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CALCULATION OF THE STRESSES OF THE ORTHOTROPIC STRIP WITH A HOLE

By Masakazu HIGUCHI

The stress of the orthotropic strip with a hole, circular or elliptic, is dealt with by use of the stress function of the type $\chi(x, y) = Z_1(z_1) + Z_2(z_2) + comp.\ conj.$ together with the stress formulas $\sigma_x = -\sum_r k_r^2 Z_r''(z_r)$, $\sigma_y = \sum Z_r''(z_r)$, and $\tau_{xy} = -\sum_r i k_r Z_r''(z_r)$, where the summations range over four terms, two of them for $r = 1, 2$ and the rest their conjugates.

As the preliminaries, the boundary-value problem of a strip without a hole is treated which has been solved in many ways already. The author's present work, however, gives the most simple process in applying to the general cases of conditions.

Numerical calculation is worked out in the case of the strips of spruce and oak with a circular hole under tension, the result of which is compared with that of the isotropic case and of the infinite plate of spruce or oak.

1. Introduction. We have now a number of analitical treatments for the subjects in the theory of anisotropic elasticity. Concerning simple cases, in particular, of external loads as well as of the forms of the body, the history of the researches started already more than half a century ago. However, more general treatments of the theory, particularly in the domain of two-dimensional problems, are the work in the past decade and it may be said that too late came out the quantitative manifestation of the characteristic features of the stress concentration on the hole boundary in the anisotropic plate[1) 2) 3)] in comparison with the analytical solution of the similar problem in the theory of isotropic elasticity. We have still so many questions as regards the features of the stress distribution in anisotropic bodies, though the sufficient investigation on the applicability of the analytical results to the estimate of the behaviors of the material under stress, is an important concern for us in spite of Hoerig's affirmative investigation on some species of wood.[4)]

The present paper deals with the stress of the orthotropic strip with a hole, circular or elliptic, using the stress function of the

1) Green, A. E., Proc. Roy. Soc., A-180, 1942.
2) Ikeda, K., J. Soc, Sci. Cul., Japan, 2,3. Aug., 1947.
3) Higuchi, M., Kyushu Univ., Rep. Res. Inst. Elas. Engng., IV, 4. 1947.
4) Hoerig, H., Ing. Arch., 6, 8. 1935.

type[5] $\chi(x, y) = Z_1(z_1) + Z_2(z_2) + comp.\ conj.$ together with the formulas for stresses $\sigma_x = -\sum_r k_r^2 Z_r''(z_r)$, $\sigma_y = \sum_r Z_r''(z_r)$, and $\tau_{xy} = -\sum_r ik_r Z_r''(z_r)$, where the summations range over four terms, two of them for $r = 1, 2$ and the rest their conjugates.

As the preliminaries, the boundary-value problem of a strip without a hole is treated. Solutions of this problem were duplicated by several research workers.[6] [7] It may be said, however, that the author's work gives the most simple process in applying to general cases of conditions. The present mode of attack on the two-dimensional boundary-value problems of the perforated strip has been made possible by the solution. An example of the perforated strip under tension is taken by way of illustration for practical calculation. Notwithstanding, applying the formulas to the case under other conditions of load on both of the straight edges and the bank of the hole will not be difficult, unlike the solutions hitherto given in the case of particular loads in the two-dimensional theory of isotropic elasticity, where it is usual that some skill is required to choice the stress function appropriate to other different external loads because the function is used there to be introduced somewhat a priori, while it may be done analytically to apply the formulas presented here.

2. The Orthotropic Strip. By use of the stress function of the form

$$\chi(x, y) = Z_1(z_1) + Z_2(z_2) + comp.\ conj.\ , \tag{1}$$

we can approach easily to the solution of our problems. Here $z_r = x + ik_r y$ and k_r $(r = 1, 2)$ are to be determined with the elastic constants of the material and the relations to be used are $k_1^2 k_2^2 = E_x/E_y$, $k_1^2 + k_2^2 = E_x/G_{xy} - 2\upsilon_x$. For the present we shall take particularly the less abstract forms

$$Z_r(z_r) = F_r(z_r) + G_r(z_r)\ , \qquad r = 1, 2 \tag{1'}$$

with

$$F_r(z_r) = \int_{-\infty}^{\infty} f_r(\gamma)\, e^{i\gamma z_r} \frac{d\gamma}{\gamma^2} \tag{1''}$$

$G_r(z_r) =$ stress functions giving the stresses at infinity.

$F_r(z_r)$ may be supposed to tend to zero according as $z_r \to \infty$ within the region of the plate. Taking the coordinate axes as shown in Fig. 1a and

[5] The Airy's stress function of such a more or less similar type as this in the two-dimensional problems of anisotropic elasticity were introduced by several research workers independently. In Japan, H. Ōkubo (1936) in a restrictive case and K. Ikeda (1942) in more general case took the initiative. The present author generalized further their works.[3] Such general treatment, however, seems to have been proposed early by S. G. Lechnitzky in USSR, whose paper could not get the author. In England, A. E. Green treated the theory with a slightly different method.

[6] Green, A.E., Proc. Roy. Soc., A-953, 1939.

[7] The author read the paper reported here as the preliminaries before the Kyushu Meeting of the Japan Society of Applied Mechanics on Dec. 5, 1948, but not printed.

denoting the external forces on the straight edges by p_x, p_y, p_x^* and p_y^*, we can regard them as the functions of S and S^* which represent the abscissae x on the edges. The boundary conditions are, therefore,

$$\left.\begin{aligned} [\sigma_y]_S &= p_y(S)\,, \\ [\tau_{xy}]_S &= p_x(S)\,, \\ [\sigma_y]_{S^*} &= -p_y^*(S^*)\,, \\ [\tau_{xy}]_{S^*} &= -p_x^*(S^*)\,, \end{aligned}\right\} \quad (2)$$

