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ELASTIC STABILITY OF THE SQUARE PLATE WITH A CENTRAL CIRCULAR HOLE UNDER EDGE THRUST

By Toyoji KUMAI

The approximate calculations and the model experiments on the elastic stability of the square plate having a central circular hole under an action of edge compression are carried out. Comparisons of the results of numerical computations of the critical load and that of the model experiments are made in the cases of simply supported edge and the clamped one. The curvatures of the buckled plates computed from assumed deflection patterns are also compared with those measured by the optical method confirming the calculations.

1. Introduction. The investigations on the effects of the presence of a lightening hole or a manhole perforated in a rectangular plate upon the critical load of the plate under an action of edge thrust are important role in the practical design of some structural elements of ship's hull or that of the aeroplane body. The theoretical investiations on the symmetrical buckling of a circular ring under radial compression have already been carried out by E. Meissner[1]. The results, therefore, probably may be applicable to the square plate with a central circular hole of the same diameter.

However, the effects of the form of outer boundary and that of deflection pattern (especially that of the higher mode) upon the critical load are unknown. Recently, the critical load of simply supported square plate with a central circular hole under edge compression has been computed by S. Levy and his coworkers[2]. However, the result of their computation of the critical load shows as much as 25 % higher than that of the corresponding circular ring plate computed by E. Meissner. Since no experimental data are dealt with in both investigations above mentioned, it is very ambiguous to apply the results in practical design. In the present paper, therefore, the approximate calculations of the critical load of the square plate with a central circular hole subjected to the edge compression of the vertical direction are carried out by the energy method using the more reasonable assumption of the deflection pattern of the perforated plate than the previous investigation, morover the critical loads are com-

1) E. Meissner, Schweizerische Bauzeitung, Bd. 101, No. 8, 1933.

2) S. Levy, R. M. Woolley, and W. D. Kroll, Jour. of Res. of National Bureau of Standards, vol. 39. Dec. 1947.

puted in three cases of simply supported edge, of clamped edge and of clamped edge with the second mode of buckling. And also the experimental observations on the critical loads in all cases above mentioned are carried out by the model specimens. The distribution of the curvature obtained by the method of direct measurement of the buckled plate with a central hole is also compared with that has been computed from the assumed deflection pattern used in the present calculations of the critical load.

2. Calculations.

Nomencrature;

x, y	cartesian co-ordinate parallel to both edges of the square plate with the origin at the centre of the plate
r, θ	polar co-ordinate with the origin at the centre of the plate
λ	ratio of diameter of a central hole with the breadth of the square plate (breadth of plate is taken as unity)
V, T	strain energy due to bending of the plate and work done produced by edge compression respectively
σ_α, σ_β, $\tau_{\alpha\beta}$	components of two dimensional stress of the plate just prior to buckle presented by the curvilinear co-ordinate
σ_x, σ_y, τ_{xy}	converted stress components of σ_α, σ_β, $\tau_{\alpha\beta}$ into cartesian co-ordinate x, y
w	deflection pattern of the perforated plate
D	bending stiffness of the plate $D = \dfrac{Et^3}{12(1-\nu^2)}$ E Young's modulus ν Poisson's ratio t thickness of the plate
$\sigma_{cr} = k\dfrac{\pi^2 D}{t^2}$	critical stress of the plate

The strain energy of the plate due to bending and the work done by the plane stress components produced by the external force just prior to buckle are computed by the well-known formulae,

$$V = \frac{D}{2}\int_A \left[\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\nu)\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\}\right]dA \quad (1)$$

$$T = \frac{\sigma_0}{2}\int_A \left[\left\{\frac{\sigma_x}{\sigma_0}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{2\tau_{xy}}{\sigma_0}\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) + \frac{\sigma_y}{\sigma_0}\left(\frac{\partial w}{\partial y}\right)^2\right\} dA \quad (2)$$

where A is the surface area of the plate except that of a hole perforated. If the deflection of the plate w in the above formulae satisfies the fundamental equation of the buckling of the plate as well as all the boundary conditions, the critical stress is determined by equating (1) and (2). Unfortunately, however, the solution of the equation which satisfies all the boundary conditions of the plate having a central circular hole is not yet known. In the present paper, the approximate value of the critical load of the plate was computed by the energy method using the deflection pattern satisfies the boundary conditions not only at the outside edge but at the periphery of a circular hole of the plate. The deflection patterns of the perforated plates satisfy the boundary conditions above mentioned are assumed to be of the forms

