\right)_0 d\sigma + \beta \int_{-1}^{\sigma} e^{-E(\sigma)} \left(\frac{\partial f}{\partial \alpha}\right)_0 d\sigma - \int_{-1}^{\sigma} \Delta_0(\sigma)\, e^{-E(\sigma)} d\sigma, \right\} \tag{24}$$

where

$$E(\sigma) = \int_{-1}^{\sigma} \left(\frac{1}{v} \frac{\partial f}{\partial v}\right)_0 d\sigma. \tag{25}$$

This easy solubility is an effect of our solution v as a function of σ, contrary to the use of Fourier series in θ, alluded in § 1. When $\sigma = \pm 1$ η must vanish, and from this condition we obtain

$$\delta \int_{-1}^{+1} e^{-E(\sigma)} \left(\frac{\partial f}{\partial \epsilon}\right)_0 d\sigma + \beta \int_{-1}^{+1} e^{-E(\sigma)} \left(\frac{\partial f}{\partial \alpha}\right)_0 d\sigma = \int_{-1}^{+1} e^{-E(\sigma)} \Delta_0(\sigma)\, d\sigma, \tag{26}$$

which determines the correction δ or β when either of them is assigned beforehand. Inserting then these values in (24) we have the correction η just determined.

This process of correction may be repeated to the higher orders step by step, until the required accuracy is reached. Useful formulae are always (24), (25), (26). The quadratures which appear in these formulae are, however, not analytically integrable and, in general, numerical or graphical methods are to be consulted. This fact prohibits also any simple analytical expression for v, and then the next integral *i.e.* the relation between θ and σ is necessarily numerical.

As an example we take the case of Van der Pol (13). We have, denoting α^2 by a and regarding a as α in the formulae above,

$$E(\sigma) = \epsilon_0 \int_{-1}^{\sigma} \frac{1 - a_0 \sigma^2}{v_0} d\sigma \tag{27}$$

and

$$\left.\begin{aligned} &\int_{-1}^{\sigma} e^{-E(\sigma)} \left(\frac{\partial f}{\partial \epsilon}\right)_0 d\sigma = \int_{-1}^{\sigma} (1 - a_0 \sigma^2)\, v_0\, e^{-E(\sigma)} d\sigma, \\ &\int_{-1}^{\sigma} e^{-E(\sigma)} \left(\frac{\partial f}{\partial a}\right)_0 d\sigma = -\epsilon_0 \int_{-1}^{\sigma} v_0 \sigma^2 e^{-E(\sigma)} d\sigma, \\ &\int_{-1}^{\sigma} \Delta_0(\sigma)\, e^{-E(\sigma)} d\sigma = \frac{1}{2} v_0^2(\sigma)\, e^{-E(\sigma)} \\ &\qquad - \frac{1}{2} \epsilon_0 \int_{-1}^{\sigma} (1 - a_0 \sigma^2)\, v_0\, e^{-E(\sigma)} d\sigma + \int_{-1}^{\sigma} \sigma\, e^{-E(\sigma)} d\sigma, \end{aligned}\right\} \tag{28}$$

in which we are required for four quadratures.

As the zeroth approximation we can perhaps use a rough one, which save much the labors of §3. But here we take (21) as v_0 and require for the correction for $\epsilon = 1$. Then

$$\left.\begin{array}{ll} \epsilon_0 = 1.0356, & \delta = -0.0356, \\ a_0 = 4.0804, & a = a_0 + \beta. \end{array}\right\} \qquad (29)$$

Quadratures are all evaluated numerically with the Simpson's rule, which presents no difficulty but the $E(\sigma)$, whose values near the ends $\sigma = \pm 1$ were estimated by power series expansions. The results are

$$\beta = -0.03856 \qquad i.e. \qquad a = 4.0418 \ (\alpha = 2.0104)\,,$$

and η, whose values are under the column η_1 in the table 2, and inscribed in figure 2 (enlarged). Now the corrected v is $v_0+\eta$; whose values are under the column v_1 and represented by black points in the same table and figure. Agreement of the direction of the integral curve with the field is much improved as will be seen under the column v_1' in the table 3. We have thus arrived at the practically correct solution.

Once the quadratures in (28) are calculated we obtain, if desired, at once the corrections η and β (or δ) for every value ϵ (or a) in the neighborhood of ϵ_0 (or a_0). In the above example if we fix α, $\alpha = \alpha_0 = 2.02$, δ is -0.1643 *i.e.* $\epsilon = 0.8713$, and the corrected v has the tangent which accord to the field within one degree of angle. Thus the neighborhood seems to be moderately wide.

With the view of examining the convergency of the iterration process, we advanced the correction once more for the case $\epsilon = 1$; the results are

$$\beta = +0.00602 \qquad i.e. \qquad a = 4.0478 \ (\alpha = 2.0119)\,,$$

and η_2, v_2 in table 2, and v_2' in table 3; in the figure 2 we also see v_2 inscribed with full line and η_2 (enlarged). Agreement of directions is almost perfect, *i.e.* within about one minute of angle.

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