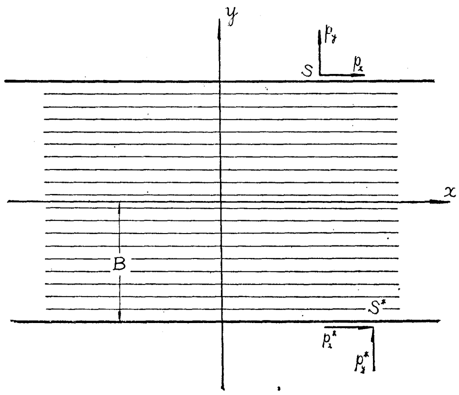

FIG. 1a.

since the stresses given by the stress function along the edges are to be no more than the external forces. By use of (1),

$$\left.\begin{aligned} &2\Re\left[\int_{-\infty}^{\infty}\left\{f_1(\gamma)e^{i\gamma z_1}+f_2(\gamma)e^{i\gamma z_2}\right\}d\gamma - G_1''(z_1) - G_2''(z_2)\right]_S = -p_y(S)\,, \\ &2\Im\left[\int_{-\infty}^{\infty}\left\{k_1 f_1(\gamma)e^{i\gamma z_1}+k_2 f_2(\gamma)e^{i\gamma z_2}\right\}d\gamma - k_1 G_1''(z_1) \right. \\ &\qquad\qquad\qquad\qquad \left. - k_2 G_2''(z_2)\right]_S = -p_x(S)\,, \\ &2\Re\left[\int_{-\infty}^{\infty}\left\{f_1(\gamma)e^{i\gamma z_1}+f_2(\gamma)e^{i\gamma z_2}\right\}d\gamma - G_1''(z_1) - G_2''(z_2)\right]_{S^*} = p_y(S^*)\,, \\ &2\Im\left[\int_{-\infty}^{\infty}\left\{k_1 f_1(\gamma)e^{i\gamma z_1}+k_2 f_2(\gamma)e^{i\gamma z_2}\right\}d\gamma - k_1 G_1''(z_1) \right. \\ &\qquad\qquad\qquad\qquad \left. - k_2 G_2''(z_2)\right]_{S^*} = p_x(S^*)\,, \end{aligned}\right\} \quad (3)$$

because

$$\sigma_y = \partial^2\chi/\partial x^2\,, \qquad \tau_{xy} = -\partial^2\chi/\partial x\,\partial y$$

and

$$\partial^2 Z_r(z_r)/\partial x^2 = Z_r''(z_r)\,, \quad \partial^2 Z_r(z_r)/\partial x\,\partial y = ik_r Z_r''(z_r)\,, \qquad r = 1, 2$$

where the simplified notations $Z_r''(z_r)$ and $G_r''(z_r)$ are used in place of $d^2 Z_r(z_r)/dz_r^2$ and $d^2 G_r(z_r)/dz_r^2$. The functions $G_r(z_r)$ $r = 1, 2$ are to be determined together with their conjugates $\overline{G_r(z_r)}$ by the given conditions at infinity and can be regarded as known functions for the present.

The equations (3) are rewritten as

$$\begin{aligned} &\int_{-\infty}^{\infty}\left\{e^{-k_1 B\gamma} f_1(\gamma) + e^{k_1 B\gamma}\overline{f_1(-\gamma)} + e^{-k_2 B\gamma} f_2(\gamma) \right. \\ &\qquad \left. + e^{k_2 B\gamma}\overline{f_2(-\gamma)}\right\} e^{i\gamma S} d\gamma = 2\Re\left[G_1''(z_1) + G_2''(z_2)\right]_S - p_y(S)\,, \end{aligned}$$

$$\left.\begin{aligned}
&\int_{-\infty}^{\infty}\Big\{k_1 e^{-k_1 B\gamma} f_1(\gamma) - k_1 e^{k_1 B\gamma}\overline{f_1(-\gamma)} + k_2 e^{-k_2 B\gamma} f_2(\gamma) \\
&\qquad - k_2 e^{k_2 B\gamma}\overline{f_2(-\gamma)}\Big\} e^{i\gamma S} d\gamma = 2i\Im\Big[k_1 G_1''(z_1) + k_2 G_2''(z_2)\Big]_S - i p_x(S), \\
&\int_{-\infty}^{\infty}\Big\{e^{k_1 B\gamma} f_1(\gamma) + e^{-k_1 B\gamma}\overline{f_1(-\gamma)} + e^{k_2 B\gamma} f_2(\gamma) \\
&\qquad + e^{-k_2 B\gamma}\overline{f_2(-\gamma)}\Big\} e^{i\gamma S^*} d\gamma = 2\Re\Big[G_1''(z_1) + G_2''(z_2)\Big]_{S^*} + p_y^*(S^*) , \\
&\int_{-\infty}^{\infty}\Big\{k_1 e^{k_1 B\gamma} f_1(\gamma) - k_1 e^{-k_1 B\gamma}\overline{f_1(-\gamma)} + k_2 e^{k_2 B\gamma} f_2(\gamma) \\
&\qquad - k_2 e^{-k_2 B\gamma}\overline{f_2(-\gamma)}\Big\} e^{i\gamma S^*} d\gamma = 2i\Im\Big[k_1 G_1''(z_1) + k_2 G_2''(z_2)\Big]_{S^*} \\
&\qquad\qquad + i p_x^*(S^*) .
\end{aligned}\right\} \quad (3')$$

Hereupon, applying Fourier-transformation to the both sides of the equations, we get

$$\left.\begin{aligned}
&e^{-k_1 B\gamma} f_1(\gamma) + e^{k_1 B\gamma}\overline{f_1(-\gamma)} + e^{-k_2 B\gamma} f_2(\gamma) + e^{k_2 B\gamma}\overline{f_2(-\gamma)} = P_y(\gamma) , \\
&k_1 e^{-k_1 B\gamma} f_1(\gamma) - k_1 e^{k_1 B\gamma}\overline{f_1(-\gamma)} + k_2 e^{-k_2 B\gamma} f_2(\gamma) \\
&\qquad - k_2 e^{k_2 B\gamma}\overline{f_2(-\gamma)} = iP_x(\gamma) , \\
&e^{k_1 B\gamma} f_1(\gamma) + e^{-k_1 B\gamma}\overline{f_1(-\gamma)} + e^{k_2 B\gamma} f_2(\gamma) \\
&\qquad + e^{-k_2 B\gamma} f_2(-\gamma) = P_y^*(\gamma) , \\
&k_1 e^{k_1 B\gamma} f_1(\gamma) - k_1 e^{-k_1 B\gamma}\overline{f_1(-\gamma)} + k_2 e^{k_2 B\gamma} f_2(\gamma) \\
&\qquad - k_2 e^{-k_2 B\gamma}\overline{f_2(-\gamma)} = iP_x^*(\gamma) .
\end{aligned}\right\} \quad (3'')$$