case I, supported edge;

$$w = -w_0\{\cos\pi x \ \cos\pi y + be^{-c\,(x^2+y^2)}\}, \tag{3}$$

case II, clamped edge;

$$w = -w_0\{\cos^2\pi x \ \cos^2\pi y + be^{-c\,(x^2+y^2)}\}, \tag{4}$$

case III, clamped edge with the second mode;

$$w = -w_0\{\cos^2\pi x \ \sin 2\pi y \ \cos\pi y + bye^{-c\,(x^2+y^2)}\}, \tag{5}$$

where w_0 is unit deflection, b and c are the constants to be determined by the conditions of the periphery of a circular hole perforated at the center of the square plate. Since the diameter of a hole is comparable small compared with the breadth of the plate, the deflections expressed in (3), (4) and (5) in the vicinity of the periphery of a hole are approximately represented by the polar co-ordinate as the following forms in each cases respectively,

case I ; $$w \doteqdot -w_0\{J_0(2\pi r) + be^{-cr^2}\}, \tag{6}$$

case II ; $$w \doteqdot -w_0\{J_0(2\pi r) + be^{-cr^2}\}, \tag{7}$$

case III; $$w \doteqdot -w_0\{2\pi J_0(2\pi r) + be^{-cr^2}\}\, r\, \sin\theta\,. \tag{8}$$

The boundary conditions at the periphery of a hole of radius $r = r_0$ in the plate are shown as follows,

$$(M_r)_{r=r_0} = 0, \quad \left(Q_r - \frac{\partial M_{r\theta}}{r\partial\theta}\right)_{r=r_0} = 0. \qquad (9),\ (10)$$

where $$M_r = -D\left\{\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\left(\frac{\partial w}{\partial r} + \frac{1}{r}\,\frac{\partial^2 w}{\partial\theta^2}\right)\right\},$$

$$M_{r\theta} = (1-\nu)D\,\frac{1}{r}\left(\frac{\partial^2 w}{\partial r\partial\theta} - \frac{1}{r}\,\frac{\partial w}{\partial\theta}\right),$$

$$Q_r = -D\frac{\partial}{\partial r}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\,\frac{\partial w}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 w}{\partial\theta^2}\right).$$

The conditions (9) and (10) mean the normal bending moment and the equivalent shear force of the plate vanish at the periphery of a hole respectively. Thus, the constants b and c in the expressions of (3), (4) and (5) are determined by the above two conditions of (9) and (10). Fortunately, the numerical value of c obtained by the above two conditions becomes very large compared with unity, and that of b becomes comparable or smaller than unity in each cases. Accordingly, the second terms in the expressions (3), (4) and (5) are neglisible small compared with the first term in the vicinity of the outside edge of the plate not only of which the deflection but the slope and the bending moment of the plate. It is a matter of course, the first terms in the equations (3), (4) and (5) satisfy the boundary conditions at the outside edge of the plates considered without the second terms. The numerical values of b and c computed by the substitutions of (6), (7) and (8) in the conditions (9) and (10) are shown as the following table.

Numerical value of b and c for given value of r_0

r_0	case I		case II		case III	
	b	c	b	c	b	c
0.05	0.0215	805.33	0.0925	810.39	1.6026	938.24
0.10	0.0807	205.49	0.1860	208.69	4.7203	334.00
0.15	0.1668	94.63	0.2802	102.10	9.4796	143.77
0.20	0.2555	56.43	0.3731	67.30	14.371	80.25
0.25	0.3376	39.43	0.4789	58.52	18.018	49.44
0.30	—	—	—	—	18.964	32.10

Now, since the exact expressions of the distributions of the plane stress components of the square plate with a central circular hole under uniform edge compression is much complicate, in the present paper, therefore, the distributions of stress components σ_α, σ_β and $\tau_{\alpha\beta}$ of the infinite plate with a circular hole under the uniform compression in y direction are used. Stress components above mentioned are obtained in connection with the curvilinear co-ordinate of

$$x = r_0 e^{\alpha} \cos\beta, \qquad y = r_0 e^{\alpha} \sin\beta, \tag{11}$$

where α being the radius and β the angle measured from x axis. The stress components σ_x, σ_y and τ_{xy} are computed from σ_α, σ_β and $\tau_{\alpha\beta}$ as the following forms by the use of conversion formulae,

$$\left.\begin{aligned}
\sigma_x &= \frac{\sigma_0}{2} e^{-2\alpha}\{\cos 2\beta + (2 - 3e^{-2\alpha})\cos 4\beta\},\\
\sigma_y &= \frac{\sigma_0}{2}\left[2 + e^{-2\alpha}\{3\cos 2\beta - (2 - 3e^{-2\alpha})\cos 4\beta\}\right],\\
\tau_{xy} &= \frac{\sigma_0}{2} e^{-2\alpha}\{-\sin 2\beta + (2 - 3e^{-2\alpha})\sin 4\beta\}.
\end{aligned}\right\} \tag{12}$$

