The singular continuous spectrum of Schrödinger operators

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https://hdl.handle.net/2324/6787431

出版情報：Kyushu University, 2022, 博士（数理学）, 課程博士
バージョン：
権利関係：

Doctoral Dissertation

The singular continuous spectrum of Schrödinger operators

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February 14, 2023

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Acknowledgment

I would like to express my sincere gratitude to my supervisor Professor Fumio Hiroshima for helpful discussion and constant encouragement. I am also grateful for support and understanding to my family.

This work was supported by JST SPRING, Grant Number JPMJSP2136.

Chapter 1

Introduction

Schrödinger operators have been studied so far in the spectral analysis of mathematics. In 1932 John von Neumann [27] constructed the mathematical foundation of quantum mechanics. In quantum mechanics, an observable is expressed by a self-adjoint operator on a Hilbert space and the spectrum of the self-adjoint operator corresponds to the observed value of the observable.

In this thesis, we consider a graph Laplacian H_d on a sparse tree and continuous Schrödinger operators H_c on $L^2([0,\infty))$ of the form $H_c f = -\Delta f + V f$, where Δ is the one-dimensional Laplacian, V a multiplication operator. One of the important problems concerning Schrödinger operators is to prove their self-adjointness. In 1951, Toshio Kato [17] proved the self-adjointness of Schrödinger operators with Coulomb type potentials. Since his work, the self-adjointness of several types of Schrödinger operators has been proven. Once we prove the self-adjointness of a Schrödinger operator, we can consider its spectral properties.

We study the singular continuous spectrum of Schrödinger operators. For a self-adjoint operator T acting in a Hilbert space $\mathcal{H}$, there exists the spectral resolution E associated with T. Let $\varphi \in \mathcal{H}$ and $\varphi \neq 0$. Then we have the probability measure $\mu_\varphi : \mathcal{B}_1 \to [0,1]$ defined by

$$\mu_\varphi(B) = \frac{(\varphi, E(B)\varphi)}{\|\varphi\|^2},$$

where $\mathcal{B}_1$ is the Borel σ-field of $\mathbb{R}$ and $(\cdot,\cdot)$ denotes the inner product on $\mathcal{H}$. The closed subspaces $\mathcal{H}_{ac}, \mathcal{H}_{sc}$ and $\mathcal{H}_{pp}$ of $\mathcal{H}$ are defined by

$$\begin{aligned} \mathcal{H}_{ac} &= \{\varphi \in \mathcal{H} \mid \mu_\varphi \text{ is absolutely continuous with respect to the Lebeasgue measure on } \mathbb{R}\}, \\ \mathcal{H}_{sc} &= \{\varphi \in \mathcal{H} \mid \mu_\varphi \text{ is singular continuous with respect to the Lebeasgue measure on } \mathbb{R}\}, \\ \mathcal{H}_{pp} &= \{\varphi \in \mathcal{H} \mid \mu_\varphi \text{ is a pure point measure}\}. \end{aligned}$$

Then $\mathcal{H}$ can be decomposed as $\mathcal{H} = \mathcal{H}_{ac} \oplus \mathcal{H}_{sc} \oplus \mathcal{H}_{pp}$ and H is reduced by the closed subspaces $\mathcal{H}_{ac}, \mathcal{H}_{sc}$ and $\mathcal{H}_{pp}$. Thus we can define $\sigma_{ac}(T), \sigma_{sc}(T)$ and $\sigma_{pp}(T)$ by

$$\begin{aligned} \sigma_{ac}(T) &= \sigma(T|_{\mathcal{H}_{ac}}), \\ \sigma_{sc}(T) &= \sigma(T|_{\mathcal{H}_{sc}}), \\ \sigma_{pp}(T) &= \sigma(T|_{\mathcal{H}_{pp}}). \end{aligned}$$

It follows that $\sigma(T) = \sigma_{ac}(T) \cup \sigma_{sc}(T) \cup \sigma_{pp}(T)$. We call $\sigma_{ac}(T)$ (resp. $\sigma_{sc}(T)$ and $\sigma_{pp}(T)$) the absolutely continuous spectrum (resp. the singular continuous spectrum and the pure point spectrum) of T. We consider which Schrödinger operators have singular continuous spectrum and we investigate the spectral properties of such Schrödinger operators.

It is known that certain random Schrödinger operators have singular continuous spectrum; the critical almost Mathieu operator [14]. Simon and Stoltz [22] show that both continuous and discrete Schrödinger operators with sparse potentials have singular continuous spectrum. In this thesis, we consider both continuous and discrete Schrödinger operators with sparse potentials.

In Chapter 2, we consider a graph Laplacian H_d on a Γ-sparse tree, $\Gamma \in (0,1)$. Our aim is to estimate the Hausdorff dimension of the spectral resolution for H_d. The Hausdorff dimension is defined for both a set and a measure. Let $A \subset \mathbb{R}$, and the diameter $d(A)$ of A is defined by

$$d(A) = \sup\{|x-y| \mid x, y \in A\}.$$

Let $\delta > 0$. A family $\{A_i\}_{i=1}^{\infty}$ of subsets of $\mathbb{R}$ is called a δ-cover of A, if $A \subset \bigcup_{i=1}^{\infty} A_i$ and $\sup_{i\in\mathbb{N}} d(A_i) \leq \delta$. For $\alpha \in [0,\infty)$, $h_\delta^\alpha, h^\alpha : 2^{\mathbb{R}} \to [0,\infty]$ are defined by

$$h_\delta^\alpha(A) = \inf\left\{\sum_{i=1}^{\infty} d(A_i)^\alpha \mid \{A_i\}_{i=1}^{\infty} \text{is a } \delta\text{-cover of } A\right\},$$
$$h^\alpha(A) = \lim_{\delta\to 0} h_\delta^\alpha(A).$$

We call h^α the α-dimensional Hausdorff measure on $\mathbb{R}$. Actually, the restriction of h^α to $\mathcal{B}_1$ is a measure on $\mathbb{R}$. For $A \subset \mathbb{R}$, $\dim A$ is defined by

$$\dim A = \sup\{\alpha \mid h^\alpha(A) \neq 0\}.$$

This is called the Hausdorff dimension of A. It follows that if $0 \leq \alpha < \dim A$, then $h^\alpha(A) = \infty$, and that if $\dim A < \alpha$, then $h^\alpha(A) = 0$. Then Hausdorff dimension is also defined for measures on $\mathbb{R}$. Let $\mu : \mathcal{B}_1 \to [0,\infty]$ be a measure. We define the lower Hausdorff dimension $\dim_*\mu$ and the upper Hausdorff dimension $\dim^*\mu$ of μ by

$$\dim_*\mu = \inf\{\dim A \mid A \in \mathcal{B}_1 \text{ such that } \mu(A) \neq 0\},$$
$$\dim^*\mu = \inf\{\dim A \mid A \in \mathcal{B}_1 \text{ such that } \mu(\mathbb{R}\setminus A) = 0\}.$$

If $\alpha = \dim_*\mu = \dim^*\mu$, then μ is said to have the exact α-Hausdorff dimension.

The Hausdorff dimension of the spectral resolution for H_d can be estimated in terms of the so-called intermittency functions. We see that H_d can be identified with the direct sum $\bigoplus_{k=1}^{\infty} H^{(k)}$ of Jacobi matrices $H^{(k)}$. Let H be a Jacobi matrix acting in $l^2(\mathbb{Z})$ and $\psi \in l^2(\mathbb{Z})$ with $\|\psi\| = 1$. For $t \in \mathbb{R}$ and $n \in \mathbb{Z}$, let

$$\psi(t) = e^{-itH}\psi,$$
$$\psi(t,n) = (\delta_n, \psi(t)),$$

where $(\cdot,\cdot)$ denotes the inner product on $l^2(\mathbb{Z})$ and $\delta_n : \mathbb{Z} \to \mathbb{C}$ is defined by

$$\delta_n(m) = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

We define the time-averaging probability measure $a_\psi(n,T)$ and the time-averaging momentum $\langle |X|^p \rangle_\psi(T)$ for $\psi \in l^2(\mathbb{Z})$, $n \in \mathbb{Z}$, $T > 0$, and $p > 0$, by

$$\begin{aligned} a_\psi(n,T) =& \frac{1}{T}\int_0^\infty e^{-\frac{t}{T}}|\psi(t,n)|^2 dt, \\ \langle |X|^p \rangle_\psi(T) =& \sum_{n\in\mathbb{Z}} |n|^p a_\psi(n,T). \end{aligned}$$

Let

$$\beta_\psi(p) = \frac{1}{p}\liminf_{T\to\infty} \frac{\log\langle |X|^p \rangle_\psi(T)}{\log T}.$$

We call β_ψ an intermittency function of ψ. Let $\alpha = \dim^* \mu_\psi$. Barbaroux, Combes, and Montcho [3, Theorem 3.1] show that for $\epsilon > 0$, there exists $C = C_{\epsilon,\psi} > 0$ such that for any $T, p > 0$,

$$\langle |X|^p \rangle_\psi(T) \geq C T^{p(\alpha-\epsilon)}.$$

This inequality implies that the intermittency function gives an upper bound of α:

$$\beta_\psi(p) \geq \alpha.$$

Breuer [5] shows that

$$\begin{aligned} \sigma_{ac}(H_d) &= \emptyset, \\ \sigma_{pp}(H_d) \cap (0,4) &= \emptyset, \\ \sigma_{sc}(H_d) \cap (0,4) &= (0,4). \end{aligned}$$

Let E be the spectral resolution of H_d and $\tilde{E}$ be the restriction of E to the interval $(0,4)$. Let

$$\begin{aligned} \dim_*\tilde{E} &= \inf\left\{\dim A \mid A \in \mathcal{B}_1 \text{ such that } \tilde{E}(A) \neq P_{\{0\}}\right\}, \\ \dim^*\tilde{E} &= \inf\left\{\dim A \mid A \in \mathcal{B}_1 \text{ such that } \tilde{E}(\mathbb{R}\setminus A) = P_{\{0\}}\right\}, \end{aligned}$$

where $P_{\{0\}}$ denotes the projection of the subspace $\{0\}$.

We show the exact Hasudorff dimension of $\tilde{E}$. Breuer [5] shows that

$$\Gamma \leq \dim_*\tilde{E} \leq \dim^*\tilde{E} \leq \frac{2\Gamma}{1+\Gamma}.$$

We can identify H_d with Jacobi matrices. Under this identification, we can consider an intermittency function of a Jacobi matrix. In order to show the intermittency function exactly, we estimate the operator kernel by Hellfer-Sjöstrand formula. Then we can show that

$$\Gamma \leq \dim_* \tilde{E} \leq \dim^* \tilde{E} \leq \Gamma.$$

and we can conclude that $\tilde{E}$ has the exact Γ-Hausdorff dimension.

In Chapter 3, we consider a one-dimensional continuous Schrödiner operator H_c with a sparse potential. Since H_c is regular at zero and in the limit point case at infinity, H_c has self-adjoint extensions H_θ which is parametrized by θ. We prove the absence of embedded eigenvalues in the singular continuous spectrum for any θ. The singular continuous spectrum for a Schrödinger operator with a sparse potential is an interval for many cases [5] [15] [22]. There are, however, few assertions about the complement of the singular continuous spectrum. We focus on the edge of the singular continuous spectrum and give a sufficient condition that the edge of the singular continuous spectrum is not an eigenvalue. Hence, we can conclude that H_θ has no embedded eigenvalues for any θ. Moreover, H_θ has purely singular continuous spectrum for $\theta \in [0, \frac{\pi}{2}]$.

Chapter 2

Exact Hausdorff dimension of the spectral resolution for the graph Laplacian on a sparse tree

2.1 Introduction and results

2.1.1 Introduction

We study the graph Laplacian on a sparse tree and its Hausdorff dimension. The Hausdorff dimension is defined for sets or measures. We estimate the Hausdorff dimension of the spectral resolution for the graph Laplacian on a sparse tree. Note that the Hausdorff dimension of a measure and that of the support of the measure are different from each other in general. If the spectra are purely point spectra, then the Hausdorff dimension of the spectral resolution is zero. If the spectrum is purely absolutely continuous, then the Hausdorff dimension of the spectral resolution is one. In this chapter, we show the Hausdorff dimension of a sparse tree exactly.

This chapter is organized as follows: In the rest of Section 2.1, we give the main result. In Section 2.2, we give a decomposition of the graph Laplacian. From this, we can identify the graph Laplacian with one-dimensional discrete Schrödinger operators with a sparse potential. In the Section 2.3, we prove that the intermittency function gives the upper bound of the upper Hausdorff dimension. In Section 2.4, we prepare to estimate the intermittency function. Here, we estimate the operator kernel, by using a quadratic form theory and Helffer-Sjöstrand formula. In Section 2.5, we estimate the intermittency function and prove the main theorem.

We define a sparse tree. We say that $G = (V, E)$ is a graph, if V is a countable set and $E \subset \{e \in 2^V \mid \#e = 2\}$. An element of V (resp. E) is called the vertex (resp. the edge). Vertices $a, b \in V$ are said to be adjacent, if $\{a, b\} \in E$. We denote by $a \sim b$, if $a, b \in V$ are adjacent. Note that this definition implies that there are no egdes which are adjacent to itself. Vertices $a, b \in V$ are said to be linked, if there exist $a_i \in V$, $i = 1, 2, ..., n-1$ such that $a_i \notin \{a, b\}$, $a \sim a_1$, $a_i \sim a_{i+1}$, $i = 1, 2, ..., n-2$, and $a_{n-1} \sim b$.

Here $\{a, a_1, ..., a_{n-1}, b\} \in 2^V$ is called a path from a to b. Let the degree $\deg(a)$ of the vertex a be defined by $\deg(a) = \#\{b \in V \mid a \sim b\}$. A graph is said to be locally finite, if $\deg(a) < \infty$ for any vertex $a \in V$. A graph G is said to be connected if any vertices $a, b \in V$ are linked. We say that a graph $G = (V, E)$ is a tree, if G is connected and for any vertices $a, b \in V$, there exists a unique path from a to b. We fix a vertex o of the tree G, and o is called the root of G. A tree G with a fixed root o is called a rooted tree. Let $p(a, b) \in 2^V$ be the unique path from the vertex a to the vertex b. The metric $d(\cdot, \cdot)$ on V is defined by

$$d(a,b) = \begin{cases} 0, & a = b, \\ \#p(a,b) - 1, & a \neq b. \end{cases}$$

Let G be a rooted tree with a root o, and let $S_n = \{a \in V \mid d(o, a) = n\}$ for $n = 0, 1, ...$. We say that a rooted tree $G = (V, E)$ is a spherically homogeneous tree if any vertices in S_n have the same degree d_n. A locally finite spherically homogeneous tree G is uniquely determined by the sequence $(g_n)_{n=0}^{\infty}$,

$$g_n = \begin{cases} d_0, & n = 0, \\ d_n - 1, & n \geq 1. \end{cases}$$

Definition 2.1.1. *Let $L_n = 2^{n^n}$, $n = 1, 2, ...,$ and $\Gamma \in (0, 1)$. We say that a locally finite spherically homogeneous tree $G = (V, E)$ is a Γ- sparse tree, if for any $n \geq 0$,*

$$g_n = \begin{cases} [n^{\frac{1-\Gamma}{\Gamma}}], & n \in \{L_m \mid m \in \mathbb{N}\}, \\ 1, & n \notin \{L_m \mid m \in \mathbb{N}\}. \end{cases}$$

We define the graph Laplacian for the locally finite graph. Let $G = (V, E)$ be a locally finite graph. Let $l^2(V)$ be the set of square summable functions on V, and this is the Hilbert space with the inner product given by

$$(f, g) = \sum_{u \in V} \overline{f(u)} g(u).$$

Let $\mathcal{D} \subset l^2(V)$ be defined by $\mathcal{D} = \{f : V \to \mathbb{C} \mid \#\text{supp}(f) < \infty\}$. Let L, A, and D be operators with its domain $\mathcal{D}$, and defined by

$$\begin{aligned} Lf(u) &= \sum_{v \sim u} (f(u) - f(v)), \\ Af(u) &= \sum_{v \sim u} f(v), \\ Df(u) &= \sum_{v \sim u} f(u) = \deg(u) f(u). \end{aligned}$$

These are called graph Laplacian, adjacency matrix, and degree matrix, respectively. The graph Laplacian L is essentially self-adjoint, if the graph is connected [16, Threom3.1].

Let (X, d_X) be a metric space, and $\mathcal{B}(X)$ be the Borel σ-field of (X, d_X). Let A be a subset of X and the diameter $d_X(A)$ of A be defined by $d_X(A) = \sup\{d_X(x,y) \mid x, y \in A\}$. Let $\delta > 0$ and a family $\{A_i\}_{i=1}^\infty$ of subsets of X is called a δ-cover of A, if $A \subset \bigcup_{i=1}^\infty A_i$ and $\sup_{1\leq i<\infty} d_X(A_i) \leq \delta$.

Definition 2.1.2. *Let $\alpha \in [0,\infty)$ and $\delta > 0$. Let $h^\alpha_\delta, h^\alpha : 2^X \to [0,\infty]$ be defined by*

$$\begin{aligned} h^\alpha_\delta(A) &= \inf\left\{\sum_{i=1}^\infty d_X(A_i)^\alpha \mid \{A_i\}_{i=1}^\infty \textit{is a } \delta\text{-cover } \textit{of } A\right\}, \\ h^\alpha(A) &= \lim_{\delta\to 0} h^\alpha_\delta(A). \end{aligned}$$

We call h^α the α-dimensional Hausdorff measure of X. Let $\dim A$ be defined for $A \subset X$ by

$$\dim A = \sup\{\alpha \mid h^\alpha(A) \neq 0\}.$$

This is called the Hausdorff dimension of A. Let $\mu : \mathcal{B}(X) \to [0,\infty]$ be a measure, and let the lower Hausdorff dimension $\dim_*\mu$ and the upper Hausdorff dimension $\dim^*\mu$ of μ be defined by

$$\begin{aligned} \dim_*\mu &= \inf\{\dim A \mid A \in \mathcal{B}(X) \text{ such that } \mu(A) \neq 0\}, \\ \dim^*\mu &= \inf\{\dim A \mid A \in \mathcal{B}(X) \text{ such that } \mu(\mathbb{R}\setminus A) = 0\}. \end{aligned}$$

If $\alpha = \dim_*\mu = \dim^*\mu$, then μ is said to have the exact α-Hausdorff dimension.

Let L be the graph Laplacian of the Γ-sparse tree. The following lemma is proved by Breuer [5].

Lemma 2.1.3. *Let $H_d = \overline{L}$, and let E be the spectral resolution of H_d and $\tilde{E}$ be the restriction of E to the interval $(0,4)$, where $\overline{L}$ is the clousre of L. Then it follows that*

(1) $\sigma_{\mathrm{ac}}(H_d) = \emptyset$, $\sigma_{\mathrm{pp}}(H_d) \cap (0,4) = \emptyset$, $\sigma_{\mathrm{sc}}(H_d) \cap (0,4) = (0,4)$,

(2) $\Gamma \leq \dim_*\tilde{E} \leq \dim^*\tilde{E} \leq \frac{2\Gamma}{1+\Gamma}$.

We obtain the main theorem below.

Theorem 2.1.4. *We suppose the same assumptions as Lemma 2.1.3. Then $\Gamma = \dim_*\tilde{E} = \dim^*\tilde{E}$, and $\tilde{E}$ has the exact Γ-Hausdorff dimension,*

This theorem implies the corollary below.

Corollary 2.1.5. *For any $\Gamma \in (0,1)$, the restriction of the spectral resolution for the graph Laplacian on the Γ-sparse tree to the interval $(0,4)$ has the exact Γ-Hausdorff dimension.*

2.1.2 Preceding results

The spectral analysis of a sparse tree bears some similarities to the theory of one-dimensional discrete Schrödinger operators with a sparse potential. In Simon-Stolz [22], Schrödinger operators with a sparse potential have singular continuous spectrum. Gilbert-Pearson [13] finds a relationship between the behavior of subordinate solutions and the spectrum of one-dimensional Schrödinger operators. Jitomirskaya and Last [15] show a relationship between the Hausdorff dimension of the spectral measure and the behavior of subordinate and non-subordinate solutions. Moreover, they estimate the Hausdorff dimension of the spectral measure by calculating the L-norm of non-subordinate solutions. This subordinate solution method is also used in [5].

On the other hand, the relationship between the type of spectra and the time-averaged behavior of Schrödinger operators is also studied. RAGE theorem implies that if the initial state is singular continuous, then the time-averaged evolution goes to infinity. Barbaroux, Combes and Montcho [3] give a lower bound of the time-averaged momentum of one-dimensional discrete Schrödinger operators, by using the upper Hausdorff dimension. This also shows an inequality between the upper Hausdorff dimension and an intermittency function. It is, however, crucial to estimate the intermittency function exactly. Tcheremchantsev [25] gives the intermittency function explicitly in the case of sparse potentials. We will apply [25] to the graph Laplacian on a sparse tree.

2.2 Preliminaries

Threre are some decomposition methods for Schrödinger operators on some trees. These methods stem from Naimark and Solomyak [20]. Breuer [6], Kostenko, and Nicolussi [2] developped this method recently. They study the case of the continuum Kirchhoff Laplacian. Allard, Froese [1] and Breuer [5] study the case of the graph Laplacian. We introduce their results as Lemma 2.2.1. Their results imply that the graph Laplacian on the spherically homogeneous tree is identified with the direct sum of Jacobi matrices. Hence, it is sufficient to study Jacobi matrices instead of the graph Laplacian.

Let $G = (V, E)$ be a Γ-sparse tree determined by the sequence $\{g_n\}_{n=0}^{\infty}$ and $H_d = \overline{L}$, where $\overline{L}$ is the closure of the graph Laplacian L on G. Let $\alpha_n = \#S_n$ for $n = 0, 1, ...$ and $\alpha_{-1} = 0$. Since $\{\alpha_n\}_{n=0}^{\infty}$ is non-decreasing, there exists a unique $N(k) \in \mathbb{N} \cup \{0\}$ such that $\alpha_{N(k)-1} < k \leq \alpha_{N(k)}$ for every $k \in \mathbb{N}$. Let $k, n \in \mathbb{N}$ and let $d_k = (d_k(n))_{n=1}^{\infty}$ and $a_k = (a_k(n))_{n=1}^{\infty}$ be defined by the following: in the case of $k = 1$,

$$\begin{aligned} d_1(n) &= \begin{cases} g_0 & (n = 1), \\ g_{n-1} + 1 & (n \geq 2), \end{cases}, \\ a_1(n) &= \sqrt{g_{n-1}}. \end{aligned}$$

In the case of $k \geq 2$,

$$\begin{aligned} d_k(n) &= g_{n+N(k)-1} + 1, \\ a_k(n) &= \sqrt{g_{n+N(k)-1}}. \end{aligned}$$

By calculatig straightfowardly, we see that for any $k, n \in \mathbb{N}$

$$d_k(n) = a_k(n)^2 + 1 - \delta_1(k)\delta_1(n), \tag{2.1}$$

where $\delta_j(k) = \begin{cases} 1 & (k = j), \\ 0 & (k \neq j). \end{cases}$ Let Jacobi matrices $H^{(k)}, A^{(k)}, D^{(k)} : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by

$$H^{(k)} = \begin{pmatrix} d_k(1) & -a_k(1) & & & \\ -a_k(1) & d_k(2) & -a_k(2) & & \\ & -a_k(2) & d_k(3) & -a_k(3) & \\ & & -a_k(3) & \ddots & \ddots \\ & & & \ddots & \end{pmatrix},$$

$$A^{(k)} = \begin{pmatrix} 0 & a_k(1) & & & \\ a_k(1) & 0 & a_k(2) & & \\ & a_k(2) & 0 & a_k(3) & \\ & & a_k(3) & \ddots & \ddots \\ & & & \ddots & \end{pmatrix}, D^{(k)} = \begin{pmatrix} d_k(1) & & & \\ & d_k(2) & & \\ & & d_k(3) & \\ & & & \ddots \\ & & & \end{pmatrix}.$$

Note that $H^{(k)} = D^{(k)} - A^{(k)}$. The next lemma shows the decomposition of graph Laplacian.