Solving these equations simultaneously, we obtain $f_1(\gamma)$ and $f_2(\gamma)$, since $P_x(\gamma)$, $P_y(\gamma)$, $P_x^*(\gamma)$ and $P_y^*(\gamma)$ are all the known functions of γ, that is,

$$\left.\begin{aligned}
&P_x(\gamma) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big\{2\Im\Big[k_1 G_1''(z_1) + k_2 G_2''(z_2)\Big]_S - p_x(S)\Big\} e^{-i\gamma S} dS , \\
&P_y(\gamma) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big\{2\Re\Big[G_1''(z_1) + G''_2(z_2)\Big]_S - p_y(S)\Big\} e^{-i\gamma S} dS , \\
&P_x^*(\gamma) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big\{2\Im\Big[k_1 G_1''(z_1) + k_2 G_2''(z_2)\Big]_{S^*} \\
&\qquad\qquad + p_x^*(S^*)\Big\} e^{-i\gamma S^*} dS^* , \\
&P_y^*(\gamma) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big\{2\Re\Big[G_1''(z_1) + G_2''(z_2)\Big]_{S^*} + p_y^*(S^*)\Big\} e^{-i\gamma S^*} dS^* .
\end{aligned}\right\} \quad (3''')$$

If the external forces are symmetrical about both axes of x and y, the relations $f_r(\gamma) = \overline{f_r(\gamma)}$ and $f_r(\gamma) = f_r(-\gamma)$ are deduced and so the equations (3″) are simplified as

$$\left.\begin{aligned} &2\cosh(k_1B\gamma)f_1(\gamma)+2\cosh(k_2B\gamma)f_2(\gamma)=P_y(\gamma)\ ,\\ &-2k_1\sinh(k_1B\gamma)f_1(\gamma)-2k_2\sinh(k_2B\gamma)f_2(\gamma)=iP_x(\gamma)\ .\end{aligned}\right\} \qquad (4)$$

3. The Orthotropic Strip with an Elliptic Hole. We must add now another term to the stress function we used in treating the strip with no hole. That is,

$$\chi(x, y) = Z_1(z_1) + Z_2(z_2) + comp.\ conj.\ , \qquad (5)$$

$$Z_r(z_r) = F_r(z_r) + G_r(z_r) + H_r(z_r)\ , \qquad r = 1, 2 \qquad (5')$$

with

$$F_r(z_r) = \int_{-\infty}^{\infty} f_r(\gamma)e^{i\gamma z_r}\frac{d\gamma}{\gamma^2} \qquad (5'')$$

and $G_r(z_r)$, with which the conditions of stresses at infinity are to be satisfied as in the previous section, while $H_r(z_r)$ is expressed as

$$c_r' \ln z_r - \frac{c_r''}{z_r^2} - \frac{c_r'''}{2z_r^2} - \cdots \qquad (5''')$$

and having a singularity at the center of the hole, is the main part of the solution to satisfy the stress or strain conditions on the hole boundary. But you need not be bored with the calculation of the coefficients of each term of $H_r(z_r)$. It is obtained easily as a closed form and that of a simple type. The following calculating method by use of Cauchy's integral is to be referred to the author's previous paper.[8)]

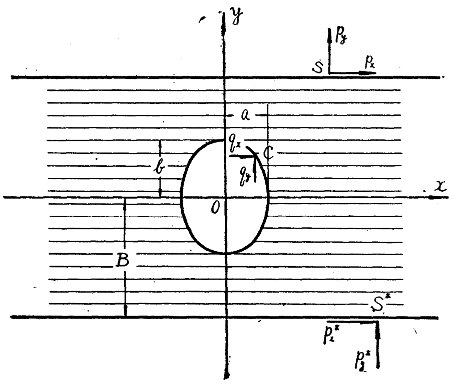

FIG. 1b.

Let us now suppose the strip under external forces as designated in Fig. 1b. Since

$$\left.\begin{aligned} \int_A^C q_y(C)dC &= \left[\frac{\partial\chi}{\partial x}\right]_A^C = \Big[Z_1'(z_1) + Z_2'(z_2) + comp.\ conj.\Big]_A^C\ ,\\ \int_A^C q_x(C)dC &= -\left[\frac{\partial\chi}{\partial y}\right]_A^C = -i\Big[k_1 Z_1'(z_1)\\ &\qquad + k_2 Z_2'(z_2) - comp.\ conj.\Big]_A^C\ ,\end{aligned}\right\} \qquad (6)$$

eliminating $Z_2'(z_2)$ from the equations (6), we get

$$\Big[(k_2-k_1)Z_1'(z_1) + (k_2+k_1)\overline{Z_1'(z_1)} + 2k_2\overline{Z_2'(z_2)}\Big]_A^C$$
$$= \int_A^C \Big\{k_2 q_y(C) - iq_x(C)\Big\}dC\ , \qquad (7)$$

8) Higuchi, M., loc. cit.

which is rewritten, using the equation (5′) and (5″), as

$$\Big[(k_1-k_2)H_1'(z_1)-(k_1+k_2)\overline{H_1'(z_1)}-2k_2\overline{H_2'(z_2)}\Big]_A^C$$

$$=-i\Big[\int_{-\infty}^{\infty}\Big\{(k_1-k_2)e^{i\gamma z_1}f_1(\gamma)+(k_1+k_2)e^{-i\gamma\bar{z}_1}\overline{f_1(\gamma)}$$

$$+2k_2e^{-i\gamma\bar{z}_2}\overline{f_2(\gamma)}\Big\}\frac{d\gamma}{\gamma}\Big]_A^C-\Big[(k_1-k_2)G_1'(z_1)-(k_1+k_2)\overline{G_1'(z_1)}$$

$$-2k_2\overline{G_2'(z_2)}\Big]_A^C-\int_A^C\Big\{k_2q_y(C)-iq_x(C)\Big\}dC\,. \qquad (8)$$