Next, the integrations of the strain energy and of the work done of the element of the plate presented in (1) and (2) respectively are numerically carried out. The method of numerical integration such as above mentioned has previously been shown using the Gauss' method by S. Levy[2]. The present author takes Gauss-Lobatto's points with $n = 5$ for both directions of α and β in the domain of the first quadrant of a ring plate, and also

[2] S. Levy, loc. cit.

takes Gauss' points with $n=3$ for x direction and with $n=2$ for y direction in the remainder part of the same quadrant in all cases of the buckling modes considered.

The critical load of the square plate with a central circular hole is presented for unit breadth as follows,

$$\sigma_{cr} = k\frac{\pi^2 D}{t^2}. \tag{13}$$

The ratios of k with k_0 (k_0 is a coefficient of $\frac{\pi^2 D}{t^2}$ in the case of the plate having nonperforation) were computed for given ratios λ of diameter of a hole with the length of the square plate in all cases. The relation between k/k_0 and λ obtained by foregoing calculations in the case of simply supported square plate is shown in Fig. 2. The result computed by S. Levy is also represented in the same figure. As will be seen in the figure, the result obtained in the present investigation differs fairly from which has been obtained by S. Levy. We shall point out that the dis-

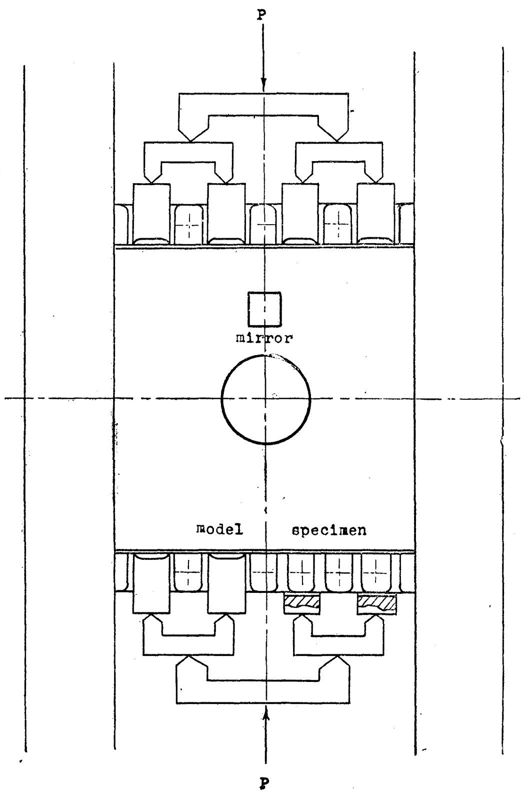

FIG. 1 Loading apparatus for measuring the load-slope relations of perforated plate under uniform edge compression.

crepancy between both results is because of the difference of assumption of the deflection patterns. In the case of the clamped edge type, the relations between k/k_0 and λ are shown in Fig. 3. Both cases of the buckling with the fundamental mode and with the second mode are plotted in the figure. It is remarkable fact in the practical application that the buckling with the second mode occur more easily than that of the fundamental one when the hole diameter ratio exceeds about 0.3.

3. Experimental Studies. Since foregoing results computed in the present investigation are that of the approximate method, the comparisons of the results of experimental observation and of calculations are shown for verification of the results.

In the present investigation mentioned previous article, the edge compression is not in the state of uniform displacement but of uniform stress along the edge even after the plate buckled. Then the special precautions are taken for the preparation of the loading apparatus for giving the uniform stress along the edges of the model specimen in the process of buckle into consideration. The apparatus giving the uniform stress on the edges of $y = \pm\frac{1}{2}$ are shown in Fig. 1.