Lemma 2.2.1. *H_d and $\bigoplus_{k=1}^{\infty} H^{(k)}$ are unitarily equivalent.*

Proof. See Appendix 4.1. □

2.3 Intermittency function and Hausdorff dimension

In this section, we introduce an intermittency function and give an important inequality in Lemma 2.3.2 which shows that the intermittency function is the upper bound of Hausdorff dimension.

Let $\psi \in l^2(\mathbb{N})$ and $E^{(k)}$ be the spectral resolution of $H^{(k)}$. We consider the time-averaged dynamics of $\exp(-itH^{(k)})\psi$. Let a finite measure $\mu_\psi^{(k)} : \mathcal{B}^1 \to [0, \infty]$ be defined by

$$\mu_\psi^{(k)}(A) = (\psi, E^{(k)}(A)\psi).$$

Definition 2.3.1. *Let* $\psi_k(t) = e^{-itH^{(k)}}\psi$ *and* $\psi_k(t,n) = (\delta_n, \psi_k(t))$ *for* $t \in \mathbb{R}$ *and* $n \in \mathbb{N}$. *Let* $a_\psi^{(k)}(n,T)$, $\langle |X|^p \rangle_\psi^{(k)}(T)$, *and* $\beta_\psi^{(k)}(p)$ *be defined by, for* $T > 0$ *and* $p > 0$,

$$\begin{aligned} a_\psi^{(k)}(n,T) &= \frac{1}{T}\int_0^\infty e^{-\frac{t}{T}}|\psi_k(t,n)|^2 dt, \\ \langle |X|^p \rangle_\psi^{(k)}(T) &= \sum_{n=1}^\infty n^p a_\psi^{(k)}(n,T), \\ \beta_\psi^{(k)}(p) &= \frac{1}{p}\liminf_{T\to\infty} \frac{\log\langle |X|^p\rangle_\psi^{(k)}(T)}{\log T}. \end{aligned}$$

We call $\beta_\psi^{(k)}$ the intermittency function. The closed subspace $\mathcal{H}_\psi$ of $l^2(\mathbb{N})$ is defined by

$$\mathcal{H}_\psi = \overline{\{p(H^{(k)})\psi \in l^2(\mathbb{N}) \mid p \text{ is a polynomial}\}},$$

and let $U_\psi : \mathcal{H}_\psi \to L^2(\mathbb{R}, d\mu_\psi^{(k)})$ be defined by

$$U_\psi(p(H^{(k)})\psi)(x) = p(x).$$

Lemma 2.3.2. *Let* $\alpha = \dim^*(\mu_\psi^{(k)})$ *and* $\epsilon > 0$. *Then there exists* $C_1 = C_1(\epsilon, \psi) > 0$ *such that for any* $T, p > 0$,

$$\langle |X|^p \rangle_\psi^{(k)}(T) \geq C_1 T^{p(\alpha-\epsilon)}.$$

In particular, for any $p > 0$,

$$\dim^*(\mu_\psi^{(k)}) \leq \beta_\psi^{(k)}(p).$$

Proof. We denote $\mu_\psi^{(k)}$, $a_\psi^{(k)}$, $\langle |X|^p\rangle_\psi^{(k)}$, and $\psi_k(t,n)$ by μ_ψ, a_ψ, $\langle |X|^p\rangle_\psi$, and $\psi(t,n)$ for simplicity of notation. Let $\epsilon > 0$ and $\gamma : \mathbb{R} \to \mathbb{R}$ be the local Hausdorff dimension of μ_ψ:

$$\gamma(x) = \liminf_{\delta\to 0} \frac{\log(\mu_\psi([x-\delta, x+\delta]))}{\log\delta}.$$

By [11, Chapter 10, Proposition 10.1], we see that $\mu\text{-}\operatorname{ess\,sup}_x \gamma(x) = \dim^*\mu_\psi = \alpha$. Thus there exists $S_\epsilon \in \mathcal{B}^1$ such that $\mu_\psi(S_\epsilon) > 0$ and $\gamma(x) > \alpha - \epsilon$ for $x \in S_\epsilon$. Let $\gamma_\delta(x) = \inf_{\delta'<\delta} \frac{\log\mu_\psi([x-\delta', x+\delta'])}{\log\delta'}$. By Egorov's theorem, there exists $S'_\epsilon \subset S_\epsilon$ such that $\mu_\psi(S'_\epsilon) > 0$ and γ_δ converges uniformly to γ on S'_ϵ. Let $\psi' = E(S'_\epsilon)\psi$. We see that $\|\psi'\|^2 = \mu_\psi(S'_\epsilon) > 0$ and $\mu_{\psi'}$ is uniformly $(\alpha-\epsilon)$-Hölder continuous. Let $\chi = \psi - \psi'$. Then we see that

$$\begin{aligned} &\sum_{n=1}^N a_\psi(n,T) \\ &= \frac{1}{T}\int_{\mathbb{R}} e^{-\frac{t}{T}} \sum_{n=1}^N |\psi'(t,n) + \chi(t,n)|^2 dt \\ &\leq \sum_{n=1}^N \frac{1}{T}\int_{\mathbb{R}} e^{-\frac{t}{T}}|\psi'(t,n)|^2 dt + 2\left(\sum_{n=1}^N \frac{1}{T}\int_{\mathbb{R}} e^{-\frac{t}{T}}|\psi'(t,n)|^2 dt\right)^{\frac{1}{2}} \|\chi\| + \|\chi\|^2. \quad (2.2) \end{aligned}$$

We assume that $c > 0$ and $N \in \mathbb{N}$ satisfy

$$\left(\sum_{n=1}^{N-1} \frac{1}{T} \int_{\mathbb{R}} e^{-\frac{t}{T}} |\psi'(t,n)|^2 dt \right)^{\frac{1}{2}} \leq c\|\psi'\|. \tag{2.3}$$

By (2.2) and (2.3), we see that

$$\sum_{n=1}^{N-1} a_\psi(n,T) \leq \left(c\|\psi'\| + \|\chi\|\right)^2.$$

Taking $c = -\frac{\|\chi\|}{\|\psi'\|} + \sqrt{\left(\frac{\|\chi\|}{\|\psi'\|}\right)^2 + \frac{1}{2}}$, we have

$$\sum_{n=1}^{N-1} a_\psi(n,T) \leq \frac{1}{2}\|\psi'\|^2 + \|\chi\|^2. \tag{2.4}$$

On the other hand, let $C_1' = -\frac{\|\chi\|}{\|\psi'\|} + \sqrt{\left(\frac{\|\chi\|}{\|\psi'\|}\right)^2 + \frac{1}{2}}$ and

$$N(T) = \max\left\{ N \in \mathbb{N} \mid \left(\sum_{n=1}^{N-1} \frac{1}{T} \int_{\mathbb{R}} e^{-\frac{t}{T}} |\psi'(t,n)|^2 dt \right)^{\frac{1}{2}} \leq C_1'\|\psi'\| \right\}. \tag{2.5}$$

Then (2.3) holds for $c = C_1'$ and $N = N(T)$. By (2.4), we have

$$\sum_{n=N(T)}^{\infty} a_\psi(n,T) \geq \frac{1}{2}\|\psi'\|^2. \tag{2.6}$$

Note that $\mu_{\psi'}$ is uniformly $(\alpha-\epsilon)$-Hölder continuous. Hence, by Lemma 4.2.6, there exists $\widetilde{C} = \widetilde{C}(\alpha-\epsilon, \mu_{\psi'}) > 0$ such that for any $T > 0$,

$$\sum_{n=1}^{N} \frac{1}{T} \int_{\mathbb{R}} e^{-\frac{t}{T}} |\psi'(t,n)|^2 dt = \sum_{n=1}^{N} \frac{1}{T} \int_0^{\infty} e^{-\frac{t}{T}} |\widehat{U_{\psi'}\delta_n \mu_{\psi'}}|^2 dt \leq \widetilde{C} N T^{-(\alpha-\epsilon)}. \tag{2.7}$$

By (2.5) and (2.7), we have

$$N(T) \geq \frac{(C_1'\|\psi'\|)^2}{\widetilde{C}} T^{\alpha-\epsilon}. \tag{2.8}$$

(2.6) and (2.8) show that there exists $C_1 = C_1(\epsilon, \psi) > 0$ such that for any $T > 0$,

$$\langle |X|^p \rangle_\psi(T) \geq \sum_{n=N(T)}^{\infty} n^p a_\psi(n,T) \geq \frac{1}{2}\|\psi'\|^2 N(T)^p \geq C_1 T^{p(\alpha-\epsilon)}.$$

Moreover, we see that for $\epsilon > 0$ and $T' > 1$,

$$\frac{1}{p} \inf_{T > T'} \frac{\log\langle |X|^p \rangle_\psi(T)}{\log T} \geq \alpha - \epsilon + \frac{1}{p} \inf_{T > T'} \frac{\log C_1}{\log T} = \alpha - \epsilon.$$

This implies our assertion. □

2.4 Estimates of operator kernel

In this section we prepare some lemmas to estimate the intermittency function. We estimate the operator kernel in Lemma 2.4.3 and 2.4.4 by using a quadratic form theory and Hellfer-Sjöstrand formula.

We denote $H^{(k)}$, $a_k(n)$, and $d_k(n)$ by H, $a(n)$, and $d(n)$, respectively for simplicity of notation. Let $\beta > 0$ and let $\mathscr{D} = \{f : \mathbb{N} \to \mathbb{C} \mid \#\mathrm{supp}(f) < \infty\}$. Let P, Δ, and $M_\beta : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ with its domain $\mathscr{D}$ be defined by

$$\begin{aligned} Pf(n) &= a(n)f(n+1), \\ \Delta f &= (P-I)f, \\ M_\beta f(n) &= \beta^n f(n), \end{aligned}$$

and let $T_\beta = M_\beta^{-1} T M_\beta$ for an operator $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$.

Lemma 2.4.1. *Let $f \in \mathscr{D}$. It follows that*

(1) $P^* f(n) = \begin{cases} 0 & (n=1), \\ a(n-1)f(n-1) & (n \geq 2), \end{cases}$

(2) $H^{(k)} f = \begin{cases} (\Delta\Delta^* - \delta_1)f & (k=1), \\ \Delta\Delta^* f & (k \geq 2), \end{cases}$

(3) $\Delta_\beta f = (\beta P - I)f,$

(4) $(\Delta^*)_\beta f = (\beta^{-1}P^* - I)f.$

Proof. Let $f, g \in \mathscr{D}$. Then we see that

$$(Pg, f) = \sum_{n=1}^{\infty} a(n)\overline{g(n+1)}f(n) = \sum_{n=2}^{\infty} \overline{g(n)}a(n-1)f(n-1).$$

This implies (1). We see that

$$\begin{aligned} \Delta\Delta^* f(n) &= a(n)\Delta^* f(n+1) - \Delta^* f(n) \\ &= \begin{cases} a(1)(a(1)f(1) - f(2)) + f(1) & (n=1) \\ a(n)(a(n)f(n) - f(n+1)) - a(n-1)f(n-1) + f(n) & (n \geq 2) \end{cases} \\ &= \begin{cases} -a(1)f(2) + \{a(1)^2+1\}f(1) & (n=1) \\ -a(n)f(n+1) + \{a(n)^2+1\}f(n) - a(n-1)f(n-1) & (n \geq 2). \end{cases} \end{aligned} \tag{2.9}$$

(2) follows from (2.9) and (2.1). We can prove (3) and (4) straightforwardly. □

Let $\beta > 0$ and the sesquilinear form $\mathfrak{h}_\beta : l^2(\mathbb{N}) \times l^2(\mathbb{N}) \to \mathbb{C}$ with its domain $\mathscr{D}$ be defined by

$$\mathfrak{h}_\beta(f, g) = ((\Delta_\beta)^* f, (\Delta^*)_\beta g) = (f, H_\beta\, g).$$

Lemma 2.4.2. *For any $t > 0$ and $f \in \mathscr{D}$,*

$$|\mathfrak{h}_\beta[f] - \mathfrak{h}_1[f]| \le C(\beta)\,\frac{t}{2}\,\mathfrak{h}_1[f] + C(\beta)(1 + \frac{1}{2t})\|f\|^2, \tag{2.10}$$

where $C(\beta) = |\beta - 1| + |\beta^{-1} - 1| = |\beta - \beta^{-1}|$.

Proof. Let $t > 0$ and $f \in \mathscr{D}$. Then we see that

$$\begin{aligned}
|\mathfrak{h}_\beta[f] - \mathfrak{h}_1[f]| &= |((\beta P^* - I)f, (\beta^{-1}P^* - I)f) - ((P - I)f, (P^* - I)f)| \\
&\le |\beta - 1||(f, Pf)| + |\beta^{-1} - 1||(f, P^* f)| \\
&\le |\beta - 1|\{(\Delta^* f, f) + (f, f)\} + |\beta^{-1} - 1|\{(f, \Delta^* f) + (f, f)\} \\
&\le C(\beta)\|f\|\|\Delta^* f\| + C(\beta)\|f\|^2 \\
&\le C(\beta)\left(\frac{t}{2}\|\Delta^* f\|^2 + \frac{1}{2t}\|f\|^2 + \|f\|^2\right).
\end{aligned}$$

□

Lemma 2.4.3. *Let $z \in \mathbb{C}^+ := \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$. Let $0 < \gamma < 1$, and let η_z, m_z, and α_z be*

$$\begin{aligned}
\eta_z &= \mathrm{dist}(z, \sigma(H)), \\
m_z &= \frac{\eta_z}{\sqrt{\eta_z + |z| + 1}}, \\
\alpha_z(\gamma) &= \frac{1}{4}(\gamma m_z + \sqrt{(\gamma m_z)^2 + 16}).
\end{aligned}$$

Then for any $i, j \in \mathbb{N}$,

$$|(\delta_i, (H - z)^{-1}\delta_j)| \le \alpha_z(\gamma)^{-|i-j|}\frac{1}{\eta_z}\left(\frac{1+\gamma}{1-\gamma}\right)^2.$$

Proof. It follows from $\|(H - z)^{-1}\| = \eta_z^{-1}$ that for any $t > 0$

$$2\left\|C(\beta)\left(\frac{t}{2}H + 1 + \frac{1}{2t}\right)(H - z)^{-1}\right\| \le C(\beta)\left\{\left(1 + \frac{|z|}{\eta_z}\right)t + \left(2 + \frac{1}{t}\right)\frac{1}{\eta_z}\right\}. \tag{2.11}$$

Let $\gamma \in (0, 1)$ and

$$\begin{aligned}
t_z &= \frac{1}{\sqrt{\eta_z + |z|}}, \\
\beta_z &= \frac{1}{4}(\gamma m_z + \sqrt{(\gamma m_z)^2 + 16}) > 1.
\end{aligned}$$

By the inequality of arithmetic and geometric means, we see that for any $z \in \mathbb{C}^+$,

$$
\begin{aligned}
C(\beta_z)\left\{\left(1+\frac{|z|}{\eta_z}\right)t_z+\left(2+\frac{1}{t_z}\right)\frac{1}{\eta_z}\right\} &= C(\beta_z)\left\{\left(1+\frac{|z|}{\eta_z}\right)t_z+\frac{1}{\eta_z}\frac{1}{t_z}+\frac{2}{\eta_z}\right\} \\
&= 2\frac{\sqrt{\eta_z+|z|}+1}{\eta_z}C(\beta_z) \\
&= \frac{2}{m_z}\left(\beta_z-\frac{1}{\beta_z}\right) \\
&= \gamma. \qquad (2.12)
\end{aligned}
$$

(2.11) and (2.12) imply that for any $z \in \mathbb{C}^+$,

$$
2\left\|C(\beta_z)\left(\frac{t_z}{2}H+1+\frac{1}{2t_z}\right)(H-z)^{-1}\right\| \le \gamma \qquad (2.13)
$$

By (2.10), (2.13) and Lemma 4.3.2, there exists the m-sectorial operator H_{β_z} associated with $\mathfrak{h}_{\beta_z}$ and for any $z \in \mathbb{C}^+$,

$$
\|(H_{\beta_z}-z)^{-1}-(H-z)^{-1}\| \le \frac{4\gamma}{(1-\gamma)^2}\|(H-z)^{-1}\|.
$$

Therefore we see that

$$
\|(H_{\beta_z}-z)^{-1}\| \le \frac{1}{\eta_z}\left(\frac{1+\gamma}{1-\gamma}\right)^2.
$$

Let $i, j \in \mathbb{N}$ with $i < j$. Then we see that

$$
\begin{aligned}
|(\delta_i,(H-z)^{-1}\delta_j)| &= |(M_{\beta_z}\delta_i,(H_{\beta_z}-z)^{-1}M_{\beta_z}^{-1}\delta_j)| \\
&\le {\beta_z}^{i-j}\|(H_\beta-z)^{-1}\| \\
&\le {\beta_z}^{-|i-j|}\frac{1}{\eta_z}\left(\frac{1+\gamma}{1-\gamma}\right)^2. \qquad (2.14)
\end{aligned}
$$

This implies our assertion in the case of $i < j$. In the case of $i \ge j$, let

$$
\beta_z = \frac{1}{4}(-\gamma m_z+\sqrt{(\gamma m_z)^2+16}).
$$

Then we can prove (2.14) similarly. □

Let $f \in C^n(\mathbb{R})$, and the norm $|||\cdot|||_n$ on $C^n(\mathbb{R})$ be defined by

$$
|||f|||_n = \sum_{r=0}^{n}\int_{\mathbb{R}}|f^{(r)}(x)|\langle x\rangle^{r-1}dx.
$$

The next lemma is used to estimate the intermittency function.

Lemma 2.4.4. *Suppose that* $f \in C^{2k+3}(\mathbb{R})$ *and* $|||f|||_{2k+3} < \infty$. *Then there exists* $C_2 = C_2(k) > 0$ *such that for any* $i, j \in \mathbb{N}$,

$$|(\delta_i, f(H)\delta_j)| \leq C_2 |||f|||_{2k+3} \langle i - j \rangle^{-k},$$

where $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$.

Proof. Let $n \geq 0$ and $\tau \in C_0^\infty(\mathbb{R})$ such that $\tau(x) = \begin{cases} 1 & (|x| \leq 1) \\ 0 & (|x| \geq 2) \end{cases}$. By Helffer-Sjöstrand formula [8, 2.2 The Helffer-Sjöstrand formula], we see that

$$f(H) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \overline{z}}(z)(H - z)^{-1} dxdy,$$
$$\tilde{f}(z) = \left\{ \sum_{r=0}^{n} f^{(r)}(x) \frac{(iy)^r}{r!} \right\} \tau \left(\frac{y}{\langle x \rangle} \right).$$

We see that

$$\begin{aligned} \left| \frac{\partial \tilde{f}}{\partial \overline{z}}(z) \right| &\leq \frac{1}{2} \left| f^{(n+1)}(x) \frac{(iy)^n}{n!} \tau \left(\frac{y}{\langle x \rangle} \right) \right| \\ &+ \frac{1}{2} \left| \sum_{r=0}^{n} f^{(r)}(x) \frac{(iy)^r}{r! \langle x \rangle} \right| \left| \left(1 + xy\langle x \rangle^{-1}\right) \tau' \left(\frac{y}{\langle x \rangle} \right) \right|. \end{aligned}$$

Let A, $B \subset \mathbb{R}^2$ be

$$A = \left\{ (x, y) \in \mathbb{R}^2 \mid \left| \frac{y}{\langle x \rangle} \right| \leq 2 \right\}, B = \left\{ (x, y) \in \mathbb{R}^2 \mid 1 \leq \left| \frac{y}{\langle x \rangle} \right| \leq 2 \right\}.$$

Then we see that there exists $C_2' = C_2'(\tau) > 0$ such that

$$\left| \frac{\partial \tilde{f}}{\partial \overline{z}}(z) \right| \leq \frac{1}{2} \left| f^{(n+1)}(x) \frac{(iy)^n}{n!} \right| \mathbb{1}_A(x, y) + C_2' \left| \sum_{r=0}^{n} f^{(r)}(x) \frac{(iy)^r}{r! \langle x \rangle} \right| \mathbb{1}_B(x, y).$$

Let $i, j \in \mathbb{N}$. Then we have

$$\begin{aligned} |(\delta_i, f(H)\delta_j)| &\leq \frac{1}{2\pi} \int_{\mathbb{C}} \left| f^{(n+1)}(x) \frac{(iy)^n}{n!} \right| |(\delta_i, (H - z)^{-1}\delta_j)| \mathbb{1}_A(x, y) dxdy \\ &+ \frac{C_2'}{\pi} \int_{\mathbb{C}} \left| \sum_{r=0}^{n} f^{(r)}(x) \frac{(iy)^r}{r! \langle x \rangle} \right| |(\delta_i, (H - z)^{-1}\delta_j)| \mathbb{1}_B(x, y) dxdy. \quad (2.15) \end{aligned}$$

Let $\gamma_z = \dfrac{1}{\sqrt{\eta_z + |z|} + 1} < 1$ and $\alpha_z = \alpha_z(\gamma_z)$. By Lemma 2.4.3, we see that

$$\int_{\mathbb{C}} \left| \sum_{r=0}^{n} f^{(r)}(x) \frac{(iy)^r}{r!\langle x \rangle} \right| |(\delta_i, (H-z)^{-1}\delta_j)| \mathbb{1}_B(x,y) dxdy$$
$$\leq \sum_{r=0}^{n} \int_{\mathbb{C}} \left| f^{(r)}(x) \frac{(iy)^r}{r!\langle x \rangle} \right| \alpha_z^{-|i-j|} \frac{1}{\eta_z} \left(\frac{1+\gamma_z}{1-\gamma_z} \right)^2 \mathbb{1}_B(x,y) dxdy. \tag{2.16}$$

We estimate the lower bound of α_z. Suppose that $(x,y) \in B$, then $1 \leq |y| \leq \eta_z$ and $|z| \leq \sqrt{2}|y|$. Therefore

$$\gamma_z m_z = \frac{\eta_z}{\eta_z + |z| + 1 + 2\sqrt{\eta_z + |z|}} \geq \frac{1}{2 + \sqrt{2} + 2\sqrt{1+\sqrt{2}}} \tag{2.17}$$

Let $b = \dfrac{1}{2 + \sqrt{2} + 2\sqrt{1+\sqrt{2}}}$. By the definition of α_z, (2.17) implies that

$$\alpha_z \geq \frac{1}{4}\left(b + \sqrt{b^2 + 16} \right) > 1. \tag{2.18}$$

Let $B = \dfrac{1}{4}\left(b + \sqrt{b^2+16} \right)$. We see that

$$\frac{1+\gamma_z}{1-\gamma_z} = 1 + \frac{2}{\sqrt{\eta_z + |z|}} \leq 1 + \sqrt{2}. \tag{2.19}$$