Now, we transform the elliptic hole on the z-plane into a unit circle on the ζ_1- and ζ_2-planes with

$$z_r=\alpha_r\zeta_r+\frac{\beta_r}{\zeta_r}\,,\qquad r=1,2 \qquad (9)$$

where $\alpha_r=(a+k_rb)/2$, $\beta_r=(a-k_rb)/2$ in the case the hole is located as in Fig. 1b. Thus, any point ($x=a\cos\theta$, $y=b\sin\theta$) on the boundary of the ellipse in the z-plane is transferred into the points in the ζ_1- and ζ_2-planes, the coordinates of which are identical with each other and is $e^{i\theta}$ ($\equiv\omega$, say), though other points than those on the boundary falls as the different points upon the ζ_1- and ζ_2-planes.

Hereupon, multiplying the both sides of (8) by $\frac{1}{2\pi i}\cdot\frac{d\omega}{\omega-\zeta_1}$ where ζ_1 is any point within the region of the plate, in other words, outside the unit circle, and integrating them in the clock-wise sense along the circumference of the unit circle Γ (see Fig. 1c), the left-hand side reduces to $-(k_1-k_2)H_1'\{z_1(\zeta_1)\}$. The full calculating procedure is omitted here because the integration can be evaluated easily by residues, considering that two of the singular points of $H_1'\{z_1(\omega)\}/(\omega-\zeta_1)$ are inside the contour and determined, as known from(5‴), by $[z_r]_C=[\alpha_r\zeta_r+\beta_r/\zeta_r]_\Gamma=\alpha_r\omega+\beta_r/\omega=0$ and the third is outside the contour and at $\omega=\zeta_1$, while $\overline{H_r'\{z_r(\omega)\}}/(\omega-\zeta_1)$ $r=1,2$ has no singular point inside the circle. On the other hand, the first integral of the right-hand side of (8) becomes

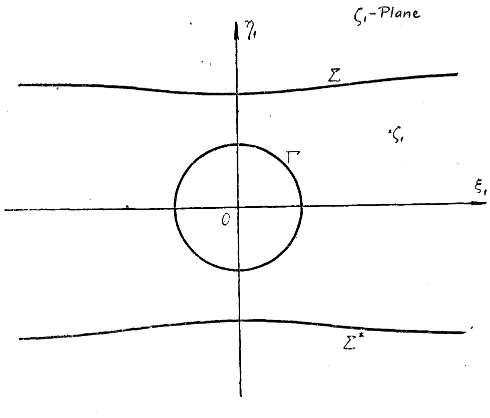

FIG. 1c.

$$-i\sum_{m=0}^{\infty}(h_1^{(2m+1)}/\zeta_1^{2m+1})$$

where

$$h_1^{(2m+1)} \equiv 2\int_0^{\infty}\Big[\Big\{(k_1-k_2)\mu_1^{-(2m+1)}+(k_1+k_2)\mu_1^{2m+1}\Big\}J_{2m+1}(\lambda_1\gamma)f_1(\gamma) + 2k_2\mu_2^{2m+1}J_{2m+1}(\lambda_2\gamma)f_2(\gamma)\Big]\frac{d\gamma}{\gamma}, \qquad (10)$$

and $J_{2m+1}(\lambda_r\gamma)$ is the Bessel's function of the order $2m+1$ derived from the integration

$$\frac{1}{2\pi}\oint_{\Gamma} e^{i\gamma z_r}\frac{d\omega}{\omega-\zeta_1} = -i\sum_{m=1}^{\infty}\frac{J_m(\lambda_r\gamma)}{(-\mu_r\zeta_1)^m},$$
$$\frac{1}{2\pi}\oint_{\Gamma} e^{i\gamma\bar{z}_r}\frac{d\omega}{\omega-\zeta_1} = -i\sum_{m=1}^{\infty}\frac{J_m(\lambda_r\gamma)}{(-\upsilon_r\zeta_1)^m}, \qquad (11)$$

where $\lambda_r = 2\sqrt{\alpha_r\beta_r}$, $\mu_r = i\sqrt{\alpha_r/\beta_r}$ and $\upsilon_r = i\sqrt{\beta_r/\alpha_r} = -1/\mu_r$ $r=1,\ 2$. Accordingly the arguments of J_{2m+1} are not always real. The rest terms of the equation (8) are the integrals of the known functions. Using another notation

$$g_1(\zeta_1) \equiv -\frac{1}{2\pi i}\oint_C\Big\{\big[(k_1-k_2)G_1'(z_1)-(k_1+k_2)\overline{G_1'(z_1)}-2k_2\overline{G_2'(z_2)}\big]_A^{C(\omega)} + \int_A^{C(\omega)}[k_2 q_y(C)-i\,q_x(C)]\,dC\Big\}\frac{d\omega}{\omega-\zeta_1}, \qquad (12)$$

we have finally

$$dH_1\big\{z_1(\zeta_1)\big\}/dz_1 = -\Big\{g_1(\zeta_1)-i\sum_{m=0}^{\infty}h_1^{(2m+1)}/\zeta_1^{2m+1}\Big\}/(k_1-k_2) \qquad (13)$$

while $H_2'\{z_2(\zeta_2)\}$ can be obtained directly from $H_1'\{z_1(\zeta_1)\}$ by interchanging its suffices 1, 2 each other, where the apostrophes signify the differentiation as regards z_2 and z_1 respectively.

We go back now to the stress function (5) in order to consider the conditions on the straight boundaries which are the same as those supposed in the previous section. The succeeding calculations also are carried out as in the previous section except for superposing another terms such as $H_r''(z_r)$, which contain the infinite numbers of the unknown constants $h_1^{(2m+1)}$ and $h_2^{(2m+1)}$, $m=0,1,2,\cdots$. These constants are the infinite integrals of $f_1(\gamma)$ and $f_2(\gamma)$ which are remaining to be determined by the conditions on these straight boundaries.

