Arithmetic Dijkgraaf-Witten Theory for Number Rings

平野, 光

出版情報：Kyushu University, 2022, 博士（数理学）, 課程博士
バージョン：

Arithmetic Dijkgraaf–Witten Theory for Number Rings

Hikaru Hirano

Graduate School of Mathematics
Kyushu University
March, 2023

Abstract

In recent years, Minhyong Kim and his collaborators ([Ki], [CK-KPY]) initiated to study an arithmetic analogue for number rings of Dijkgraaf–Witten theory for 3-manifolds, based on the analogies between 3-manifolds and number rings, knots and primes in arithmetic topology. In this thesis, we present basic constructions and properties in the arithmetic Dijkgraaf–Witten theory along the lines of topological quantum field theory. For a finite set S of finite primes of a number field k, we construct arithmetic analogues of the Chern–Simons 1-cocycle, the prequantization bundle for a surface and the Chern–Simons functional for a 3-manifold. We then construct arithmetic analogues for k and S of the quantum Hilbert space (space of conformal blocks) and the Dijkgraaf–Witten partition function in (2+1)-dimensional Chern–Simons TQFT. We show some basic and functorial properties of those arithmetic analogues, in paticular, we establish the decomposition and gluing formulas for arithmetic Chern–Simons invariants and arithmetic Dijkgraaf–Witten partition functions. Furthermore, we give explicit formulas of mod 2 arithmetic Dijkgraaf–Witten invariants for number rings $\overline{\mathrm{Spec}(\mathcal{O}_K)}$, where $K = \mathbb{Q}(\sqrt{p_1 p_2 \cdots p_r})$, p_i's being distinct prime numbers congruent to 1 mod 4, in terms of the Legendre symbols of p_i's. We also show topological analogues of our formulas for 3-manifolds.

Acknowledgements

The author would like to thank his supervisor Professor Masanori Morishita for suggesting the problem studied in this thesis. He would like to thank Professors Kiyonori Gomi, Tomoki Mihara, Yuji Terashima, Masahito Yamazaki and Michihisa Wakui for useful communications. He was supported by Grant-in-Aid for JSPS Fellow (DC1) Grant Number 17J02472.

Notations and convention

$\mathbb{Z}$: the ring of rational integers
$\mathbb{Q}$: the field of rational numbers
$\mathbb{R}$: the field of real numbers
$\mathbb{C}$: the field of complex numbers
$R^\times$: the group of units in a commutative ring R
$\mathcal{O}_K$: the ring of integers of a number field K
I_K : the group of fractional ideals of K
$\mathrm{N}\mathfrak{a} \in \mathbb{Q}$: the norm of $\mathfrak{a} \in I_K$
Cl_K : the ideal class group of K
Cl_K^+ : the narrow ideal class group of K
For a G-equivariant fiber bundle $\varpi : E \to B$
$\Gamma(B, E)$: the set of sections of ϖ
$\Gamma_G(B, E)$: the set of G-equivariant sections of ϖ

Contents

1 Introduction

In this thesis, we study *arithmetic Dijkgraaf–Witten theory* for number ring, based on the analogies between knots and primes, number rings and 3-manifolds in arithmetic topology.

Dijkgraaf–Witten theory is a 3-dimensional Chern–Simons gauge theory with finite gauge group. It can be interpreted as a toy model of (2+1)-dimensional topological quantum field theory, TQFT for short, in the sense of Atiyah ([At1]). TQFT is a framework to produce topological invariants for manifolds. So Dijkgraaf–Witten TQFT gives the Dijkgraaf–Witten invariants (partition functions) for 3-manifolds. For the construction of Dijkgraaf–Witten TQFT, we consult [DW], [FQ], [Gm], [Wa], and [Ye].

In recent years, Minhyong Kim initiated to study *arithmetic Chern–Simons theory* for number rings, which is based on the ideas of Dijkgraaf–Witten theory for 3-manifolds ([DW]) and the analogies between 3-manifolds and number rings, knots and primes in arithmetic topology ([Mo2]). In [Ki] he constructed an arithmetic analogue of the Chern–Simons functional, which is defined on a space of Galois representations over a totally imaginary number field. In the subsequent paper [CKKPY] Kim and his collaborators showed a decomposition formula for arithmetic Chern–Simons invariants and applied it to concrete computations for some examples. Computations of arithmetic Chern–Simons invariants have also been carried out for some examples, by employing number-theoretic considerations in [AC], [BCGKPT], [CKKPPY], and [LP]. In [Hi], we extended Kim's construction over arbitrary number fields, which may have real primes (see also [LP] for another construction), and we gave an explicit formula for the mod 2 arithmetic Dijkgraaf–Witten invariants for certain real quadratic number fields. In [HKM], we showed a TQFT structure of arithmetic Dijkgraaf–Witten theory for the complements of finite number of finite primes in number rings, along the line of topological Dijkgraaf–Witten TQFT.

In this thesis, after reviewing the topological Dijkgraaf–Witten theory, we present our results in arithmetic Dijkgraaf–Witten theory, based on [Hi] and [HKM]. The more precise contents of this thesis are organized as follows.

In Chapter 2, we recall Dijkgraaf–Witten theory for closed 3-manifolds ([DW]). For a finite group A and a 3-cocycle $c \in Z^3(A, \mathbb{Z}/n\mathbb{Z})$, we introduce the Chern–Simons invariant and the Dijkgraaf–Witten invariant for a 3-manifold. We then show explicit formulas for mod 2 Dijkgraaf–Witten invariants for double covers of the 3-sphere.

In Chapter 3, we recall the definition of topological quantum field theory, due to Atiyah, and construct the TQFT structure for Dijkgraaf–Witten theory following Gomi ([Gm]). To be precise, for a oriented closed surface Σ, we construct the Chern–Simons 1-cocycle and the prequantization bundle. For a oriented compact 3-manifold M, we introduce the Chern–Simons functional. We then construct the quantum Hilbert space and the Dijkgraaf–Witten partition function in (2+1)-dimensional Chern–Simons TQFT.

In Chapter 4, we study an arithmetic analogues for number rings of Chern–Simons functional and Dijkgraaf–Witten invariant for 3-manifolds in Chapter 2. Namely we introduce the definition of the arithmetic Chern–Simons invariant to any number rings, by using the modified étale cohomology groups and fundamental groups which take real primes into account, and introduce the arithmetic Dijkgraaf–Witten invariant. We then show explicit formulas of mod 2 arithmetic Dijkgraaf–Witten invariants for real quadratic fields $\mathbb{Q}(\sqrt{p_1 p_2 \cdots p_r})$, where p_i's are distinct prime numbers congruent to 1 mod 4, in terms of the Legendre symbols of p_i's.

In Chapter 5, we present arithmetic Dijkgraaf–Witten theory for number rings, which may be regarded as an arithmetic analogue of Dijkgraaf–Witten theory in Chapter 3, along the line of topological quantum field theory. To be precise, for a finite set S of finite primes of a number field k, we construct arithmetic analogues of the Chern–Simons 1-cocycle, the prequantization bundle for a surface and the Chern–Simons functional for a 3-manifold. We then construct arithmetic analogues for k and S of the quantum Hilbert space and the Dijkgraaf–Witten partition function in (2+1)-dimensional Chern–Simons TQFT. We show some basic and functorial properties of those arithmetic analogues. Finally we show decomposition and gluing formulas for arithmetic Chern–Simons invariants and arithmetic Dijkgraaf–Witten partition functions.

2 Chern–Simons functionals and Dijkgraaf–Witten invariants for closed 3-manifolds

In this chapter, we recall Dijkgraaf–Witten theory for closed 3-manifolds ([DW]). For a finite group A and a 3-cocycle $c \in Z^3(A, \mathbb{Z}/n\mathbb{Z})$, we introduce the Chern–Simons invariant and the Dijkgraaf–Witten invariant for a 3-manifold. We then show explicit formulas for mod 2 Dijkgraaf–Witten invariants for double covers of the 3-sphere. The contents of this chapter are based on [Hi].

2.1 Dijkgraaf–Witten invariants for 3-manifolds

In this section, we introduce the Dijkgraaf–Witten invariants in a manner slightly different from the original one [DW] to clarify the analogy between the Dijkgraaf–Witten invariant for a 3-manifold and that for a number ring, which will be discussed in Chapter 4. In order to define the invariant, we show the following proposition.

Proposition 2.1.1. *Let M be a connected compact 3-manifold. Then, for $n \geq 2$, there is a cohomological spectral sequence*

$$\mathrm{H}^p(\pi_1(M), \mathrm{H}^q(\widetilde{M}, \mathbb{Z}/n\mathbb{Z})) \Rightarrow \mathrm{H}^{p+q}(M, \mathbb{Z}/n\mathbb{Z}),$$

where $\widetilde{M}$ denotes the universal covering of M.

Proof. Although this may be well known, we give a proof for the sake of readers. Since M is compact, the singular cohomology $\mathrm{H}^i(M, \mathbb{Z}/n\mathbb{Z})$ can be identified with the cohomology of the constant sheaf $\mathbb{Z}/n\mathbb{Z}$ on M. So we show the assertion for the cohomology of the constant sheaf. We denote by $\mathrm{Gal}(\widetilde{M}/M)$-mod the category of $\mathrm{Gal}(\widetilde{M}/M)$-modules. We consider the functors

$$F_1 : \mathrm{Sh}(M) \to \mathrm{Gal}(\widetilde{M}/M)\text{-mod}, S \mapsto S(\widetilde{M})$$
$$F_2 : \mathrm{Gal}(\widetilde{M}/M)\text{-mod} \to \mathrm{Ab}, R \mapsto R^{\mathrm{Gal}(\widetilde{M}/M)},$$

where the action of $G = \mathrm{Gal}(\widetilde{M}/M)$ on $S(\widetilde{M})$ is defined by $\sigma.x = S(\sigma)(x)$ for $x \in S(\widetilde{M})$ and $\sigma \in G$. We can easily check $(F_2 \circ F_1)(S) = S(\widetilde{M})^G = S(M)$ and that F_1 sends any injective object I to a F_2-acyclic object. Therefore we have the expected spectral sequence by the Grothendieck spectral sequence and $\pi_1(M) \cong \mathrm{Gal}(\widetilde{M}/M)$. □

Now we define the Dijkgraaf–Witten invariant for a 3-manifold.

Definition 2.1.2. Let M be a connected oriented closed 3-manifold and let $c \in \mathrm{H}^3(A, \mathbb{Z}/n\mathbb{Z})$ for a finite group A and $n \geq 2$. Let $\mathcal{M}(M, A) = \mathrm{Hom}(\pi_1(M), A)/A$ denote the set of conjugacy classes of all homomorphisms $\pi_1(M) \to A$. Note that the fundamental class $[M]$ generates $\mathrm{H}_3(M, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. For each $\rho \in \mathcal{M}(M, A)$, the *Chern–Simons invariant* $CS_c(\rho)$ of ρ associated to c is defined by the image of c under the composition of the maps

$$\mathrm{H}^3(A, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\rho^*} \mathrm{H}^3(\pi_1(M), \mathbb{Z}/n\mathbb{Z}) \xrightarrow{j_3} \mathrm{H}^3(M, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{< \ ,[M]>} \mathbb{Z}/n\mathbb{Z},$$

where j_3 denotes the edge homomorphisms in the spectral sequence

$$\mathrm{H}^p(\pi_1(M), \mathrm{H}^q(\widetilde{M}, \mathbb{Z}/n\mathbb{Z})) \Rightarrow \mathrm{H}^{p+q}(M, \mathbb{Z}/n\mathbb{Z})$$

of Proposition 2.1.1. The *Dijkgraaf–Witten invariant* of M associated to c is then defined by

$$Z_c(M) = \frac{1}{\#A} \sum_{\rho \in \mathcal{M}(M,A)} \exp\left(\frac{2\pi i}{n} CS_c(\rho)\right).$$

When $A = \mathbb{Z}/m\mathbb{Z}$, we call $CS_c(\rho)$ and $Z_c(M)$ the *mod m Chern–Simons invariant* and the *mod m Dijkgraaf–Witten invariant* respectively.

Remark 2.1.3. The Dijkgraaf–Witten invariant was originally defined as follows [DW]. Let M and A be as in Definition 2.1.2. Let $\mathrm{B}A$ denotes a classifying space for A. Consider $U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$ and let $c \in \mathrm{H}^3(A, U(1))$. Then the Dijkgraaf–Witten invariant $DW_c(M)$, is defined by

$$DW_c(M) = \frac{1}{\#A} \sum_{\rho \in \mathrm{Hom}(\pi_1(M),A)} < {f_\rho}^* c, [M] >,$$

where $f_\rho : M \to \mathrm{B}A$ denotes the classifying map with respect to ρ and $< \ , \ > : \mathrm{H}^3(M, U(1)) \times \mathrm{H}_3(M, \mathbb{Z}) \to U(1)$ denotes the natural pairing.

The relation between this definition and Definition 2.1.2 is given as follows. Suppose that $A = \mathbb{Z}/n\mathbb{Z}$, so that there is an isomorphism $\mathrm{H}^3(A, U(1)) \cong \mu_n \subset U(1)$ sending c to an n-th root of unity $\zeta_{n,c}$ in $U(1)$. Then, we may verify that for any $\rho \in \mathrm{Hom}(\pi_1(M), A)$, the equality

$$\zeta_{n,c}^{CS_{id \cup \beta(id)}(\rho)} =< {f_\rho}^* c, [M] >$$

holds, where $id \cup \beta(id)$ is a natural generator of $\mathrm{H}^3(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ (see Lemma 5.5.2). In particular, when $\zeta_{n,c} = \exp(\frac{2\pi i}{n})$, we have

$$DW_c(M) = Z_{\mathrm{id} \cup \beta(\mathrm{id})}(M).$$

2.2 A formula for Chern–Simons invariants using the Hurewicz isomorphism

In this section, we show a formula for Chern–Simons invariants using the Hurewicz isomorphism. Let us describe the setting in this section. Keeping the same notations as in the section 2.1, we set $A = \mathbb{Z}/n\mathbb{Z}$ and $c = \mathrm{id} \cup \beta(\mathrm{id}) \in \mathrm{H}^3(A, \mathbb{Z}/n\mathbb{Z})$, where $id \in \mathrm{H}^1(A, \mathbb{Z}/n\mathbb{Z})$ is the identity map and

$$\beta^i : \mathrm{H}^i(A, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}^{i+1}(A, \mathbb{Z}/n\mathbb{Z}) \ (i = 0, 1, 2, \cdots)$$

is the Bockstein map induced by the short exact sequence

$$(*) \qquad 0 \to \mathbb{Z}/n\mathbb{Z} \overset{\times n}{\to} \mathbb{Z}/n^2\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

In addition, for $i = 1, 2, \cdots$, let $\beta^i : \mathrm{H}^i(M, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}^{i+1}(M, \mathbb{Z}/n\mathbb{Z})$ and $\beta_i : \mathrm{H}_i(M, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}_{i-1}(M, \mathbb{Z}/n\mathbb{Z})$ denote the Bockstein maps of the singular homology and cohomology induced by $(*)$, Furthermore, for $i = 1, 2, \cdots$, let $\tilde{\beta}_i : \mathrm{H}_i(M, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}_{i-1}(M, \mathbb{Z})$ denotes the Bockstein map of the singular homology induced by the short exact sequence

$$0 \to \mathbb{Z} \overset{\times n}{\to} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

Let $j_i : \mathrm{H}^i(\pi_1(M), \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}^i(M, \mathbb{Z}/n\mathbb{Z})$ $(i = 0, 1, 2, 3, \cdots)$ denote the edge homomorphisms in the spectral sequence of Proposition 2.1.1. We will abbreviate $j_i \circ \rho^*$ to ρ_M^* for $\rho \in \mathcal{M}(M, A) = \mathrm{Hom}(\pi_1(M), A)/A$. We denote by $\Phi^i : \mathrm{H}^i(M, \mathbb{Z}/n\mathbb{Z}) \overset{\sim}{\to} \mathrm{H}_{3-i}(M, \mathbb{Z}/n\mathbb{Z})$ $(i = 0, 1, 2, 3)$ the isomorphism of the Poincaré duality defined by $u \mapsto u \cap [M]$, where

$$\cap : \mathrm{H}^i(M, \mathbb{Z}/n\mathbb{Z}) \times \mathrm{H}_3(M, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}_{3-i}(M, \mathbb{Z}/n\mathbb{Z})$$

denotes the cap product. Note that, by the universal coefficient theorems, we have

$$\mathrm{H}_1(M, \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{H}_1(M) \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathrm{H}_1(M)/n\mathrm{H}_1(M).$$

Together with the Hurewicz isomorphism, we obtain the isomorphisms

$$\begin{aligned} \mathrm{Hom}(\pi_1(M), \mathbb{Z}/n\mathbb{Z}) &\cong \mathrm{Hom}(\mathrm{H}_1(M), \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Hom}(\mathrm{H}_1(M, \mathbb{Z}/n\mathbb{Z}), \mathbb{Z}/n\mathbb{Z}) \\ &\cong \mathrm{H}^1(M, \mathbb{Z}/n\mathbb{Z}). \end{aligned}$$

We see that each $\rho \in \mathrm{Hom}(\pi_1(M), \mathbb{Z}/n\mathbb{Z})$ corresponds to $\rho_M^*(\mathrm{id}) \in \mathrm{H}^1(M, \mathbb{Z}/n\mathbb{Z})$ via these isomorphisms. We denote by $\tilde{\rho} \in \mathrm{Hom}(\mathrm{H}_1(M, \mathbb{Z}/n\mathbb{Z}), \mathbb{Z}/n\mathbb{Z})$ the homomorphism corresponding to ρ and $\rho_M^*(\mathrm{id})$.