By (2.16), (2.18), and (2.19), we see that

$$\int_{\mathbb{C}} \left| \sum_{r=0}^{n} f^{(r)}(x) \frac{(iy)^r}{r!\langle x \rangle} \right| |(\delta_i, (H-z)^{-1}\delta_j)| \mathbb{1}_B(x,y) dxdy$$
$$\leq \left(1+\sqrt{2}\right)^2 B^{-|i-j|} \sum_{r=0}^{n} \frac{1}{r!} \int_{\mathbb{C}} |f^{(r)}(x)| \left| \frac{y^{r-1}}{\langle x \rangle} \right| \mathbb{1}_B(x,y) dxdy$$
$$\leq \left(1+\sqrt{2}\right)^2 B^{-|i-j|} \sum_{r=0}^{n} \frac{2^{r-1}}{r!} \int_{\mathbb{C}} |f^{(r)}(x)| \langle x \rangle^{r-2} \mathbb{1}_B(x,y) dxdy$$
$$\leq \left(1+\sqrt{2}\right)^2 B^{-|i-j|} \sum_{r=0}^{n} \int_{\mathbb{R}} |f^{(r)}(x)| \langle x \rangle^{r-1} dx. \tag{2.20}$$

By Lemma 2.4.3, we see that

$$\int_{\mathbb{C}} \left| f^{(n+1)}(x) \frac{(iy)^n}{n!} \right| |(\delta_i, (H-z)^{-1}\delta_j)| \mathbb{1}_A(x,y) dxdy$$
$$\leq \int_{\mathbb{C}} \left| f^{(n+1)}(x) \frac{(iy)^n}{n!} \right| \alpha_z^{-|i-j|} \frac{1}{\eta_z} \left(\frac{1+\gamma_z}{1-\gamma_z} \right)^2 \mathbb{1}_A(x,y) dxdy. \tag{2.21}$$

Note that for any $k \in \mathbb{Z}_{\geq 0}$ and $t > 0$,

$$e^{-t} \leq \frac{e^{-k} k^k}{t^k}.$$

This implies that, for $i, j \in \mathbb{N}$ with $i \neq j$,

$$\alpha_z^{-|i-j|} \leq \left(1 + \left(\frac{\gamma_z m_z}{4}\right)^2\right)^{-\frac{|i-j|}{2}} \leq \frac{e^{-k}(2k)^k}{|i-j|^k \left(\log\left(1 + \left(\frac{\gamma_z m_z}{4}\right)^2\right)\right)^k}. \tag{2.22}$$

Suppose that $(x, y) \in A$, then $|y| \leq 2\langle x \rangle$ and $|z| \leq \sqrt{5}\langle x \rangle$. We see that

$$\gamma_z m_z \geq \frac{1}{2} \frac{\eta_z}{\eta_z + |z| + 1} \geq \frac{1}{2} \frac{|y|}{|y| + |z| + 1} \geq \frac{3 - \sqrt{5}}{8} \frac{|y|}{\langle x \rangle}. \tag{2.23}$$

(2.22) and (2.23) imply

$$\alpha_z^{-|i-j|} \leq \frac{e^{-k}(2k)^k}{|i-j|^k \left(\log\left(1 + \left(\frac{3-\sqrt{5}}{32} \frac{|y|}{\langle x \rangle}\right)^2\right)\right)^k}. \tag{2.24}$$

We see that

$$\frac{1 + \gamma_z}{1 - \gamma_z} \leq 1 + \sqrt{\frac{2}{|y|}}. \tag{2.25}$$

By (2.21), (2.24), and (2.25), we see that

$$\begin{aligned}
&\int_{\mathbb{C}} \left| f^{(n+1)}(x) \frac{(iy)^n}{n!} \right| |(\delta_i, (H-z)^{-1}\delta_j)| \mathbb{1}_A(x, y) dx dy \\
&\leq \frac{2e^{-k}(2k)^k}{n!|i-j|^k} \int_{\mathbb{C}} \frac{|f^{(n+1)}(x)||y|^{n-1}}{\left(\log\left(1 + \left(\frac{3-\sqrt{5}}{32} \frac{|y|}{\langle x \rangle}\right)^2\right)\right)^k} \left(1 + \frac{2}{|y|}\right) \mathbb{1}_A(x, y) dx dy \\
&\leq \frac{8e^{-k}(2k)^k}{n!|i-j|^k} \int_{\mathbb{R}} dx |f^{(n+1)}(x)| \int_0^{2\langle x \rangle} dy \frac{|y|^{n-1} + |y|^{n-2}}{\left(\log\left(1 + \left(\frac{3-\sqrt{5}}{32} \frac{|y|}{\langle x \rangle}\right)^2\right)\right)^k} \\
&\leq \frac{8e^{-k}(2k)^k}{n!|i-j|^k} \int_{\mathbb{R}} dx |f^{(n+1)}(x)| \langle x \rangle^n \int_0^2 dt \frac{t^{n-1} + t^{n-2}}{\left(\log\left(1 + \left(\frac{3-\sqrt{5}}{32} t\right)^2\right)\right)^k}.
\end{aligned} \tag{2.26}$$

If $n > 2k + 1$,

$$C_2''(n) := \int_0^2 dt \frac{t^{n-1} + t^{n-2}}{\left(\log\left(1 + \left(\frac{3-\sqrt{5}}{32} t\right)^2\right)\right)^k} < \infty. \tag{2.27}$$

Let $n = 2k+2$. Then (2.15), (2.20), (2.26), and (2.27) imply that there exists $C_2 = C_2(k) > 0$ such that for any $i, j \in \mathbb{N}$,

$$
\begin{aligned}
&|(\delta_i, f(H)\delta_j)| \\
&\leq \frac{C_2'}{\pi}\left(1+\sqrt{2}\right)^2 B^{-|i-j|}|||f|||_{2k+2} + \frac{1}{2\pi}C_2''(2k+2)\frac{8e^{-k}(2k)^k}{(2k+2)!|i-j|^k}|||f|||_{2k+3} \\
&\leq C_2|||f|||_{2k+3}\langle i-j\rangle^{-k}.
\end{aligned}
$$

This implies our assertion. □

2.5 Intermittency function and proof of the main result

In this section, we consider the distribution of $a_\psi^{(k)}(n,T)$ and estimate the lower and upper bounds the momentum $\langle |X|^p\rangle_\psi^{(k)}(T)$. From this, we calculate the intermittency function exactly. Finally, we prove Theorem 2.1.4 by using the intermittency function.

2.5.1 Lower bound of intermittency function

Let $k \in \mathbb{N}$, $\psi \in l^2(\mathbb{N})$, and $T > 0$. We define for $S \in 2^{\mathbb{N}}$,

$$
P_\psi^{(k)}(S,T) = \sum_{n\in S} a_\psi^{(k)}(n,T).
$$

For $M \geq N \geq 1$, let subsets $\{N \sim M\}$ and $\{M \sim \infty\}$ of $\mathbb{N}$ be

$$
\begin{aligned}
\{N \sim M\} &= \{n \in \mathbb{N} \mid N \leq n \leq M\}, \\
\{M \sim \infty\} &= \{n \in \mathbb{N} \mid n \geq M\}.
\end{aligned}
$$

Lemma 2.5.1. *Let $T > 0$ and $\epsilon > 0$. Suppose that $B \in \mathcal{B}^1$ and $A := \mu_\psi^{(k)}(B) > 0$. Let*

$$
\begin{aligned}
M_T &= \frac{A^2}{16J_\psi^{(k)}(T^{-1},B)}, \\
J_\psi^{(k)}(\epsilon,B) &= \int_B \mu_\psi^{(k)}(dx)\int_{\mathbb{R}}\mu_\psi^{(k)}(dy)\frac{\epsilon^2}{(x-y)^2+\epsilon^2}.
\end{aligned}
$$

Then for any $T > 0$

$$
P_\psi^{(k)}(\{M_T \sim \infty\},T) \geq \frac{A}{2} > 0.
$$

Proof. We denote $P_\psi^{(k)}$, $\mu_\psi^{(k)}$, and $J_\psi^{(k)}$ by P_ψ, μ_ψ, and J_ψ, respectively for simplicity of notation. Let $\rho = E^{(k)}(B)\psi$ and $\chi = \psi - \rho$. Note that $\rho \neq 0$. We see that

$$\begin{aligned} P_\psi(\{1\sim M\},T) &= \sum_{n=1}^{M}\frac{1}{T}\int_{\mathbb{R}} dt\, e^{-\frac{t}{T}}|\chi(n,t)+\rho(n,t)|^2 \\ &= P_\chi(\{1\sim M\},T)+P_\rho(\{1\sim M\},T)+2\sum_{n=1}^{M}\frac{1}{T}\int_{\mathbb{R}} dt\, e^{-\frac{t}{T}}\mathrm{Re}(\chi(n,t)\overline{\rho(n,t)}) \\ &= P_\chi(\{1\sim M\},T)-P_\rho(\{1\sim M\},T)+2\sum_{n=1}^{M}\frac{1}{T}\int_{\mathbb{R}} dt\, e^{-\frac{t}{T}}\mathrm{Re}(\psi(n,t)\overline{\rho(n,t)}) \\ &\leq P_\chi(\{1\sim M\},T)+2\sum_{n=1}^{M}\frac{1}{T}\int_{\mathbb{R}} dt\, e^{-\frac{t}{T}}\mathrm{Re}(\psi(n,t)\overline{\rho(n,t)}). \end{aligned} \tag{2.28}$$

Since $P_\psi(\{M\sim\infty\},T) = \|\psi\|^2 - P_\psi(\{1, M-1\},T)$ and $\|\psi\|^2 = \|\rho\|^2 + \|\chi\|^2$, (2.28) implies that

$$P_\psi(\{M\sim\infty\},T) \geq \|\rho\|^2 - 2|D(M-1,T)|, \tag{2.29}$$

where

$$D(M,T) = \sum_{n=1}^{M}\frac{1}{T}\int_{\mathbb{R}} dt\, e^{-\frac{t}{T}}\psi(n,t)\overline{\rho(n,t)} = \sum_{n=1}^{M}\frac{1}{T}\int_{\mathbb{R}} dt\, e^{-\frac{t}{T}}(\delta_n,\psi(t))(\rho(t),\delta_n).$$

Since $U_\psi : \mathcal{H}_\psi \to L^2(\mathbb{R}, d\mu_\psi^{(k)})$ is unitary, by Schwarz inequality we see that

$$\begin{aligned} &|D(M,T)| \\ &= \left|\sum_{n=1}^{M}\frac{1}{T}\int_{\mathbb{R}} dt\, e^{-\frac{t}{T}}\int_{\mathbb{R}}\mu_\psi(dx)\int_B \mu_\psi(dy) e^{-it(x-y)}\overline{U_\psi\delta_n(x)}U_\psi\delta_n(y)\right| \\ &= \left|\int_{\mathbb{R}}\mu_\psi(dx)\int_B\mu_\psi(dy)\,\frac{1-iT(x-y)}{1+T^2(x-y)^2}\,\sum_{n=1}^{M}\overline{U_\psi\delta_n(x)}U_\psi\delta_n(y)\right| \\ &\leq \int_{\mathbb{R}}\mu_\psi(dx)\int_B\mu_\psi(dy)\,\frac{1}{\sqrt{1+T^2(x-y)^2}}\left|\sum_{n=1}^{M}\overline{U_\psi\delta_n(x)}U_\psi\delta_n(y)\right| \\ &\leq \int_B\mu_\psi(dy)\left(\int_{\mathbb{R}}\frac{\mu_\psi(dx)}{1+T^2(x-y)^2}\right)^{\frac12}\left(\int_{\mathbb{R}}\mu_\psi(dx)\left|\sum_{n=1}^{M}\overline{U_\psi\delta_n(x)}U_\psi\delta_n(y)\right|^2\right)^{\frac12}. \\ &\leq \left(\int_B\mu_\psi(dy)\int_{\mathbb{R}}\frac{\mu_\psi(dx)}{1+T^2(x-y)^2}\right)^{\frac12}\left(\int_B\mu_\psi(dy)\int_{\mathbb{R}}\mu_\psi(dx)\left|\sum_{n=1}^{M}\overline{U_\psi\delta_n(x)}U_\psi\delta_n(y)\right|^2\right)^{\frac12} \end{aligned} \tag{2.30}$$

Since $U_\psi : \mathcal{H}_\psi \to L^2(\mathbb{R}, d\mu_\psi^{(k)})$ is unitary, we have

$$\begin{aligned}
\int_{\mathbb{R}} \mu_\psi(dx) \left| \sum_{n=1}^{M} U_\psi \delta_n(x) \overline{U_\psi \delta_n(y)} \right|^2 &= \int_{\mathbb{R}} \mu_\psi(dx) \left| U_\psi \left(\sum_{n=1}^{M} \overline{U_\psi \delta_n(y)} \delta_n \right)(x) \right|^2 \\
&= \left\| \sum_{n=1}^{M} \overline{U_\psi \delta_n(y)} \delta_n \right\|_{l^2(\mathbb{N})}^2 \\
&= \sum_{n=1}^{M} |U_\psi \delta_n(y)|^2 . \qquad (2.31)
\end{aligned}$$

By (2.30) and (2.31), we see that

$$\begin{aligned}
|D(M-1,T)|^2 &\le \left(\int_B \mu_\psi(dy) \int_{\mathbb{R}} \frac{\mu_\psi(dx)}{1 + T^2(x-y)^2} \right) \left(\sum_{n=1}^{M} \int_B \mu_\psi(dy) \, |U_\psi \delta_n(y)|^2 \right) \\
&\le J_\psi(T^{-1}, B) \sum_{n=1}^{M} \|U_\psi \delta_n\|_{l^2}^2 \\
&\le M J_\psi(T^{-1}, B)
\end{aligned}$$

Let $M = M_T$. Then

$$|D(M_T - 1, T)| \le \sqrt{M_T J_\psi(T^{-1}, B)} = \frac{\|\rho\|^2}{4}. \qquad (2.32)$$

By (2.29) and (2.32), we obtain that

$$P_\psi(\{M_T \sim \infty\}, T) \ge \frac{\|\rho\|^2}{2} = \frac{A}{2}.$$

This implies our assertion. □

For $\psi \in l^2(\mathbb{N})$, let an analytic function $m_\psi^{(k)} : \mathbb{C}^+ \to \mathbb{C}^+$ be defined by

$$m_\psi^{(k)}(z) = \int_{\mathbb{R}} \frac{\mu_\psi^{(k)}(d\lambda)}{\lambda - z} = (\psi, (H^{(k)} - z)^{-1} \psi).$$

Let $\epsilon > 0$ and $B \in \mathcal{B}^1$. We define

$$I_\psi^{(k)}(\epsilon, B) = \epsilon \int_B dE |\mathrm{Im}\, m_\psi^{(k)}(E + i\epsilon)|^2.$$

Lemma 2.5.2. *Let $B = [a,b] \subset \mathbb{R}$. Then there exists $C_3 = C_3(a,b) > 0$ such that for any $\epsilon \in (0,1)$*

$$J_\psi^{(k)}(\epsilon, B) \le C_3 I_\psi^{(k)}(\epsilon, B). \qquad (2.33)$$

Proof. We denote $J_\psi^{(k)}$, $I_\psi^{(k)}$, and $\mu_\psi^{(k)}$ by J_ψ, I_ψ, and μ_ψ, respectively for simplicity of notation. We see that

$$\begin{aligned} I_\psi(\epsilon, B) &= \epsilon^3 \int_B dE \left(\int_{\mathbb{R}} \frac{\mu_\psi(dx)}{\epsilon^2 + (E-x)^2} \right)^2 \\ &= \epsilon^3 \int_B dE \int_{\mathbb{R}} \frac{\mu_\psi(dx)}{\epsilon^2 + (E-x)^2} \int_{\mathbb{R}} \frac{\mu_\psi(dy)}{\epsilon^2 + (E-y)^2} \\ &\geq \int_B \mu_\psi(dx) \int_{\mathbb{R}} \mu_\psi(dy)\, \epsilon^3 \int_B \frac{dE}{(\epsilon^2 + (E-x)^2)(\epsilon^2 + (E-y)^2)}. \end{aligned} \tag{2.34}$$

Let $s = \frac{x-y}{\epsilon}$. Since $x \in B = [a, b]$ and $0 < \epsilon < 1$, we have

$$\begin{aligned} \epsilon^3 \int_B \frac{dE}{(\epsilon^2 + (E-x)^2)(\epsilon^2 + (E-y)^2)} &= \int_{\frac{a-x}{\epsilon}}^{\frac{b-x}{\epsilon}} \frac{dt}{(1+t^2)(1+(t+s)^2)} \\ &\geq \int_{a-x}^{b-x} \frac{dt}{(1+t^2)(1+(|t|+|s|)^2)}. \end{aligned}$$

If $|s| \leq 1$, then

$$\int_{a-x}^{b-x} \frac{dt}{(1+t^2)(1+(|t|+|s|)^2)} \geq \int_{a-x}^{b-x} \frac{dt}{(1+t^2)(1+(|t|+1)^2)}. \tag{2.35}$$

If $|s| \geq 1$, then there exists $C_3' = C_3'(a, b) > 0$ such that for any $x \in B$,

$$\begin{aligned} \int_{a-x}^{b-x} \frac{dt}{(1+t^2)(1+(|t|+|s|)^2)} &= \int_{a-x}^{b-x} dt \left(\frac{1}{1+t^2} - \frac{1}{1+(|t|+|s|)^2} \right) ((|t|+|s|)^2 - t^2)^{-1} \\ &\geq \int_{a-x}^{b-x} dt \left(\frac{1}{1+t^2} - \frac{1}{1+(|t|+1)^2} \right) \frac{C_3'}{1+s^2} \end{aligned} \tag{2.36}$$

By (2.35) and (2.36), there exists $C_3 = C_3(a, b) > 0$ such that for any $x \in B$ and any $y \in \mathbb{R}$,

$$\epsilon^3 \int_B \frac{dE}{(\epsilon^2 + (E-x)^2)(\epsilon^2 + (E-y)^2)} \geq \frac{C_3}{1+s^2}, \qquad s = \frac{x-y}{\epsilon}. \tag{2.37}$$

(2.34) and (2.37) imply our assertion. □

Definition 2.5.3. *Let* $f : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ *and* $n \in \mathbb{N}$. *Let* $\left(\tilde{H}^{(k)} f\right)(n)$ *be defined by*

$$\left(\tilde{H}^{(k)} f\right)(n) = -a_k(n) f(n+1) + d_k(n) f(n) - a_k(n-1) f(n-1),$$

where $a_k(0) = 1$.

Let $z \in \mathbb{C}^+$, and $n, m \in \mathbb{N}$ such that $n \geq m$. We define

$$\begin{aligned} T_z(n) &= \begin{cases} \begin{pmatrix} 0 & 1 \\ -\sqrt{\frac{g_{n-1}}{g_n}} & \frac{g_n+1-z}{\sqrt{g_n}} \end{pmatrix} & (n \geq 1), \\ \begin{pmatrix} 0 & 1 \\ -1 & 1-z \end{pmatrix} & (n = 0), \end{cases} \\ S_z(n,m) &= T_z(n)T_z(n-1)\cdots T_z(m), \\ S_z(n) &= S_z(n,0). \end{aligned}$$

Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ and $z \in \mathbb{C}^+$. Suppose that $\left(\tilde{H}^{(k)} f\right)(n) = zf(n)$ for each $n \in \mathbb{N}$. Then

$$\begin{aligned} \begin{pmatrix} f(n) \\ f(n+1) \end{pmatrix} &= T_z(n+N(k)-1) \begin{pmatrix} f(n-1) \\ f(n) \end{pmatrix} \\ &= S_z(n+N(k)-1, N(k)) \begin{pmatrix} f(0) \\ f(1) \end{pmatrix}. \end{aligned}$$

Lemma 2.5.4. *Let $K > 0$ and $z = E + i\epsilon$ with $0 < E < 4$ and $\epsilon > 0$. Then there exists $C_4 = C_4(E, K) > 0$ such that*

(1) *if $L_m + 1 \leq n < L_{m+1}$ and $n\epsilon < K$, then*

$$\|S_z(n)^{-1}\| \leq C_4^{m+1} \prod_{j=1}^{m} L_j^{\frac{1-\Gamma}{2\Gamma}},$$

(2) *if $n \leq L_m$ and $n\epsilon < K$, then*

$$\|S_z(n)^{-1}\| \leq C_4^{m} \prod_{j=1}^{m-1} L_j^{\frac{1-\Gamma}{2\Gamma}}.$$

Proof. By Definition 2.1.1, we see that

$$T_z(n) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -[n^{\frac{1-\Gamma}{\Gamma}}]^{-\frac{1}{2}} & \frac{[n^{\frac{1-\Gamma}{\Gamma}}]+1-z}{[n^{\frac{1-\Gamma}{\Gamma}}]^{\frac{1}{2}}} \end{pmatrix} & (n \in \{L_m \mid m \in \mathbb{N}\}), \\ \begin{pmatrix} 0 & 1 \\ -[n^{\frac{1-\Gamma}{\Gamma}}]^{\frac{1}{2}} & 2-z \end{pmatrix} & (n \in \{L_m + 1 \mid m \in \mathbb{N}\}), \\ \begin{pmatrix} 0 & 1 \\ -1 & 2-z \end{pmatrix} & (\text{otherwise}). \end{cases}$$

If $L_m+1\le n<L_{m+1}$, then

$$\begin{aligned} S_z(n) &= R^{n-L_m-1}S(L_m+1)\\ &= R^{n-L_m-1}S(L_m+1,L_m)S(L_m-1)\\ &= R^{n-L_m-1}S(L_m+1,L_m)R^{L_m-L_{m-1}-2}S(L_{m-1}+1)\\ &= \cdots\\ &= R^{n-L_m-1}S(L_m+1,L_m)R^{L_m-L_{m-1}-2}\cdots S(L_1+1,L_1)R^2 \end{aligned}$$

where $R=\begin{pmatrix} 0 & 1\\ -1 & 2-z\end{pmatrix}$. Let $L_0=-2$. Then we see that

$$\|S_z(n)^{-1}\|\le\|R^{-n+L_m+1}\|\|T_z(0)\|\prod_{j=1}^{m}\|S(L_j+1,L_j)^{-1}\|\|R^{-L_j+L_{j-1}+2}\|. \tag{2.38}$$

Note that $R^{-1}=\begin{pmatrix} 2-z & -1\\ 1 & 0\end{pmatrix}$ and $\|R^m\|=\|R^{-m}\|$ for any $m\in\mathbb{N}$.

$$R=\begin{pmatrix} 0 & 1\\ -1 & 2-E\end{pmatrix}-i\begin{pmatrix} 0 & 0\\ 0 & \epsilon\end{pmatrix}.$$

Since $E\in(0,4)$, there exist invertible matrix A_E and $\lambda_\pm\in\mathbb{C}$ with $|\lambda_\pm|=1$ such that

$$A_E^{-1}\begin{pmatrix} 0 & 1\\ -1 & 2-E\end{pmatrix}A_E=\begin{pmatrix} \lambda_+ & 0\\ 0 & \lambda_-\end{pmatrix}.$$

Therefore we see that

$$\begin{aligned} \|A_E^{-1}R^nA_E\| &\le \left\|\begin{pmatrix} \lambda_+ & 0\\ 0 & \lambda_-\end{pmatrix}-i\epsilon A_E^{-1}\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}A_E\right\|^n\\ &\le \left(1+\epsilon\|A_E^{-1}\|\|A_E\|\right)^n. \end{aligned}$$

If $\epsilon<\frac{K}{n}$, then

$$\|A_E^{-1}R^nA_E\|\le 2\exp(K\|A_E^{-1}\|\|A_E\|).$$

Therefore we obtain that for $\epsilon<\frac{K}{n}$,

$$\|R^n\|=\|A_EA_E^{-1}R^nA_EA_E^{-1}\|\le 2\|A_E\|\|A_E^{-1}\|\exp(K\|A_E^{-1}\|\|A_E\|). \tag{2.39}$$

If $0<\epsilon<1$, then there exists $C_4'>0$ such that

$$\|S_z(L_j+1,L_j)^{-1}\|\le C_4'[L_j^{\frac{1-\Gamma}{\Gamma}}]^{\frac{1}{2}}\le C_4'L_j^{\frac{1-\Gamma}{2\Gamma}}. \tag{2.40}$$

By (2.38), (2.39) and (2.40), we see that for $L_m+1\le n<L_{m+1}$,

$$\|S_z(n)^{-1}\|\le(2C_4'\|A_E\|\|A_E^{-1}\|)^{m+1}\exp((m+1)K\|A_E^{-1}\|\|A_E\|)\prod_{j=1}^{m}L_j^{\frac{1-\Gamma}{2\Gamma}}.$$

This implies our first part of the assertion. The second part of the assertion can be proved similarly. □

Lemma 2.5.5. *Let $\psi \in l^2(\mathbb{N})$ and $n \in \mathbb{N}$. Then for any $T > 0$,*

$$\frac{1}{T}\int_0^\infty e^{-\frac{t}{T}}|\psi_k(t,n)|^2\,dt = \frac{\epsilon}{\pi}\int_{\mathbb{R}} |(H^{(k)} - (E+i\epsilon))^{-1}\psi(n)|^2\,dE,$$

where $\epsilon = \frac{1}{2T}$.