Following the procedure taken in the previous section and additionally evaluating the Fourier-transforms $\frac{1}{2\pi}\int_{-\infty}^{\infty}[H_r''(z_r)+\overline{H_r''(z_r)}]_S\, e^{-i\gamma S}\,dS$, $\frac{1}{2\pi}\int_{-\infty}^{\infty}[k_r H_r''(z_r)-k_r\overline{H_r''(z_r)}]_S\, e^{-i\gamma S}\,dS$ etc, we get the simultaneous Fredholm's integral equations of the 2nd kind as regards $f_1(\gamma)$, $\overline{f_1(-\gamma)}$, $f_2(\gamma)$ and $f_2(-\gamma)$. The equations are troublesome to solve generally, because Bessel's coefficients of odd order result again from the above Fourier-transformations. We must, therefore, solve them by the method of successive approximation or other appropriate method as the case may require.

4. The Orthotropic Strip with an Elliptic Hole under Tension Uniform at the Remote Ends. We shall calculate the stresses of the orthotropic strip with an elliptic hole under tension uniform and parallel to the x-axis at the remote ends. We have need to take the functions

$$G_r(z_r) = -\frac{z_r^2}{4(k_r^2 - k_s^2)}, \qquad r, s = 1, 2 \quad \text{but} \quad r \neq s \tag{14}$$

which yield the unit tension at infinity, that is, $\sigma_{x\infty} = 1$.

Since $p_x, p_y, p_x^*, p_y^*, q_x, q_y = 0$ and $f_r(\gamma) = \overline{f_r(\gamma)}$, $f_r(\gamma) = f_r(-\gamma)$

$r = 1, 2$, the equation (13) becomes

$$H_r'\left\{z_r(\zeta_r)\right\} = \frac{-b}{2(k_r - k_s)\zeta_r} + \frac{i}{k_r - k_s} \sum_{m=0}^{\infty} \frac{h_r^{(2m+1)}}{\zeta_r^{2m+1}}, \qquad r,\ s = 1, 2 \quad \text{but} \quad r \neq s \tag{13'}$$

and the equations (4) are

$$\left.\begin{aligned}
&\cosh k_1 B\gamma \cdot f_1(\gamma) + \cosh k_2 B\gamma \cdot f_2(\gamma) \\
&= \frac{ib}{4(k_1 - k_2)}\left\{\mu_1 e^{-k_1 B\gamma} J_1(\lambda_1\gamma) - \mu_2 e^{-k_2 B\gamma} J_1(\lambda_2\gamma)\right\} \\
&\quad - \frac{1}{k_1 - k_2}\Bigg\{\int_0^{\infty}\Bigg[e^{-k_1 B\gamma} \sum_{m=0}^{\infty}(2m+1)\{(k_2 - k_1) \\
&\quad - (k_2 + k_1)\mu_1^{4m+2}\} J_{2m+1}(\lambda_1\gamma') J_{2m+1}(\lambda_1\gamma) \\
&\quad + 2k_1 e^{-k_2 B\gamma} \sum_{m=0}^{\infty}(2m+1)(\mu_1\mu_2)^{2m+1} J_{2m+1}(\lambda_1\gamma') J_{2m+1}(\lambda_2\gamma)\Bigg] f_1(\gamma')\frac{d\gamma'}{\gamma'} \\
&\quad - \int_0^{\infty}\Bigg[e^{-k_2 B\gamma} \sum_{m=0}^{\infty}(2m+1)\{(k_1 - k_2) \\
&\quad - (k_1 + k_2)\mu_2^{4m+2}\} J_{2m+1}(\lambda_2\gamma') J_{2m+1}(\lambda_2\gamma) \\
&\quad + 2k_2 e^{-k_1 B\gamma} \sum_{m=0}^{\infty}(2m+1)(\mu_1\mu_2)^{2m+1} J_{2m+1}(\lambda_2\gamma') J_{2m+1}(\lambda_1\gamma)\Bigg] f_2(\gamma')\frac{d\gamma'}{\gamma'}\Bigg\}, \\
&-k_1 \sinh k_1 B\gamma \cdot f_1(\gamma) - k_2 \sinh k_2 B\gamma \cdot f_2(\gamma) \\
&= \frac{ib}{4(k_1 - k_2)}\left\{k_1\mu_1 e^{-k_1 B\gamma} J_1(\lambda_1\gamma) - k_2\mu_2 e^{-k_2 B\gamma} J_1(\lambda_2\gamma)\right\} \\
&\quad - \frac{1}{k_1 - k_2}\Bigg\{\int_0^{\infty}\Bigg[k_1 e^{-k_1 B\gamma} \sum_{m=0}^{\infty}(2m+1)\{(k_2 - k_1) \\
&\quad - (k_2 + k_1)\mu_1^{4m+2}\} J_{2m+1}(\lambda_1\gamma') J_{2m+1}(\lambda_1\gamma)
\end{aligned}\right\} \tag{4'}$$

$$
\left.\begin{aligned}
&+ 2k_1k_2e^{-k_2B\gamma}\sum_{m=0}^{\infty}(2m+1)(\mu_1\mu_2)^{2m+1}J_{2m+1}(\lambda_1\gamma')J_{2m+1}(\lambda_2\gamma)\Big]f_1(\gamma')\frac{d\gamma'}{\gamma'} \\
&-\int_0^{\infty}\Big[k_2e^{-k_2L\gamma}\sum_{m=0}^{\infty}(2m+1)\{(k_1-k_2) \\
&-(k_1+k_2)\mu_2^{4m+2}\}J_{2m+1}(\lambda_2\gamma')J_{m2+1}(\lambda_2\gamma) \\
&+2k_1k_2e^{-k_1B\gamma}\sum_{m=0}^{\infty}(2m+1)(\mu_1\mu_2)^{2m+1}J_{2m+1}(\lambda_2\gamma')J_{2m+1}(\lambda_1\gamma)\Big]f_2(\gamma')\frac{d\gamma'}{\gamma'}\Big\} \,.
\end{aligned}\right)
$$