We recall some calculations of the cohomology of groups.

Lemma 2.2.1. (1) *We have an isomorphism* $\mathrm{H}^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ *for every* $i \geq 0$.
(2) *The cohomology class* $c = \mathrm{id} \cup \beta(\mathrm{id}) \in \mathrm{H}^3(A, \mathbb{Z}/n\mathbb{Z})$ *is represented by a cochain* $\alpha : (\mathbb{Z}/n\mathbb{Z})^3 \to \mathbb{Z}/n\mathbb{Z}$ *defined by*

$$\alpha(g_1, g_2, g_3) = \frac{1}{n}\overline{g_1}(\overline{g_2} + \overline{g_3} - (\overline{g_2 + g_3})) \bmod n,$$

where $\overline{g} \in \{0, 1, \ldots, n-1\}$ *is a representative element of* $g \in \mathbb{Z}/n\mathbb{Z}$.
(3) *The 3rd cohomology group* $\mathrm{H}^3(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ *is generated by* $c = \mathrm{id} \cup \beta(\mathrm{id})$.

Proof. Consider the projective resolution of $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$-modules over $\mathbb{Z}$

$$\cdots \overset{\times p}{\to} \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] \overset{\times q}{\to} \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] \overset{\times p}{\to} \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] \overset{\times q}{\to} \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] \overset{\epsilon}{\to} \mathbb{Z},$$

where $p = \sum_{g \in \mathbb{Z}/n\mathbb{Z}} g$, $q = -(0 \bmod n) + (1 \bmod n)$, and $\epsilon(\sum_{g \in \mathbb{Z}/n\mathbb{Z}} a_g g) = \sum_{g \in \mathbb{Z}/n\mathbb{Z}} a_g$,
By taking the functor $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]}(-, \mathbb{Z}/n\mathbb{Z})$, we obtain the first assertion

(2.2.1.1) $$\mathrm{H}^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z} \ (i \geq 0).$$

By applying the snake lemma to the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\times n} & C^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n^2\mathbb{Z}) & \longrightarrow & C^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \longrightarrow & C^2(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\times n} & C^2(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n^2\mathbb{Z}) & \longrightarrow & C^2(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) & \longrightarrow & 0,
\end{array}$$

we obtain the second assertion. For the third assertion, by (2.2.1.1), it suffices to show that for each $n' = 1, 2, \ldots, n-1$, the cohomology class $n'c$ is not zero in $\mathrm{H}^3(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$. Assume that there is a cochain $b \in C^2(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ such that $db = n'\alpha$. Then, for each $(g_1, g_2, g_3) \in (\mathbb{Z}/n\mathbb{Z})^3$, we have

$$(n'\alpha)(g_1, g_2, g_3) = b(g_2, g_3) - b(g_1 + g_2, g_3) + b(g_1, g_2 + g_3) - b(g_1, g_2).$$

So we obtain

(2.2.1.2) $$\sum_{i=0}^{n-1} (n'\alpha)(1 \bmod n, i \bmod n, 1 \bmod n) = 0.$$

By the definition of α, we also have

$$(n'\alpha)(g_1, g_2, g_3) = \frac{n'}{n}\overline{g_1}(\overline{g_2} + \overline{g_3} - (\overline{g_2 + g_3})) \bmod n.$$

So we obtain

$$\sum_{i=0}^{n-1} (n'\alpha)(1 \bmod n, i \bmod n, 1 \bmod n) = n' \bmod n.$$

This contradicts the equation (2.2.1.2). □

Now we show the main assertion in this section.

Theorerm 2.2.2. *Notations being as above, let $u \in Z_2(M, \mathbb{Z}/n\mathbb{Z})$ be a 2-cycle that represents $\Phi^1(\rho_M^*(\mathrm{id})) \in \mathrm{H}_2(M, \mathbb{Z}/n\mathbb{Z})$. Then there is a 2-chain $D \in C_2(M, \mathbb{Z})$ such that $D \bmod n = u$ and there is a 1-cycle $\mathfrak{a} \in Z_1(M, \mathbb{Z})$ satisfying $\partial D = n\mathfrak{a}$. Let $[\mathfrak{a}]$ denote the homology class in $\mathrm{H}_1(M, \mathbb{Z}/n\mathbb{Z})$ defined by $\mathfrak{a}$. Then we have*

$$CS_c(\rho) = \tilde{\rho}([\mathfrak{a}]).$$

Proof. We consider the following commutative diagram,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_2(M,\mathbb{Z}) & \xrightarrow{\times n} & C_2(M,\mathbb{Z}) & \xrightarrow{\bmod n} & C_2(M,\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\
0 & \longrightarrow & C_1(M,\mathbb{Z}) & \xrightarrow{\times n} & C_1(M,\mathbb{Z}) & \xrightarrow{\bmod n} & C_1(M,\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & 0.
\end{array}$$

By the upper short exact sequence, there is a 2-chain $D \in C_2(M, \mathbb{Z})$ such that $D \bmod n = u$. Hence, we have

$$(\partial D)\mathrm{mod}\ n = \partial(D\mathrm{mod}\ n) = \partial u = 0.$$

Therefore, by the lower short exact sequence, there is a 1-cycle $\mathfrak{a} \in Z_1(M, \mathbb{Z})$ such that $\partial D = n\mathfrak{a}$. For the latter assertion, by direct calculation, we can check $\Phi^2 \circ \beta^1 = \beta_2 \circ \Phi^1$. Then, by Definition 2.1.2, we have,

$$\begin{aligned}
CS_c(\rho) &= < \rho_M^*(\mathrm{id}) \cup \beta^1(\rho_M^*(\mathrm{id})), [M] > \\
&= < \rho_M^*(\mathrm{id}), \Phi^2(\beta^1(\rho_M^*(\mathrm{id}))) > \\
&= \tilde{\rho}(\beta_2(\Phi^1(\rho_M^*(\mathrm{id})))).
\end{aligned}$$

Next, we consider the following commutative diagram,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times n} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow p_1 & & \downarrow p_2 & & \downarrow id & & \\
0 & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \xrightarrow{\times n} & \mathbb{Z}/n^2\mathbb{Z} & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0,
\end{array}$$

where p_1 and p_2 are natural projections, and id is the identity map. By considering the connecting homomorphism with respect to the singular homologies for each row, we see that $\beta_2 = {p_1}_* \circ \tilde{\beta}_2$. Then the required statement immediately follows by the definition of $\tilde{\beta}_2$. □

2.3 Explicit formulas of the mod 2 Dijkgraaf–Witten invariants for double branched covers of the 3-sphere

In this section, we prove explicit formulas of the mod 2 Dijkgraaf–Witten invariants for double branched covers of the 3-sphere. Keeping the notations as in the sections 2.1 and 2.2, we consider the case $A = \mathbb{Z}/2\mathbb{Z}$ and $c = \mathrm{id} \cup \beta(\mathrm{id}) \in \mathrm{H}^3(A, \mathbb{Z}/2\mathbb{Z})$ in Definition 2.1.2. A tame knot $\mathcal{K}$ is the image of a continuous embedding $S^1 \to S^3$ which extends to an embedding of a solid torus. Let $\mathcal{L} = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \cdots \cup \mathcal{K}_r$ be a tame link in the 3-sphere S^3. Let $h : M \to S^3$ denote the double covering ramified along $\mathcal{L}$, that is, h is obtained by the Fox completion [Fo] of the unramified covering $Y \to X := S^3 \backslash \mathcal{L}$ corresponding to the kernel of the surjective homomorphism $\mathrm{H}_1(X) \to \mathbb{Z}/2\mathbb{Z}$ that maps any meridian of $\mathcal{K}_i$ to $1 \in \mathbb{Z}/2\mathbb{Z}$. Recall that T_+ denotes the abelian group defined by

$$T_+ = \{(x_1, x_2, \cdots, x_r) \in (\mathbb{Z}/2\mathbb{Z})^r \mid \sum_{i=1}^{r} x_i = 0\}$$

and we put

$$e_{ij}^+ = (0, \cdots, 0, \overset{i\text{-th}}{1}, 0, \cdots, 0, \overset{j\text{-th}}{1}, 0, \cdots, 0) \in T_+$$

for each (i, j) with $1 \leqq i < j \leqq r$. By the topological analogue of Gauss's genus theory [Mo3; Corollary], there is an isomorphism

$$g : \mathrm{H}_1(M)/2\mathrm{H}_1(M) \overset{\sim}{\to} T_+ \tag{2.3.1}$$

given by

$$[\mathfrak{a}] \mapsto (\mathrm{lk}(h_*(\mathfrak{a}), \mathcal{K}_i) \bmod 2),$$

where lk(,) denotes the linking number. Hence we obtain the isomorphisms

$$\begin{aligned} \mathrm{Hom}(\pi_1(M), \mathbb{Z}/2\mathbb{Z}) &\cong \mathrm{Hom}(\mathrm{H}_1(M), \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}(\mathrm{H}_1(M, \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \\ &\cong \mathrm{Hom}(T_+, \mathbb{Z}/2\mathbb{Z}) \\ &\cong \mathrm{H}^1(M, \mathbb{Z}/2\mathbb{Z}). \end{aligned}$$

Now we prove the following formula.

Theorerm 2.3.2. *Notations being as above, for $\rho \in \mathrm{Hom}(T_+, \mathbb{Z}/2\mathbb{Z})$, we have*

$$CS_c(\rho) = \sum_{i<j} \rho(e_{ij}^+) \mathrm{lk}(\mathcal{K}_i, \mathcal{K}_j) \bmod 2.$$

Proof. Define elements $b_1, b_2, \cdots, b_{r-1} \in T_+$ by

$$b_1 = (1, 0, 0, \cdots, 1), b_2 = (0, 1, 0, 0, \cdots, 1), \cdots, b_{r-1} = (0, 0, \cdots, 0, 1, 1)$$

so that the tuple $(b_1, b_2, \cdots, b_{r-1})$ is a basis of T_+. Let $J = \{j_1, j_2 \cdots, j_m\} \subset \{1, 2, \cdots, r-1\}$ with $j_1 < j_2 \cdots < j_m$ and suppose that $\rho(b_i) = 1$ if and only if $i \in J$ For each $i = 1, 2, \cdots, r$, let $\mathcal{S}_i$ be a Seifert surface of $\mathcal{K}_i$ in S^3, and put $\tilde{\mathcal{K}}_i := h^{-1}(\mathcal{K}_i)$ and $\tilde{\mathcal{S}}_i := h^{-1}(\mathcal{S}_i)$. Let $u \in Z_2(M, \mathbb{Z}/n\mathbb{Z})$ be a 2-cycle that represents $\Phi^1(\rho_M^*(\mathrm{id})) \in \mathrm{H}_2(M, \mathbb{Z}/n\mathbb{Z})$. There is a 2-chain $D \in C_2(M, \mathbb{Z})$ such that $D \bmod 2 = u$ and a 1-cycle $\mathfrak{a}_\rho \in Z_1(M, \mathbb{Z})$ satisfying $\partial D = 2\mathfrak{a}_\rho$. In order to apply Theorem 2.2.2, let us explicitly find such a D. Let $\mathbf{a} = (a_1, a_2, \cdots, a_r) \in T_+$ and let $\mathfrak{a} \in Z_1(M, \mathbb{Z})$ whose image $[\mathfrak{a}]$ corresponds to $\mathbf{a}$ via the isomorphism $\mathrm{H}_1(M)/2\mathrm{H}_1(M) \xrightarrow{\sim} T_+$ of the topological analogue of Gauss's genus theory. We note that the mod 2 linking number $(\mathrm{lk}(h_*(\mathfrak{a}), h_*(\partial D)) \bmod 2)$ is equal to the mod 2 intersection number of $\mathfrak{a}$ and D. Therefore, by the Poincaré duality, a 2-chain $D \in C_2(M, \mathbb{Z})$ satisfies $u = D \bmod 2 \in Z_2(M, \mathbb{Z}/n\mathbb{Z})$ for some u with $[u] = \Phi^1(\rho_M^*(\mathrm{id}))$ if and only if

$$\mathrm{lk}(h_*(\mathfrak{a}), h_*(\partial D)) \bmod 2 = \rho(\mathbf{a}).$$

Therefore, we may put

$$D = \sum_{i=1}^{r} \tilde{\mathcal{S}}_i - \sum_{i \in \{j_1, j_2, \cdots, j_m\}} \tilde{\mathcal{S}}_i.$$

In this case, the 1-cycle

$$\mathfrak{a}_\rho = \sum_{i=1}^{r} \tilde{\mathcal{K}}_i - \sum_{i \in \{j_1, j_2, \cdots, j_m\}} \tilde{\mathcal{K}}_i$$

satisfies $\partial D = 2\mathfrak{a}_\rho$. By Theorem 2.2.2, we obtain

$$\begin{aligned} CS_c(\rho) &= \tilde{\rho}([\mathfrak{a}_\rho]) \\ &= \rho(g([\mathfrak{a}_\rho])) \\ &= \rho((\mathrm{lk}(h_*(\mathfrak{a}_\rho), \mathcal{K}_i) \bmod 2) \\ &= \sum_{l=1}^{m} \mathrm{lk}(h_*(\mathfrak{a}_\rho), \mathcal{K}_{j_l}) \bmod 2 \\ &= \sum_{l=1}^{m} \sum_{i \notin \{j_1, j_2, \cdots, j_m\}} \mathrm{lk}(\mathcal{K}_i, \mathcal{K}_{j_l}) \bmod 2 \\ &= \sum_{i<j} \rho(e_{ij}^+) \mathrm{lk}(\mathcal{K}_i, \mathcal{K}_j) \bmod 2. \end{aligned}$$

□

By Definition 2.1.2, the mod 2 Dijkgraaf–Witten invariant is given by

$$Z_c(M) = \frac{1}{2} \sum_{\rho \in \mathrm{Hom}(T_+, \mathbb{Z}/2\mathbb{Z})} \exp\left(\pi i \; CS_c(\rho)\right).$$

Hence we obtain the following.