Proof. Abbreviate $H^{(k)}$ and ψ_k to H and ψ, respectively. We see that

$$\begin{aligned}\frac{1}{T}\int_0^\infty e^{-\frac{t}{T}}|\psi(t,n)|^2\,dt &= \frac{1}{T}\int_0^\infty dt e^{-\frac{t}{T}}(\delta_n, e^{-itH}\psi)(e^{-itH}\psi,\delta_n)\\ &= \int_{\mathbb{R}}(\delta_n, E(dx)\psi)\int_{\mathbb{R}}(E(dy)\psi,\delta_n)\frac{1}{T}\int_0^\infty dt e^{-\frac{t}{T}-it(x-y)}\\ &= \int_{\mathbb{R}}(\delta_n, E(dx)\psi)\int_{\mathbb{R}}(E(dy)\psi,\delta_n)(1+iT(x-y))^{-1},\end{aligned}$$

and

$$\begin{aligned}&\frac{\epsilon}{\pi}\int_{\mathbb{R}}|(H^{(k)}-(E+i\epsilon))^{-1}\psi(n)|^2\,dE\\ &= \int_{\mathbb{R}}(\delta_n, E(dx)\psi)\int_{\mathbb{R}}(E(dy)\psi,\delta_n)\frac{\epsilon}{\pi}\int_{\mathbb{R}}(E-x+i\epsilon)^{-1}(E-y-i\epsilon)^{-1}\\ &= \int_{\mathbb{R}}(\delta_n, E(dx)\psi)\int_{\mathbb{R}}(E(dy)\psi,\delta_n)(1+iT(x-y))^{-1}.\end{aligned}$$

These imply our assertion. □

Definition 2.5.6. *Let $f:\mathbb{R}\to\mathbb{C}$ be measurable and $B_\nu = [\nu, 4-\nu]$ with $0<\nu<1$. We say that f is the first kind, if there exist $\nu>0$ and $x_0\in B_\nu$ such that $f\in C_0^\infty(B_\nu)$ and $f(x_0)\neq 0$, and we say that f is the second kind, if f is bounded and there exist $E_0\in(0,4)$ and $\nu>0$ with $[E_0-\nu, E_0+\nu]\subset B_\nu$ such that $f\in C^\infty([E_0-\nu,E_0+\nu])$ and $|f(x)|\geq c>0$ for $x\in[E_0-\nu,E_0+\nu]$.*

Lemma 2.5.7. *Let $f:\mathbb{R}\to\mathbb{C}$ be the second kind and $\psi = f(H^{(k)})\delta_1$. Let N be sufficiently large. Then there exists $C_5 = C_5(\nu) > 1$ such that*

(1) *if $L_N \leq T \leq \dfrac{L_{N+1}}{4}$, then*

$$P_\psi^{(k)}(\{T\sim\infty\},T) \geq C_5^{-(N+1)}T\left(\frac{1}{T}+I_{\delta_1}^{(k)}(T^{-1},B_\nu)\right)\prod_{j=1}^N L_j^{\frac{\Gamma-1}{\Gamma}}, \tag{2.41}$$

(2) *if $\dfrac{L_N}{4}\leq T\leq L_N$, then*

$$P_\psi^{(k)}(\{T\sim\infty\},T) \geq C_5^{-(N+1)}L_N\left(\frac{1}{T}+I_{\delta_1}^{(k)}(T^{-1},B_\nu)\right)\prod_{j=1}^N L_j^{\frac{\Gamma-1}{\Gamma}}, \tag{2.42}$$

(3) *if* $\dfrac{L_N}{4} \leq T$, *then*

$$P_\psi^{(k)}(\{\frac{L_N}{4} \sim \frac{L_N}{2}\}, T) \geq C_5^{-N} L_N \left(\frac{1}{T} + I_{\delta_1}^{(k)}(T^{-1}, B_\nu) \right) \prod_{j=1}^{N-1} L_j^{\frac{1-\Gamma}{\Gamma}}. \tag{2.43}$$

Proof. Firstly, we prove the lemma in the case of $\psi = \delta_1$. Let $z \in \mathbb{C}^+$ and $f_k : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ be

$$f_k(n) = \begin{cases} (H^{(k)} - z)^{-1}\delta_1(n) & (n \in \mathbb{N}), \\ 1 & (n = 0). \end{cases}$$

Let $g = (H^{(k)} - z)^{-1}\delta_1 \in l^2(\mathbb{N})$. We see that $g(n) = f_k(n)$ for each $n \in \mathbb{N}$ and that

$$\begin{aligned} &(H^{(k)} - z)g(n) = \delta_1(n) \\ \Leftrightarrow &\begin{cases} -a_k(n)g(n+1) + d_k(n)g(n) - a_k(n-1)g(n-1) - zg(n) = 0 & (n \geq 2) \\ -a_k(1)g(2) + d_k(1)g(1) - zg(1) = 1 & (n = 1) \end{cases} \\ \Leftrightarrow &\begin{cases} -a_k(n)f_k(n+1) + d_k(n)f_k(n) - a_k(n-1)f_k(n-1) = zf_k(n) & (n \geq 2) \\ -a_k(1)f_k(2) + d_k(1)f_k(1) - 1 = zf_k(1) & (n = 1) \end{cases} \\ \Leftrightarrow &\begin{cases} -a_k(n)f_k(n+1) + d_k(n)f_k(n) - a_k(n-1)f_k(n-1) = zf_k(n) & (n \geq 2) \\ -a_k(1)f_k(2) + d_k(1)f_k(1) - a_k(0)f_k(0) = zf_k(1) & (n = 1) \end{cases}. \end{aligned}$$

This implies that f_k satisfies the equation $\left(\tilde{H}^{(k)} f_k\right)(n) = zf_k(n)$ for each $n \in \mathbb{N}$. We see that

$$\begin{pmatrix} f_k(n) \\ f_k(n+1) \end{pmatrix} = S_z(n + N(k) - 1, N(k)) \begin{pmatrix} f_k(0) \\ f_k(1) \end{pmatrix}. \tag{2.44}$$

Note that $f_k(1) = m_{\delta_1}^{(k)}(z)$. Let $z = E + i\epsilon$. By (2.44), we obtain that

$$|f_k(n)|^2 + |f_k(n+1)|^2 \geq \frac{1 + |m_{\delta_1}^{(k)}(E + i\epsilon)|^2}{\|S_z(n + N(k) - 1, N(k))^{-1}\|^2}.$$

Suppose that $L_N + 1 \leq n + N(k) \leq L_{N+1}$ and $\epsilon < \dfrac{K}{n}$. By Lemma 2.5.4, we see that

$$|f_k(n)|^2 + |f_k(n+1)|^2 \geq C_4^{-(N+1)} \prod_{j=1}^{N} L_j^{\frac{\Gamma-1}{\Gamma}} \left(1 + |\mathrm{Im}\, m_{\delta_1}^{(k)}(E + i\epsilon)|^2\right). \tag{2.45}$$

By Lemma 2.5.5, we see that $f_k(n) = (H^{(k)} - z)^{-1}\delta_1(n)$ and that

$$\frac{\epsilon}{\pi} \int_{\mathbb{R}} |(H^{(k)} - (E + i\epsilon))^{-1}\delta_1(n)|^2 \, dE = \frac{1}{T} \int_0^\infty e^{-\frac{t}{T}} |\delta_{1k}(t, n)|^2 \, dt = a_{\delta_1}^{(k)}(n, T), \qquad \epsilon = (2T)^{-1}.$$

Let $(2T)^{-1} < \dfrac{K}{n}$. By (2.45), we see that there exist C_4 and $C_4' > 0$ such that

$$\begin{aligned}
a_{\delta_1}^{(k)}(n,T) + a_{\delta_1}^{(k)}(n+1,T) &\geq C_4^{-(N+1)} \prod_{j=1}^{N} L_j^{\frac{\Gamma-1}{\Gamma}} \frac{1}{2T} \int_{B_\nu} dE \left(1 + |\mathrm{Im}\, m_{\delta_1}^{(k)}(E + i(2T)^{-1})|^2\right) \\
&\geq C_4^{-(N+1)} \prod_{j=1}^{N} L_j^{\frac{\Gamma-1}{\Gamma}} \left(\frac{1}{2T} + I_{\delta_1}^{(k)}((2T)^{-1}, B_\nu)\right) \\
&\geq C_4'^{-(N+1)} \prod_{j=1}^{N} L_j^{\frac{\Gamma-1}{\Gamma}} \left(\frac{1}{T} + I_{\delta_1}^{(k)}(T^{-1}, B_\nu)\right).
\end{aligned}$$

Let $L_N < T < \dfrac{L_{N+1}}{4}$, and K sufficiently large. Then we have

$$\sum_{T \leq n \leq 2T} a_\psi^{(k)}(n,T) + a_\psi^{(k)}(n+1,T) \geq C_4'^{-(N+1)} \prod_{j=1}^{N} L_j^{\frac{\Gamma-1}{\Gamma}} T \left(\frac{1}{T} + I_{\delta_1}^{(k)}(T^{-1}, B_\nu)\right),$$

and hence

$$P_\psi^{(k)}(\{T \sim 2T\}, T) \geq C_4'^{-(N+1)} \prod_{j=1}^{N} L_j^{\frac{\Gamma-1}{\Gamma}} T \left(\frac{1}{T} + I_{\delta_1}^{(k)}(T^{-1}, B_\nu)\right). \tag{2.46}$$

Let $\dfrac{L_N}{4} < T < L_N$. Then we see that

$$P_\psi^{(k)}(\{2L_N \sim 3L_N\}, T) \geq C_4'^{-(N+1)} \prod_{j=1}^{N} L_j^{\frac{\Gamma-1}{\Gamma}} L_N \left(\frac{1}{T} + I_{\delta_1}^{(k)}(T^{-1}, B_\nu)\right).$$

Let $\dfrac{L_N}{4} \leq T$. Then we see that

$$P_\psi^{(k)}(\{\frac{L_N}{4} \sim \frac{L_N}{2}\}, T) \geq C_4'^{-N} \prod_{j=1}^{N-1} L_j^{\frac{\Gamma-1}{\Gamma}} L_N \left(\frac{1}{T} + I_{\delta_1}^{(k)}(T^{-1}, B_\nu)\right).$$

Therefore, we can prove the lemma in the case of $\psi = \delta_1$.

Next we take $g \in C_0^\infty([0,4])$ such that $g(x) = 1$ on $B_{\frac{\nu}{2}}$. We prove the lemma in the case of $\psi = g(H^{(k)})\delta_1$. Let $\chi = \delta_1 - \psi$ and $z \in \mathbb{C}^+$. Then

$$|(H^{(k)} - z)^{-1}\psi(n)|^2 \geq \frac{1}{2}|(H^{(k)} - z)^{-1}\delta_1(n)|^2 - |(H^{(k)} - z)^{-1}\chi(n)|^2.$$

Let $L_N < T < \dfrac{L_{N+1}}{4}$. Then we see that, by (2.46),

$$
\begin{aligned}
P_\psi^{(k)}(\{T\sim\infty\},T) \;\geq\; & \frac{1}{2} C_4'^{-(N+1)} \prod_{j=1}^{N} L_j^{\frac{\Gamma-1}{\Gamma}} T \left(\frac{1}{T} + I_{\delta_1}^{(k)}(T^{-1},B_\nu) \right) \\
& - \frac{1}{T}\int_{B_\nu} dE \sum_{T\leq n\leq 2T} |(H^{(k)}-(E+i\epsilon))^{-1}\chi(n)|^2. \qquad (2.47)
\end{aligned}
$$

Let $f_z(x) = \dfrac{1-g(x)}{x-z}$. Then $(H^{(k)}-z)^{-1}\chi(n) = f_z(H^{(k)})\delta_1(n)$ and Lemma 2.4.4 implies that for $l>1$,

$$
|(H^{(k)}-z)^{-1}\chi(n)| = |f_z(H^{(k)})\delta_1(n)| \;\leq\; C_2 |||f_z|||_{2l+3} n^{-l},
$$

and that

$$
\sum_{T\leq n\leq 2T} |(H^{(k)}-z)^{-1}\chi(n)| = \sum_{T\leq n\leq 2T} |f_z(H^{(k)})\delta_1(n)| \;\leq\; C_2 |||f_z|||_{2l+3} T^{-(l-1)}. \quad (2.48)
$$

Let $z = E+i\epsilon$. Note that there exists $C_5' = C_5'(g,\nu,l) > 0$ such that $\sup_{E\in B_\nu, 0<\epsilon<1} |||f_z|||_{2l+3} \leq C_5'$. By (2.47) and (2.48), we obtain

$$
\begin{aligned}
P_\psi^{(k)}(n\geq T,T) \;\geq\; & \frac{1}{2} C_4'^{-(N+1)} \prod_{j=1}^{N} L_j^{\frac{\Gamma-1}{\Gamma}} T \left(\frac{1}{T} + I_{\delta_1}^{(k)}(T^{-1},B_\nu) \right) - \frac{4}{T} C_2 C_5' T^{-(l-1)} \\
\geq\; & \left\{ \frac{1}{2} C_4'^{-(N+1)} - 4C_2C_5' T^{-l} \prod_{j=1}^{N} L_j^{\frac{1-\Gamma}{\Gamma}} \right\} \prod_{j=1}^{N} L_j^{\frac{\Gamma-1}{\Gamma}} T \left(\frac{1}{T} + I_{\delta_1}^{(k)}(T^{-1},B_\nu) \right).
\end{aligned}
$$

Let l sufficiently large, then we can prove (2.41) in the case of $\psi = g(H^{(k)})$. We can prove (2.42) and (2.43) in the case of $\psi = g(H^{(k)})$ similarly.

Finally, let f be the second kind, and we prove the lemma in the case of $\psi = f(H^{(k)})$. Let ν satisfy $f \in C^\infty([E_0-\nu, E_0+\nu])$ and $|f(x)| \geq c > 0$ for $x \in [E_0-\nu, E_0+\nu]$. We take $g \in C_0^\infty([E_0-\nu, E_0+\nu])$ such that $g(x) = 1$ on $[E_0 - \frac{3\nu}{4}, E_0 + \frac{3\nu}{4}]$. Then there exists $h \in C_0^\infty([E_0-\nu, E_0+\nu])$ such that $g(x) = h(x)f(x)$. Since $\sup_n \sum_{m=1}^\infty \langle n-m\rangle^{-l} < \infty$, by

Lemma 2.4.4, we see that

$$\begin{aligned}
&|(H^{(k)}-z)^{-1}g(H^{(k)})\delta_1(n)|^2 \\
&= |(h(H^{(k)})\delta_n, (H^{(k)}-z)^{-1}f(H^{(k)})\delta_1)|^2 \\
&= \left|\sum_{m=1}^{\infty}(h(H^{(k)})\delta_n, \delta_m)(\delta_m, (H^{(k)}-z)^{-1}f(H^{(k)})\delta_1)\right|^2 \\
&\leq C_2\left|\sum_{m=1}^{\infty}\langle n-m\rangle^{-l}|(\delta_m, (H^{(k)}-z)^{-1}f(H^{(k)})\delta_1)|\right|^2 \\
&\leq C_2\left(\sum_{m=1}^{\infty}\langle n-m\rangle^{-l}\right)\left(\sum_{m=1}^{\infty}\langle n-m\rangle^{-l}|(H^{(k)}-z)^{-1}f(H^{(k)})\delta_1(m)|^2\right) \\
&\leq C_2'\sum_{m=1}^{\infty}\langle n-m\rangle^{-l}|(H^{(k)}-z)^{-1}f(H^{(k)})\delta_1(m)|^2.
\end{aligned}$$

This implies the inequality

$$\begin{aligned}
A(2L,T) &:= \epsilon\sum_{n\geq 2L}\int_{B_\nu}dE\,|(H^{(k)}-z)^{-1}g(H^{(k)})\delta_1(n)|^2 \\
&= C_2'\epsilon\sum_{m=1}^{\infty}\sum_{n\geq 2L}\langle n-m\rangle^{-l}\int_{B_\nu}dE\,|(H^{(k)}-z)^{-1}f(H^{(k)})\delta_1(m)|^2 \\
&\leq \epsilon\sum_{m=1}^{\infty}h_l(m,L)\int_{B_\nu}dE\,|(H^{(k)}-z)^{-1}f(H^{(k)})\delta_1(m)|^2, \qquad (2.49)
\end{aligned}$$

where $z = E + i\epsilon$, $\epsilon = \dfrac{1}{2T}$ and $h_l(m,L) = \displaystyle\sum_{n\geq 2L}\frac{C_2'}{1+|n-m|^l}$. It follows for $\phi \in l^2(\mathbb{N})$ and $\epsilon > 0$ that

$$\epsilon\sum_{n=1}^{\infty}\int_{\mathbb{R}}dE|(H^{(k)}-z)^{-1}\phi(n)|^2 = \pi\|\phi\|^2, \qquad z = E+i\epsilon.$$

There exists $C_2'' > \max\{C_2', \sup_{m\geq T}h_l(m,T)\}$. By (2.49), we obtain that

$$\begin{aligned}
A(2T,T) &\leq \epsilon\sum_{m<T}h_l(m,T)\int_{B_\nu}dE\,|(H^{(k)}-z)^{-1}f(H^{(k)})\delta_1(m)|^2 \\
&+ \epsilon\sum_{m\geq T}h_l(m,T)\int_{B_\nu}dE\,|(H^{(k)}-z)^{-1}f(H^{(k)})\delta_1(m)|^2 \\
&\leq \pi\|f(H^{(k)})\delta_1\|^2C_2''T^{1-l} + C_2''\epsilon\sum_{m\geq T}\int_{B_\nu}dE\,|(H^{(k)}-z)^{-1}f(H^{(k)})\delta_1(m)|^2 \\
&= \pi\|f(H^{(k)})\delta_1\|^2C_2''T^{1-l} + C_2''P_\psi^{(k)}(\{T\sim\infty\},T), \qquad (2.50)
\end{aligned}$$

where $z = E + i\epsilon$ and $\epsilon = \dfrac{1}{2T}$. Let $L_N < T < \dfrac{L_{N+1}}{4}$. Then the previous argument shows that

$$A(2T,T) = \pi P^{(k)}_{g(H^{(k)})\delta_1}(\{2T \sim \infty\},T) \geq C_5^{-(N+1)} \prod_{j=1}^{N} L_j^{\frac{\Gamma-1}{\Gamma}} T \left(\frac{1}{T} + I^{(k)}_{\delta_1}(T^{-1},B_\nu) \right). \quad (2.51)$$

We take l sufficiently large. Then (2.50) and (2.51) imply (2.41) in the case of $\psi = f(H^{(k)})$. We can also prove (2.42) and (2.43) similarly. □

Lemma 2.5.8. *Let f be the first kind with $\sup_x |f(x)| \leq 1$ and $\psi = f(H^{(k)})\delta_1$. If N is sufficient large, then there exist $C_6 > 0$ and $q_N \in \mathbb{R}$ such that $\lim_{N\to\infty} q_N = 0$ and it follows for $\dfrac{L_N}{4} < T < \dfrac{L_{N+1}}{4}$ that*

$$\begin{aligned} \langle |X|^p \rangle^{(k)}_\psi(T) &\geq C_6 I^{(k)}_{\delta_1}(T^{-1},B_\nu)^{-p} \\ &+ C_6 \left(L_N^{p+1+q_N} + T^{p+1} L_N^{\frac{\Gamma-1}{\Gamma}+q_N} \right) I^{(k)}_{\delta_1}(T^{-1},B_\nu). \end{aligned}$$

Proof. Let $M \in \mathbb{N}$. Then it follows that $\langle |X|^p \rangle^{(k)}_\psi(T) \geq M^p P^{(k)}_\psi(\{M \sim \infty\},T)$. Lemma 2.5.1 implies that

$$\langle |X|^p \rangle^{(k)}_\psi(T) \geq M_T^p P^{(k)}_\psi(\{M_T \sim \infty\},T) \geq C_6' J^{(k)}_\psi(T^{-1},B_\nu)^{-p} \geq C_6' J^{(k)}_{\delta_1}(T^{-1},B_\nu)^{-p}.$$

By (2.33), we have

$$\langle |X|^p \rangle^{(k)}_\psi(T) \geq C_6'' I^{(k)}_{\delta_1}(T^{-1},B_\nu)^{-p}. \quad (2.52)$$

Note that f is the second kind. For $\frac{L_N}{4} \leq T \leq L_N$, by (2.42) we have

$$\begin{aligned} \langle |X|^p \rangle^{(k)}_\psi(T) &\geq T^p P^{(k)}_\psi(\{T \sim \infty\},T) \\ &\geq 4C_5^{-(N+1)} T^{p+1} I^{(k)}_{\delta_1}(T^{-1},B_\nu) \prod_{j=1}^{N} L_j^{\frac{\Gamma-1}{\Gamma}}. \end{aligned}$$

Let $q_N > 0$ satisfy

$$L_N^{q_N} = C_5^{-(N+1)} \prod_{j=1}^{N-1} L_j^{\frac{\Gamma-1}{\Gamma}}.$$

Then $\lim_{N\to\infty} q_N = 0$ and we see that for $\frac{L_N}{4} \leq T \leq L_N$,

$$\langle |X|^p \rangle^{(k)}_\psi(T) \geq 4T^{p+1} L_N^{\frac{\Gamma-1}{\Gamma}+q_N} I^{(k)}_{\delta_1}(T^{-1},B_\nu). \quad (2.53)$$

For $L_N \le T \le \dfrac{L_{N+1}}{4}$, by (2.41) we have

$$\begin{aligned}
\langle |X|^p \rangle_\psi^{(k)}(T) &\ge T^p P_\psi^{(k)}(\{T \sim \infty\}, T) \\
&\ge C_5^{-(N+1)} T^{p+1} I_{\delta_1}^{(k)}(T^{-1}, B_\nu) \prod_{j=1}^{N} L_j^{\frac{\Gamma-1}{\Gamma}} \\
&\ge T^{p+1} L_N^{\frac{\Gamma-1}{\Gamma}+q_N} I_{\delta_1}^{(k)}(T^{-1}, B_\nu). \qquad (2.54)
\end{aligned}$$

By (2.53) and (2.54), we see that for $\dfrac{L_N}{4} \le T \le \dfrac{L_{N+1}}{4}$,

$$\langle |X|^p \rangle_\psi^{(k)}(T) \ge T^{p+1} L_N^{\frac{\Gamma-1}{\Gamma}+q_N} I_{\delta_1}^{(k)}(T^{-1}, B_\nu). \qquad (2.55)$$