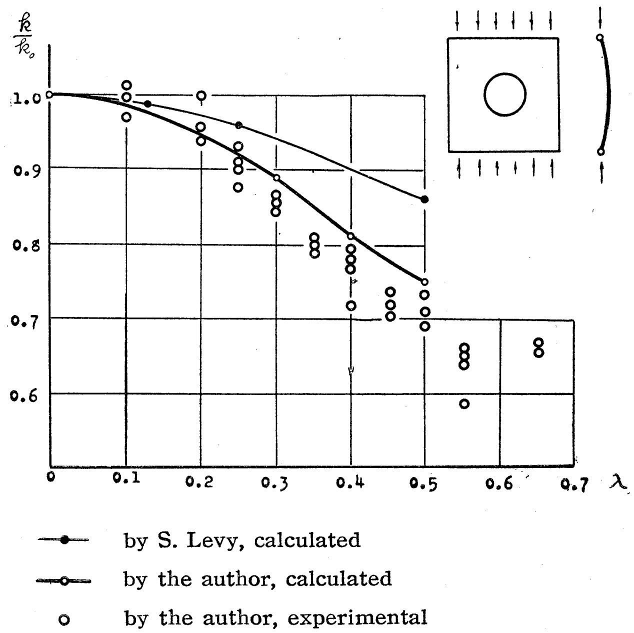

FIG. 2 Relations between critical load ratio and diameter ratio of circular hole in the case of supported edge.

The small angles attached to the upper and lower edges are flexible in the x-y plane by cutting off the vertical flanges of the angles, but are rigid in the direction of the deformation of the plate. The material of the model specimen is made of plexi-glass ($E = 290 \sim 300$ kg/mm^2) of 100 mm in breadth with 1 mm thickness, and the materials of loading frame are made of steel and alminium alloy. The tests were carried out by enlarging the area of a central hole using the same specimen. The slope of a point for given load in the specimen is measured by the scale-telescope through the mirror attached on the point. The critical stress of the model plate is, thus determined by Southwell-Donnell's method[3] from the load-slope curve obtained by above observation.

The computed results are compared with that of the observations as shown in Figs. 2 and 3. The critical load ratio decreases gradually for increment of the diameter ratio in the case of simply supported edge as will be seen in the Fig. 2. The critical loads obtained forgoing calculations are slightly higher than that of the observation. The result obtained by S. Levy, however, is found fairly higher than the present results of computations and of experiments. In the case of clamped edge, as will be seen in the Fig. 3, the results of calculations are also shown slightly higher than that of observation. Both of the critical loads obtained by calculation and by observation in the case of buckling with the second mode are found lower than that of the fundamental one when the hole diameter ratio exceeds about 0.3 in the clamped type.

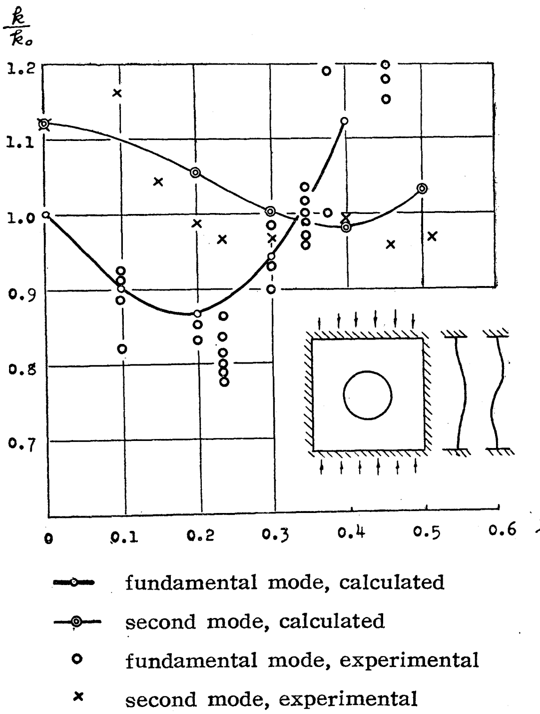

FIG. 3 Relations between critical load ratio and diameter ratio of circular hole in the case of clamped edge.

On the other hand, the measurements of the distribution of the curvature of the perforated plate after buckled were carried out for verify the deflection pattern assumed in the present calculation by way of

3) L. H. Donnell, S. Timoshenko 60th Anniversary Volume, 1938.

precaution. The optical method for direct measurement of the curvature of slightly curved plate is applied in the present paper. The principle of the method have already been examined by H Cranz[4] for the measurement of the plane stress by the flexural bending of the plate using the analogy of the second derivative of Airy's stress function and the corresponding curvature of the plate in the fundamental equations. The apparatus for measuring the curvature of the plate is

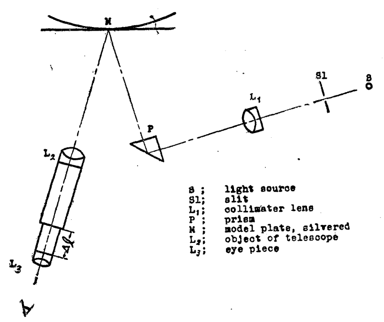

FIG. 4 Apparatus for measuring the curvature of buckled plate.