Corollary 2.3.3. *Notations being as above, we have*

$$Z_c(M) = \frac{1}{2} \sum_{\rho \in \mathrm{Hom}(T_+, \mathbb{Z}/2\mathbb{Z})} \prod_{i<j} (-1)^{\rho(e_{ij}^{+})\mathrm{lk}(\mathcal{K}_i, \mathcal{K}_j)}$$

Example 2.3.4. Let L be a two-bridge link $B(a,b)$ ($0 < a < b$, b: even, $(a,b) = 1$). So we have $r = 2$ and $\mathrm{Hom}(T_+, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Then, the double branched cover M is the lens space $L(a,b)$. By Theorem 2.3.2 and [Tu; p.540 and p.543], for each $0 \neq \rho \in \mathrm{Hom}(T_+, \mathbb{Z}/2\mathbb{Z})$, we have

$$CS_c(\rho) = \sum_{k=1}^{b/2} (-1)^{\lfloor (2k-1)a/b \rfloor} \bmod 2,$$

where $\lfloor\ \rfloor$ denotes the floor function. Therefore, we also have

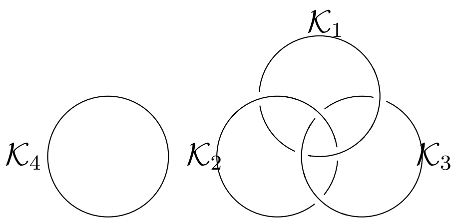
$$Z_c(M) = \begin{cases} 1, & \text{if } \displaystyle\sum_{k=1}^{b/2} (-1)^{\lfloor (2k-1)a/b \rfloor} \text{ is even,} \\ 0, & \text{if otherwise.} \end{cases}$$

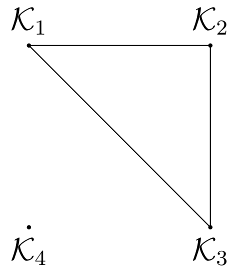
Now We introduce the mod 2 *linking diagram* $D_{\mathcal{L}}$ of $\mathcal{L}$ as follows. The diagram $D_{\mathcal{L}}$ consists of r vertices and edges. Each vertex represents each component knot $\mathcal{K}_i$ and two vertices $\mathcal{K}_i$ and $\mathcal{K}_j$ are adjacent by an edge if and only if the linking number $\mathrm{lk}(\mathcal{K}_i, \mathcal{K}_j) \equiv 1 \mod 2$. The diagrram is called a *circuit* (or *closed trail*) if it can be written in one-stroke. A graph consisting of a single vertex is considered to be a circuit. The following formula can be proved by using genus theory for M.

Theorerm 2.3.5 ([Hi], [DK])**.** *Notations being as above, we have*

$$Z(M) = \begin{cases} 2^{r-2} & \textit{if any connected component of } D_{\mathcal{L}} \textit{ is a circuit} \\ 0 & \textit{otherwise.} \end{cases}$$

Example 2.3.6. Let $\mathcal{L} = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3 \cup \mathcal{K}_4$ be the following link (left figure) in S^3 so that the mod 2 linking diagram $D_{\mathcal{L}}$ is given by the right figure. Let M be the double covering of S^3 ramified along $\mathcal{L}$. By Theorem 2.3.5, we have $Z(M) = 2^2 = 4$.





Remark 2.3.7. In the context of quantum topology, Murakami, Ohtsuki and Okada calculated the mod n Dijkgraaf–Witten invariant for the 3-manifold obtained by a Dehn surgery on S^3 along a framed link and expressed the mod n Dijkgraaf–Witten invariant in terms of Gaussian sums and the linking matrix of the framed link [MOO; Proposition 9.1]. In Chapter 4, for number fields, we will show a formula given in a form similar to Gaussian sums (Theorem 4.5.2). So we may expect that the cases with non-abelian gauge groups would be given by a non-abelian generalization of Gaussian sums.

3 Dijkgraaf–Witten TQFT for 3-manifolds with boundaries

In this chapter, we recall the definition of topological quantum field theory, for the case of (2+1)-dimension, and construct the TQFT structure for Dijkgraaf–Witten theory following Gomi ([Gm]). To be precise, for an oriented closed surface Σ, we construct the Chern–Simons 1-cocycle and the prequantization bundle. For a oriented compact 3-manifold M, we introduce the Chern–Simons functional. We then construct the quantum Hilbert space and the Dijkgraaf–Witten partition function in (2+1)-dimensional Chern–Simons TQFT.

3.1 The definition of (2+1)-dimensional TQFT

We start to recall the definition of topological quantum field theory, due to Atiyah, for the case of (2+1)-dimension.

Definition 3.1.1 ([At1], [At2]). A (2+1)-dimensional topological quantum field theory, called *TQFT* for short, consists of the following correspondences (a functor from the cobordism category of surfaces to the category of $\mathbb{C}$-vector spaces)

$$\begin{array}{rcl} \text{oriented closed surface } \Sigma & \rightsquigarrow & \mathbb{C}\text{-vector space } \mathcal{H}_\Sigma, \\ \text{oriented compact 3-manifold } M & \rightsquigarrow & \text{vector } Z_M \in \mathcal{H}_{\partial M}, \end{array}$$

where $\mathcal{H}_\Sigma$ is called the *state space* or *the quantum Hilbert space* and Z_M is called the *partition function*. These correspondences must satisfy the following axioms.
($A1$) *functoriality.* An orientation preserving homeomorphism $f : \Sigma \overset{\approx}{\to} \Sigma'$ induces an isomorphism $\mathcal{H}_\Sigma \overset{\sim}{\to} \mathcal{H}_{\Sigma'}$ of Hilbert quantum spaces. Moreover, if f extends to an orientation preserving homeomorphism $M \overset{\approx}{\to} M'$, with $\partial M = \Sigma, \partial M' = \Sigma'$, then Z_M is sent to $Z_{M'}$ under the induced isomorphism $\mathcal{H}_{\partial M} \overset{\sim}{\to} \mathcal{H}_{\partial M'}$.
($A2$) *multiplicativity.* For disjoint surfaces Σ_1, Σ_2, we require and the surface $\Sigma^* = \Sigma$ with the opposite orientation, we require

$$\mathcal{H}_{\Sigma_1 \sqcup \Sigma_2} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}.$$

This multiplicative property is indicative of the quantum feature of the theory.
($A3$) *involutority.* For a surface Σ^*, which is Σ with opposite orientation, we require

$$\mathcal{H}_{\Sigma^*} = (\mathcal{H}_\Sigma)^*,$$

where $(\mathcal{H}_\Sigma)^*$ is the dual vector space of $\mathcal{H}_\Sigma$.
$(A3)$ *Gluing formula.* If $\partial M_1 = \Sigma_1 \sqcup \Sigma_2, \partial M_2 = \Sigma_2^* \sqcup \Sigma_3$ and M is the 3-manifold obtained by gluing M_1 and M_2 along Σ_2, then we require

$$< Z_{M_1}, Z_{M_2} >= Z_M,$$

where $< \cdot, \cdot >: \mathcal{H}_{\Sigma_1 \sqcup \Sigma_2} \times \mathcal{H}_{\Sigma_2^* \sqcup \Sigma_3} \to \mathcal{H}_{\Sigma_1 \sqcup \Sigma_3}$ is the natural gluing pairing of state spaces obtained by (A2), (A3).
When $\Sigma = \emptyset$ (empty), we suppose $H_\emptyset = \mathbb{C}$, and so we suppose $Z_M = Z(M) \in \mathbb{C}$ when M is closed. we also suppose $Z_{\Sigma \times [0,1]} = \mathrm{id}_{H_\Sigma}$.

3.2 The TQFT structure for Dijkgraaf–Witten Theory

In this section, we recall the construction of the TQFT structure for Dijkgraaf–Witten theory, following [Gm]. We refer to [DW], [FQ], [Wa], [Ye] for other constructions. We fix a finite group and a 3-cocycle $c \in Z^3(G, \mathbb{R}/\mathbb{Z})$ once and for all. Let X be an oriented compact manifold X with a fixed finite triangulation $\mathcal{T}$. Let $\mathcal{T}^{(n)}$ denotes the set of n-simplices in $\mathcal{T}$. Each $\sigma_{i_0 \cdots i_n} \in \mathcal{T}^{(n)}$ for $i_0 < i_1 < \cdots < i_n$ has the orientation determined by the numberings $i_0 \cdots i_n$ assigned to vertices and $\sigma_{\pi(i_0) \cdots \pi(i_n)}$ for a permutation π of $i_0 \cdots i_n$ is defined by $\mathrm{sgn}(\pi) \sigma_{i_0 \cdots i_n}$. We then define the gauge group $\mathcal{G}_X$ on X associated to G by

$$\mathcal{G} := \{\gamma : \mathcal{T}^{(0)} \to G\}$$

and define the space $\mathcal{F}_X$ of gauge fields on X associated to G by

$$\mathcal{F}_X := \{\varrho : \mathcal{T}^{(1)} \to G \mid \varrho(\sigma_{i_0 i_1}) \varrho(\sigma_{i_1 i_2}) = \varrho(\sigma_{i_0 i_2}) \text{ for } i_0 < i_1 < i_2\}$$

on which $\mathcal{G}_X$ acts from the right by

$$(\varrho.\gamma)(\sigma_{ij}) := \gamma(\sigma_i)^{-1} \varrho(\sigma_{ij}) \gamma(\sigma_j).$$

Note that $\mathcal{F}_X$ and $\mathcal{G}_X$ are finite sets. We remark that the quotient space $\mathcal{F}_X / \mathcal{G}_X$ is identified with the set $\mathcal{M}_X$ of isomorphism classes principal G-bundles (G-torsors) on X by the parallel transport along 1-simplices σ_{ij} and that the holonomy gives the bijection between $\mathcal{M}_X$ and $\mathrm{Hom}(\pi_1(X), G)/G$ if X is connected, where $\mathrm{Hom}(\pi_1(X), G)/G$ is the quotient of $\mathrm{Hom}(\pi_1(X), G)$ by the conjugate action of G from the right:

$$\mathrm{Hom}(\pi_1(X), G) \times G \to \mathrm{Hom}(\pi_1(X), G); \ (\varrho, g) \mapsto g^{-1} \varrho g.$$

We construct the classical theory in the sense of physics. The key ingredient is the transgression homomorphism for an oriented compact d-manifold X and $m \geq d$

$$\mathrm{trans}_X^m : C^m(G, \mathbb{R}/\mathbb{Z}) \longrightarrow C^{m-d}(\mathcal{G}_X, \mathrm{Map}(\mathcal{F}_X, \mathbb{R}/\mathbb{Z})),$$

where $\mathrm{Map}(\mathcal{F}_X, \mathbb{R}/\mathbb{Z}))$ is the additive group of maps $\mathcal{F}_X \to \mathbb{R}/\mathbb{Z}$, on which $\mathcal{G}_X$ acts from the left by $(\gamma.\psi)(\varrho) := \psi(\varrho.\gamma)$ for $\gamma \in \mathcal{G}_X$, $\psi \in \mathrm{Map}(\mathcal{F}_X, \mathbb{R}/\mathbb{Z})$, and $\varrho \in \mathcal{F}_X$. The explicit expressions of trans_X for the cases that $d = m, m{+}1$ are given as follows. For $\gamma \in \mathcal{G}_X, \varrho \in \mathcal{F}_X$ and $\sigma_i \in \mathcal{T}^{(0)}, \sigma_{ij} \in \mathcal{T}^{(1)}$, we set $\gamma_i := \gamma(\sigma_i), \varrho_{ij} := \varrho(\sigma_{ij})$.

・ Case that $d = m$. For $\alpha \in C^m(G, \mathbb{R}/\mathbb{Z}), \varrho \in \mathcal{F}_X$, we have

$$\mathrm{trans}_X^m(\alpha)(\varrho) = \sum_{\sigma_{01\cdots m} \in \mathcal{T}^{(m)}} \varepsilon_{01\cdots m}\alpha(\varrho_{01}, \ldots, \varrho_{m-1\ m}),$$

where $\varepsilon_{01\cdots m} = 1$ if the orientations of $\sigma_{01\cdots m}$ and X coincides, and $\varepsilon_{01\cdots m} = -1$ otherwise.

・ Case that $d = m+1$. For $\alpha \in C^m(G, \mathbb{R}/\mathbb{Z}), \gamma \in \mathcal{G}_X$, and $\varrho \in \mathcal{F}_X$, we have

$$(\mathrm{trans}_X^m(\alpha))(\gamma)(\varrho) = \sum_{\sigma_{01\cdots m-1} \in \mathcal{T}^{(m-1)}} \varepsilon_{01\cdots m-1} \sum_{j=0}^{m-1} (-1)^j$$

$$\alpha({\gamma_0}^{-1}\varrho_{01}\gamma_1, \cdots, {\gamma_{j-1}}^{-1}\varrho_{j-1\ j}\gamma_j, \varrho_{j\ j+1}, \cdots, \varrho_{m-2\ m-1}),$$

where the signature $\varepsilon_{01\cdots m-1}$ is defined as above. It can be shown that the following Stokes type formula holds:

$$\mathrm{trans}_X^{m+1} \circ \delta = (-1)^d \delta \circ \mathrm{trans}_X^m + \mathrm{res}^*(\mathrm{trans}_{\partial X}^m),$$

where δ denotes the coboundary map of group cochains and res denotes the map on the cochain induced by the restriction $\mathcal{F}_X \to \mathcal{F}_{\partial X}$ (resp. $\mathcal{G}_X \to \mathcal{G}_{\partial X}$).

Now, let $m = 3$ and consider the cases that X is an oriented closed surface Σ ($d = 2$) or X is an oriented compact 3-manidold M ($d = 3$), and we set

$$\lambda_\Sigma := \mathrm{trans}_\Sigma^3(c) \in C^1(\mathcal{G}_\Sigma, \mathrm{Map}(\mathcal{F}_\Sigma, \mathbb{R}/\mathbb{Z})) \tag{3.2.1}$$

$$CS_M := \mathrm{trans}_M^3(c) \in \mathrm{Map}(\mathcal{F}_M, \mathbb{R}/\mathbb{Z}), \tag{3.2.2}$$

which are explicitly given as follows:

$$\begin{aligned}\lambda_\Sigma(\varrho) = \sum_{\sigma_{012} \in \mathcal{T}^{(2)}} \varepsilon_{\sigma_{012}} \{ & c(\gamma_0, \varrho_{01}, \varrho_{12}) - c(\gamma_0\varrho_{01}{\gamma_1}^{-1}, \gamma_1, \varrho_{12}) \\ & + c(\gamma_0\varrho_{01}{\gamma_1}^{-1}, \gamma_1\varrho_{12}{\gamma_2}^{-1}, \gamma_2)\} \quad (\gamma \in \mathcal{G}_\Sigma, \varrho \in \mathcal{F}_\Sigma),\end{aligned} \tag{3.2.3}$$

$$CS_M(\varrho) = \sum_{\sigma_{0123} \in \mathcal{T}^{(3)}} \varepsilon_{\sigma_{0123}} c(\varrho_{01}, \varrho_{12}, \varrho_{23}) \quad (\varrho \in \mathcal{F}_M), \tag{3.2.4}$$

where $\varepsilon_\sigma := 1$ is the orientations of σ and X coincides, and $\varepsilon_\sigma := -1$ otherwise. We suppose that the triangulation $\mathcal{T}_{\partial M}$ on ∂M is the restriction of the triangulation $\mathcal{T}_M$ of M. Since c is a $3-$*cocycle*, by (3.2.1), (3.2.2), (3.2.3), and (3.2.4), we have

$$\delta CS_M = \mathrm{res}^*\lambda_{\partial M}, \delta\lambda_\Sigma = 0. \tag{3.2.5}$$

We call λ_Σ the *Chern–Simons 1-cocycle* associated to c for an oriented closed surface Σ. The cohomology class of λ_Σ is independent of the choice of a finite triangulation $\mathcal{T}$. We call CS_M the *Chern–Simons functional* associated to c for an oriented compact 3-manifold M.