Let us take as the first approximate solutions $f_1^{(1)}(\gamma)$ and $f_2^{(1)}(\gamma)$ which satisfy the equations

$$
\left.\begin{aligned}
&\cosh k_1B\gamma\cdot f_1^{(1)}(\gamma)+\cosh k_2B\gamma\cdot f_2^{(1)}(\gamma) \\
&=\frac{ib}{4(k_1-k_2)}\{\mu_1e^{-k_1B\gamma}J(\lambda_1\gamma)-\mu_2e^{-k_2B\gamma}J_1(\lambda_2\gamma)\} \\
&\quad-\frac{1}{k_1-k_2}\Big\{\int_0^{\infty}\Big[e^{-k_1B\gamma}\{(k_2-k_1)-(k_2+k_1)\mu_1^2\}J_1(\lambda_1\gamma')J_1(\lambda_1\gamma) \\
&\quad+2k_1e^{-k_2B\gamma}\mu_1\mu_2J_1(\lambda_1\gamma')J_1(\lambda_2\gamma)\Big]f_1^{(1)}(\gamma')\frac{d\gamma'}{\gamma'} \\
&\quad-\int_0^{\infty}\Big[e^{-k_2L\gamma}\{(k_1-k_2)-(k_1+k_2)\mu_2^2\}J_1(\lambda_2\gamma')J_1(\lambda_2\gamma) \\
&\quad+2k_2e^{-k_1B\gamma}\mu_1\mu_2J_1(\lambda_2\gamma')J_1(\lambda_1\gamma)\Big]f_2^{(1)}(\gamma')\frac{d\gamma'}{\gamma',}\Big\}\,, \\
&-k_1\sinh k_1B\gamma\cdot f_1(\gamma)-k_2\sinh k_2B\gamma\cdot f_2(\gamma) \\
&=\frac{ib}{4(k_1-k_2)}\Big\{k_1\mu_1e^{-k_1B\gamma}J_1(\lambda_1\gamma)-k_2\mu_2e^{-k_2B\gamma}J_1(\lambda_2\gamma)\Big\} \\
&\quad-\frac{1}{k_1-k_2}\Big\{\int_0^{\infty}\Big[k_1e^{-k_1B\gamma}\{(k_2-k_1)-(k_2+k_1)\mu_1^2\}J_1(\lambda_1\gamma')J_1(\lambda_1\gamma) \\
&\quad+2k_1k_2e^{-k_2L\gamma}\mu_1\mu_2J_1(\lambda_1\gamma')J_1(\lambda_2\gamma)\Big]f_1^{(1)}(\gamma')\frac{d\gamma'}{\gamma',} \\
&\quad-\int_0^{\infty}\Big[k_2e^{-k_2L\gamma}\{(k_1-k_2)-(k_1+k_2)\mu_2^2\}J_1(\lambda_2\gamma')J_1(\lambda_2\gamma) \\
&\quad+2k_1k_2e^{-k_1B\gamma}\mu_1\mu_2J_1(\lambda_2\gamma')J_1(\lambda_1\gamma)\Big]f_2^{(1)}(\gamma')\frac{d\gamma'}{\gamma'}\Big\}\,.
\end{aligned}\right\}\quad(15)
$$

that is, the equations which leave the terms for $m\geqq 1$ in the equations (4′). Solving the equations (15) we get

$$
\left.\begin{aligned}
f_1^{(1)}&=4\Big\{(k_1+k_2)\frac{\alpha_1h_1}{\lambda_1}J_1(\lambda_1\gamma)\Big[e^{-2k_1B\gamma}+\frac{k_1-k_2}{k_1+k_2}e^{-2(k_1+k_2)L\gamma}\Big] \\
&\qquad-\frac{2k_2\alpha_2h_2}{\lambda_2}J_1(\lambda_2\gamma)e^{-(k_1+k_2)L\gamma}\Big\}/D(\gamma)\,, \\
f_2^{(1)}&=4\Big\{(k_1+k_2)\frac{\alpha_2h_2}{\lambda_2}J_1(\lambda_2\gamma)\Big[e^{-2k_2B\gamma}+\frac{k_2-k_1}{k_2+k_1}e^{-2(k_2+k_1)L\gamma}\Big] \\
&\qquad-\frac{2k_1\alpha_1h_1}{\lambda_1}J_1(\lambda_1\gamma)e^{-(k_1+k_2)L\gamma}\Big\}/D(\gamma)\,,
\end{aligned}\right\}\quad(16)
$$

where

$$h_1 = \Big\{(k_1 + k_2)\alpha_1 - (k_1 - k_2)\beta_1\Big\} j_1 + 2k_2\alpha_2 j_2 + \frac{b}{4} , \tag{16'}$$

$$h_2 = \Big\{(k_1 + k_2)\alpha_2 - (k_2 - k_1)\beta_2\Big\} j_2 + 2k_1\alpha_1 j_1 + \frac{b}{4} , \tag{16''}$$

and

$$D(\gamma) = (k_1^2 - k_2^2)\left\{\frac{k_1 - k_2}{k_1 + k_2} - e^{-2k_1 B\gamma} + e^{-2k_1 B\gamma} - \frac{k_1 - k_2}{k_1 + k_2} e^{-2(k_1+k_2)B\gamma}\right\} .$$