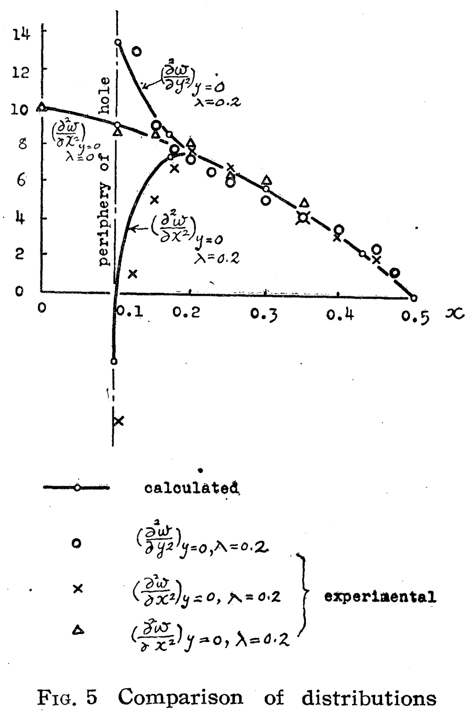

FIG. 5 Comparison of distributions of curvatures along x-axis obtained by assumed deflection and by measurements in both cases of the plate with a central circular hole ($\lambda = 0.2$) and of the plate without hole ($\lambda = 0$).

shown in Fig. 6. In this figure, the displacement of the eyepiece (Δl) for focussing to the image of a slit through the collimater is proportional to the change of the curvature of the point on the silvered specimen which reflect it, thus the distribution of curvature along the axis considered is directly measured by travelling the set except the model plate and reading Δl when the slit has been focussed.

The distributions of the curvatures with respect to x and to y, for example, on the positive side of x-axis are measured and compared with which have been computed from the assumed deflection pattern in both cases of the plate having no perforation and having a perforation of $\lambda = 0.2$ in the case of simply supported plate as shown in the Fig. 5. As will be seen in the figure, the tendencies of the distributions of the curvature obtained in both results of the assumption and the observation are found in good

[4] H. Cranz, Ingenieur Archiv Juni, 1933.

agreement. It is remarkable matter of fact that the distributions of the curvatures of the plate having a perforation are greatly different from that having no perforation in the vicinity of a periphery of a hole by virtue of taking the boundary conditions of the free edge along the periphery of a hole into consideration. In the consequence of the above experimental observations, the assumed patterns of the deflection of the perforated plate may reasonably be considered.

4. Conclusions. The important conclusions that can be drawn from the present investigations are shown as follows;

1. As the hole diameter ratio is made up larger, lower and lower the critical load ratio becomes in the case of simply supported square plate with a central circular hole under edge thrust, for example, the critical load of the plate with a hole diameter of half length of the breadth of the square plate shows 75 % of that of nonperforated plate by the calculation and is 70 % by the experiment. This value computed by S. Levey is 86 % in the same case.

2. The minimum critical load exists at about 20 % of the diameter ratio and its value is 87 % in the case of clamped edge with the fundamental buckling mode, and when the diameter ratio exceeds this value, the critical load increases, and when it reaches to about 34 % of the diameter ratio, the critical load is almost equal to the load of the plate having no perforation. After the diameter ratio goes over this point, more and more the value increases. The same tendency of the curve is shown as in the case of the circular ring computed by E. Meissner.

3. It is remarkable fact that the buckling with the second mode occures more easily than that of the fundamental mode in the case of clamped plate with the presence of a perforation of more than about 30 % in diameter ratio. As will be seen in the practical matters,[5] for example, the buckling failure of the floor plates having a manhole in the structure of ship's hull produced by the panting action due to sea wave frequently occur with the pattern of the second mode. The reason of the failure state of buckling above mentioned can qualitatively be interpreted by the present investigation.

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[5] Society of Naval Architecture, Japan, "Reports of Inquiry Committee of Damage of Motor Ships." Jour. of N. A. Japan, Vol. 66 No. 2, 1936. (Zōsen Kyōkai Zassan).

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