Using λ_Σ, we define a $\mathcal{G}$-equivariant principal $\mathbb{R}/\mathbb{Z}$-bundle $\mathcal{L}_\Sigma$ by the product bundle

$$\mathcal{L}_\Sigma := \mathcal{F}_\Sigma \times \mathbb{R}/\mathbb{Z},$$

on which $\mathcal{G}_\Sigma$ acts by $(\varrho, m).\gamma = (\varrho.\gamma, m + \lambda_\Sigma(\gamma)(\varrho))$ for $\varrho \in \mathcal{F}_\Sigma, m \in \mathbb{R}/\mathbb{Z}$ and $\gamma \in \mathcal{G}_\Sigma$. It depends on the cohomology class of λ_Σ up to isomorphism of $\mathcal{G}_\Sigma$-equivariant principal $\mathbb{R}/\mathbb{Z}$-bundles. We call $\mathcal{L}_\Sigma$ the *prequantization principal* $\mathbb{R}/\mathbb{Z}$*-bundle* over $\mathcal{F}_\Sigma$. Let L_Σ be the complex line bundle associated to $\mathcal{L}_\Sigma$ via the homomorphism $\mathbb{R}/\mathbb{Z} \to \mathbb{C}^\times$; $m \mapsto e^{2\pi\sqrt{-1}m}$, and we have the complex line bundle $\overline{L}_\Sigma$ over $\mathcal{M}_X$. The line bundle L_Σ (or $\overline{L}_\Sigma$) is called the *prequantization complex line bundle* for a surface Σ. By (3.2.5), we see that CS_M (resp. $e^{2\pi\sqrt{-1}CS_M}$) is a $\mathcal{G}_M$-equivariant section of $\mathrm{res}^*\mathcal{L}_{\partial M}$ (resp. $\mathrm{res}^*L_{\partial M}$) over $\mathcal{F}_M$.

We construct the quantum theory in the sense of physics, namely, the correspondences in Definition 3.1.1 of (2+1)-dimensional TQFT. We define the state space $\mathcal{H}_\Sigma$ for an oriented closed surface Σ by the space of sections of the prequantization bundle $\overline{L}_\Sigma$ over $\mathcal{M}_\Sigma$, equivalently, the space of $\mathcal{G}_\Sigma$-equivariant sections of the prequantization line bundle L_Σ over $\mathcal{F}_\Sigma$:

(3.2.6)

$$\begin{aligned}\mathcal{H}_\Sigma &:= \Gamma(\mathcal{M}_\Sigma, \overline{L}_\Sigma)\\ &= \{\theta : \mathcal{F}_\Sigma \to \mathbb{C} \,|\, \theta(\varrho.\gamma) = e^{2\pi\sqrt{-1}\lambda_\Sigma(\gamma)(\varrho)}\theta(\varrho) \text{ for } \varrho \in \mathcal{F}_\Sigma, \gamma \in \mathcal{G}_\Sigma\}.\end{aligned}$$

We call $\mathcal{H}_\Sigma$ the *Dijkgraaf–Witten state space* and the above construction is along the line similar to the geometric quantization. We define the *Dijkgraaf–Witten partition function* by the following sum fixing the boundary condition

$$Z_M(\varrho_{\partial M}) = \frac{1}{\#'\mathcal{T}_M^{(0)}} \sum_{\substack{\varrho\in\mathcal{F}_M\\ \mathrm{res}(\varrho)=\varrho_{\partial M}}} e^{2\pi\sqrt{-1}CS_M(\varrho)} \tag{3.2.7}$$

for $\varrho_{\partial M} \in \mathcal{F}_{\partial M}$, where $\#'\mathcal{T}_M^{(0)}$ is the number of 0-simplices in the interior of M. By (3.2.5), we see $Z_M \in \mathcal{H}_{\partial M}$. The value $Z_M(\varrho_{\partial M})$ is called the *Dijkgraaf–Witten invariant* of $\varrho_{\partial M} \in \mathcal{F}_{\partial M}$.

Remark 3.2.8. The above constructions depends only on the cohomology class of the fixed cocycle c. So we may take c to be nomalized. Furthermore, the above constructions turn out to be independent of the choice of triangulations of Σ and M. Suppose that another choice of triangulation $\mathcal{T}'_\Sigma$ of Σ yields $\mathcal{H}_\Sigma^{\mathcal{T}'_\Sigma}$ as above. Then it can be shown that there is an isomorphism $\Theta : \mathcal{H}_\Sigma^{\mathcal{T}_\Sigma} \xrightarrow{\sim} \mathcal{H}_\Sigma^{\mathcal{T}'_\Sigma}$. So taking the colimit of $\mathcal{H}_\Sigma^{\mathcal{T}_\Sigma}$'s with respect to triangulations $\mathcal{T}_\Sigma$ of Σ, we obtain the state space $\mathcal{H}_\Sigma$ which is independent of $\mathcal{T}_\Sigma$. Suppose that another choice of triangulation $\mathcal{T}'_M$ of M yields $Z_M^{\mathcal{T}'_M} \in \mathcal{H}_{\partial M}^{\mathcal{T}'_{\partial M}}$. Then we can show $\Theta(Z_M^{\mathcal{T}_M}) = Z_M^{\mathcal{T}'_M}$ and so we have a topological invariant $Z_M \in \mathcal{H}_{\partial M}$ ([Gm]).

Theorerm 3.2.9. *The above correspondences*

$$\begin{array}{rcl} \textit{oriented closed surface } \Sigma & \rightsquigarrow & \mathbb{C}\textit{-vector space } \mathcal{H}_\Sigma, \\ \textit{oriented compact 3-manifold } M & \rightsquigarrow & \textit{vector } Z_M \in \mathcal{H}_{\partial M}, \end{array}$$

satisfy the axioms (A1) ∼ (A4) in the Definition 3.1.1 of the (2+1)-dimensional TQFT.

For the proof of Theorem 3.2.9, we consult the references [Gm], [DW], [Wa], [FQ], [Ye].

Remark 3.2.10. For Chern–Simons theory with a compact Lie gauge group, it is known that the state space $\mathcal{H}_\Sigma$ is isomorphic to the space of conformal blocks ([Ko]) and its element is called an non-Abelian theta function ([BL]). The dimension of $\mathcal{H}_\Sigma$ is given by Verlinde's formula([Ve]). Dijkgraaf–Witten theory is a finite analogue and an element of $\mathcal{H}_\Sigma$ may be regarded as a sort of non-Abelian finite theta function or non-Abelian Gaussian sum.

4 Arithmetic Chern–Simons functionals and arithmetic Dijkgraaf–Witten invariants for number fields

In this chapter, we study arithmetic analogues for number rings of Chern–Simons functional and Dijkgraaf–Witten invariant for 3-manifolds in Chapter 2. We list herewith some analogies which will be used in this section.

3-dimensional topology	number theory
connected, oriented, and closed 3-manifold M	compactified spectrum of a number ring $\overline{X} = \overline{\text{Spec } \mathcal{O}_K}$
knot $\mathcal{K} : S^1 \to M$	maximal ideal $\mathfrak{p} : \text{Spec } (\mathcal{O}_K/\mathfrak{p}) \to \text{Spec } (\mathcal{O}_K)$
link $L = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \cdots \cup \mathcal{K}_r$	finite set of maximal ideals $S = \{\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_r\}$
fundamental group $\pi_1(M)$	modified étale fundamental group $\pi_1(\overline{X})$
1-cycle group $Z_1(M)$	ideal group I_K
mod 2 linking number $\text{lk}(\mathcal{K}_1, \mathcal{K}_2)$ mod 2	Legendre symbol $\left(\dfrac{p_1}{p_2}\right)$
1-boundary group $B_1(M)$ $\partial : C_2(M) \to Z_1(M); S \mapsto \partial S$	principal ideal group P_K $\partial : K^\times \to I_K; a \mapsto (a)$
1st integral homology group $\text{H}_1(M) = Z_1(M)/B_1(M)$	ideal class group $\text{Cl}_K = I_K/P_K$
Hurewicz isomorphism $\pi_1(M)^{ab} \cong \text{Gal}(M^{ab}/M) \cong \text{H}_1(M)$	Artin reciprocity $\pi_1(\overline{X})^{ab} \cong \text{Gal}(\tilde{K}^{ab}/K) \cong \text{Cl}_K$
Poincaré duality $\text{H}^i(M, \mathbb{Z}/n\mathbb{Z}) \cong \text{H}_{3-i}(M, \mathbb{Z}/n\mathbb{Z})$	Artin–Verdier duality $\text{H}^i(\overline{X}, \mathbb{Z}/n\mathbb{Z}) \cong \text{Ext}^{3-i}_{\overline{X}}(\mathbb{Z}/n\mathbb{Z}, \phi_* G_{m,X})^{\sim}$

Based on the analogies recalled above, we give the definition of the arithmetic Chern–Simons invariant for any number rings, by using the modified étale cohomology groups and fundamental groups which take real primes into account, and introduce the arithmetic Dijkgraaf–Witten invariant. We then show explicit formulas of mod 2 arithmetic Dijkgraaf–Witten invariants for real quadratic fields $\mathbb{Q}(\sqrt{p_1 p_2 \cdots p_r})$, where p_i's are distinct prime numbers congruent to 1 mod 4, in terms of the Legendre symbols of p_i's. The contents of this chapter are based on [Hi].

4.1 The Artin–Verdier site and the modified étale fundamental group

Let K be a finite algebraic number field and let $X = \text{Spec } \mathcal{O}_K$ be the prime spectrum of the ring $\mathcal{O}_K$ of integers of K. Let X_∞ denote the set of infinite primes, namely, real primes and pairs of conjugate complex primes of K, and we set $\overline{X} = X \sqcup X_\infty$. Let Y be a scheme which is étale over X. A real prime of Y is defined by a point $y : \text{Spec } \mathbb{C} \to Y$ which factors through Spec $\mathbb{R}$. A complex prime of Y is defined to be a pair of complex conjugate points $y, \overline{y} : \text{Spec } \mathbb{C} \to Y$ such that $y \neq \overline{y}$. An infinite prime of Y is a real prime or a complex prime of Y. Let Y_∞ denote the set of infinite primes of Y. Note that an étale morphism $f : Y \to X$ induces $f_\infty : Y_\infty \to X_\infty$. We say that f_∞ is unramified at $y_\infty \in Y_\infty$ if y_∞ is a real prime or if $(y_\infty, f_\infty(y_\infty))$ is a complex prime. Regarding Grothendieck topologies, we refer to [Ar] and [Ta].

Definition 4.1.1 ([AC; Definition 2.1], [Bi; Proposition 1.2])**.** The *Artin–Verdier site* $\overline{X}_{\text{ét}}$ of $\overline{X}$ is the Grothendieck topology consisting of the category $\text{Cat}(\overline{X}_{\text{ét}})$ and a set $\text{Cov}(\overline{X}_{\text{ét}})$ of coverings defined as follows.

- An object in $\text{Cat}(\overline{X}_{\text{ét}})$ is a pair (Y, M), where $f : Y \to X$ is a scheme étale over X and $M \subset Y_\infty$ such that $f_\infty| : M \to X_\infty$ is unramified. A morphism $\varphi : (Y_1, M_1) \to (Y_2, M_2)$ in $\text{Cat}(\overline{X}_{\text{ét}})$ is a morphism of shemes $\varphi : Y_1 \to Y_2$ over X such that the induced map $\varphi_\infty : (Y_1)_\infty \to (Y_2)_\infty$ satisfies $\varphi_\infty(M_1) \subset M_2$.

- A covering in $\text{Cov}(\overline{X}_{\text{ét}})$ is a family of morphisms $\{\varphi_i : (Y_i, M_i) \to (Z, N)\}_{i \in I}$ in $\text{Cat}(\overline{X}_{\text{ét}})$ which satisfies $\underset{i}{\cup}\varphi_i(Y_i) = Z$ and $\underset{i}{\cup}\varphi_i(M_i) = N$.

Remark 4.1.2. In $\text{Cat}(\overline{X}_{\text{ét}})$, the fiber product of morphisms $\varphi_i : (Y_i, M_i) \to (Z, N)$ $(i = 1, 2)$ is defined by $(Y_1 \underset{Z}{\times} Y_2, M_3)$, where $Y_1 \underset{Z}{\times} Y_2$ is the fiber product in the category of schemes and M_3 is the set consisting of points of $(Y_1 \underset{Z}{\times} Y_2)_\infty$ whose images are in M_i under the projections $(Y_1 \underset{Z}{\times} Y_2)_\infty \to Y_{i\infty}$ for $i = 1, 2$. We can check easily that M_3 is isomorphic to $M_1 \underset{N}{\times} M_2$ in the category of sets.

Next, we introduce a Galois category to define the modified étale fundamental group.

We say that $(Y, M) \in \text{Cat}(\overline{X}_{\text{ét}})$ is *finite étale* if $Y \to X$ is a finite étale morphism of schemes over X and $M = Y_\infty$. We denote by $\text{FEt}_{\overline{X}}$ the full subcategory of $\overline{X}_{\text{ét}}$ whose objects are finite étale, and denote by FSets the category of finite sets.

In the following, we often abbreviate (Y, Y_∞) to $\overline{Y}$ for a scheme Y étale over X. Let $\overline{K}$ be an algebraic closure of K and let $\tilde{\eta} : \mathrm{Spec}\ \overline{K} \to X$ be a geometric point. Then we have functors

$$F_{\tilde{\eta}} : \mathrm{FEt}_{\overline{X}} \to \mathrm{FSets};\ \overline{Y} \mapsto \mathrm{Hom}_X(\tilde{\eta}, Y),$$

$$U : \mathrm{FEt}_{\overline{X}} \to \mathrm{FEt}_X;\ \overline{Y} \mapsto Y.$$

We note that the forgetful functor U is fully faithful.

Definition 4.1.3 ([SGA I ; V.4]). Let $\mathscr{C}$ be a category and let $F : \mathscr{C} \to \mathrm{FSets}$ be a covariant functor. $\mathscr{C}$ is called *a Galois category* with *a fiber functor F* if $\mathscr{C}$ and F satisfy the following axioms.

(G1) $\mathscr{C}$ has a final object and finite fiber products.

(G2) Finite direct sums exist in $\mathscr{C}$. The quotient of an object by a finite group of automorphisms exist in $\mathscr{C}$.

(G3) Let $u : A_1 \to A_2$ be a morphism in $\mathscr{C}$. Then u factors into a composition $A_1 \xrightarrow{f} A' \xrightarrow{g} A_2$, where f is a strict epimorphism and g is a monomorphism which is an isomorphism on a direct summand of A_2.