For $\dfrac{L_N}{4} \le T$, by (2.43) we see that

$$\begin{aligned}
\langle |X|^p \rangle_\psi^{(k)}(T) &\ge T^p P_\psi^{(k)}(\{T \sim \infty\}, T) \\
&\ge C_5^{-N} L_N^{p+1} I_{\delta_1}^{(k)}(T^{-1}, B_\nu) \prod_{j=1}^{N-1} L_j^{\frac{\Gamma-1}{\Gamma}} \\
&\ge L_N^{p+1+q_N} I_{\delta_1}^{(k)}(T^{-1}, B_\nu). \qquad (2.56)
\end{aligned}$$

(2.52), (2.55), and (2.56) imply our assertion. □

Lemma 2.5.9. *Let f be the first kind with $\sup_x |f(x)| \le 1$ and $\psi = f(H^{(k)})\delta_1$. Then*

$$\beta_\psi^{(k)}(p) \ge \frac{p+1}{p+\frac{1}{\Gamma}}. \qquad (2.57)$$

Proof. By Lemma 2.5.8, for $x = I_{\delta_1}^{(k)}(T^{-1}, B_\nu)$, we obtain that

$$\langle |X|^p \rangle_\psi^{(k)}(T) \ge C_6 x^{-p} + C_6 \left(L_N^{p+1-q_N} + T^{p+1} L_N^{\frac{\Gamma-1}{\Gamma}-q_N} \right) x.$$

Let $f(x) = x^{-p} + Kx$. Then $\inf\limits_{x>0} f(x) = c(p) K^{\frac{p}{p+1}}$, where $c(p) = p^{-\frac{p}{p+1}} + p^{\frac{1}{p+1}}$. Let $\dfrac{L_N}{4} \le T \le \dfrac{L_{N+1}}{4}$. Then there exists $C_6' = C_6'(p) > 0$ such that

$$\begin{aligned}
\langle |X|^p \rangle_\psi^{(k)}(T) &\ge c(p) C_6 \left(L_N^{p+1-q_N} + T^{p+1} L_N^{\frac{\Gamma-1}{\Gamma}-q_N} \right)^{\frac{p}{p+1}} \\
&\ge C_6' L_N^{-\frac{p}{p+1} q_N} \left(L_N^{p+1} + T^{p+1} L_N^{\frac{\Gamma-1}{\Gamma}} \right)^{\frac{p}{p+1}}.
\end{aligned}$$

For $\frac{L_N}{4} \le T \le L_N^A$ with $A = \frac{p+\frac{1}{\Gamma}}{p+1}$, we have

$$\langle |X|^p \rangle_\psi^{(k)}(T) \ge C_6' L_N^{-\frac{p}{p+1}q_N} L_N^p \ge C_6' L_N^{-\frac{p}{p+1}q_N} T^{\frac{p}{A}}. \tag{2.58}$$

For $L_N^A \le T \le \frac{L_{N+1}}{4}$, we have

$$\langle |X|^p \rangle_\psi^{(k)}(T) \ge C_6' L_N^{-\frac{p}{p+1}q_N} T^p L_N^{\frac{\Gamma-1}{\Gamma}\frac{p}{p+1}} \ge C_6' L_N^{-\frac{p}{p+1}q_N} T^{\frac{p}{A}}. \tag{2.59}$$

(2.58) and (2.59) imply that for sufficiently large $T > 0$ and any $\epsilon > 0$,

$$\langle |X|^p \rangle_\psi^{(k)}(T) \ge C_6' T^{\frac{p}{A}-\epsilon}.$$

Therefore we obtain that

$$\beta_\psi^{(k)}(p) = \frac{1}{p} \liminf_{T\to\infty} \frac{\log\langle |X|^p \rangle_\psi^{(k)}(T)}{\log T} \ge \frac{p+1}{p+\frac{1}{\Gamma}} - \frac{\epsilon}{p}.$$

This implies our assertion. □

2.5.2 Upper bound of intermittency function

Lemma 2.5.10. *Let f be the first kind, $\psi = f(H^{(k)})\delta_1$, and $p > 0$. Then there exists $C_7 = C_7(p) > 0$ such that for $L_N \le T \le L_N^{\frac{1}{\Gamma}}$ with N sufficiently large,*

$$\sum_{n\ge 2L_N} n^p a_\psi^{(k)}(n,T) \le C_7 T^{p+1} L_N^{-\frac{1}{\Gamma}}. \tag{2.60}$$

Proof. We have

$$\sum_{n\ge 2L_m} n^p a_\psi^{(k)}(n,T) = \sum_{n=2L_m}^{T^3} n^p a_\psi^{(k)}(n,T) + \sum_{n>T^3} n^p a_\psi^{(k)}(n,T).$$

Let $G_t(x) = e^{-itx}$. Lemma 2.4.4 shows that for any $l > 1$, there exists $C_7^{(4)} = C_7^{(4)}(l) > 0$ such that

$$\begin{aligned}
\sum_{n>T^3} n^p a_\psi^{(k)} &= \sum_{n>T^3} n^p \frac{1}{T}\int_{\mathbb{R}} dt\, e^{-\frac{t}{T}} |(\delta_n, G_t(H^{(k)}) f(H^{(k)})\delta_1)|^2 \\
&\le \sum_{n>T^3} n^p \frac{C_2}{T}\int_{\mathbb{R}} dt\, e^{-\frac{t}{T}} |||G_t f|||_{2l+3} n^{-l} \\
&\le \sum_{n>T^3} n^p \frac{C_7^{(4)}}{T}\int_{\mathbb{R}} dt\, e^{-\frac{t}{T}}\ t^{2l+3} n^{-l} \\
&\le C_7^{(4)} T^{2l+3} \sum_{n>T^3} n^{p-l} \\
&\le C_7^{(4)} T^{-l+3p+6}.
\end{aligned}$$

We take l large enough, so it is sufficient to prove $\sum_{n=2L_N}^{T^3} n^p a_\psi^{(k)}(n,T) \le C_7 T^{p+1} L_N^{-\frac{1}{\Gamma}}$. Since f is the first kind, $f \in C_0^\infty(B_\nu)$. By Lemma 2.5.5, we have

$$\begin{aligned}\sum_{n=2L_N}^{T^3} n^p a_\psi^{(k)}(n,T) &= \sum_{n=2L_N}^{T^3} n^p \frac{\epsilon}{\pi}\int_{B_{\frac{\nu}{2}}} dE\ |(H^{(k)}-E-i\epsilon)^{-1}\psi(n)|^2 \\ &+ \sum_{n=2L_N}^{T^3} n^p \frac{\epsilon}{\pi}\int_{\mathbb{R}\setminus B_{\frac{\nu}{2}}} dE\ |(H^{(k)}-E-i\epsilon)^{-1}\psi(n)|^2, \qquad \epsilon=(2T)^{-1}.\end{aligned}$$

Let $\chi_z(x) = (x-z)^{-1}$. Then Lemma 2.4.4 shows that for any $l>0$, there exists $C_7^{(3)} = C_7^{(3)}(l) > 0$ such that

$$\begin{aligned}\frac{\epsilon}{\pi}\sum_{n=2L_N}^{T^3} n^p \int_{-\infty}^{\frac{\nu}{2}} dE\ |(H^{(k)}-E-i\epsilon)\psi(n)|^2 &\le \frac{\epsilon}{\pi}\sum_{n=2L_N}^{T^3} n^p \int_{-\infty}^{\frac{\nu}{2}} dE\ |(\delta_n, \chi_{E+i\epsilon}(H^{(k)})f(H^{(k)})\delta_1)|^2 \\ &\le C_2\frac{\epsilon}{\pi}\sum_{n=2L_N}^{T^3} n^p \int_{-\infty}^{\frac{\nu}{2}} dE\ |||\chi_{E+i\epsilon}f|||_{2l+3}^2 n^{-2l} \\ &\le C_2\frac{\epsilon}{\pi}\sum_{n=2L_N}^{T^3} n^{p-2l} \int_{-\infty}^{\frac{\nu}{2}} dE\ \frac{C_7^{(3)}}{(E-\nu)^2+\epsilon^2} \\ &\le C_2 C_7^{(3)} L_N^{-2l+p+1}. \qquad (2.61)\end{aligned}$$

Similarly, there exists $D_7^{(3)} = D_7^{(3)}(l) > 0$ such that

$$\frac{\epsilon}{\pi}\sum_{n=2L_N}^{T^3} n^p \int_{4-\frac{\nu}{2}}^{\infty} dE\ |(H^{(k)}-E-i\epsilon)^{-1}\psi(n)|^2 \le C_2 D_7^{(3)} L_N^{-2l+p+1}. \qquad (2.62)$$

(2.61) and (2.62) imply that

$$\sum_{n=2L_N}^{T^3} n^p \frac{\epsilon}{\pi}\int_{\mathbb{R}\setminus B_{\frac{\nu}{2}}} dE\ |(H^{(k)}-E-i\epsilon)^{-1}\psi(n)|^2 \le \max\{C_2C_7^{(3)}, C_2D_7^{(3)}\} L_N^{-2l+p+1}.$$

We take l large enough, so it is sufficient to prove

$$\frac{\epsilon}{\pi}\sum_{n=2L_N}^{T^3} n^p \int_{B_{\frac{\nu}{2}}} dE\ |(H^{(k)}-E-i\epsilon)^{-1}\psi(n)|^2 \le C_7 T^{p+1} L_N^{-\frac{1}{\Gamma}}, \qquad \epsilon=(2T)^{-1}.$$

Lemma 2.4.4 implies that there exists $C = C(l,f) > 0$ such that

$$|(H^{(k)}-E-i\epsilon)^{-1}\psi(n)|^2 \le C\sum_{m=1}^{\infty}(1+|n-m|^2)^{-l}|\chi_{E+i\epsilon}(H^{(k)})\delta_1(m)|^2.$$

Therefore it is sufficient to prove

$$\frac{\epsilon}{\pi}\sum_{n=2L_N}^{T^3} n^p \int_{B_{\frac{\nu}{2}}} dE \sum_{m=1}^{\infty}(1+|n-m|^2)^{-l}|\chi_{E+i\epsilon}(H^{(k)})\delta_1(m)|^2 \le C_7 T^{p+1} L_N^{-\frac{1}{\Gamma}}, \qquad \epsilon=(2T)^{-1}.$$

We see that for any $l>1$, by Lemma 2.5.5, there exists $C_7^{(2)} = C_7^{(2)}(l) > 0$ such that

$$\begin{aligned}
&\frac{\epsilon}{\pi}\sum_{n=2L_N}^{T^3} n^p \int_{B_{\frac{\nu}{2}}} dE \sum_{m=1}^{L_N}(1+|n-m|^2)^{-l}|\chi_{E+i\epsilon}(H^{(k)})\delta_1(m)|^2 \\
&\le \sum_{n=2L_N}^{T^3} n^p L_N^{-2l} \sum_{m=1}^{L_N} \frac{\epsilon}{\pi}\int_{B_{\frac{\nu}{2}}} dE\, |\chi_{E+i\epsilon}(H^{(k)})\delta_1(m)|^2 \\
&\le \sum_{n=2L_N}^{T^3} n^p L_N^{-2l} \\
&\le C_7^{(2)} T^{3(p+1)} L_N^{-2l}, \qquad \epsilon=(2T)^{-1}. \qquad (2.63)
\end{aligned}$$

Similarly, we have

$$\frac{\epsilon}{\pi}\sum_{n=2L_N}^{T^3} n^p \int_{B_{\frac{\nu}{2}}} dE \sum_{m=T^3+L_N}^{\infty}(1+|n-m|^2)^{-l}|\chi_{E+i\epsilon}(H^{(k)})\delta_1(m)|^2 \le D_7^{(2)} T^{3(p+1)} L_N^{-2l}. \quad (2.64)$$

By (2.63) and (2.64), it is sufficient to prove

$$\frac{\epsilon}{\pi}\sum_{n=2L_N}^{T^3} n^p \int_{B_{\frac{\nu}{2}}} dE \sum_{m=L_N}^{T^3+L_N}(1+|n-m|^2)^{-l}|\chi_{E+i\epsilon}(H^{(k)})\delta_1(m)|^2 \le C_7 T^{p+1} L_N^{-\frac{1}{\Gamma}}, \qquad \epsilon=(2T)^{-1}.$$

Let $A_N^{(k)}$ and $D_N^{(k)} : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be

$$A_N^{(k)} = \left(\begin{array}{ccccc|cccc}
0 & a_k(1) & & & & & & & \\
a_k(1) & 0 & a_k(2) & & & & & & \\
 & a_k(2) & \ddots & \ddots & & & & & \\
 & & \ddots & 0 & a_k(L_N) & & & & \\
 & & & a_k(L_N) & 0 & 1 & & & \\
\hline
 & & & & 1 & 0 & 1 & & \\
 & & & & & 1 & 0 & 1 & \\
 & & & & & & 1 & \ddots & \ddots \\
 & & & & & & & \ddots &
\end{array}\right)$$

and

$$D_N^{(k)} = \left(\begin{array}{cccc|ccc} d_k(1) & & & & & & \\ & d_k(2) & & & & & \\ & & \ddots & & & & \\ & & & d_k(L_N) & & & \\ \hline & & & & 2 & & \\ & & & & & 2 & \\ & & & & & & \ddots \end{array}\right).$$

Let $H_N^{(k)} = D_N^{(k)} - A_N^{(k)}$. We see that for any $l > 1$ there exists $C' = C'(l) > 0$ such that

$$\begin{aligned}
&\frac{\epsilon}{\pi}\sum_{n=2L_N}^{T^3} n^p \int_{B_{\frac{\nu}{2}}} dE \sum_{m=L_N}^{T^3+L_N} (1+|n-m|^2)^{-l} |\chi_{E+i\epsilon}(H^{(k)})\delta_1(m)|^2 \\
&= \frac{\epsilon}{\pi}\sum_{m=L_N}^{T^3+L_N} \left(\sum_{n=2L_N}^{T^3} n^p (1+|n-m|^2)^{-l}\right) \int_{B_{\frac{\nu}{2}}} dE\, |\chi_{E+i\epsilon}(H^{(k)})\delta_1(m)|^2 \\
&\le C'\frac{\epsilon}{\pi}\sum_{m=L_N}^{T^3+L_N} m^p \int_{B_{\frac{\nu}{2}}} dE\, |\chi_{E+i\epsilon}(H^{(k)})\delta_1(m)|^2 \\
&\le 2C'\frac{\epsilon}{\pi}\sum_{m=L_N}^{T^3+L_N} m^p \int_{B_{\frac{\nu}{2}}} dE\, |\chi_{E+i\epsilon}(H^{(k)})\delta_1(m) - \chi_{E+i\epsilon}(H_N^{(k)})\delta_1(m)|^2 \\
&+2C'\frac{\epsilon}{\pi}\sum_{m=L_N}^{T^3+L_N} m^p \int_{B_{\frac{\nu}{2}}} dE\, |\chi_{E+i\epsilon}(H_N^{(k)})\delta_1(m)|^2. \qquad (2.65)
\end{aligned}$$

By the resolvent equation, we have

$$\begin{aligned}
&\|(\chi_{E+i\epsilon}(H^{(k)}) - \chi_{E+i\epsilon}(H_N^{(k)}))\delta_1\| \\
&\le \frac{1}{\epsilon}\|(H^{(k)} - H_N^{(k)})\chi_{E+i\epsilon}(H_N^{(k)})\delta_1\| \\
&\le \frac{1}{\epsilon}\|(D^{(k)} - D_N^{(k)})\chi_{E+i\epsilon}(H_N^{(k)})\delta_1\| + \frac{1}{\epsilon}\|(A^{(k)} - A_N^{(k)})\chi_{E+i\epsilon}(H_N^{(k)})\delta_1\|. \qquad (2.66)
\end{aligned}$$

Let $z = E + i\epsilon$ and $\phi : \mathbb{Z}_{\ge 0} \to \mathbb{C}$ be $\phi(0) = 1$, $\phi(n) = \chi_z(H_N^{(k)})\delta_1(n)$ $(n \ge 1)$. Then it follows for $n > L_N$ that

$$-\phi(n+1) + 2\phi(n) - \phi(n-1) = z\phi(n).$$

This implies for $n > L_N$,

$$\begin{pmatrix} \phi(n) \\ \phi(n+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2-z \end{pmatrix} \begin{pmatrix} \phi(n-1) \\ \phi(n) \end{pmatrix}.$$

Let $\lambda_\pm = \dfrac{2 - z \pm \sqrt{(2-z)^2+4}}{2}$, then there exists $C_\pm$ such that

$$\phi(n) = C_+ \lambda_+^{n-L_N} + C_- \lambda_-^{n-L_N}.$$

Since $\epsilon > 0$, we get $|\lambda_-| < 1$ and $|\lambda_+| > 1$. Since $\|\phi\|_{l^2} < \infty$, $C_+ = 0$, $C_- = \phi(L_N)$ and there exists $c, c' > 0$ such that it follows for $0 < E < 4$ and $0 < \epsilon < 1$ that

$$e^{-c'\epsilon} \leq |\lambda_-| \leq e^{-c\epsilon}.$$

If $\beta > 1$, then it follows for sufficiently large N that

$$L_{N+1} - L_N^\beta > \frac{1}{2} L_{N+1}.$$

Therefore, we see that there exist $C'' > 0$ such that

$$\begin{aligned}
\|(D^{(k)} - D_N^{(k)})\chi_{E+i\epsilon}(H_N^{(k)})\delta_1\|^2 &= \sum_{j=N+1}^{\infty} |d_k(L_j) - 2|^2 |\phi(L_j)|^2 \\
&\leq 2|\phi(L_N)|^2 \sum_{j=N+1}^{\infty} L_j^{\frac{2(1-\Gamma)}{\Gamma}} \exp(-2c\epsilon(L_j - L_N)) \\
&\leq 2\epsilon^{-2} \sum_{j=N+1}^{\infty} L_j^{\frac{2(1-\Gamma)}{\Gamma}} \exp(-2c\epsilon(L_j - L_N^\beta))\exp(-2c\epsilon(L_N^\beta - L_N)) \\
&\leq 4T^2 \sum_{j=N+1}^{\infty} L_j^{\frac{2(1-\Gamma)}{\Gamma}} \exp(-c\epsilon L_j)\exp(-c\epsilon L_N^\beta) \\
&\leq C''\exp(-c\epsilon L_N^\beta), \qquad (2.67)
\end{aligned}$$

and

$$\begin{aligned}
\|(A^{(k)} - A_N^{(k)})\chi_{E+i\epsilon}(H_N^{(k)})\delta_1\|^2 &\leq \sum_{j=N+1}^{\infty} |1 - a_k(L_j)|^2 \{|\phi(L_j)|^2 + |\phi(L_j+1)|^2\} \\
&\leq C''T^2 \sum_{j=N+1}^{\infty} \exp(-2c\epsilon(L_j - L_N)) \\
&\leq C''T^2\exp(-c\epsilon L_N^\beta). \qquad (2.68)
\end{aligned}$$

By (2.66), (2.67), and (2.68), we see that there exists $C_7^{(1)} > 0$ such that

$$\begin{aligned}
&\epsilon \sum_{m=L_N}^{T^3+L_N} m^p \int_{B_{\frac{\nu}{2}}} dE\, |\chi_{E+i\epsilon}(H^{(k)})\delta_1(m) - \chi_{E+i\epsilon}(H_N^{(k)})\delta_1(m)|^2 \\
&\leq \epsilon \sum_{m=L_N}^{T^3+L_N} m^p \int_{B_{\frac{\nu}{2}}} dE\, \|\chi_{E+i\epsilon}(H^{(k)})\delta_1 - \chi_{E+i\epsilon}(H_N^{(k)})\delta_1\|^2 \\
&\leq C_7^{(1)} T^{3(p+1)} \exp(-c\epsilon L_N^{\beta}), \qquad \epsilon = (2T)^{-1}.
\end{aligned} \tag{2.69}$$

By (2.65) and (2.69), it is sufficient to prove

$$\epsilon \sum_{m=L_N}^{T^3+L_N} m^p \int_{B_{\frac{\nu}{2}}} dE\, |\chi_{E+i\epsilon}(H_N^{(k)})\delta_1(m)|^2 \leq C_7 T^{p+1} L_N^{-\frac{1}{\Gamma}}.$$

Let $F_N(z) = (\delta_1, \chi_z(H_N^{(k)})\delta_1) = (\delta_1, (H_N^{(k)} - z)^{-1}\delta_1)$. We see that there exists $C''' > 0$ such that

$$\frac{1}{\epsilon}\mathrm{Im}F_N(E+i\epsilon) = \|\chi_{E+i\epsilon}(H_N^{(k)})\delta_1\|^2 \geq \sum_{m>L_N} |\phi(m)|^2 \geq \sum_{m>L_N} e^{-c'\epsilon(m-L_N)}|\phi(L_N)|^2 \geq \frac{C'''}{\epsilon}|\phi(L_N)|^2.$$

This implies that $|\phi(L_N)| \leq C'''\mathrm{Im}F_N(E+i\epsilon)$. Let $L_{N-1} < n < L_{N+1}$. Then

$$\begin{cases}
-\phi(n+1) + 2\phi(n) - \phi(n-1) = z\phi(n) & (n \neq L_N, L_N+1) \\
-\sqrt{[L_N^{\frac{1-\Gamma}{\Gamma}}]}\phi(L_N+1) + ([L_N^{\frac{1-\Gamma}{\Gamma}}]+2)\phi(L_N) - \phi(L_N-1) = z\phi(L_N) & (n = L_N) \\
-\phi(L_N+2) + 2\phi(L_N+1) - \sqrt{[L_N^{\frac{1-\Gamma}{\Gamma}}]}\phi(L_N) = z\phi(L_N+1) & (n = L_N+1).
\end{cases}$$

Let $R = \begin{pmatrix} 0 & 1 \\ -1 & 2-z \end{pmatrix}$. Then it follows for $L_N + 1 < n < L_{N+1}$ that

$$\begin{pmatrix} \phi(n) \\ \phi(n+1) \end{pmatrix} = R^{n-L_N} \begin{pmatrix} \phi(L_N) \\ \phi(L_N+1) \end{pmatrix}.$$

Similary, for $L_{N-1} < n < L_N - 1$, we have

$$\begin{pmatrix} \phi(n) \\ \phi(n+1) \end{pmatrix} = R^{n-L_N+1} \begin{pmatrix} \phi(L_N-1) \\ \phi(L_N) \end{pmatrix}.$$

There exists $B = B(K) > 0$ such that $\|R^n\| < B$ for $\epsilon < \frac{K}{|n|}$. It follows for $L_N < n < 2L_N$ that

$$|\phi(n)|^2 + |\phi(n+1)|^2 \geq B^{-1}(|\phi(L_N)|^2 + |\phi(L_N+1)|^2).$$

Therefore, we have

$$\frac{1}{\epsilon}\,\mathrm{Im}F_N(E+i\epsilon)=\|\phi\|^2\geq B^{-1}L_N(|\phi(L_N)|^2+|\phi(L_N+1)|^2). \tag{2.70}$$

Similarly, it follows that for $\dfrac{L_N}{2}<n<L_N$,

$$|\phi(n)|^2+|\phi(n+1)|^2\geq B^{-1}(|\phi(L_N)|^2+|\phi(L_N+1)|^2),$$

and that

$$\frac{1}{\epsilon}\mathrm{Im}F_N(E+i\epsilon)=\|\phi\|^2\geq B^{-1}L_N(|\phi(L_N-1)|^2+|\phi(L_N)|^2). \tag{2.71}$$

(2.70) and (2.71) imply that

$$|\phi(L_N-1)|^2+|\phi(L_N+1)|^2\leq\frac{2B}{\epsilon L_N}\mathrm{Im}F_N(E+i\epsilon). \tag{2.72}$$

We see that

$$([L_N^{\frac{1-\Gamma}{\Gamma}}]+2-z)\phi(L_N)=\phi(L_N-1)+\sqrt{[L_N^{\frac{1-\Gamma}{\Gamma}}]}\phi(L_N+1).$$