In order to calculate h_1 and h_2, we must evaluate j_1 and j_2 by use of the formulas

$$j_1 = \frac{(1 - l_2)n_1 + m_1 n_2}{(1 - l_1)(1 - l_2) - m_1 m_2} , \qquad j_2 = \frac{(1 - l_1)n_2 + m_2 n_1}{(1 - l_1)(1 - l_2) - m_1 m_2} , \tag{16'''}$$

where

$$\left.\begin{aligned}
l_1 &= 4(k_1 + k_2)\int_0^\infty\Big[(k_1^2 b + k_2 a)\alpha_1\Big\{\frac{k_1 + k_2}{2} e^{-(k_1-k_2)B\gamma} \\
&\quad + \frac{k_1 - k_2}{2} e^{-(k_1+k_2)B\gamma}\Big\}\frac{J_1(\lambda_1\gamma)}{\lambda_1} - 2k_1k_2\alpha_1\alpha_2\frac{J_1(\lambda_2\gamma)}{\lambda_2}\Big]\frac{J_1(\lambda_1\gamma)}{D(\gamma)\lambda_1\gamma}\, d\gamma , \\
m_1 &= 4(k_1 + k_2)\int_0^\infty\Big[2k_2\alpha_1\alpha_2\Big\{\frac{k_1 + k_2}{2} e^{-(k_1-k_2)B\gamma} \\
&\quad + \frac{k_1 - k_2}{2} e^{-(k_1+k_2)B\gamma}\Big\}\frac{J_1(\lambda_1\gamma)}{\lambda_1} - k_2(k_2^2 b + k_1 a)\alpha_2\frac{J_1(\lambda_2\gamma)}{\lambda_2}\Big]\frac{J_1(\lambda_1\gamma)}{D(\gamma)\lambda_1\gamma}\, d\gamma , \\
n_1 &= (k_1 + k_2)b\int_0^\infty\Big[\alpha_1\Big\{\frac{k_1 + k_2}{2} e^{-(k_1-k_2)B\gamma} \\
&\quad + \frac{k_1 - k_2}{2} e^{-(k_1+k_2)B\gamma}\Big\}\frac{J_1(\lambda_1\gamma)}{\lambda_1} - k_2\alpha_2\frac{J_1(\lambda_2\gamma)}{\lambda_2}\Big]\frac{J_1(\lambda_1\gamma)}{D(\gamma)\lambda_1\gamma}\, d\gamma ,
\end{aligned}\right\} \tag{16''''}$$

and l_2, m_2 and n_2 are what we get by interchanging the suffices 1, 2 each other in the formulas of l_1, m_1 and n_1, though $J_1(\lambda_1\gamma)$ and $J_1(\lambda_2\gamma)$ are to be replaced by $J_1(\lambda_2\gamma)$ and $J_1(\lambda_1\gamma)$ respectively.

Next, substituting $f_1^{(1)}$ and $f_2^{(1)}$ in place of f_1 and f_2 in the equations (4') and putting the second approximate solutions $f_1^{(1)} + f_1^{(2)}$ and $f_2^{(1)} + f_2^{(2)}$, we can obtain $f_1^{(2)}$ and $f_2^{(2)}$ from (4') algebraically, that is

$$\left.\begin{aligned}
f_1^{(2)}(\gamma) &= \frac{e^{-(k_1+k_2)B\gamma}}{D(\gamma)}\sum_{m=1}^{\infty}(2m + 1)\Big\{(k_2S_2M_1^{(m)} + k_1C_2N_1^{(m)}) \\
&\quad \times\int_0^\infty J_{2m+1}(\lambda_1\gamma')f_1^{(1)}(\gamma')\frac{d\gamma'}{\gamma'} - (k_2S_2M_2^{(m)} + k_2C_2N_2^{(m)}) \\
&\qquad \times\int_0^\infty J_{2m+1}(\lambda_2\gamma')f_2^{(1)}(\gamma')\frac{d\gamma'}{\gamma'}\Big\} , \\
f_2^{(2)}(\gamma) &= \frac{e^{-(k_1+k_2)B\gamma}}{D(\gamma)}\sum_{m=1}^{\infty}(2m + 1)\Big\{(k_1S_1M_2^{(m)} + k_2C_1N_2^{(m)})
\end{aligned}\right\} \tag{17}$$

$$
\times\int_0^\infty J_{2m+1}(\lambda_2\gamma')f_2^{(1)}(\gamma')\frac{d\gamma'}{\gamma'}-(k_1S_1M_1^{(m)}+k_1C_1N_1^{(m)})
\times\int_0^\infty J_{2m+1}(\lambda_1\gamma')f_1^{(1)}(\gamma')\frac{d\gamma'}{\gamma'}\Big\}\,.
$$

Generally, we can obtain $f_1^{(n+1)}$ and $f_2^{(n+1)}$ by the following recurrence formulas, for $n \geqq 2$;

$$
\left.\begin{aligned}
f_1^{(n+1)}(\gamma) &= \frac{1}{D(\gamma)}\sum_{m=0}^{\infty}(2m+1)\Big\{(k_2S_2M_1^{(m)}+k_1C_2N_1^{(m)})\\
&\quad\times\int_0^\infty J_{2m+1}(\lambda_1\gamma')\,f_1^{(n)}(\gamma')\,\frac{d\gamma'}{\gamma'}-(k_1S_2M_2^{(m)}+k_2C_2N_2^{(m)})\\
&\qquad\times\int_0^\infty J_{2m+1}(\lambda_2\gamma')\,f_2^{(n)}(\gamma')\frac{d\gamma'}{\gamma'}\Big\}\,,\\
f_2^{(n+1)}(\gamma) &= \frac{1}{D(\gamma)}\sum_{m=0}^{\infty}(2m+1)\Big\{(k_1S_1M_2^{(m)}+k_2C_1N_2^{(m)})\\
&\quad\times\int_0^\infty J_{2m+1}(\lambda_2\gamma')\,f_2^{(n)}(\gamma')\,\frac{d\gamma'}{\gamma'}-(k_1S_1M_1^{(m)}+k_1C_1N_1^{(m)})\\
&\qquad\times\int_0^\infty J_{2m+1}(\lambda_1\gamma')f_1^{(n)}(\gamma')\frac{d\gamma'}{\gamma'}\Big\}\,,
\end{aligned}\right\}\quad(18)
$$

where we must be careful for the difference between the summations in (16) and (17). Those of the equations for $f_1^{(2)}$ and $f_2^{(2)}$ have not the terms for $m=0$. Besides, the following notations are, through the equations (17) and (18), used;