(G4) F is a left exact functor.

(G5) F commutes with finite direct sums and the quotient of an object by a finite group of automorphisms. F sends strict epimorphisms to epimorphisms.

(G6) Let $u : A_1 \to A_2$ be a morphism in $\mathscr{C}$ such that $F(u)$ is an isomorphism. Then u is an isomorphism.

Proposition 4.1.4. $\mathrm{FEt}_{\overline{X}}$ *is a Galois category with a fiber functor $F_{\tilde{\eta}}$.*

Proof. We check the six axioms (G1)∼(G6) of Galois categories for $\mathrm{FEt}_{\overline{X}}$ and $F_{\tilde{\eta}}$. It is well-known that the category FEt_X of schemes finite étale over X is a Galois category with a fiber functor $F'_{\tilde{\eta}} : \mathrm{FEt}_X \to \mathrm{FSets}\ Y \mapsto \mathrm{Hom}_X(\tilde{\eta}, Y)$ [SGA I ; V.7], so that FEt_X and $F'_{\tilde{\eta}}$ admit the axioms (G1)∼(G6) and $F'_{\tilde{\eta}}$. Let us verify (G1)∼(G6) for $\mathrm{FEt}_{\overline{X}}$.
(G1) $\mathrm{FEt}_{\overline{X}}$ has a final object $(id : X \to X, X_\infty)$. For $\overline{Y_i} \in \mathrm{FEt}_{\overline{X}}$ $(i = 1, 2, \cdots, m)$, we see that $\prod_i Y_i \in \mathrm{FEt}_X$. So we have $\prod_i \overline{Y_i} = \overline{\prod_i Y_i}$ by the universal property of fiber products.
(G2) $\mathrm{FEt}_{\overline{X}}$ has an initial object $(\mathrm{Spec}\ 0, (\mathrm{Spec}\ 0)_\infty) = (\emptyset, \emptyset)$. In a similar way to (G1), we see that $\mathrm{FEt}_{\overline{X}}$ admits finite direct sums. For $\overline{Y} \in \mathrm{FEt}_{\overline{X}}$

and a finite subgroup $G \subset \mathrm{Aut}_{\overline{X}}(\overline{Y})$, we have $\mathrm{Aut}_{\overline{X}}(\overline{Y}) = \mathrm{Aut}_X(Y)$ by the definition of morphisms of $\mathrm{Cat}(\overline{X}_{\text{ét}})$. So we have the quotient of $Y \to X \in \mathrm{FEt}_X$ by $G \subset \mathrm{Aut}_X(Y)$ and then one can check $\overline{Y}/G = \overline{Y/G}$.
(G3) For any morphism $\overline{Y_1} \to \overline{Y_2}$ in $\mathrm{FEt}_{\overline{X}}$, $Y_1 \to Y_2$ factors as
$Y_1 \xrightarrow{f} Y' \xrightarrow{g} Y' \bigsqcup Y'' \cong Y_2$ in FEt_X, where f is a strict epimorphism and g is a monomorphism. This sequence induces $\overline{Y_1} \xrightarrow{f} \overline{Y'} \xrightarrow{g} \overline{Y'} \bigsqcup \overline{Y''} \cong \overline{Y_2}$.
(G4) and (G5) are obvious since U is fully faithful and $U \circ F'_{\tilde{\eta}} = F_{\tilde{\eta}}$.
(G6) Let $u : \overline{Y_1} \to \overline{Y_2}$ be a morphism in $\mathrm{FEt}_{\overline{X}}$. If $F_{\tilde{\eta}}(u) : F_{\tilde{\eta}}(\overline{Y_1}) \to F_{\tilde{\eta}}(\overline{Y_2})$ is an isomorphism, then $U(u) : Y_1 \to Y_2$ is an isomorphism. Since the forgetful functor U is fully faithful, u is an isomorphism. □

Now we define the modified étale fundamental group.

Definition 4.1.5. The *modified étale fundamental group* $\pi_1(\overline{X}) = \pi_1(\overline{X}, \tilde{\eta})$ with geometric basepoint $\tilde{\eta}$ is defined by the fundamental group of the Galois category $\mathrm{FEt}_{\overline{X}}$ associated to the fiber functor $F_{\tilde{\eta}}$, namely, the group of automorphisms of $F_{\tilde{\eta}}$.

The fundamental theorem of Galois categories is stated as follows.

Theorerm 4.1.6. *There is an equivalence of categories between* $\mathrm{FEt}_{\overline{X}}$ *and the category of finite discrete sets equipped with continuous left actions of* $\pi_1(\overline{X})$.

Next, in order to describe $\pi_1(\overline{X})$ more explicitly, we observe which object is Galois in the Galois category $\mathrm{FEt}_{\overline{X}}$. By the definitions of a connected object and a Galois object in a Galois category, one can see that $\overline{Y} \in \mathrm{FEt}_{\overline{X}}$ is connected in $\mathrm{FEt}_{\overline{X}}$ if and only if $Y \to X$ is connected in FEt_X, and that a connected object $\overline{Y}$ is Galois in $\mathrm{FEt}_{\overline{X}}$ if and only if $\mathrm{Aut}_{\overline{X}}(\overline{Y}) = \mathrm{Aut}_X(Y) \to F'_{\tilde{\eta}}(Y) = F_{\tilde{\eta}}(\overline{Y})$ is bijective, i.e., Y is Galois in FEt_X. Therefore, we have the following Proposition.

Proposition 4.1.7. *Let $\tilde{K}$ (resp. $\tilde{K}^{ab}$) denote the maximal Galois (resp. abelian) extension of K which is unramified over all finite and infinite primes. Then we have the following.*
(1) *There is a natural isomorphisms* $\mathrm{Gal}(\tilde{K}/K) \cong \pi_1(\overline{X})$.
(2) *The abelianization* $\pi_1^{ab}(\overline{X})$ *of* $\pi_1(\overline{X})$ *admits natural isomorphisms*

$$\mathrm{Cl}_K \xrightarrow{\sim} \mathrm{Gal}(\tilde{K}^{ab}/K) \cong \pi_1^{ab}(\overline{X}) \ ; \ [\mathfrak{a}] \mapsto \left(\frac{\tilde{K}^{ab}/K}{\mathfrak{a}} \right)$$

given by the Artin reciprocity law.

4.2 The Artin–Verdier topos and the modified étale cohomology groups

Let $\mathrm{Sh}(\overline{X}_{\text{ét}})$ denote the *Artin–Verdier étale topos*, namely, the category of abelian sheaves on the site $\overline{X}_{\text{ét}}$. Let us recall the decomposition lemma for $\mathrm{Sh}(\overline{X}_{\text{ét}})$ following [AC] and [Bi]. We fix an algebraic closure $\overline{K}$ of K. For each $x \in X_\infty$, we fix an extension $\overline{x}$ of x to $\overline{K}$ and denote by $I_{\overline{x}}$ the inertia group of $\overline{x}$. If x is a real prime, then we have $I_{\overline{x}} \cong \mathbb{Z}/2\mathbb{Z}$; if x is a complex prime, then $I_{\overline{x}}$ is trivial. Let $\eta : \mathrm{Spec}\ K \to X$ denote the generic point. Then, for $F \in \mathrm{Sh}(\overline{X}_{\text{ét}})$, we can regard $\eta^*F = F_\eta$ as a $\mathrm{Gal}(\overline{K}/K)$-module and $I_{\overline{x}} \subset \mathrm{Gal}(\tilde{K}/K)$ acts on η^*F. We define a site TX_∞ as follows. An object in TX_∞ is a pair (M, m) where M is a finite set and $m : M \to X_\infty$ is a map. A morphism $(M_1, m_1) \to (M_2, m_2)$ in TX_∞ is a map $f : M_1 \to M_2$ such that $m_2 = f \circ m_1$. A covering in TX_∞ is a family of morphisms $\{\varphi_i : (M_i, m_i) \to (M, m)\}_{i \in I}$ in TX_∞ such that m_i is surjective and $M = \underset{i}{\cup}\varphi(M_i)$. Hence, each G on TX_∞ is identified with a family of abelian groups $\{G_x\}_{x\in X_\infty}$. We define maps of sites $p : \overline{X}_{\text{ét}} \to TX_\infty$ and $q : \overline{X}_{\text{ét}} \to X_{\text{ét}}$ by the forgetful functors. Then we have functors

$$\mathrm{Sh}(TX_\infty) \underset{p^*}{\overset{p_*}{\leftrightarrows}} \mathrm{Sh}(\overline{X}_{\text{ét}}) \underset{q^*}{\overset{q_*}{\leftrightarrows}} \mathrm{Sh}(X_{\text{ét}}).$$

Next, we define the category $\mathrm{Sh}(\overline{X}_{\text{ét}})'$ as follows. An object in $\mathrm{Sh}(\overline{X}_{\text{ét}})'$ is a triple $(\{G_x\}_{x\in X_\infty}, F, \{\sigma_x : G_x \to (\eta^*F)^{I_x}\}_{x\in X_\infty})$, where $\{G_x\}_{x\in X_\infty} \in \mathrm{Sh}(TX_\infty)$, $F \in \mathrm{Sh}(X_{\text{ét}})$ and $\{\sigma_x : G_x \to (\eta^*F)^{I_x}\}_{x\in X_\infty}$ is a family of homomorphisms of abelian groups. A morphism $(\{G_x\}, F, \{\sigma_x\}) \to (\{G'_x\}, F', \{\sigma'_x\})$ is a pair of morphisms $\{G_x\} \to \{G'_x\}$, and $F \to F'$ such that the induced diagram

$$\begin{array}{ccc} G_x & \xrightarrow{\sigma_x} & (\eta^*F)^{I_x} \\ \downarrow & & \downarrow \\ G'_x & \xrightarrow{\sigma'_x} & (\eta^*F')^{I_x} \end{array}$$

is commutative for each $x \in X_\infty$.

Now we state the decomposition lemma, which was previously proved for $\mathrm{Sh}(\overline{X}_{\text{ét}})$ ([AC: Proposition 2.3] and [Bi; Proposition 1.2]).

Lemma 4.2.1. *There is an equivalence of categories given by the functors*

$$\mathrm{Sh}(\overline{X}_{\text{ét}}) \underset{\Psi}{\overset{\Phi}{\leftrightarrows}} \mathrm{Sh}(\overline{X}_{\text{ét}})'$$

defined by

$$\Phi : S \mapsto (q_*S, p_*S, p_*S \to p_*q^*q_*S),\ \Psi : (\{G_x\}, F, \{\sigma_x\}) \mapsto q^*F \times_{p^*q_*q^*F} p^*\{G_x\}.$$

Proof. We may check the following properties (1)–(4), so that [Ar; Proposition 2.4] yields the assertion.
(1) q_*(resp. p_*) is left adjoint to q^* (resp. p^*).
(2) q_*, p_* are exact.
(3) p^*, q^* are fully faithful.
(4) For any $S \in \mathrm{Sh}(\overline{X}_{\text{ét}})$, $q_*S = 0$ holds if and only if there exists $G \in \mathrm{Sh}(TX_\infty)$ such that $S = p^*G$.
We refer to [Zi; Proposition 1.3.3] for (1), (3), and (4). The property (2) follows from the fact that $\overline{X}_{\text{ét}}$, $X_{\text{ét}}$ and TX_∞ have final objects and finite fiber products preserved by p and q. □

Remark 4.2.2. (1) Via the equivalence of categories in Lemma 4.2.1, we identify p_*, p^*, q_*, q^* with the functors ψ_*, ψ^*, ϕ^*, ϕ_* defined by

$$\phi^*(\{G_x\}, F, \{\sigma_x\}) = F,\ \phi_*F = (\{(\eta^*F)^{I_x}\}, F, \{\mathrm{id}\}),$$
$$\psi^*(\{G_x\}, F, \{\sigma_x\}) = \{G_x\},\ \ \psi_*\{G_x\} = (\{G_x\}, 0, \{0\}).$$

respectively.
(2) The constant sheaf $\underline{A}_{\overline{X}_{\text{ét}}}$ on $\overline{X}_{\text{ét}}$ associated to an abelian group A satisfies $\underline{A}_{\overline{X}_{\text{ét}}} = \phi_*(\underline{A}_{X_{\text{ét}}})$. In the following, if there is no confusion, we will abbreviate $\underline{A}_{\overline{X}_{\text{ét}}}$ to A.
(3) For $S = (\{G_x\}, F, \{\sigma_x\}) \in \mathrm{ObSh}(\overline{X}_{\text{ét}})$, the section of S at $(Y, M) \in \overline{X}_{\text{ét}}$ is given by $F(Y) \times_{\eta^*F} G_{x_1} \times_{\eta^*F} G_{x_2} \times_{\eta^*F} \cdots \times_{\eta^*F} G_{x_r}$, where $\{x_1, x_2, \cdots, x_r\}$ is the image of M by $Y_\infty \to X_\infty$.

Definition 4.2.3. For each $S \in \mathrm{Sh}(\overline{X}_{\text{ét}})$, the cohomology group $\mathrm{H}^i(\overline{X}, S)$ is called the i-th *modified étale cohomology group* of $\overline{X}$ with values in S.

The group $\mathrm{H}^i(\overline{X}, S)$ of the constant sheaf $\mathbb{Z}/n\mathbb{Z}$ is calculated in [Bi; Proposition 2.13] and [AC; Corollary 2.15]. Let us recall the Artin–Verdier duality.

Proposition 4.2.4 (The Artin–Verdier duality [Bi; Theorem 5.1])**.** *Let F be a constructible sheaf on $X = \mathrm{Spec}\ \mathcal{O}_K$. We fix an algebraic closure $\overline{K}$ of K. For each $x \in X_\infty$, we fix an extension $\overline{x}$ of x to $\overline{K}$. Let $\eta : \mathrm{Spec}\ K \to X$ denote the generic point. Let $G_{m,X}$ denote the étale sheaf of units on X. Then we have the following.*

(a) $\mathrm{H}^i(\overline{X}, \phi_* F) = \mathrm{Ext}^i_{\overline{X}}(\phi_* F, \phi_* G_{m,X}) = 0$ *for* $i > 3$.
(b) *The Yoneda pairing*

$$\mathrm{H}^i(\overline{X}, \phi_* F) \times \mathrm{Ext}^{3-i}_{\overline{X}}(\phi_* F, \phi_* G_{m,X}) \to \mathrm{H}^3(\overline{X}, \phi_* G_{m,X}) \cong \mathbb{Q}/\mathbb{Z}$$

is a perfect duality of finite groups for $i \geq 2$.
(c) *If for every* $x \in X_\infty$ *the inertia group* $I_{\overline{x}}$ *of* $\overline{x}$ *acts trivially on the* $\mathrm{Gal}(\overline{K}/K)$*-module* $\eta^* F = F_\eta$*, then the pairing in* (b) *is perfect for any* $i \geq 0$.

Applying Proposition 4.2.4 to the constant sheaf $F = \mathbb{Z}/n\mathbb{Z}$ on X, we obtain the following proposition, where we denote by $\mu_n(K)$ the group of n-th roots of unity in K and put $Z_1 = \{(a, \mathfrak{a}) \in K^\times \oplus I_K \mid (a)^{-1} = \mathfrak{a}^n\}$, $B_1 = \{(b^n, (b)^{-1}) \in K^\times \oplus I_K \mid b \in K^\times\}$.