This shows that

$$|[L_N^{\frac{1-\Gamma}{\Gamma}}]+2-z|^2|\phi(L_N)|^2\leq 2[L_N^{\frac{1-\Gamma}{\Gamma}}](|\phi(L_N-1)|^2+|\phi(L_N+1)|^2).$$

Let $|z|<5$. Then there exists $B'>0$ such that

$$|\phi(L_N)|^2\leq B'L_N^{\frac{\Gamma-1}{\Gamma}}(|\phi(L_N-1)|^2+|\phi(L_N+1)|^2). \tag{2.73}$$

(2.72) and (2.73) imply that

$$|\phi(L_N)|^2\leq 2BB'\frac{L_N^{-\frac{1}{\Gamma}}}{\epsilon}\mathrm{Im}F_N(E+i\epsilon).$$

Therefore, there exists $C_7=C_7(p)$ such that

$$\begin{aligned}
&\epsilon\sum_{m=L_N}^{T^3+L_N}m^p\int_{B_{\frac{\nu}{2}}}dE\ |\chi_{E+i\epsilon}(H_N^{(k)})\delta_1(m)|^2\\
&\leq C_7\epsilon\sum_{m=L_N}^{T^3+L_N}m^p\exp(-2c\epsilon(m-L_N))\int_{B_{\frac{\nu}{2}}}dE\ |\chi_{E+i\epsilon}(H_N^{(k)})\delta_1(L_N)|^2\\
&\leq C_7\epsilon^{-p}\int_{B_{\frac{\nu}{2}}}dE\ |\phi(L_N)|^2\\
&\leq 2BB'C_7\epsilon^{-p-1}L_N^{-\frac{1}{\Gamma}}\int_{B_{\frac{\nu}{2}}}dE\ \mathrm{Im}F_N(E+i\epsilon)\\
&\leq 2BB'C_7T^{p+1}L_N^{-\frac{1}{\Gamma}},\qquad \epsilon=(2T)^{-1}.
\end{aligned}$$

□

Corollary 2.5.11. *Let $p > 0$, f be the first kind, and $\psi = f(H^{(k)})\delta_1$. Then there exists $C_8 = C_8(p) > 0$ such that for $L_N \leq T \leq L_N^{\frac{1}{\Gamma}}$ with N sufficiently large,*

$$\langle |X|^p \rangle_\psi^{(k)}(T) \leq C_8 L_N^p + C_8 T^{p+1} L_N^{-\frac{1}{\Gamma}}.$$

Lemma 2.5.12. *Let f be the first kind and $\psi = f(H^{(k)})\delta_1$. Then*

$$\beta_\psi^{(k)}(p) = \frac{p+1}{p+\frac{1}{\Gamma}}.$$

Proof. Let $L_N \leq T = L_N^A \leq L_N^{\frac{1}{\Gamma}}$, where $A = \dfrac{p+\frac{1}{\Gamma}}{p+1}$. Then Corollary 2.5.11 shows that

$$\langle |X|^p \rangle_\psi^{(k)}(L_N^A) \leq C_8 L_N^p.$$

Therefore we have

$$\beta_\psi^{(k)}(p) \leq \frac{1}{p} \lim_{N\to\infty} \frac{\log \langle |X|^p \rangle_\psi^{(k)}(L_N^A)}{\log L_N^A} \leq A^{-1} = \frac{p+1}{p+\frac{1}{\Gamma}}.$$

Since f is the first kind, (2.57) holds. □

2.5.3 Proof of the main result

Lemma 2.5.13. *Let $A \in \mathcal{B}^1$. Then $E(A) = 0$ if and only if $\mu_{\delta_1}^{(k)}(A) = 0$ for any $k \in \mathbb{N}$.*

Proof. Assume that $E(A) = 0$. Then we see that $E^{(k)}(A) = 0$ and

$$\mu_{\delta_1}^{(k)}(A) = (\delta_1, E^{(k)}(A)\delta_1) = 0.$$

Conversely, assume that $\mu_{\delta_1}^{(k)}(A) = 0$ for any $k \in \mathbb{N}$. It is sufficient to prove that $E^{(k)}(A) = 0$ for any $k \in \mathbb{N}$. Let p ba a polynomial, then we see that

$$\mu_{p(H^{(k)})\delta_1}^{(k)}(A) = (p(H^{(k)})\delta_1, E^{(k)}(A)p(H^{(k)})\delta_1) = \int_A |p(\lambda)|^2 \mu_{\delta_1}^{(k)}(d\lambda) = 0.$$

This implies that $E^{(k)}(A)p(H^{(k)})\delta_1 = 0$. Since δ_1 is a cyclic vector for $H^{(k)} : l^2(\mathbb{N}) \to l^2(\mathbb{N})$, $\{p(H^{(k)})\delta_1 \in l^2(\mathbb{N}) \mid p \text{ is a polynomial}\}$ is dense in $l^2(\mathbb{N})$. Therefore $E^{(k)}(A) = 0$ follows. □

Lemma 2.5.14. *Let $A \in \mathcal{B}^1$ and $A \subset (0, 4)$. Then $\tilde{E}(A) = 0$ if and only if $\mu_\psi^{(k)}(A) = 0$ for any $k \in \mathbb{N}$ and any $\psi = f(H^{(k)})\delta_1$, with the first kind f. Moreover, $\dim_* \tilde{E} = \dim_* \mu_\psi^{(k)}$ and $\dim^* \tilde{E} = \dim^* \mu_\psi^{(k)}$ follow for any $k \in \mathbb{N}$ and any $\psi = f(H^{(k)})\delta_1$, with the first kind f.*

Proof. Assume that $\tilde{E}(A) = 0$. Then we see that $E^{(k)}(A) = 0$ and

$$\mu_\psi^{(k)}(A) = (\psi, E^{(k)}(A)\psi) = 0.$$

Assume that $\mu_\psi^{(k)}(A) = 0$ for any $k \in \mathbb{N}$ and $\psi = f(H^{(k)})\delta_1$, where f is the first kind. Let $f_n \in C_0^\infty(\frac{1}{n}, 4 - \frac{1}{n})$, $|f_n| \leq 1$, and $f_n = 1$ on the interval $(\frac{2}{n}, 4 - \frac{2}{n})$, $n = 1, 2, \ldots$. Let $\psi_n = f_n(H^{(k)})\delta_1$. Since f_n is the first kind, $\mu_{\psi_n}^{(k)}(A) = 0$ for any $k \in \mathbb{N}$. It is sufficient to prove that $E^{(k)}(A) = 0$ for any $k \in \mathbb{N}$. We see that

$$\mu_{\psi_n}^{(k)}(A) = (f_n(H^{(k)})\delta_1, E^{(k)}(A) f_n(H^{(k)})\delta_1) = \int_A |f_n(\lambda)|^2 \mu_{\delta_1}^{(k)}(d\lambda) = 0.$$

By the Lebeasgue's dominated convergence theorem,

$$0 = \lim_{n\to\infty} \mu_{\psi_n}^{(k)}(A) = \lim_{n\to\infty} \int_A |f_n(\lambda)|^2 \mu_{\delta_1}^{(k)}(d\lambda) = \mu_{\delta_1}^{(k)}(A).$$

By Lemma 2.5.13, we see that $E^{(k)}(A) = 0$. Then we prove the first part of our assertion. The second part is straightforward to prove by the first part and the definition of the lower and upper Hausdorff dimensions. $\square$

Proof of Theorem 2.1.4. By Lemma 2.5.14, it is sufficient to prove that $\dim_* \mu_\psi^{(k)} = \dim^* \mu_\psi^{(k)} = \Gamma$ for any $k \in \mathbb{N}$ and any $\psi = f(H^{(k)})\delta_1$, with the first kind f. By Lemma 2.1.3, we see that

$$\Gamma \leq \dim_* \mu_\psi^{(k)} \leq \dim^* \mu_\psi^{(k)}.$$

By Lemma 2.3.2 and Lemma 2.5.12, for any $p > 0$,

$$\dim^*(\mu_\psi^{(k)}) \leq \beta_\psi^{(k)}(p) = \frac{p+1}{p + \frac{1}{\Gamma}}.$$

This imlies that $\dim_* \mu_\psi^{(k)} = \dim^* \mu_\psi^{(k)} = \Gamma$. $\square$

Chapter 3

No eigenvectors embedded in the singular continuous spectrum of Schrödinger operators

3.1 Introduction and a result

We investigate one-dimensional Schrödinger operators with sparse potentials. It is known that the spectrum of Schrödinger operators with sparse potential consists of singular continuous spectrum. Simon and Spencer [23] show the absence of absolutely continuous spectrum of Schrödinger operators with sparse potentials. Simon and Stoltz [22] also show that $-\frac{d^2}{dx^2}f + Vf = Ef$ has no L^2-solutions for any $E > 0$. We have the question whether the edge of the singular continuous spectrum is an eigenvalue or not. We give a sufficient condition for the absence of embedded eigenvalues and give examples.

Definition 3.1.1. *A function $V : [0, \infty) \to \mathbb{R}$ is called a sparse potential, if there exist positive sequences $\{x_n\}_{n=1}^{\infty}, \{\alpha_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ such that $x_{n+1} > x_n$ for $n = 1, 2, 3, ...$,*

(i) $\displaystyle\lim_{n\to\infty} \frac{x_{n+1} - x_n}{\alpha_{n+1} + \alpha_n + 1} = \infty$,

(ii) $|V(x)| \leq h_n$, *if* $x \in [x_n - \alpha_n, x_n + \alpha_n]$ *for* $n = 1, 2, 3, ...$,

(iii) $V(x) = 0$, *if* $\displaystyle x \in \left(\bigcup_{n=1}^{\infty} [x_n - \alpha_n, x_n + \alpha_n] \right)^c$

We define $L_n = x_{n+1} - x_n - \alpha_{n+1} - \alpha_n$ for $n \geq 1$ and $L_0 = x_1 - \alpha_1$. By (i), $L_n \to \infty$ as $n \to \infty$. By Strum-Liouville theory [26, Theorem 9.1.], there exists a unique solution $f \in AC_{loc}([0, \infty))$ of the equation $-\frac{d^2}{dx^2}f + Vf = 0$ with $\frac{d}{dx}f \in AC_{loc}([0, \infty))$ and the boundary condition $f(0) = \alpha, \frac{d}{dx}f(0) = \beta, \alpha, \beta \in \mathbb{C}$. We give a sufficient condition of the existence of a non L^2-integrable solution.

Theorem 3.1.2. *Let V be the sparse potential and f a weak solution of $-\frac{d^2}{dx^2}f + Vf = 0$. If*

$$\frac{L_n}{4^n}\left(\prod_{m=1}^{n}(L_{m-1}^2+2)\right)^{-1}\left(\prod_{m=1}^{n}(2\alpha_m^2+1)\right)^{-1}\exp\left(-\frac{2}{3}\sum_{m=1}^{n}h_m(4\alpha_m^3+3\alpha_m)\right)\to\infty, \quad (3.1)$$

as $n \to \infty$, then $f \notin L^2([0,\infty))$.

We give an example for one-dimensional Schrödinger operators with singular continuous spectrum which has no embedded eigenvalues. Let $x_n = \exp(n^n)$ for $n = 1, 2, 3, ...$, and

$$V(x) = \begin{cases} e^n, & \text{if } |x - x_n| \le \frac{1}{2} \text{ for } n = 1, 2, 3, ..., \\ 0, & \text{otherwise.} \end{cases}$$

Let $H_c = -\frac{d^2}{dx^2} + V : L^2([0,\infty)) \to L^2([0,\infty))$ with the domain $D(H_c) = C_0^\infty((0,\infty))$. We see that H_c is regular at zero and in the limit point case at infinity. This implies that H_c has self-adjoint extensions H_θ which can be parametrized by boundary conditions. Hence, H_θ is the restriction of H^* to $D_\theta = \{f \in D(H^*) \mid f(0)\sin\theta - \frac{d}{dx}f(0)\cos\theta = 0\}$. By [22], we have $\sigma_{sc}(H_\theta) = [0,\infty)$, $\sigma_{pp}(H_\theta) \cap (0,\infty) = \emptyset$ and $\sigma_{ac}(H_\theta) = \emptyset$ for all $\theta \in (\frac{\pi}{2}, \frac{\pi}{2}]$. See section 4.4 for the proof. Theorem 3.1.2 implies the next corollary. This also implies that H_θ has purely singular continuous spectrum for some θ.

Corollary 3.1.3. *It follows that*

(1) $\sigma_{sc}(H_\theta) = [0,\infty)$, $\sigma_{pp}(H_\theta) \cap [0,\infty) = \emptyset$ *and* $\sigma_{ac}(H_\theta) = \emptyset$ *for all* $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}]$,

(2) H_θ *has purely singular continuous spectrum for* $\theta \in [0, \frac{\pi}{2}]$,

(3) H_θ *has a single negative eigenvalue for* $\theta \in (-\frac{\pi}{2}, \arctan(-\frac{1+\sqrt{3}}{2})]$.

3.2 Proof of Theorem 3.1.2.

We calculate a lower bound of Wronskian matrices. For a 2×2-real matrix M, let

$$\mathrm{Low}\, M = \inf\left\{\left|M\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\right| \;\middle|\; \theta \in [0, 2\pi)\right\}.$$

We see that $|Mu| \ge \mathrm{Low}\, M|u|$ for $u \in \mathbb{R}^2$ and $\mathrm{Low}\, M = \sqrt{\inf\sigma({}^tMM)}$. Let V be a sparse potential and $f \in AC_{loc}([0,\infty))$ be a weak solution of $-\frac{d^2}{dx^2}f + Vf = 0$ with $\frac{d}{dx}f \in AC_{loc}([0,\infty))$ and the boundary condition $f(0) = \cos\theta$, $\frac{d}{dx}f(0) = \sin\theta$. We can represent the weak solution concretely as follows. Let $J_0 = [0, x_1 - \alpha_1]$, $I_n = [x_n - \alpha_n, x_n + \alpha_n]$ and $J_n = [x_n + \alpha_n, x_{n+1} - \alpha_{n+1}]$ for $n = 1, 2, ...$ Let $p_n, q_n : J_n \to \mathbb{R}$ be defined by $p_n(x) = 1, q_n(x) = x - x_n - \alpha_n$, and $p_0, q_0 : J_0 \to \mathbb{R}$ by $p_0(x) = 1, q_0(x) = x$. Let $\varphi_n, \psi_n : I_n \to \mathbb{R}$ be the weak solution of $-\frac{d^2}{dx^2}f + Vf = 0$ on I_n with the boundary condition $\varphi_n(x_n-\alpha_n)=1$, $\frac{d}{dx}\varphi_n(x_n-\alpha_n)=0, \psi_n(x_n-\alpha_n)=0$ and $\frac{d}{dx}\psi_n(x_n-\alpha_n)=1$. We define $\tilde{f}:[0,\infty)\to\mathbb{R}$ by

$$\begin{aligned}\tilde{f}(x)&=c_n^{(1)}p_n(x)+c_n^{(2)}q_n(x) \qquad \text{if } x\in J_n,\\ \tilde{f}(x)&=d_n^{(1)}\varphi_n(x)+d_n^{(2)}\psi_n(x) \qquad \text{if } x\in I_n,\end{aligned}$$

where $c_n^{(1)}, c_n^{(2)}, d_n^{(1)}$ and $d_n^{(2)}$ are inductively determined by for $n=1,2,...$

$$\begin{aligned}
\lim_{x\downarrow 0}\begin{pmatrix} p_0(x) & q_0(x)\\ \frac{d}{dx}p_0(x) & \frac{d}{dx}q_0(x)\end{pmatrix}\begin{pmatrix} c_0^{(1)}\\ c_0^{(2)}\end{pmatrix} &= \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix},\\
\lim_{x\uparrow x_n-\alpha_n}\begin{pmatrix} p_{n-1}(x) & q_{n-1}(x)\\ \frac{d}{dx}p_{n-1}(x) & \frac{d}{dx}q_{n-1}(x)\end{pmatrix}\begin{pmatrix} c_{n-1}^{(1)}\\ c_{n-1}^{(2)}\end{pmatrix} &= \lim_{x\downarrow x_n-\alpha_n}\begin{pmatrix} \varphi_n(x) & \psi_n(x)\\ \frac{d}{dx}\varphi_n(x) & \frac{d}{dx}\psi_n(x)\end{pmatrix}\begin{pmatrix} d_n^{(1)}\\ d_n^{(2)}\end{pmatrix}, \quad (3.2)\\
\lim_{x\uparrow x_n+\alpha_n}\begin{pmatrix} \varphi_n(x) & \psi_n(x)\\ \frac{d}{dx}\varphi_n(x) & \frac{d}{dx}\psi_n(x)\end{pmatrix}\begin{pmatrix} d_n^{(1)}\\ d_n^{(2)}\end{pmatrix} &= \lim_{x\downarrow x_n+\alpha_n}\begin{pmatrix} p_n(x) & q_n(x)\\ \frac{d}{dx}p_n(x) & \frac{d}{dx}q_n(x)\end{pmatrix}\begin{pmatrix} c_n^{(1)}\\ c_n^{(2)}\end{pmatrix}. \quad (3.3)
\end{aligned}$$

By the definition, $\tilde{f}$ and $\frac{d}{dx}\tilde{f}$ are continuous, and $\begin{pmatrix}\tilde{f}(0)\\ \frac{d}{dx}\tilde{f}(0)\end{pmatrix}=\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$. The coefficients $c_n^{(1)}, c_n^{(2)}, d_n^{(1)}$ and $d_n^{(2)}$ are uniquely determined by θ. We see that

$$\int_0^\infty \tilde{f}(x)\frac{d^2}{dx^2}g(x)dx=\int_0^\infty \tilde{f}(x)V(x)g(x)dx$$

for $g\in C_0^\infty((0,\infty))$ straightforwardly. This implies that $\tilde{f}$ satisfies $-\frac{d^2}{dx^2}f+Vf=0$ in the sence of the weak derivative. Therefore $f=\tilde{f}$ by the uniqueness of the solution.

Lemma 3.2.1. *Let* $c_n=\begin{pmatrix} c_n^{(1)}\\ c_n^{(2)}\end{pmatrix}$. *Then we see that*

$$\int_{J_n}|f(x)^2|dx \geq \frac{1}{4}\frac{L_n^3}{L_n^2+3}|c_n|^2. \qquad (3.4)$$

Proof. We obtain

$$\begin{aligned}\int_{J_n}|f(x)|^2dx &= \int_{J_n}|c_n^{(1)}p_n(x)+c_n^{(2)}q_n(x)|^2dx\\ &= \begin{pmatrix} c_n^{(1)} & c_n^{(2)}\end{pmatrix}\begin{pmatrix} L_n & \frac{1}{2}L_n^2\\ \frac{1}{2}L_n^2 & \frac{1}{3}L_n^3\end{pmatrix}\begin{pmatrix} c_n^{(1)}\\ c_n^{(2)}\end{pmatrix}.\end{aligned}$$

The matrix $\begin{pmatrix} L_n & \frac{1}{2}L_n^2\\ \frac{1}{2}L_n^2 & \frac{1}{3}L_n^3\end{pmatrix}$ can be diagonalized and its eigenvalues $\lambda_\pm$ are

$$\lambda_\pm=\frac{1}{2}\left(L_n+\frac{1}{3}L_n^3\pm\sqrt{(L_n+\frac{1}{3}L_n^3)^2-\frac{1}{3}L_n^4}\right).$$

Since $1-\sqrt{1-t} \geq \frac{1}{2}t$ for $0<t<1$, we have

$$\lambda_- = \frac{L_n + \frac{1}{3}L_n^3}{2}\left(1-\sqrt{1-\frac{L_n^4}{3}\left(L_n+\frac{1}{3}L_n^3\right)^{-2}}\right) \geq \frac{1}{4}\frac{L_n^3}{L_n^2+3}.$$

This implies our assertion. □

By (3.2) and (3.3), we obtain $c_n = R_n W_{n-1} c_{n-1}$ for $n \geq 1$, where

$$\begin{aligned} R_m &= \lim_{x\uparrow x_m+\alpha_m} \begin{pmatrix} \varphi_m(x) & \psi_m(x) \\ \frac{d}{dx}\varphi_m(x) & \frac{d}{dx}\psi_m(x) \end{pmatrix}, \\ W_m &= \lim_{x\uparrow x_{m+1}-\alpha_{m+1}} \begin{pmatrix} p_m(x) & q_m(x) \\ \frac{d}{dx}p_m(x) & \frac{d}{dx}q_m(x) \end{pmatrix} = \begin{pmatrix} 1 & L_m \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

We shall estimate $\mathrm{Low}\, R_m$ and $\mathrm{Low}\, W_m$.