$$
\begin{aligned}
S_1 &= \sinh k_1B\gamma\,, \qquad C_1=\cosh k_1B\gamma\,,\\
M_1^{(m)} &= \{(k_2-k_1)-(k_2+k_1)\mu_1^{4m+2}\}e^{-k_1B\gamma}J_{2m+1}(\lambda_1\gamma)\\
&\qquad +2k_1(\mu_1\mu_2)^{2m+1}e^{-k_1B\gamma}J_{2m+1}(\lambda_2\gamma)\,,\\
N_1^{(m)} &= \{(k_2-k_1)-(k_2+k_1)\mu_1^{4m+2}\}e^{-k_1B\gamma}J_{2m+1}(\lambda_1\gamma)\\
&\qquad +2k_2(\mu_1\mu_2)^{2m+1}e^{-k_1B\gamma}J_{2m+1}(\lambda_2\gamma)\,,\qquad m=0,\,1,\,2,\cdots
\end{aligned}
$$

while, for $M_2^{(m)}$ and $N_2^{(m)}$, the same formulas serve to calculation when the suffices 1 and 2 in $M_1^{(m)}$ and $N_1^{(m)}$ are replaced each other. Thus, we have $f_r(\gamma)=\sum_{n=1}^{\infty}f_r^{(n)}(\gamma)$, $h_r^{(2m+1)}$ for $r=1,2$ and consequently $F_r''(z_r)$ as well as $H_r''(z_r)$.

The convergency of the solutions $\sum_{n=1}^{\infty}f_r^{(n)}(\gamma)$ as well as the series involved in each function $f_r^{(n)}(\gamma)$ $n=1,2,\cdots$ may be studied mathematically. In the following, however, we shall be satisfied with the simple reasoning that the external force $2B\sigma_{x\infty}$ must be equal to twice the total stress $\int_b^B\sigma_x dy$ on the minimum section where $\sigma_x=\sum_{n=1}^{\infty}\sigma_x^{(n)}$, $\sigma_x^{(n)}$ being ascribed to the functions $f_r^{(n)}(\gamma)$, and when the equality can be ascertained enough to some extent with the several terms of $\sigma_x^{(1)}+\sigma_x^{(2)}+\cdots\cdots$and the higher terms are negligibly small, the summation of the finite terms gives the figure utilizable enough.

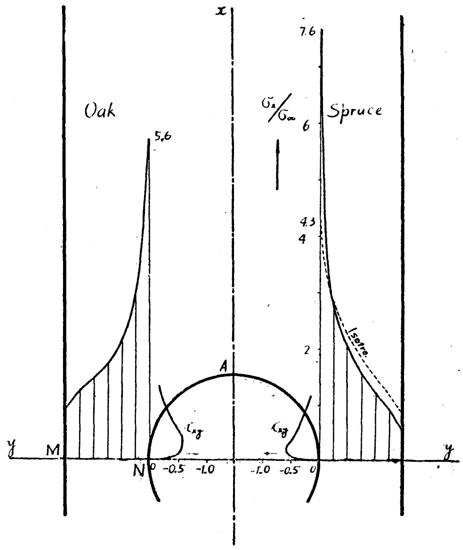

FIG. 2 a.

Numerical calculation has been worked out in the case that the strips, cut from spruce and oak, have a circular hole as $a/B = b/B = 0.5$.

Shearing stresses τ_{xy} on the hole boundary as well as normal stresses σ_x on the minimum section concentrate more strikingly than in the perforated infinite plates of the same materials.

In Fig. 2, the results of calculation are reproduced. τ_{xy} in Fig. 2a is the shearing stress along the line tangent to the hole in the direction of the external force, and σ_x the normal stress on the minimum section of the strip. Howland's result is also reproduced with a dotted line for the sake of compar-

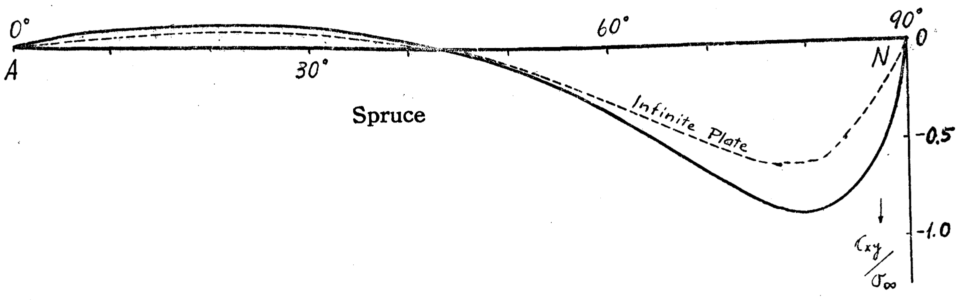

Fig. 2 b.

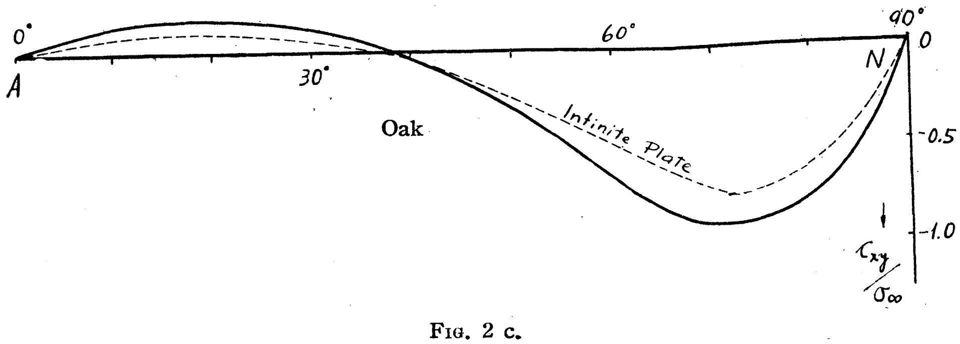

FIG. 2 c.

ison. In Fig. 2 b and c, τ_{xy} on the edge of the hole are reproduced together with Green's results which are shown with broken lines.

The Elastic Constants of Spruce and Oak
(by Hoerig)

	E kg/mm²	E' kg/mm²	G kg/mm²	ν_x	k_1	k_2
Spruce	1,704	64.5	87.0	0.562	4.112	1.249
Oak	581	98.5	78.1	0.506	2.306	1.054

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