Proposition 4.2.5 ([Bi; Proposition 2.13], [AC; Corollary 2.15])**.** *We have*

$$\mathrm{Ext}^i_{\overline{X}}(\mathbb{Z}/n\mathbb{Z}, \phi_* G_{m,X}) \cong \begin{cases} \mu_n(K) & (i = 0) \\ Z_1/B_1 & (i = 1) \\ \mathrm{Cl}_K/n\mathrm{Cl}_K & (i = 2) \\ \mathbb{Z}/n\mathbb{Z} & (i = 3) \\ 0 & (i > 3), \end{cases}$$

where $G_{m,X}$ *is the étale sheaf of units on* X*. Then we have, by the Artin–Verdier duality,*

$$\mathrm{H}^i(\overline{X}, \mathbb{Z}/n\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} & (i = 0) \\ (\mathrm{Cl}_K/n\mathrm{Cl}_K)^\sim & (i = 1) \\ (Z_1/B_1)^\sim & (i = 2) \\ (\mu_n(K))^\sim & (i = 3) \\ 0 & (i > 3), \end{cases}$$

where $(-)^\sim$ *denotes* $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$*.*

Remark 4.2.6. Assume that K contains primitive n-th roots of unity. For each $v' \in K^\times$, we choose a primitive n-th root $v'^{\frac{1}{n}}$ of v'. By Theorem 4.1.6, for a continuous and surjective homomorphism $\rho : \pi_1(\overline{X}) \to \mathbb{Z}/n\mathbb{Z}$, there is a corresponding Galois object $\overline{Y} \to \overline{X}$ ($Y = \mathrm{Spec}\, \mathcal{O}_L$) whose Galois group is $\mathbb{Z}/n\mathbb{Z}$. Since L is a cyclic extension of degree n unramified at all finite and infinite primes, there exists $v \in K^\times$ such that $L = K(v^{\frac{1}{n}})$ and there exists $\mathfrak{a} \in I_K$ which satisfies $\mathfrak{a}^n = (v)^{-1}$. By the definition of L and the Galois correspondence, there is an isomorphism $\chi : \mathrm{Gal}(L/K) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$ such that the following diagram

$$\begin{array}{ccc} \pi_1(\overline{X}) & \xrightarrow{res} & \mathrm{Gal}(L/K) \\ {\scriptstyle\rho}\big\downarrow & \swarrow{\scriptstyle\chi} & \\ \mathbb{Z}/n\mathbb{Z} & & \end{array}$$

commutes, where $res : \pi_1(\overline{X}) \to \mathrm{Gal}(L/K)$ denotes the restriction map. By Proposition 4.1.7, we also have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Cl}_K & \xrightarrow{\left(\frac{\tilde{K}^{ab}/K}{}\right)} & {\pi_1}^{ab}(\overline{X}) \\ {\scriptstyle\left(\frac{L/K}{}\right)}\big\downarrow & \swarrow{\scriptstyle res} & \\ \mathrm{Gal}(L/K), & & \end{array}$$

where $\left(\frac{L/K}{}\right) : \mathrm{Cl}_K \to \mathrm{Gal}(L/K)$ denotes the Artin map.

Now we state the extension of Hochschild–Serre spectral sequence.

Theorerm 4.2.7. *Let $\overline{Y} \to \overline{X}$ be a Galois object in $\mathrm{FEt}_{\overline{X}}$. Then for any $S \in \mathrm{Sh}(\overline{X}_{\text{ét}})$, there is a cohomological spectral sequence*

$$\mathrm{H}^p(\mathrm{Gal}(\overline{Y}/\overline{X}), \mathrm{H}^q(\overline{Y}, S|\overline{Y})) \Rightarrow \mathrm{H}^{p+q}(\overline{X}, S).$$

Proof. Let $\mathrm{Gal}(\overline{Y}/\overline{X})$-mod denote the category of $\mathrm{Gal}(\overline{Y}/\overline{X})$-modules. We consider the functors

$$F_1 : \mathrm{Sh}(\overline{X}_{\text{ét}}) \to \mathrm{Gal}(\overline{Y}/\overline{X})\text{-mod}, S \mapsto S(\overline{Y})$$
$$F_2 : \mathrm{Gal}(\overline{Y}/\overline{X})\text{-mod} \to \mathrm{Ab}, M \mapsto M^{\mathrm{Gal}(\overline{Y}/\overline{X})},$$

where the action of $G = \mathrm{Gal}(\overline{Y}/\overline{X})$ on $S(\overline{Y})$ is defined by $\sigma.x = S(\sigma)(x)$ for $x \in S(\overline{Y})$ and $\sigma \in G$. In the same manner as in [Mi1; Remark5.4] and [Mi1; Proposition 1.4], we can easily check $(F_2 \circ F_1)(S) = S(\overline{Y})^G = S(\overline{X})$. Let I be an injective object in $\mathrm{Sh}(\overline{X}_{\text{ét}})$. By replacing Y and X with $\overline{Y}$ and $\overline{X}$ in the argument of [Mi1; Example2.6], one can see that $\mathrm{H}^i(G, I(\overline{Y})) \cong \check{\mathrm{H}}^i(\overline{Y}/\overline{X}, I)$ for any $i \geq 1$. Since I is injective, we have $\check{\mathrm{H}}^i(\overline{Y}/\overline{X}, I) = 0$ by the definition of Čech cohomologies. Therefore, $F_1(I) = I(\overline{Y})$ is a F_2-acyclic object. By the Grothendieck spectral sequence, we obtain the assertion. □

Let $(\overline{Y_i} \to \overline{X}, \overline{Y_i} \to \overline{Y_j})$ denote the inverse system of finite Galois coverings over $\overline{X}$ and put $\widetilde{X} = \varprojlim_i Y_i$, $\widetilde{\overline{X}} = \varprojlim_i \overline{Y_i}$. By $\mathrm{H}^p(\widetilde{X}, \mathbb{Z}/n\mathbb{Z}) = \varinjlim_i \mathrm{H}^p(Y_i, \mathbb{Z}/n\mathbb{Z})$

and the local cohomology sequence [Bi; Proposition 1.4], we have $\mathrm{H}^p(\widetilde{\overline{X}}, \mathbb{Z}/n\mathbb{Z}) = \varprojlim_i \mathrm{H}^p(\overline{Y_i}, \mathbb{Z}/n\mathbb{Z})$. So on passing to the inverse limit, we obtain the following.

Corollary 4.2.8. *There is a cohomological spectral sequence*

$$\mathrm{H}^p(\pi_1(\overline{X}), \mathrm{H}^q(\widetilde{\overline{X}}, \mathbb{Z}/n\mathbb{Z})) \Rightarrow \mathrm{H}^{p+q}(\overline{X}, \mathbb{Z}/n\mathbb{Z}).$$

4.3 Arithmetic Dijkgraaf–Witten invariants for a number ring

Let $X = \operatorname{Spec} \mathcal{O}_K$ denote the prime spectrum of the ring of integers of a number field K containing primitive n-th roots of unity. We choose a primitive n-th root of unity ζ_n in K, which induces an isomorphism $\mathbb{Z}/n\mathbb{Z} \cong \mu_n$. Let A be a finite group and let $c \in \mathrm{H}^3(A, \mathbb{Z}/n\mathbb{Z})$. Let $\mathcal{M}(\overline{X}, A) = \mathrm{Hom}_c(\pi_1(\overline{X}), A)/A$ denote the set of conjugacy classes of all continuous homomorphisms $\pi_1(\overline{X}) \to A$. Recall that by Proposition 4.2.5 we have the fundamental class isomorphism $\mathrm{H}^3(\overline{X}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ that depends on the choice of ζ_n.

Definition 4.1. For $\rho \in \mathcal{M}(\overline{X}, A)$, the *arithmetic Chern–Simons invariant* $CS_c(\rho)$ associated to c is defined by the image of c under the composition of maps

$$\mathrm{H}^3(A, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\rho^*} \mathrm{H}^3(\pi_1(\overline{X}), \mathbb{Z}/n\mathbb{Z}) \xrightarrow{j_3} \mathrm{H}^3(\overline{X}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z},$$

where j_3 is the edge homomorphisms in the modified Hochschild–Serre spectral sequence $\mathrm{H}^p(\pi_1(\overline{X}), \mathrm{H}^q(\widetilde{\overline{X}}, \mathbb{Z}/n\mathbb{Z})) \Rightarrow \mathrm{H}^{p+q}(\overline{X}, \mathbb{Z}/n\mathbb{Z})$ of Corollary 4.2.8 We can easily see that $CS_c(\rho)$ is independent of the choice of ρ in its conjugacy class. The map

$$CS_c : \mathcal{M}(\overline{X}, A) \to \mathbb{Z}/n\mathbb{Z}$$

is called *the arithmetc Chern–Simons functional* associated to c. *The arithmetic Dijkgraaf–Witten invariant* of $\overline{X}$ associated to c is then defined by

$$Z_c(\overline{X}) = \frac{1}{\#A} \sum_{\rho \in \mathcal{M}(\overline{X}, A)} \exp\left(\frac{2\pi i}{n} CS_c(\rho)\right).$$

When $A = \mathbb{Z}/m\mathbb{Z}$, we call $CS_c(\rho)$ and $Z_c(\overline{X})$ *the mod m arithmetic Chern–Simons invariant* and *the mod n arithmetic Dijkgraaf–Witten invariant*, respectively.

Remark 4.3.1. (1) If K is totally imaginary, so that K has no ramification at infinite primes, then we have $\pi_1(\overline{X}) = \pi_1(X)$ and $\mathrm{H}^i(\overline{X}, \mathbb{Z}/n\mathbb{Z}) = \mathrm{H}^i(X, \mathbb{Z}/n\mathbb{Z})$. Therefore Definition 4.3.1 is indeed an extension of Kim's definition [Ki].
(2) When A is abelian, by Proposition 4.1.7, we have

$$\mathcal{M}(\overline{X}, A) = \mathrm{Hom}_c(\pi_1(\overline{X}), A) \cong \mathrm{Hom}(\mathrm{Cl}_K, A).$$

4.4 A formula for arithmetic Chern–Simons invariants using the Artin Symbols

Let us describe the setting in this section. We continue to work over any number field K containing primitive n-th roots of unity. Keeping the same notations as in the section 4.3 and 4.4, we set $A = \mathbb{Z}/n\mathbb{Z}$ and $c = \mathrm{id} \cup \beta(\mathrm{id}) \in \mathrm{H}^3(A, \mathbb{Z}/n\mathbb{Z})$. Here, $id \in \mathrm{H}^1(A, \mathbb{Z}/n\mathbb{Z})$ denotes (the image of) the identity map and

$$\beta : \mathrm{H}^1(A, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}^2(A, \mathbb{Z}/n\mathbb{Z})$$

denotes the Bockstein map (connecting homomorphism) induced by the short exact sequence

$$0 \to \mathbb{Z}/n\mathbb{Z} \stackrel{\times n}{\to} \mathbb{Z}/n^2\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

Let $j_i : \mathrm{H}^i(\pi_1(\overline{X}), \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}^i(\overline{X}, \mathbb{Z}/n\mathbb{Z})$ ($i = 0, 1, 2, 3, \cdots$) denote the edge homomorphisms in the modified Hochschild–Serre spectral sequence (Corollary 4.2.8). For each $\rho \in \mathcal{M}(\overline{X}, A) = \mathrm{Hom}_c(\pi_1(\overline{X}), A)$, let ρ_X^* denote / ρ^* also denote the composition

$$\mathrm{H}^1(A, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\rho^*} \mathrm{H}^1(\pi_1(\overline{X}), \mathbb{Z}/n\mathbb{Z}) \xrightarrow{j_1} \mathrm{H}^1(\overline{X}, \mathbb{Z}/n\mathbb{Z})$$

of the natural map j_1 and the induced map ρ^*. Then we have

$$CS_c(\rho) = \rho_X^*(\mathrm{id}) \cup \tilde{\beta}(\rho_X^*(\mathrm{id})) \in \mathrm{H}^3(\overline{X}, \mathbb{Z}/n\mathbb{Z}),$$

where $\cup : \mathrm{H}^1(\overline{X}, \mathbb{Z}/n\mathbb{Z}) \times \mathrm{H}^2(\overline{X}, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}^3(\overline{X}, \mathbb{Z}/n\mathbb{Z})$ is the cup product and $\tilde{\beta} : \mathrm{H}^1(\overline{X}, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}^2(\overline{X}, \mathbb{Z}/n\mathbb{Z})$ is the Bockstein map.

Remark 4.4.1. For the definition of the cup product in the category of sheaves on any site, we refer to [Sw; Corollary 3.7].

Now we show the main assertion of this section. We keep the same notations as in Remark 4.2.6.

Theorerm 4.4.2. *Let $X = \mathrm{Spec}\ \mathcal{O}_K$ denote the prime spectrum of the ring of integers of a number field K containing primitive n-th roots of unity. Let $\rho : \pi_1(\overline{X}) \to \mathbb{Z}/n\mathbb{Z}$ be a continuous and surjective homomorphism. Set $A = \mathbb{Z}/n\mathbb{Z}$ and $c = \mathrm{id} \cup \beta(\mathrm{id}) \in \mathrm{H}^3(A, \mathbb{Z}/n\mathbb{Z})$. Let $L = K(v^{\frac{1}{n}})$ denote the Kummer extension corresponding to Ker ρ as in Remark 4.2.6, so that L/K is unramified at all finite and infinite primes and there exist some $\mathfrak{a} \in I_K$ and $v \in K^\times$ with $\mathfrak{a}^n = (v)^{-1}$. Let $\chi : \mathrm{Gal}(L/K) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$ denote the natural isomorphism induced by ρ. Then we have*

$$CS_c(\rho) = \chi\left(\left(\frac{L/K}{\mathfrak{a}}\right)\right).$$

Proof. When K is totally imaginary, the assertion holds by [BCGKPT; Theorem 1.3]. So we consider the case K has real primes and $n = 2$. By direct calculation, we see that $\rho^*(\mathrm{id}) \in \mathrm{H}^1(\pi_1(\overline{X}), \mathbb{Z}/2\mathbb{Z})$ corresponds to $\rho \in \mathrm{Hom}_c(\pi_1(\overline{X}), \mathbb{Z}/2\mathbb{Z})$ via the natural isomorphism

$$\mathrm{H}^1(\pi_1(\overline{X}), \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}_c(\pi_1(\overline{X}), \mathbb{Z}/2\mathbb{Z}).$$

Then, by Proposition 4.2.5 and Remark 4.2.6 , $\rho^*_{\overline{X}}(\mathrm{id}) = j_1 \circ \rho^*(\mathrm{id}) \in \mathrm{H}^1(\overline{X}, \mathbb{Z}/2\mathbb{Z})$ corresponds to the composition $\chi \circ \left(\frac{L/K}{}\right) \in \mathrm{Hom}_c(\mathrm{Cl}_K, \mathbb{Z}/2\mathbb{Z})$ via the natural isomorphism

$$\mathrm{H}^1(\overline{X}, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}_c(\mathrm{Cl}_K, \mathbb{Z}/2\mathbb{Z}).$$

We regard $\tilde{\beta}(\rho^*_{\overline{X}}(\mathrm{id}))$ as an element in $\mathrm{Ext}^1_{\overline{X}}(\mathbb{Z}/2\mathbb{Z}, \phi_* G_{m,X})^\sim = (Z_1/B_1)^\sim$ through Artin–Verdier duality. Then by [AC; Corollary 3.13], we have

$$\rho^*_{\overline{X}}(\mathrm{id}) \cup \tilde{\beta}(\rho^*_{\overline{X}}(\mathrm{id})) = \tilde{\beta}(\rho^*_{\overline{X}}(\mathrm{id}))([(v, \mathfrak{a})]) = \rho^*_{\overline{X}}(\mathrm{id})(\tilde{\beta}'([(v, \mathfrak{a})])),$$

where $\tilde{\beta}' : \mathrm{Ext}^1_{\overline{X}}(\mathbb{Z}/2\mathbb{Z}, \phi_* G_{m,X}) \to \mathrm{Ext}^2_{\overline{X}}(\mathbb{Z}/2\mathbb{Z}, \phi_* G_{m,X})$ is the connecting homomorphism induced by the short exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/2^2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

By replacing X with $\overline{X}$ in the proof of [AC; Lemma 4.1], one can see $\tilde{\beta}'([(v, \mathfrak{a})]) = [\mathfrak{a}]$. Hence we see that $CS_c(\rho) = 0$ holds if and only if $\left(\frac{L/K}{\mathfrak{a}}\right) \in \mathrm{Gal}(L/K)$ is trivial. Therefore, we obtain the assertion. □

4.5 Explicit formulas of the mod 2 arithmetic Dijkgraaf–Witten invariants for real quadratic number fields $\mathbb{Q}(\sqrt{p_1 p_2 \cdots p_r})$ with $p_i \equiv 1 \bmod 4$

In the following, we consider the case $K = \mathbb{Q}(\sqrt{p_1 p_2 \cdots p_r})$, where each p_i is a prime number such that $p_i \equiv 1 \bmod 4$. We keep the notation as in the previous section and suppose that $n = 2$, $A = \mathbb{Z}/2\mathbb{Z}$, and $c = \mathrm{id} \cup \beta(\mathrm{id}) \in \mathrm{H}^3(A, \mathbb{Z}/2\mathbb{Z})$. Assume that the norm of the fundamental unit in $\mathcal{O}_K^\times$ is -1. Then the narrow ideal class group Cl_K^+ is the same as Cl_K.