Lemma 3.2.2. *It follows that*

$$\mathrm{Low}\, W_m \geq \frac{1}{\sqrt{L_m^2+2}}. \tag{3.5}$$

Proof. The eigenvalues $\xi_\pm$ of ${}^tW_mW_m$ are

$$\xi_\pm = \frac{1}{2}\left(L_m^2 + 2 \pm \sqrt{(L_m^2+2)^2-4}\right).$$

Since $1-\sqrt{1-t} \geq \frac{1}{2}\,t$ for $0<t<1$, we have

$$\xi_- = \frac{L_m^2+2}{2}\left(1-\sqrt{1-4(L_m^2+2)^{-2}}\right) \geq \frac{1}{L_m^2+2}.$$

Since $\mathrm{Low}\, M = \sqrt{\inf \sigma({}^tMM)}$, we have our assertion. □

Let $\tilde{\varphi}_m, \tilde{\psi}_m : I_m \to \mathbb{R}$ be defined by $\tilde{\varphi}_m(x) = 1, \tilde{\psi}_m(x) = x - x_m + \alpha_m$. We see that $\tilde{\varphi}_m, \tilde{\psi}_m$ satisfy $-\frac{d^2}{dx^2}f = 0$. There exist $u_m^{(j)}, v_m^{(j)} \in AC(I_m)$, $j=1,2$ such that

$$\begin{aligned} \begin{pmatrix} \varphi_m(x) \\ \frac{d}{dx}\varphi_m(x) \end{pmatrix} &= u_m^{(1)}(x)\begin{pmatrix} \tilde{\varphi}_m(x) \\ \frac{d}{dx}\tilde{\varphi}_m(x) \end{pmatrix} + u_m^{(2)}(x)\begin{pmatrix} \tilde{\psi}_m(x) \\ \frac{d}{dx}\tilde{\psi}_m(x) \end{pmatrix}, \\ \begin{pmatrix} \psi_m(x) \\ \frac{d}{dx}\psi_m(x) \end{pmatrix} &= v_m^{(1)}(x)\begin{pmatrix} \tilde{\varphi}_m(x) \\ \frac{d}{dx}\tilde{\varphi}_m(x) \end{pmatrix} + v_m^{(2)}(x)\begin{pmatrix} \tilde{\psi}_m(x) \\ \frac{d}{dx}\tilde{\psi}_m(x) \end{pmatrix}. \end{aligned}$$

We see that $u_m^{(1)}(x_m-\alpha_m) = 1, u_m^{(2)}(x_m-\alpha_m) = 0$, $v_m^{(1)}(x_m-\alpha_m) = 0, v_m^{(2)}(x_m-\alpha_m) = 1$, and

$$\begin{pmatrix} \varphi_m & \psi_m \\ \frac{d}{dx}\varphi_m & \frac{d}{dx}\psi_m \end{pmatrix} = \begin{pmatrix} \tilde{\varphi}_m & \tilde{\psi}_m \\ \frac{d}{dx}\tilde{\varphi}_m & \frac{d}{dx}\tilde{\psi}_m \end{pmatrix}\begin{pmatrix} u_m^{(1)} & v_m^{(1)} \\ u_m^{(2)} & v_m^{(2)} \end{pmatrix}. \tag{3.6}$$

Note that φ_n and ψ_n satisfy the equation $-\frac{d^2}{dx^2}f + Vf = 0$ which is equivalent to

$$\begin{pmatrix} \frac{d}{dx}f \\ \frac{d^2}{dx^2}f \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ V & 0 \end{pmatrix} \begin{pmatrix} f \\ \frac{d}{dx}f \end{pmatrix}.$$

Differentiating both sides of (3.6), we obtain

$$\begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix} \begin{pmatrix} \tilde{\varphi}_m & \tilde{\psi}_m \\ \frac{d}{dx}\tilde{\varphi}_m & \frac{d}{dx}\tilde{\psi}_m \end{pmatrix} \begin{pmatrix} u_m^{(1)} & v_m^{(1)} \\ u_m^{(2)} & v_m^{(2)} \end{pmatrix} = \begin{pmatrix} \tilde{\varphi}_m & \tilde{\psi}_m \\ \frac{d}{dx}\tilde{\varphi}_m & \frac{d}{dx}\tilde{\psi}_m \end{pmatrix} \begin{pmatrix} \frac{d}{dx}u_m^{(1)} & \frac{d}{dx}v_m^{(1)} \\ \frac{d}{dx}u_m^{(2)} & \frac{d}{dx}v_m^{(2)} \end{pmatrix}.$$

Thus we have

$$\begin{pmatrix} \frac{d}{dx}u_m^{(1)} & \frac{d}{dx}v_m^{(1)} \\ \frac{d}{dx}u_m^{(2)} & \frac{d}{dx}v_m^{(2)} \end{pmatrix} = -V \begin{pmatrix} \tilde{\varphi}_m\tilde{\psi}_m & \tilde{\psi}_m^2 \\ -\tilde{\varphi}_m^2 & -\tilde{\varphi}_m\tilde{\psi}_m \end{pmatrix} \begin{pmatrix} u_m^{(1)} & v_m^{(1)} \\ u_m^{(2)} & v_m^{(2)} \end{pmatrix}. \tag{3.7}$$

Let $u_m = \begin{pmatrix} u_m^{(1)} \\ u_m^{(2)} \end{pmatrix}$ and $v_m = \begin{pmatrix} v_m^{(1)} \\ v_m^{(2)} \end{pmatrix}$. By (3.7), we see that u_m and v_m satisfy that

$$\frac{d}{dx}u_m = -V \begin{pmatrix} \tilde{\varphi}_m\tilde{\psi}_m & \tilde{\psi}_m^2 \\ -\tilde{\varphi}_m^2 & -\tilde{\varphi}_m\tilde{\psi}_m \end{pmatrix} u_m, \tag{3.8}$$

$$\frac{d}{dx}v_m = -V \begin{pmatrix} \tilde{\varphi}_m\tilde{\psi}_m & \tilde{\psi}_m^2 \\ -\tilde{\varphi}_m^2 & -\tilde{\varphi}_m\tilde{\psi}_m \end{pmatrix} v_m. \tag{3.9}$$

Lemma 3.2.3. *For $x \in I_m$, $u_m^{(1)}(x)v_m^{(2)}(x) - v_m^{(1)}(x)u_m^{(2)}(x) = 1$.*

Proof. By (3.8) and (3.9), we obtain

$$\begin{aligned} \frac{d}{dx}(u_m^{(1)}v_m^{(2)} - v_m^{(1)}u_m^{(2)}) &= \frac{d}{dx}u_m^{(1)}v_m^{(2)} + u_m^{(1)}\frac{d}{dx}v_m^{(2)} - \frac{d}{dx}v_m^{(1)}u_m^{(2)} - v_m^{(1)}\frac{d}{dx}u_m^{(2)} \\ &= -V(\tilde{\varphi}_m\tilde{\psi}_m u_m^{(1)} + \tilde{\psi}_m^2 u_m^{(2)})v_m^{(2)} + u_m^{(1)}V(\tilde{\varphi}_m^2 v_m^{(1)} + \tilde{\varphi}_m\tilde{\psi}_m v_m^{(2)}) \\ &\quad +V(\tilde{\varphi}_m\tilde{\psi}_m v_m^{(1)} + \tilde{\psi}_m^2 v_m^{(2)})u_m^{(2)} - v_m^{(1)}V(\tilde{\varphi}_m^2 u_m^{(1)} + \tilde{\varphi}_m\tilde{\psi}_m u_m^{(2)}) \\ &= 0. \end{aligned}$$

Since $u_m^{(1)}(x_m-\alpha_m)v_m^{(2)}(x_m-\alpha_m) - v_m^{(1)}(x_m-\alpha_m)u_m^{(2)}(x_m-\alpha_m) = 1$, we have our assertion. □

Lemma 3.2.4. *Assume $u_j, v_j \in \mathbb{R}$, $j = 1, 2$ and $u_1v_2 - v_1u_2 = 1$. Then*

$$\text{Low} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \geq \frac{1}{\sqrt{u_1^2 + u_2^2 + v_1^2 + v_2^2}}.$$

Proof. Let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, and $u \cdot v = u_1 v_1 + u_2 v_2$. The eigenvalues $\xi_\pm$ of the matrix $\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$ are

$$\xi_\pm = \frac{1}{2}\left(|u|^2 + |v|^2 \pm \sqrt{(|u|^2+|v|^2)^2 - 4(|u|^2|v|^2 - u\cdot v)^2} \right).$$

Since $1 - \sqrt{1-t} \geq \frac{1}{2}t$ for $0 < t < 1$ and $|u|^2|v|^2 - u \cdot v = (u_1 v_2 - u_2 v_1)^2 = 1$, we obtain

$$\xi_- \geq \frac{1}{|u|^2+|v|^2}.$$

This implies our assertion. □

Lemma 3.2.5. *It follows that*

$$\text{Low}\, R_m \geq \frac{1}{2\sqrt{2\alpha_m^2+1}} \exp\left(-\frac{1}{3} h_m (4\alpha_m^3 + 3\alpha_m) \right). \tag{3.10}$$

Proof. Note that

$$R_m = \lim_{x \uparrow x_m + \alpha_m} \begin{pmatrix} \varphi_m(x) & \psi_m(x) \\ \frac{d}{dx}\varphi_m(x) & \frac{d}{dx}\psi_m(x) \end{pmatrix} = \lim_{x \uparrow x_m + \alpha_m} \begin{pmatrix} \tilde{\varphi}_m(x) & \tilde{\psi}_m(x) \\ \frac{d}{dx}\tilde{\varphi}_m(x) & \frac{d}{dx}\tilde{\psi}_m(x) \end{pmatrix} \begin{pmatrix} u_m^{(1)}(x) & v_m^{(1)}(x) \\ u_m^{(2)}(x) & v_m^{(2)}(x) \end{pmatrix}.$$

It is straightfoward to see

$$\begin{aligned} \lim_{x \uparrow x_m + \alpha_m} \begin{pmatrix} \tilde{\varphi}_m(x) & \tilde{\psi}_m(x) \\ \frac{d}{dx}\tilde{\varphi}_m(x) & \frac{d}{dx}\tilde{\psi}_m(x) \end{pmatrix} &= \begin{pmatrix} 1 & 2\alpha_m \\ 0 & 1 \end{pmatrix}, \\ \text{Low} \begin{pmatrix} 1 & 2\alpha_m \\ 0 & 1 \end{pmatrix} &\geq \frac{1}{\sqrt{4\alpha_m^2+2}}. \end{aligned} \tag{3.11}$$

By Lemmas 3.2.3 and 3.2.4, we have

$$\text{Low} \begin{pmatrix} u_m^{(1)}(x) & v_m^{(1)}(x) \\ u_m^{(2)}(x) & v_m^{(2)}(x) \end{pmatrix} \geq \frac{1}{\sqrt{|u_m(x)|^2 + |v_m(x)|^2}}.$$

We see that for $a, b \in \mathbb{R}$,

$$\sup_{\theta \in [0,\pi)} \left| \left(\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \right) \right| = \frac{a^2+b^2}{2}. \tag{3.12}$$

By (3.8) and (3.12), we have

$$\begin{aligned} \frac{d}{dx}(|u_m|^2) &= 2\left(u_m, \frac{d}{dx} u_m \right) \\ &\leq 2|V| \left| \left(u_m, \begin{pmatrix} \tilde{\varphi}_m \tilde{\psi}_m & \tilde{\psi}_m^2 \\ -\tilde{\varphi}_m^2 & -\tilde{\varphi}_m \tilde{\psi}_m \end{pmatrix} u_m \right) \right| \\ &\leq h_m (\tilde{\varphi}_m^2 + \tilde{\psi}_m^2) |u_m|^2. \end{aligned}$$

Thus, by Gronwall's inequality, we obtain

$$|u_m(x)|^2 \le \exp\left(h_m \int_{x_m-\alpha_m}^{x} \left(\tilde{\varphi}_m(y)^2 + \tilde{\psi}_m(y)^2 \right) dy \right).$$

In particular

$$\begin{aligned} |u_m(x_m+\alpha_m)|^2 &\le \exp\left(h_m \int_{x_m-\alpha_m}^{x_m+\alpha_m} \left(1 + (y - x_m + \alpha_m)^2\right) dy \right). \\ &= \exp\left(h_m (2\alpha_m + \frac{8}{3}\alpha_m^3) \right). \end{aligned}$$

We can estimate $|v_m|^2$ in a similar way:

$$|v_m(x_m+\alpha_m)|^2 \le \exp\left(\frac{2}{3} h_m (4\alpha_m^3 + 3\alpha_m) \right).$$

Therefore, we see that

$$\begin{aligned} \mathrm{Low}\begin{pmatrix} u_m^{(1)}(x) & v_m^{(1)}(x) \\ u_m^{(2)}(x) & v_m^{(2)}(x) \end{pmatrix}\Bigg|_{x=x_m+\alpha_m} &\ge \frac{1}{\sqrt{|u_m(x)|^2 + |v_m(x)|^2}}\Bigg|_{x=x_m+\alpha_m} \\ &\ge \frac{1}{\sqrt{2}} \exp\left(-\frac{1}{3} h_m (4\alpha_m^3 + 3\alpha_m) \right). \qquad (3.13) \end{aligned}$$

By (3.11) and (3.13), we have our assertion. □

Proof of Theorem 3.1.2. By (3.5), (3.10), and $c_n = R_n W_{n-1} c_{n-1}$ for $n \ge 1$, we obtain

$$|c_n| \ge \frac{1}{2^n} \left(\prod_{m=1}^{n} (L_{m-1}^2 + 2)(2\alpha_m^2 + 1) \right)^{-\frac{1}{2}} \exp\left(-\frac{1}{3} \sum_{m=1}^{n} h_m (4\alpha_m^3 + 3\alpha_m) \right). \qquad (3.14)$$

If n is sufficiently large, then

$$\frac{L_n^3}{L_n^2 + 3} \ge \frac{L_n}{4}.$$

Thus, by (3.4) and (3.14), if n is sufficiently large, then we have

$$\int_{J_n} |f(x)^2| dx \ge \frac{L_n}{4^{n+2}} \left(\prod_{m=1}^{n} (L_{m-1}^2 + 2)(2\alpha_m^2 + 1) \right)^{-1} \exp\left(-\frac{2}{3} \sum_{m=1}^{n} h_m (4\alpha_m^3 + 3\alpha_m) \right).$$

Suppose (3.1). Then we obtain

$$\int_{J_n} |f(x)^2| dx \to \infty.$$

This implies that any solutions f of $-\frac{d^2}{dx^2} f + V f = 0$ do not belong to $L^2([0, \infty))$. □

3.3 Proof of Corollary 3.1.3.

Note that $x_n = \exp(n^n)$, $\alpha_n = \frac{1}{2}$, $h_n = e^n$, $L_0 = x_1 - \frac{1}{2}$ and $L_n = x_{n+1} - x_n - 1$ for $n \geq 1$.

Lemma 3.3.1. *If there exists $f \in D(H_\theta)$ such that $(f, H_\theta f) < 0$, then H_θ has a single negative eigenvalue.*

Proof. $\mathcal{R}[A]$ denotes the range of a map A. We see that $\dim \mathcal{R}[E_\theta((-\infty, 0))] \leq 1$. For its proof, see Lemma 4.4.1. Let $f \in D(H_\theta)$ such that $(f, H_\theta f) < 0$. Then this implies $\mathcal{R}[E_\theta((-\infty, 0))] \neq \{0\}$ and $\dim \mathcal{R}[E_\theta((-\infty, 0))] = 1$. This implies our assertion. □

Lemma 3.3.2. *It follows that*

(1) *H_θ has no negative eigenvalues for $\theta \in [0, \frac{\pi}{2}]$,*

(2) *H_θ has a single negative eigenvalue for $\theta \in (-\frac{\pi}{2}, \arctan(-\frac{1+\sqrt{3}}{2})]$.*

Proof. Let $f \in D(H_\theta)$. Then f satisfies the boundary condition $f(0)\sin\theta - \frac{d}{dx}f(0)\cos\theta = 0$. We obtain that

$$\begin{aligned}(f, H_\theta f) &= \int_0^\infty \overline{f(x)}(-\frac{d^2}{dx^2}f(x) + V(x)f(x))dx \\ &= \overline{f(0)}\frac{d}{dx}f(0) + \int_0^\infty \left(\left|\frac{d}{dx}f(x)\right|^2 + V(x)|f(x)|^2\right)dx.\end{aligned}$$

If $\theta = 0$ or $\frac{\pi}{2}$, then we have $\overline{f(0)}\frac{d}{dx}f(0) = 0$ and $(f, H_\theta f) \geq 0$. If $0 < \theta < \frac{\pi}{2}$, then by the boundary condition, we obtain

$$(f, H_\theta f) = |f(0)|^2\tan\theta + \int_0^\infty \left(\left|\frac{d}{dx}f(x)\right|^2 + V(x)|f(x)|^2\right)dx \geq 0.$$

Thus $(f, H_\theta f) \geq 0$ for $\theta \in [0, \frac{\pi}{2}]$. This implies the first part of our assertion.

We shall prove that there exists $f \in D(H_\theta)$ such that $(f, H_\theta f) < 0$ for any $\theta \in (-\frac{\pi}{2}, \arctan(-\frac{1+\sqrt{3}}{2})]$. It is sufficient to prove there exists $f \in L^2([0,\infty))$ such that $(-\frac{d^2}{dx^2}f + Vf, f) < 0$ for any boundray conditions $\frac{\frac{d}{dx}f(0)}{f(0)} = -\lambda$, $\lambda \geq \frac{1+\sqrt{3}}{2}$. Let $\lambda \geq 1$. Define $f_\lambda : [0,\infty) \to \mathbb{R}$ by

$$f_\lambda(x) = \begin{cases} \exp\left(\dfrac{\lambda}{x-1}\right), & \text{if } 0 \leq x < 1 \\ 0, & \text{otherwise.}\end{cases}$$

We see that $f_\lambda \in L^2([0,\infty))$ and that $f_\lambda(0) = \exp(-\lambda)$, $\frac{d}{dx}f_\lambda(0) = -\lambda\exp(-\lambda)$. Then we have

$$\begin{aligned}\int_0^\infty |\frac{d}{dx}f_\lambda(x)|^2 dx &= \int_0^1 \left|\frac{\lambda}{(x-1)^2}\exp\left(\frac{\lambda}{x-1}\right)\right|^2 dx \\ &= \frac{1}{4\lambda}(2\lambda^2 + 2\lambda + 1)\exp(-2\lambda).\end{aligned}$$

Since $V(x)f_\lambda(x) = 0$ for $x \geq 0$, we obtain that, for $\lambda \geq \frac{1+\sqrt{3}}{2}$,

$$\left(-\frac{d^2}{dx^2}f_\lambda + Vf_\lambda, f_\lambda\right) = \frac{1}{4\lambda}(-2\lambda^2 + 2\lambda + 1)\exp(-2\lambda) < 0.$$

By Lemma 3.3.1, we have our assertion. □

Lemma 3.3.3. *For any $p > 0$, it follows that*

$$\lim_{n\to\infty} x_n \left(\prod_{m=1}^{n-1} x_m\right)^{-p} = \infty.$$

Proof. We obtain

$$\begin{aligned} x_{n+1}\left(\prod_{m=1}^{n} x_m\right)^{-p} &= \exp\left((n+1)^{n+1} - pn^n - p\sum_{m=1}^{n-1} m^m\right) \\ &\geq \exp\left((n+1)^{n+1} - pn^n - p(n-1)^n\right) \\ &= \exp\left((n+1-2p)(n+1)^n\right) \\ &\to \infty, \qquad \text{as } n \to \infty. \end{aligned}$$

□

Proof of Corollary 3.1.3. By Lemma 3.3.2 it is sufficient to prove that $0 \notin \sigma_{pp}(H_\theta)$ for all θ. We see that for all $n \geq 1$,

$$\frac{L_n^2 + 2}{L_n^2} < 2.$$

Thus we have for all $n \geq 1$,

$$\left(\prod_{m=1}^{n}(L_{m-1}^2 + 2)\right)^{-1} \geq \frac{2^{n-1}}{L_0^2 + 2}\left(\prod_{m=1}^{n-1} L_m\right)^{-2}.$$

We see that $L_n < x_{n+1}$ for all $n \geq 1$ and $L_n > \frac{1}{2}x_{n+1}$ for sufficiently large $n \geq 1$. Therefore we have

$$L_n\left(\prod_{m=1}^{n-1} L_m\right)^{-2} \geq \frac{1}{2}x_{n+1}\left(\prod_{m=1}^{n} x_m\right)^{-2}.$$

By Lemma 3.3.3, we obtain

$$
\begin{aligned}
&\frac{L_n}{4^n}\left(\prod_{m=1}^{n}(L_{m-1}^2+2)\right)^{-1}\left(\prod_{m=1}^{n}(2\alpha_m^2+1)\right)^{-1}\exp\left(-\frac{2}{3}\sum_{m=1}^{n}h_m(4\alpha_m^3+3\alpha_m)\right)\\
&\geq \frac{1}{2(L_0^2+2)3^n}L_n\left(\prod_{m=1}^{n-1}L_m\right)^{-2}\exp\left(-\frac{4}{3}\sum_{m=1}^{n}e^m\right)\\
&\geq \frac{1}{4(L_0^2+2)3^n}x_{n+1}\left(\prod_{m=1}^{n}x_m\right)^{-2}\exp\left(-\frac{4}{3}e^{n+1}\right)\\
&= \frac{\exp(\frac{1}{3}(n+1)^{n+1})}{4(L_0^2+2)3^n}{x_{n+1}}^{\frac{1}{3}}\left(\prod_{m=1}^{n}x_m\right)^{-2}\exp\left(\frac{1}{3}\left((n+1)^{n+1}-4e^{n+1}\right)\right)\\
&\to \infty, \qquad \text{as } n\to\infty.
\end{aligned}
$$

By Theorem 3.1.2, we see that $0 \notin \sigma(H_\theta)$. □

Chapter 4

Appendix

4.1 Decomposition of the graph Laplacian

We discuss the decomposition of the graph Laplacian and represent the graph Laplacian as a Jacobi matrix. See [1].

We assume that $G = (V, E)$ is a spherically homogeneous tree. Let $\pi_n : l^2(S_n) \to l^2(S_{n+1})$, $n = 0, 1, ...,$ be defined by

$$\pi_n f(u) = \sum_{v \in S_n : v \sim u} f(v), \ \ u \in S_{n+1}.$$

The adjoint $\pi_n^* : l^2(S_{n+1}) \to l^2(S_n)$ is given by

$$\pi_n^* g(u) = \sum_{v \in S_{n+1} : v \sim u} g(v), \ \ u \in S_n.$$

Lemma 4.1.1. *Let f, $g \in l^2(S_n)$. Then $(\pi_n f, \pi_n g) = g_n(f, g)$.*

Proof. Let f, $g \in l^2(S_n)$. Since G is a spherically homogeneous,

$$\langle \pi_n f, \pi_n g \rangle = \sum_{u \in S_{n+1}} \overline{\pi_n f(u)} \pi_n g(u) = g_n \sum_{u \in S_n} \overline{f(u)} g(u).$$

□

We see that V is a disjoint union $V = \cup_{n=0}^{\infty} S_n$, and that $l^2(V) = \bigoplus_{n=0}^{\infty} l^2(S_n)$. Let $\Pi : l^2(V) \to l^2(V)$ be defined by $\Pi = \bigoplus_{n=0}^{\infty} \pi_n$.

Lemma 4.1.2. *Let $f \in \mathcal{D}$. Then $Af = (\Pi + \Pi^*)f$.*

Proof. Let $f \in \mathcal{D}$ and $u \in S_n$. Since G is a spherically homogeneous tree, u is adjacent with only vertices in S_{n-1} and S_{n+1}. Therefore we see that

$$(\Pi + \Pi^*)f(u) = \sum_{v \in S_{n-1}; v \sim u} f(v) + \sum_{v \in S_{n+1}; v \sim u} f(v) = \sum_{v \in V; v \sim u} f(v) = Af(u).$$

□

Let $\alpha_n = \#S_n = \dim(l^2(S_n))$, $n = 0, 1, \ldots$. Suppose that $\{e_k^{(n)}\}_{k=1}^{\alpha_n}$ is a CONS of $l^2(S_n)$. Then we can construct a CONS $\{e_k^{(n+1)}\}_{k=1}^{\alpha_{n+1}}$ of $l^2(S_{n+1})$ by the following procedure. Let $e_k^{(n+1)} = \|\pi_n e_k^{(n)}\|^{-1}\pi_n e_k^{(n)}$, $k = 1, 2, \ldots, \alpha_n$. By Lemma 4.1.1, $\{e_k^{(n+1)}\}_{k=1}^{\alpha_n}$ is an ONS of $l^2(S_{n+1})$. If $\alpha_n = \alpha_{n+1}$, then $\{e_k^{(n+1)}\}_{k=1}^{\alpha_n}$ is a CONS of $l^2(S_{n+1})$. If $\alpha_n < \alpha_{n+1}$, by the Gram-Schmidt process, we can obtain $e_k^{(n+1)} \in l^2(S_{n+1})$, $k = \alpha_n + 1, \ldots, \alpha_{n+1}$, such that $\{e_k^{(n+1)}\}_{k=1}^{\alpha_n} \cup \{e_k^{(n+1)}\}_{k=\alpha_n+1}^{\alpha_{n+1}}$ is a CONS of $l^2(S_{n+1})$.

Suppose that a CONS of $l^2(S_0)$ is given. Then we can costruct a CONS $\{e_k^{(n)}\}_{k=1}^{\alpha_n}$ of $l^2(S_n)$, $n = 0, 1, \ldots$, inductively. Hence, $\bigcup_{n=0}^{\infty}\{e_k^{(n)}\}_{k=1}^{\alpha_n}$ is a CONS of $l^2(V)$.

Assume that $\sup_{n=0,1,\ldots} \alpha_n = \infty$, and let $\alpha_{-1} = 0$. Since $\{\alpha_n\}_{n=0}^{\infty}$ is non-decreasing, there exists a unique $N(k) \in \mathbb{N} \cup \{0\}$ such that $\alpha_{N(k)-1} < k \leq \alpha_{N(k)}$ for every $k \in \mathbb{N}$. We see that

$$\bigcup_{n=0}^{\infty}\{e_k^{(n)} \mid k = 1, 2, \ldots, \alpha_n\} = \bigcup_{k=1}^{\infty}\{e_k^{(n)} \mid n = N(k), N(k)+1, \ldots\}.$$

Lemma 4.1.3. *Let the closed subspace M_k of $l^2(V)$, $k = 1, 2, \ldots$, be defined by*

$$M_k = \overline{\langle\{e_k^{(n)} \mid n = N(k), N(k)+1, \ldots\}\rangle}.$$

Then M_k is invariant under A, D and L.