Remark 4.5.1. The fundamental unit of $\mathbb{Q}(\sqrt{5 \cdot 13 \cdot 61})$ is $\epsilon = \dfrac{63 + \sqrt{5 \cdot 13 \cdot 61}}{2}$ with $\mathrm{Nr}\,\epsilon = 1$. We eliminate such cases to use Gauss's genus theory.

By $p_i \equiv 1 \bmod 4$, the discriminant of K is $p_1 p_2 \cdots p_r$. We define the abelian multiplicative 2-group $T_\times$ by

$$T_\times = \{(x_1, x_2, \cdots, x_r) \in \{\pm 1\}^r \mid \prod_{i=1}^{r} x_i = 1\},$$

and put $e_{ij}^\times = (1, \cdots, 1, \overset{i\text{-th}}{-1}, 1, \cdots, 1, \overset{j\text{-th}}{-1}, 1, \cdots, 1) \in T_\times$ for each (i, j) with $1 \leqq i < j \leqq r$. In addition, we define an additive 2-group T_+ by

$$T_+ = \{(x_1, x_2, \cdots, x_r) \in (\mathbb{Z}/2\mathbb{Z})^r \mid \sum_{i=1}^{r} x_i = 0\}.$$

and put $e_{ij}^+ \overset{def}{=} (0, \cdots, 0, \overset{i\text{-th}}{1}, 0, \cdots, 0, \overset{j\text{-th}}{1}, 0, \cdots, 0) \in T_+$ for each (i, j) with $1 \leqq i < j \leqq r$. Then we have a standard isomorphism $T_+ \to T_\times$; $(x_i)_i \mapsto ((-1)^{x_i})_i$. By Gauss's genus theory [O; §4.7], there is an isomorphism

$$\mathrm{Cl}_K^+/2\mathrm{Cl}_K^+ \xrightarrow{\sim} T_\times,$$

given by

$$[\mathfrak{a}] \mapsto \left(\left(\frac{\mathrm{N}\mathfrak{a}}{p_1}\right), \left(\frac{\mathrm{N}\mathfrak{a}}{p_2}\right), \cdots, \left(\frac{\mathrm{N}\mathfrak{a}}{p_r}\right) \right),$$

where $\left(\frac{-}{p_i}\right)$ denotes the Legendre symbol. Therefore, by Proposition 4.1.7, we obtain the following isomorphisms

$$\begin{aligned} \mathrm{Hom}_c(\pi_1(\overline{X}), \mathbb{Z}/2\mathbb{Z}) &\cong \mathrm{Hom}(\mathrm{Cl}_K^+/2\mathrm{Cl}_K^+, \mathbb{Z}/2\mathbb{Z}) \\ &\cong \mathrm{Hom}(T_\times, \{\pm 1\}) \cong \mathrm{Hom}(T_+, \mathbb{Z}/2\mathbb{Z}). \end{aligned}$$

We denote the corresponding elements in those groups by the same letters.

Now we prove the following formula.

Theorerm 4.5.2. *Notations being as above, for each nontrivial $\rho \in \mathrm{Hom}(T_\times, \{\pm 1\})$, the arithmetic Chern–Simons invariant satisfies*

$$(-1)^{CS_c(\rho)} = \prod_{\substack{i<j \\ \rho(e_{ij}^\times) = -1}} \left(\frac{p_j}{p_i}\right).$$

Proof. Define elements $b_1, b_2, \cdots, b_{r-1} \in T_\times$ by

$$b_1 = (-1, 1, 1, \cdots, -1), b_2 = (1, -1, 1, 1, \cdots, -1), \cdots, b_{r-1} = (1, 1, \cdots, 1, -1, -1)$$

so that the tuple $(b_1, b_2, \cdots, b_{r-1})$ is a basis of $T_\times$. Let $J = \{j_1, j_2 \cdots, j_m\} \subset \{1, 2, \cdots, r-1\}$ with $j_1 < j_2 \cdots < j_m$ and suppose that $\rho(b_i) = -1$ if and only if $i \in J$. Note that $\rho(e_{ij}^\times) = -1$ holds if and only if the intersection $\{i, j\} \cap J$ consists of one element. Let L denote the abelian unramified extension of K corresponding to $2\mathrm{Cl}_K$ via the class field theory (Proposition 4.1.7), namely, we put

$$L = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \cdots \sqrt{p_r}).$$

Let L_ρ denote the unramified Kummer extension of K corresponding to $\mathrm{Ker}\rho \subset \pi_1(\overline{X})$, so that we have

$$L_\rho = K(\sqrt{v}), \quad \mathfrak{a}_v^2 = (v)^{-1} \tag{4.5.2.1}$$

for some $v \in K^\times$ and $\mathfrak{a}_v \in I_K$. In order to apply Theorem 4.4.2, let us explicitly find such v. Let $\mathbf{a} = (a_1, a_2, \cdots, a_r) \in T_\times$ and let $\mathfrak{a} \in I_K$ whose image $[\mathfrak{a}]$ corresponds to $\mathbf{a}$ via the isomorphism $\mathrm{Cl}_K^+/2\mathrm{Cl}_K^+ \xrightarrow{\sim} T_\times$ of Gauss's genus theory. Then the Artin symbol $\left(\frac{L/K}{\mathfrak{a}}\right) \in \mathrm{Gal}(L/K)$ is characterized by

$$\left(\frac{L/K}{\mathfrak{a}}\right)(\sqrt{p_i}) = a_i\sqrt{p_i} \ \ (i = 1, 2, \cdots, r).$$

Let $u : K^\times \to K^\times/(K^\times)^2$ denote the natural projection. By Remark 4.2.6, the class $u(v) \in K^\times/(K^\times)^2$ is characterized by

$$\left(\frac{L_\rho/K}{\mathfrak{a}}\right)(\sqrt{v})/\sqrt{v} = \rho(\mathbf{a}).$$

Since $\left(\frac{L_\rho/K}{\mathfrak{a}}\right)$ is the restriction of $\left(\frac{L/K}{\mathfrak{a}}\right)$ to L_ρ, we may put

$$v = p_{j_1}p_{j_2}\cdots p_{j_m}/p_1p_2\cdots p_r.$$

Since the minimal polynomial of $(1 + \sqrt{p_1p_2\cdots p_r})/2$ over $\mathbb{Q}$ is congruent to $(2X-1)^2 \bmod p_i$, we have

$$(p_i) = \mathfrak{p}_i{}^2,$$

where $\mathfrak{p}_i = (p_i, \sqrt{p_1p_2\cdots p_r})$ is the prime ideal of $\mathcal{O}_K$. Hence we have

$$\mathfrak{a}_v = \mathfrak{p}_1\mathfrak{p}_2\cdots\mathfrak{p}_r/\mathfrak{p}_{j_1}\mathfrak{p}_{j_2}\cdots\mathfrak{p}_{j_m}.$$

We see that the composite map

$$\chi' : \mathrm{Gal}(L/K) \overset{\sim}{\to} \mathrm{Cl}_K/2\mathrm{Cl}_K \overset{\sim}{\to} T_\times \overset{\rho}{\twoheadrightarrow} \{\pm 1\}$$

sends $\left(\frac{L/K}{\mathfrak{a}_v}\right) \in \mathrm{Gal}(L/K)$ to $\displaystyle\prod_{i=1}^{r}\left(\frac{\mathrm{N}\mathfrak{a}_v}{p_i}\right) \in \{\pm 1\}$. By the quadratic residue and the assumption that $p_l \equiv 1 \bmod 4$ $(l = 1, \ldots, r)$, we have $\left(\dfrac{p_j}{p_i}\right) = \left(\dfrac{p_i}{p_j}\right)$ for any distinct $i, j \in \{1, 2, \ldots, r\}$. So we obtain

$$\begin{aligned}
\prod_{i=1}^{r}\left(\frac{\mathrm{N}\mathfrak{a}_v}{p_i}\right) &= \prod_{i=1}^{r}\prod_{\substack{1\le j\le r\\ j\notin J}}\left(\frac{p_j}{p_i}\right)\\
&= \prod_{\substack{1\le i\le r\\ i\in J}}\prod_{\substack{1\le j\le r\\ j\notin J}}\left(\frac{p_i}{p_j}\right)\\
&= \prod_{\substack{i<j\\ \rho(e_{ij}^{\times})=-1}}\left(\frac{p_j}{p_i}\right).
\end{aligned}$$

The last equation follows from the fact that $\rho(e_{ij}^{\times}) = -1$ holds if and only if the intersection $\{i, j\} \cap J$ consists of one element. On the other hand, since L_ρ is the unramified Kummer extension of K corresponding to $\mathrm{Ker}\rho \subset \pi_1(\overline{X})$, the composite map $\chi' : \mathrm{Gal}(L/K) \overset{\sim}{\to} \mathrm{Cl}_K/2\mathrm{Cl}_K \overset{\sim}{\to} T_\times \overset{\rho}{\twoheadrightarrow} \{\pm 1\}$ induces the natural isomorphism $\chi'' : \mathrm{Gal}(L_\rho/K) = \mathrm{Gal}(L/K)/(\mathrm{Ker}\rho) \overset{\sim}{\to} \{\pm 1\}$. Let $\chi : \mathrm{Gal}(L_\rho/K) \overset{\sim}{\to} \mathbb{Z}/2\mathbb{Z}$ denote the natural isomorphism induced by $\rho : \pi_1(\overline{X}) \to \mathbb{Z}/2\mathbb{Z}$. We see that χ is equal to the composite map $\mathrm{Gal}(L_\rho/K) \overset{\chi''}{\to} \{\pm 1\} \overset{\sim}{\to} \mathbb{Z}/2\mathbb{Z}$. Therefore, by Theorem 4.4.2, we have

$$(-1)^{CS_c(\rho)} = \chi''\left(\left(\frac{L_\rho/K}{\mathfrak{a}_v}\right)\right) = \prod_{\substack{i<j\\ \rho(e_{ij}^{\times})=-1}}\left(\frac{p_j}{p_i}\right).$$

□

Since the invariant $CS_c(0)$ of the trivial representation $0 \in \mathrm{Hom}(T_+, \mathbb{Z}/2\mathbb{Z})$ is zero, we have the following

Corollary 4.5.3. *For $\rho \in \mathrm{Hom}(T_+, \mathbb{Z}/2\mathbb{Z})$, we have*

$$CS_c(\rho) = \sum_{i<j} \rho(e_{ij}^{+})\mathrm{lk}_2(p_i, p_j),$$

where $\mathrm{lk}_2(p_i, p_j)$ *denotes the modulo* 2 *linking number of* p_i *and* p_j *defined by* $(-1)^{\mathrm{lk}_2(p_i,p_j)} = \left(\frac{p_i}{p_j}\right)$.

By Definition 4.3.1, the mod 2 arithmetic Dijkgraaf–Witten invariant is given by

$$Z_c(\overline{X}) = \frac{1}{2} \sum_{\rho \in \mathrm{Hom}(T_+, \mathbb{Z}/2\mathbb{Z})} (-1)^{CS_c(\rho)}.$$

Hence we obtain the following.

Corollary 4.5.4. *The mod* 2 *arithmetic Dijkgraaf–Witten invariant is given by*

$$Z_c(\overline{X}) = \frac{1}{2} \sum_{\rho \in \mathrm{Hom}(T_+, \mathbb{Z}/2\mathbb{Z})} \left(\prod_{i<j} \left(\frac{p_i}{p_j}\right)^{\rho(e_{ij}^+)} \right).$$

Example 4.5.5. Here are some numerical examples of $CS_c(\rho)$ and $Z_c(\overline{X})$ for the case $r = 3$. We define ρ_0, ρ_1, ρ_2 and ρ_3 in $\mathrm{Hom}(T_+, \mathbb{Z}/2\mathbb{Z})$ by

$$\rho_0(1,1,0) = 0,\ \rho_0(0,1,1) = 0,\ \rho_0(1,0,1) = 0,$$

$$\rho_1(1,1,0) = 1,\ \rho_1(0,1,1) = 0,\ \rho_1(1,0,1) = 1,$$
$$\rho_2(1,1,0) = 0,\ \rho_2(0,1,1) = 1,\ \rho_2(1,0,1) = 1,$$
$$\rho_3(1,1,0) = 1,\ \rho_3(0,1,1) = 1,\ \rho_3(1,0,1) = 0,$$

so that $\mathrm{Hom}(T_+, \mathbb{Z}/2\mathbb{Z}) = \{\rho_0, \rho_1, \rho_2, \rho_3\}$.
(1) $K = \mathbb{Q}(\sqrt{5 \cdot 29 \cdot 37})$:

$$\mathrm{lk}_2(5, 29) = 0,\ \mathrm{lk}_2(29, 37) = 1,\ \mathrm{lk}_2(37, 5) = 1,$$

$$CS_c(\rho_0) = 0,\ CS_c(\rho_1) = 1,\ CS_c(\rho_2) = 0,\ CS_c(\rho_3) = 1,$$
$$Z_c(\overline{X}) = 0.$$

(2) $K = \mathbb{Q}(\sqrt{5 \cdot 13 \cdot 73})$:

$$\mathrm{lk}_2(5, 13) = \mathrm{lk}_2(13, 73) = \mathrm{lk}_2(73, 5) = 1,$$

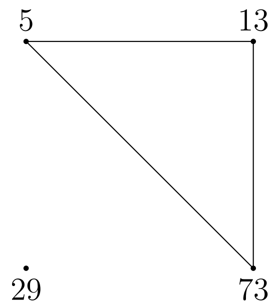
$$CS_c(\rho_0) = CS_c(\rho_1) = CS_c(\rho_2) = CS_c(\rho_3) = 0,$$
$$Z_c(\overline{X}) = 2.$$

Let us consider the case $N = 2$, $G \cong \mathbb{Z}/2\mathbb{Z}$ and take $c \in Z^3(G, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ the non-trivial cocycle. Let $S := \{(p_1), \cdots, (p_r)\}$ be a finite set of primes of $\mathbb{Q}$ ($r \geq 2$) such that $p_i \equiv 1 \mod 4$ and let $k := \mathbb{Q}(\sqrt{p_1 \cdots p_r})$ be the quadratic extension of Q ramified over $(p_1), (p_2), \cdots, (p_r)$. In order to describe the arithmetic Dijkgraaf–Witten invariant $Z(\overline{X})$, we describe the mod 2 arithmetic linking diagram D_S of S, following the mod 2 linking diagram and the analogy between the linking number and the Legendre symbol. The mod 2 arithmetic linking diagram D_S of S consists of r vertices and edges. Each vertex represents each prime (p_i) and two vertices (p_i) and (p_j) are adjacent by an edge if and only if $\left(\frac{p_j}{p_i}\right) = -1$. since $p_i \equiv 1 \mod 4$, D_S is well defined by the quadratic reciprocity law. The following formula can be proved by using genus theory for $\overline{X}$.