Proof. By the definition of $e_k^{(n)}$ and Lemma 4.1.1, we see that

$$\begin{aligned}
\Pi e_k^{(n)} &= \|\pi_n e_k^{(n)}\| e_k^{(n+1)}, \\
\Pi^* e_k^{(n)} &= \begin{cases} \dfrac{g_{n-1}}{\|\pi_{n-1} e_k^{(n-1)}\|}\, e_k^{(n-1)} & (n \geq N(k)+1), \\ \boldsymbol{o} & (n = N(k)). \end{cases}
\end{aligned}$$

This implies that M_k is invariant under Π and Π^*, and hence, by Lemma 4.1.2, we see that M_k is invariant under A. Since G is a spherically homogeneous tree, we have

$$De_k^n = \begin{cases} (g_n+1)e_k^{(n)} & (n \geq 1), \\ g_0 e_1^{(0)} & (n = 0). \end{cases}$$

Hence, M_k is also inavariant under D. Since $L = D - A$, we see that M_k is invariant under L. □

By Lemma 4.1.3, let $H^{(k)}, A^{(k)}, D^{(k)} : M_k \to M_k$, $k = 1, 2, \ldots$, be defined by the restriction of H_d, A and D to M_k, respectively. We see that $H^{(k)}$ is self-adjoint and $H = \bigoplus_{k=1}^{\infty} H^{(k)}$.

We consider the matrix representation of $H^{(k)}$ with respect to the CONS $\{e_k^{(n)} \mid n = N(k), N(k)+1, \ldots\}$ of M_k for $k = 2, 3, \ldots$. Then it follows for $n, m \geq N(k)$ that

$$\begin{aligned}
(e_k^{(n)}, H^{(k)} e_k^{(n)}) &= (e_k^{(n)}, D^{(k)} e_k^{(n)}) = g_n + 1, \\
(e_k^{(n)}, H^{(k)} e_k^{(n+1)}) &= -(e_k^{(n)}, A^{(k)} e_k^{(n+1)}) = -\sqrt{g_n}, \\
(e_k^{(n)}, H^{(k)} e_k^{(m)}) &= 0, \text{ if } |n-m| \geq 2\ .
\end{aligned}$$

We have the matrix representation of $H^{(1)}$. It follows for $n, m \geq N(1) = 0$ that

$$\begin{aligned}
(e_1^{(n)}, H^{(1)} e_1^{(n)}) &= (e_1^{(n)}, D^{(1)} e_1^{(n)}) = \begin{cases} g_0 & (n=0), \\ g_n + 1 & (n \geq 1), \end{cases} \\
(e_1^{(n)}, H^{(1)} e_1^{(n+1)}) &= -(e_1^{(n)}, A^{(1)} e_1^{(n+1)}) = -\sqrt{g_n}, \\
(e_1^{(n)}, H^{(1)} e_1^{(m)}) &= 0, \text{ if } |n-m| \geq 2.
\end{aligned}$$

Let $k, n \in \mathbb{N}$ and let $d_k = (d_k(n))_{n=1}^{\infty}$ and $a_k = (a_k(n))_{n=1}^{\infty}$ be defined by

$$\begin{aligned}
d_k(n) &= (e_k^{(n+N(k)-1)}, D^{(k)} e_k^{(n+N(k)-1)}), \\
a_k(n) &= (e_k^{(n+N(k))}, A^{(k)} e_k^{(n+N(k)-1)}).
\end{aligned}$$

We can identify $H^{(k)} : l^2(\mathbb{N}) \to l^2(\mathbb{N})$, $k = 1, 2, \ldots$, with the following Jacobi matrix :

$$H^{(k)} = \begin{pmatrix}
d_k(1) & -a_k(1) & & & \\
-a_k(1) & d_k(2) & -a_k(2) & & \\
 & -a_k(2) & d_k(3) & -a_k(3) & \\
 & & -a_k(3) & \ddots & \ddots \\
 & & & \ddots &
\end{pmatrix}.$$

This implies our assertion of Lemma 2.2.1. Similary, we can identify $A^{(k)}, D^{(k)} : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ with the following Jacobi matrices :

$$A^{(k)} = \begin{pmatrix}
0 & a_k(1) & & & \\
a_k(1) & 0 & a_k(2) & & \\
 & a_k(2) & 0 & a_k(3) & \\
 & & a_k(3) & \ddots & \ddots \\
 & & & \ddots &
\end{pmatrix}, D^{(k)} = \begin{pmatrix}
d_k(1) & & & \\
 & d_k(2) & & \\
 & & d_k(3) & \\
 & & & \ddots \\
 & & &
\end{pmatrix}.$$

4.2 Fourier analysis and fractal measures

In this section, we introduce a result about the Fourier analysis of the fractal measure. Let $B_r(x) = [x-r, x+r] \subset \mathbb{R}$. Let $\mathcal{L}$ be the Lebeage measure on $\mathbb{R}$, and $\mu : \mathcal{B}^1 \to [0, \infty]$ be a locally finite measure. Let $M_\mu f : \mathbb{R} \to \mathbb{R}$ be defined by

$$M_\mu f(x) = \sup_{r>0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f| d\mu$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}, d\mu)$, where we take $\frac{0}{0} = 0$ if $\mu(B_r(x)) = 0$. $M_\mu f$ is measurable and called the Maximal function.

Lemma 4.2.1. *$M_\mu : L^p(\mathbb{R}, d\mu) \to L^p(\mathbb{R}, d\mu)$ is bounded for any $p \in (1, \infty)$.*

Proof. Let $E^n_s = \{x \in \mathbb{R} \mid |x| \le n, M_\mu f(x) > s\}$ and $x \in E^n_s$. There exists $r_x > 0$ such that

$$\int_{B_{r_x}(x)} |f| d\mu \ge s\mu(B_{r_x}(x)).$$

Note that $\{B_{r_x}(x) \mid x \in E^n_s\}$ is a Besicovitch covering of E^n_s. By [9, II, 18 The Besicovitch covering theorem, Theorem18.1], we see that there exists a countable subcollections $\{B^n_j\}_{j=1}^\infty$ of $\{B_{r_x}(x) \mid x \in E^n_s\}$ such that $\{B^n_j\}_{j=1}^\infty$ is a closed covering of E^n_s and there exist $C > 0$ which is independent of E^n_s such that for any $x \in \mathbb{R}$,

$$\mathbb{1}_{E^n_s}(x) \le \sum_{j=1}^\infty \mathbb{1}_{B^n_j}(x) \le C.$$

Hence we have

$$s\ \mu(E^n_s) \le s\sum_{j=1}^\infty \mu(B^n_j) \le \sum_{j=1}^\infty \int_{B^n_j} |f| d\mu \le C \int_{\mathbb{R}} |f| d\mu.$$

Let $n \to \infty$. Then we see that for any $f \in L^1(\mathbb{R}, d\mu)$ and $s > 0$,

$$\mu(\{x \in \mathbb{R} \mid M_\mu f(x) > s\}) \le Cs^{-1}\|f\|_{L^1}.$$

This implies that $M_\mu : L^1(\mathbb{R}, d\mu) \to L^1(\mathbb{R}, d\mu)$ is weak $(1, 1)$ type. We also see that $M_\mu : L^\infty(\mathbb{R}, d\mu) \to L^\infty(\mathbb{R}, d\mu)$ is weak (∞, ∞) type. Thus we have our assertion by [9, VIII, 9 The Marcinkiewicz interpolation theorem, Theorem 9.1]. □

We consider the Fourier transformation of the fractal measure. Let $f \in L^1(\mathbb{R}, d\mu)$ and let $\widehat{f\mu}(\xi)$, $\xi \in \mathbb{R}$, be defined by

$$\widehat{f\mu}(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x}\mu(dx).$$

Lemma 4.2.2. *Suppose μ be a finite measure. Then*

$$\int_{\mathbb{R}} |\widehat{f\mu}(\xi)|^2 e^{-t\xi^2} d\xi < \infty.$$

for any $f \in L^2(\mathbb{R}, d\mu)$ and $t > 0$.

Proof. Let $f \in L^1(\mathbb{R}, d\mu)$. We see that

$$\int_{\mathbb{R}} |\widehat{f\mu}(\xi)| e^{-t\xi^2} d\xi \quad \le \quad \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x)|\mu(dx)e^{-t\xi^2} d\xi < \infty.$$

This implies that $L^1(\mathbb{R}, d\mu) \ni f \to \widehat{f\mu} \in L^1(\mathbb{R}, e^{-t\xi^2}d\xi)$ is bounded. Since μ is a finite measure, $L^\infty(\mathbb{R}, d\mu) \ni f \to \widehat{f\mu} \in L^\infty(\mathbb{R}, e^{-t\xi^2}d\xi)$ is bounded. We have our assertion by the Riesz interpolation theorem. □

Definition 4.2.3. *Let $\alpha \in (0,1)$. We say that a measure μ is uniformly α-Hölder continuous, if there exists $\widetilde{C}_1 > 0$ such that $\mu(I) < \widetilde{C}_1 \mathcal{L}(I)^\alpha$ for any interval $I \subset \mathbb{R}$ with $\mathcal{L}(I) < 1$.*

Lemma 4.2.4. *Let μ be a uniformly α-Hölder continuous and finite measure. Then there exists $\widetilde{C}_2 = \widetilde{C}_2(\alpha, \mu) > 0$ such that for any $f \in L^2(\mathbb{R}, d\mu)$,*

$$\sup_{0<t\leq 1} t^{\frac{1-\alpha}{2}} \int_{\mathbb{R}} |\widehat{f\mu}(\xi)|^2 e^{-t\xi^2} d\xi < \widetilde{C}_2 \|f\|_{L^2}^2.$$

Proof. Let $f \in L^2(\mathbb{R}, d\mu)$. We see that

$$\begin{aligned}
t^{\frac{1-\alpha}{2}} \int_{\mathbb{R}} |\widehat{f\mu}(\xi)|^2 e^{-t\xi^2} d\xi &= t^{\frac{1-\alpha}{2}} \int_{\mathbb{R}} \mu(dx) \int_{\mathbb{R}} \mu(dy) f(x)\overline{f(y)} \int_{\mathbb{R}} e^{-t\xi^2 - i\xi(x-y)} d\xi \\
&= \pi t^{-\frac{\alpha}{2}} \int_{\mathbb{R}} \mu(dx) \int_{\mathbb{R}} \mu(dy) f(x)\overline{f(y)} e^{-\frac{(x-y)^2}{4}} \\
&= \pi t^{-\frac{\alpha}{2}} \int_{\mathbb{R}} \mu(dx) \int_{\mathbb{R}} \mu(dy) f(x)\overline{f(y)} \int_{|x-y|}^{\infty} \frac{r}{2t} e^{-\frac{r^2}{4t}} dr \\
&\leq \pi t^{-\frac{\alpha}{2}} \int_{\mathbb{R}} \mu(dx)\, |f(x)| \int_0^\infty dr \frac{r}{2t} e^{-\frac{r^2}{4t}} \int_{B_r(x)} \mu(dy)|f(y)| \\
&\leq \pi t^{-\frac{\alpha}{2}} \int_{\mathbb{R}} \mu(dx)\, |f(x)| \int_0^\infty dr \frac{r}{2t} e^{-\frac{r^2}{4t}} \mu(B_r(x)) M_\mu f(x).
\end{aligned}$$

Since μ is uniformly α-Hölder continuous and finite, there exists $\widetilde{C}_2' = \widetilde{C}_2'(\alpha, \mu) > 0$ such that for any $t \in (0,1]$,

$$\begin{aligned}
t^{-\frac{\alpha}{2}} \int_0^\infty dr \frac{r}{t} e^{-\frac{r^2}{4t}} \mu(B_r(x)) &\leq \mu(\mathbb{R}) t^{-\frac{\alpha}{2}} \int_1^\infty dr \frac{r}{t} e^{-\frac{r^2}{4t}} + \widetilde{C}_1 t^{-\frac{\alpha}{2}} \int_0^1 dr \frac{r^{1+\alpha}}{t} e^{-\frac{r^2}{4t}} \\
&\leq 2\mu(\mathbb{R}) t^{-\frac{\alpha}{2}} e^{-\frac{1}{4t}} + 2^{2+\alpha} \widetilde{C}_1 \int_0^\infty s^{1+\alpha} e^{-s^2} ds \\
&\leq \widetilde{C}_2'.
\end{aligned}$$

Let $\widetilde{C}_2 = \pi \widetilde{C}_2' > 0$. By Schwarz inequality, we see that for any $t \in (0,1)$,

$$t^{\frac{1-\alpha}{2}} \int_{\mathbb{R}} |\widehat{f\mu}(\xi)|^2 e^{-t\xi^2} d\xi \leq \widetilde{C}_2 \int_{\mathbb{R}} \mu(dx)\, |f(x)| M_\mu f(x) \leq D_2 \|f\|_{L^2} \|M_\mu f\|_{L^2}.$$

By Lemma 4.2.1, we have our assertion. □

Lemma 4.2.5. *Let μ be a uniformly α-Hölder continuous and finite measure. Then there exists $\widetilde{C}_3 = \widetilde{C}_3(\alpha, \mu) > 0$ such tha for any $f \in L^2(\mathbb{R}, d\mu)$,*

$$\sup_{T\geq 1} T^{\alpha-1} \int_0^T |\widehat{f\mu}(\xi)|^2 d\xi \leq \widetilde{C}_3 \|f\|^2.$$

Proof. Let $t \in (0,1)$ and $T = t^{-\frac{1}{2}}$. By Lemma 4.2.4, we see that for any $T > 1$,

$$\widetilde{C}_2\|f\|^2 \geq T^{\alpha-1}\int_{\mathbb{R}} |\widehat{f\mu}(\xi)|^2 e^{-(\frac{\xi}{T})^2} d\xi \geq e^{-1}T^{\alpha-1}\int_0^T |\widehat{f\mu}(\xi)|^2 d\xi.$$

This implies our assertion. □

Lemma 4.2.6. *Let μ be a uniformly α-Hölder continuous and finite measure. Then there exists $\widetilde{C}_4 = \widetilde{C}_4(\alpha,\mu) > 0$ such that for any $f \in L^2(\mathbb{R}, d\mu)$,*

$$\sup_{T\geq 1} T^{\alpha-1}\int_0^\infty e^{-\frac{t}{T}}|\widehat{f\mu}(t)|^2 dt \leq \widetilde{C}_4\|f\|^2.$$

Proof. Let $f \in L^2(\mathbb{R}, d\mu)$. Then, by Lemma 4.2.5, we see that for any $T > 1$,

$$\begin{aligned} T^{\alpha-1}\int_0^\infty e^{-\frac{t}{T}}|\widehat{f\mu}(t)|^2 dt &= \lim_{N\to\infty} T^{\alpha-1}\int_0^{T(N+1)} e^{-\frac{t}{T}}|\widehat{f\mu}(t)|^2 dt \\ &= \lim_{N\to\infty} T^{\alpha-1}\sum_{n=0}^{N}\int_{Tn}^{T(n+1)} e^{-\frac{t}{T}}|\widehat{f\mu}(t)|^2 dt \\ &= \sum_{n=0}^{\infty}(n+1)^{1-\alpha}e^{-n}\{T(n+1)\}^{\alpha-1}\int_{Tn}^{T(n+1)} |\widehat{f\mu}(t)|^2 dt \\ &\leq \widetilde{C}_3\|f\|^2\sum_{n=0}^{\infty}(n+1)^{1-\alpha}e^{-n}. \end{aligned}$$

This implies our assertion. □

4.3 Quadratic forms

In this section we give some lemmas about quadratic form theory which is used in Seciotin 4. Let $\mathcal{H}$ ba a complex Hilbert space. Let $\mathfrak{s} : \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ be a closed sesquilinear form, and $T : \mathcal{H} \to \mathcal{H}$ be a closed linear operator. We say that $\mathfrak{s}$ is symmetric, if $\mathfrak{s}(f,g) = \overline{\mathfrak{s}(g,f)}$ for $f, g \in \mathcal{D}(\mathfrak{s})$, and that $\mathfrak{s}$ is sectorial, if there exist $r \in \mathbb{R}$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that for $f \in \mathcal{D}(\mathfrak{s})$ with $\|f\| = 1$,

$$\arg(\mathfrak{s}[f] - r) \leq \theta,$$

where $\mathfrak{s}[f] = \mathfrak{s}(f,f)$. We say that T is sectorial, if there exist $r \in \mathbb{R}$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that for $f \in \mathcal{D}(\mathfrak{s})$ with $\|f\| = 1$,

$$\arg((f, Tf) - r) \leq \theta,$$

and that T is m-accretive, if $\mathrm{Re}(f, Tf) \geq 0$ for $f \in \mathcal{D}(T)$ and $(T+\lambda)^{-1}$ is bounded and $\|(T+\lambda)^{-1}\| \leq (\mathrm{Re}\lambda)^{-1}$ for $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 0$. In particular, T is said to be quasi m-accretive, if there exists $\gamma \in \mathbb{R}$ such that $T + \gamma$ is m-accretive, and T is said to be m-sectorial, if T s quasi m-accretive and sectorial.

Lemma 4.3.1. *Let $\mathfrak{s} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ be a densely defined, closed, and sectorial sesquilinear form. Then there exist a unique m-sectorial operator $S : \mathcal{H} \to \mathcal{H}$ such that for $f \in \mathcal{D}(\mathfrak{s}), g \in \mathcal{D}(S)$,*

$$\mathfrak{s}(f, g) = (f, Sg).$$

Proof. [18, VI, §2, Theorem 2.1]

Lemma 4.3.2. *Let $\mathfrak{t} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ be a densely defined, closed, and symmetric form bounded from below, and let T be the self-adjoint operator associated with $\mathfrak{t}$. Suppose that $\mathfrak{s} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is a relatively bounded sesquilinear form with respect to $\mathfrak{t}$ such that for any $f \in \mathcal{D}(\mathfrak{t}) \subset \mathcal{D}(\mathfrak{s})$,*

$$|\mathfrak{s}[f]| \leq a\mathfrak{t}[f] + b\|f\|^2, \qquad 0 < a < 1,\ b \geq 0.$$

Then $\mathfrak{t}' = \mathfrak{s} + \mathfrak{t}$ is sectorial and closed. Let T' be the m-sectorial operators associated with $\mathfrak{t}'$. If $0 < \gamma < 1$, $z \in \rho(T)$ and

$$2\|(aT + b)(T - z)^{-1}\| \leq \gamma < 1,$$

then $z \in \rho(T')$ and

$$\|(T' - z)^{-1} - (T - z)^{-1}\| \leq \frac{4\gamma}{(1-\gamma)^2}\|(T - z)^{-1}\|.$$

Proof. [18, VI, §3, Theorem 3.9] □

4.4 Negative eigenvalue and singular continuous spectrum

By [22], we see that $\sigma_{ac}(H_\theta) = \emptyset$ and $\sigma_{pp}(H_\theta) \cap (0, \infty) = \emptyset$. In this section, we prove that H_θ has a single negative eigenvalue for some θ and that $\sigma_{sc}(H_\theta) = [0, \infty)$ for all θ. Let V ba a sparse potential with $x_n = \exp(n^n)$ for $n = 1, 2, ...,$ and

$$V(x) = \begin{cases} e^n, & \text{if } |x - x_n| \leq \frac{1}{2} \text{ for } n = 1, 2, ..., \\ 0, & \text{otherwise.} \end{cases}$$

By [22] and [23], we see that $\sigma_{pp}(H_\theta) \cap (0, \infty) = \emptyset$ and $\sigma_{ac}(H_\theta) = \emptyset$ for all θ.

Lemma 4.4.1. *Let E_θ be the spectral resolution of H_θ. For all θ, $\dim \mathcal{R}[E_\theta((-\infty, 0))] \leq 1$.*

Proof. We prove this by a contradiction. Suppose that $\dim \mathcal{R}[E_\theta((-\infty, 0))] > 1$. Then we can take $\varphi, \psi \in E_\theta((-\infty, 0))$ such that φ and ψ are orthogonal to each other. Let $\varphi_n = E_\theta((-n, -\frac{1}{n}))\varphi$ and $\psi_n = E_\theta((-n, -\frac{1}{n}))\psi$. We see that $\varphi_n, \psi_n \in D(H_\theta)$, $\varphi_n \to \varphi$ and $\psi_n \to \psi$. Let $N \geq 1$ be sufficiently large such that φ_N and ψ_N are linearly independent. Since $\alpha\varphi_N + \beta\psi_N \in \mathcal{R}[E_\theta((-N, -\frac{1}{N}))]$ for $\alpha, \beta \in \mathbb{C}$, we have, for $(\alpha, \beta) \neq (0, 0)$,

$$(\alpha\varphi_N + \beta\psi_N, H_\theta(\alpha\varphi_N + \beta\psi_N)) < 0. \tag{4.1}$$

On the other hand, since the deficiency indices of H are equal to one, there exists an ismometric operator $U_\theta : \ker(H^* - i) \to \ker(H^* + i)$ and $w \in \ker(H^* - i)$ such that

$$D(H_\theta) = \left\{ v + \alpha(w + U_\theta w) \middle| v \in D(\overline{H}), \alpha \in \mathbb{C} \right\},$$

where $\overline{H}$ is the closure of H. Let $u_\theta = w + U_\theta w$. There exist $v_1, v_2 \in D(\overline{H})$ and $\alpha_1, \alpha_2 \in \mathbb{C}$ such that

$$\begin{aligned} \varphi_N &= v_1 + \alpha_1 u_\theta, \\ \psi_N &= v_2 + \alpha_2 u_\theta. \end{aligned}$$

If $\alpha_1 = 0$, then $(\varphi_N, H_\theta \varphi_N) = (v_1, \overline{H} v_1) \geq 0$. (4.1) implies $\alpha_1 \neq 0$. Similarly we have $\alpha_2 \neq 0$. We obtain

$$(\alpha_2 \varphi_N + \alpha_1 \psi_N, H_\theta(\alpha_2 \varphi_N + \alpha_1 \psi_N)) = (\alpha_2 v_1 + \alpha_1 v_2, \overline{H}(\alpha_2 v_1 + \alpha_1 v_2)) \geq 0.$$

This contradicts with (4.1). Thus we have our assertion. □

Let $E < 0$. We see that $\dim \ker(H^* - E) = 1$. This implies that there exists a boundary condition $\theta(E)$ such that $H_{\theta(E)}$ has a single negative eigenvalue E.

Lemma 4.4.2. *For all $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}]$, it follows that $\sigma_{sc}(H_\theta) = [0, \infty)$.*

Proof. Since $\sigma_{pp}(H_\theta) \cap (0, \infty) = \emptyset$ and $\sigma_{ac}(H_\theta) = \emptyset$ for all θ, we see that $\sigma(H_\theta) \cap (0, \infty) = \sigma_{sc}(H_\theta) \cap (0, \infty)$. We prove $(0, \infty) \subset \sigma(H_\theta)$ for all θ by contradition. Suppose that there exist $\theta \in [0, \pi), E > 0$ such that $E \in (0, \infty) \setminus \sigma(H_\theta)$. Since H is regular at zero and limit-circle case at infinity, the deficiency indices $\dim \ker(H^* \pm i)$ are equal to one. Thus $\dim \ker(H^* - E) = 1$. This implies that there exists an L^2-solution of $-\frac{d^2}{dx^2} f + Vf = Ef$. By [22, Theorem 2.3.], however, $-\frac{d^2}{dx^2} f + Vf = pf$ has no solutions with $f \in L^2([0, \infty))$ for any $p > 0$. This is a contradiction and we get $(0, \infty) \subset \sigma_{sc}(H_\theta)$ for all θ. Since $\sigma_{sc}(H_\theta) \cap (-\infty, 0) = \emptyset$, we get our assertion. □

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