Theorerm 4.5.6 ([Hi], [DK])**.** *Notations being as above, we have*

$$Z(\overline{X}) = \begin{cases} 2^{r-2} & \textit{if any connected component of } D_S \textit{ is a circuit} \\ 0 & \textit{otherwise.} \end{cases}$$

Example 4.5.7. Let $S = \{5, 13, 29, 73\}$ so that $(\frac{5}{13}) = (\frac{5}{73}) = (\frac{13}{73}) = -1$, $(\frac{5}{29}) = (\frac{13}{29}) = (\frac{73}{29}) = 1$. Then the mod 2 linking diagram D_S is given by the following figure. Let $K := \mathbb{Q}(\sqrt{5 \cdot 13 \cdot 29 \cdot 73}) = \mathbb{Q}(\sqrt{137605})$ and $\overline{X} := \mathrm{Spec}(\mathbb{Z}[\frac{1+\sqrt{137605}}{2}])$. By Theorem 4.5.6, we have $Z(\overline{X}) = 2^2 = 4$.



5 Arithmetic Dijkgraaf–Witten TQFT for S-integer number rings

In this chapter, we present arithmetic Dijkgraaf–Witten theory for number rings, which may be regarded as an arithmetic analogue of Dijkgraaf–Witten theory in Chapter 3, along the line of topological quantum field theory. We list herewith some analogies which will be used in this section.

oriented, connected, closed 3-manifold M	compactified spectrum of number ring $\overline{X}_k = \overline{\mathrm{Spec}(\mathcal{O}_k)}$
knot $\mathcal{K} : S^1 \hookrightarrow M$	prime $\{\mathfrak{p}\} = \mathrm{Spec}(\mathcal{O}_k/\mathfrak{p}) \hookrightarrow \overline{X}_k$
link $\mathcal{L} = \mathcal{K}_1 \sqcup \cdots \sqcup \mathcal{K}_r$	finite set of maximal ideals $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$
tubular n.b.d of a knot $V_\mathcal{K}$ boundary torus $\partial V_\mathcal{K}$ peripheral group $\pi_1(\partial V_\mathcal{K})$	$\mathfrak{p}$-adic integer ring $V_\mathfrak{p} = \mathrm{Spec}(\mathcal{O}_\mathfrak{p})$ $\mathfrak{p}$-adic field $\partial V_\mathfrak{p} = \mathrm{Spec}(k_\mathfrak{p})$ local absolute Galois group $\Pi_\mathfrak{p} = \mathrm{Gal}(\overline{k}_\mathfrak{p}/k_\mathfrak{p})$
tubular n.b.d of a link $V_\mathcal{L} = V_{\mathcal{K}_1} \sqcup \cdots \sqcup V_{\mathcal{K}_r}$ boundary tori $\partial V_\mathcal{L} = \partial V_{\mathcal{K}_1} \sqcup \cdots \sqcup \partial V_{\mathcal{K}_r}$	union of $\mathfrak{p}_i$-adic integer rings $V_S = \mathrm{Spec}(\mathcal{O}_{\mathfrak{p}_1}) \sqcup \cdots \sqcup \mathrm{Spec}(\mathcal{O}_{\mathfrak{p}_r})$ union of $\mathfrak{p}_i$-adic fields $\partial V_S = \mathrm{Spec}(k_{\mathfrak{p}_1}) \sqcup \cdots \sqcup \mathrm{Spec}(k_{\mathfrak{p}_r})$
link complement $X_\mathcal{L} = M \setminus \mathrm{Int}(V_\mathcal{L})$ link group $\Pi_\mathcal{L} = \pi_1(X_\mathcal{L})$	complement of a finite set of primes $\overline{X}_S = \overline{X}_k \setminus S$ maximal Galois group with given ramification $\Pi_S = \mathrm{Gal}(k_S/k)$

Based on the analogies recalled above, for a finite set S of finite primes of a number field k, we construct arithmetic analogues of the Chern–Simons 1-cocycle, the prequantization bundle for a surface and the Chern–Simons functional for a 3-manifold. We then construct arithmetic analogues for k and S of the quantum Hilbert space (space of conformal blocks) and the Dijkgraaf–Witten partition function in (2+1)-dimensional Chern–Simons TQFT. We show some basic and functorial properties of those arithmetic analogues. Finally we show a decomposition formula for arithmetic Chern–Simons invariants and a gluing formula for arithmetic Dijkgraaf–Witten partition functions. The contents of this chapter are based on [HKM].

5.1 Torsors for an additive group

Let A be an additive group, where the identity element of A is denoted by 0. An A-*torsor* is defined by a non-empty set T equipped with action of A from the right

$$T \times A \longrightarrow T;\ (t, a) \mapsto t.a,$$

which is simply transitive. So, for any elements $s, t \in T$, there exists uniquely $a \in A$ such that $s = t.a$. We denote such an a by $s - t$:

$$a = s - t \overset{\text{def}}{\Longleftrightarrow} s = t.a. \tag{5.1.1}$$

For A-torsors T and T', a morphism $f : T \to T'$ is defined by a map of sets, which satisfies

$$f(t.a) = f(t).a \tag{5.1.2}$$

for all $t \in T$ and $a \in A$. We easily see that any morphism of A-torsors is an isomorphism.

Defining the action of A on A by $(t, a) \in A \times A \mapsto t + a \in A$, A itself becomes an A-torsor. We call it a *trivial A-torsor*. A morphism $f : A \to A$ of trivial A-torsors is given by $f(a) = a + \lambda$ for any $a \in A$ with $\lambda = f(0)$. Choosing an element $t \in T$, any A-torsor T is isomorphic to the trivial A-torsor by the morphism

$$\varphi_t : T \xrightarrow{\sim} A;\ s \mapsto \varphi_t(s) := s - t. \tag{5.1.3}$$

We call φ_t the *trivialization* at t.

Here are some properties concerning A-torsors, which will be used in the subsequent sections.

Theorem 5.1.4. (1) *Let T be an A-torsor. For $s, t, u \in T$ and $a \in A$, we have the following equality in A:*

$$s - s = 0,\ \ s - u = (s - t) + (t - u),\ \ s.a - t = (s - t) + a.$$

(2) *T, T' be A-torsors and let $f : T \to T'$ be a morphism of A-torsors. Then, for $s, t \in T$, we have the following equality in A:*

$$s - t = f(s) - f(t).$$

(3) *Let T, T' be A-torsors and let $f : T \to T'$ be a morphism of A-torsors. Fix $t \in T$ and $t' \in T'$, and let $\lambda(f; t, t') := f(t) - t'$. Then we have the*

following commutative diagram:

$$\begin{array}{ccc} T & \xrightarrow{f} & T' \\ \varphi_t \downarrow & & \downarrow \varphi_{t'} \\ A & \xrightarrow{+\lambda(f;t,t')} & A. \end{array}$$

For other choices $s \in T$ and $s' \in T'$, we have

$$\lambda(f; s, s') = \lambda(f; t, t') + (s - t) - (s' - t').$$

(4) *For an A-torsor T and a subgroup B of A, we note that the quotient set T/B is an A/B-torsor by $(t \bmod B).(a \bmod B) := (t.a \bmod B)$ for $t \in T$ and $a \in A$.*

Proof. (1) These equalities follow from the definition of group action and (5.1.1).
(2) This follows from (5.1.1) and (5.1.2).
(3) The former assertion follows from (5.1.3). For the latter assertion, we note the following commutative diagram.

$$\begin{array}{ccccccc} T & \xrightarrow{\mathrm{id}} & T & \xrightarrow{f} & T' & \xrightarrow{\mathrm{id}} & T' \\ \downarrow \varphi_s & & \downarrow \varphi_t & & \downarrow \varphi_{t'} & & \downarrow \varphi_{s'} \\ A & \xrightarrow{+(s-t)} & A & \xrightarrow{+\lambda(f;t,t')} & A & \xrightarrow{-(s'-t')} & A. \end{array}$$

Since the composite map in the lower row is $+\lambda(f; s, s')$ by the former assertion, the latter assertion follows.
(4) This is easily seen. □

5.2 Conjugate action on group cochains

Let Π be a profinite group and let M be an additive discrete group on which Π acts continuously from the left. Let $C^n(\Pi, M)$ $(n \geq 0)$ be the group of continuous n-cochains of Π with coefficients in M and let $d^{n+1} : C^n(\Pi, M) \to C^{n+1}(\Pi, M)$ be the coboundary homomorphisms defined by

$$\begin{aligned} &(d^{n+1}\alpha^n)(\gamma_1, \dots, \gamma_{n+1}) \\ &\quad := \gamma_1\alpha^n(\gamma_2, \dots, \gamma_{n+1}) \\ &\qquad + \sum_{i=1}^{n} (-1)^i \alpha^n(\gamma_1, \dots, \gamma_{i-1}, \gamma_i\gamma_{i+1}, \gamma_{i+2}, \dots, \gamma_{n+1}) \\ &\qquad + (-1)^{n+1}\alpha^n(\gamma_1, \dots, \gamma_n) \end{aligned} \tag{5.2.1}$$

for $\alpha^n \in C^n(\Pi, M)$ and $\gamma_1, \dots, \gamma_{n+1} \in \Pi$. Let $Z^n(\Pi, M) := \mathrm{Ker}(d^{n+1})$ and $B^n(\Pi, M) := \mathrm{Im}(d^n)$ be the subgroups of $C^n(\Pi, M)$ consisting of n-cocycles and n-coboundaries, respectively, and let $H^n(\Pi, M) := Z^n(\Pi, M)/B^n(\Pi, M)$, the n-th cohomology group of Π with coefficients in M. By convention, we put $C^n(\Pi, M) = 0$ for $n < 0$. We sometimes write d for d^n simply if no misunderstanding is caused.

Note that Π acts on $C^n(\Pi, M)$ from the left by

$$(\sigma.\alpha^n)(\gamma_1, \ldots, \gamma_n) := \sigma\alpha^n(\sigma^{-1}\gamma_1\sigma, \ldots, \sigma^{-1}\gamma_n\sigma) \tag{5.2.2}$$

for $\alpha^n \in C^n(\Pi, M)$ and $\sigma, \gamma_1, \ldots, \gamma_n \in \Pi$. By (5.2.1) and (5.2.2), we see that this action commutes with the coboundary homomorphisms:

$$d^{n+1}(\sigma.\alpha^i) = \sigma.d^{n+1}(\alpha^i) \quad (\alpha^i \in C^i(\Pi, M)). \tag{5.2.3}$$

Now we shall describe the action of Π on $C^n(\Pi, M)$ in a concrete manner. For σ, σ_1, $\sigma_2 \in \Pi$, $0 \le i \le j \le n$ $(n \ge 1)$, and $1 \le k \le n-1$, we define the maps $s_i = s_i^n(\sigma) : \Pi^n \to \Pi^{n+1}$, $s_{i,j} = s_{i,j}^n(\sigma_1, \sigma_2) : \Pi^n \to \Pi^{n+2}$ and $t_k = t_k^n : \Pi^n \to \Pi^{n-1}$ by

$$\begin{aligned} s_i(g_1, g_2, \ldots, g_n) &:= (g_1, \ldots, g_i, \sigma, \sigma^{-1}g_{i+1}\sigma, \ldots, \sigma^{-1}g_n\sigma), \\ s_{i,j}(g_1, g_2, \ldots, g_n) &:= (g_1, \ldots, g_i, \sigma_1, {\sigma_1}^{-1}g_{i+1}\sigma_1, \ldots, {\sigma_1}^{-1}g_j\sigma_1, \\ &\qquad \sigma_2, (\sigma_1\sigma_2)^{-1}g_{j+1}\sigma_1\sigma_2, \ldots, (\sigma_1\sigma_2)^{-1}g_n\sigma_1\sigma_2), \\ t_k(g_1, g_2, \ldots, g_n) &:= (g_1, \ldots, g_{k-1}, g_kg_{k+1}, g_{k+2}, \ldots, g_n) \end{aligned} \tag{5.2.4}$$

for $(g_1, g_2, \ldots, g_n) \in \Pi^n$. We note that $s_{j+1}^{n+1}(\sigma_2) \circ s_i^n(\sigma_1) = s_{i,j}^n(\sigma_1, \sigma_2)$. We define the homomorphisms

$$\begin{aligned} h_\sigma^n &: C^{n+1}(\Pi, M) \longrightarrow C^n(\Pi, M), \\ H_{\sigma_1,\sigma_2}^n &: C^{n+2}(\Pi, M) \longrightarrow C^n(\Pi, M) \end{aligned}$$

by

$$\begin{aligned} h_\sigma^n(\alpha^{n+1}) &:= \sum_{0 \le i \le n} (-1)^i(\alpha^{n+1} \circ s_i^n(\sigma)), \\ H_{\sigma_1,\sigma_2}^n(\alpha^{n+2}) &:= \sum_{0 \le i \le j \le n} (-1)^{i+j}(\alpha^{n+2} \circ s_{i,j}^n(\sigma_1, \sigma_2)) \end{aligned} \tag{5.2.5}$$

for $\alpha^{n+1} \in C^{n+1}(\Pi, M)$ and $\alpha^{n+2} \in C^{n+2}(\Pi, M)$. For example, explicit forms of $h_\sigma^n(\alpha^{n+1}), H_{\sigma_1,\sigma_2}^n(\alpha^{n+2})$ for $n = 1, 2$ are given as follows:

$h_\sigma^1(\alpha^2)(g) = \alpha^2(\sigma, \sigma^{-1}g\sigma) - \alpha^2(g, \sigma)$.
$h_\sigma^2(\alpha^3)(g_1, g_2) = \alpha^3(\sigma, \sigma^{-1}g_1\sigma, \sigma^{-1}g_2\sigma) - \alpha^3(g_1, \sigma, \sigma^{-1}g_2\sigma) + \alpha^3(g_1, g_2, \sigma)$.
$H_{\sigma_1,\sigma_2}^1(\alpha^3)(g) = \alpha^3(\sigma_1, \sigma_2, (\sigma_1\sigma_2)^{-1}g\sigma_1\sigma_2) - \alpha^3(\sigma_1, \sigma_1^{-1}g\sigma_1, \sigma_2) + \alpha^3(g, \sigma_1, \sigma_2)$
$H_{\sigma_1,\sigma_2}^2(\alpha^4)(g_1, g_2) = \alpha^4(\sigma_1, \sigma_2, (\sigma_1\sigma_2)^{-1}g_1\sigma_1\sigma_2, (\sigma_1\sigma_2)^{-1}g_2\sigma_1\sigma_2)$
$-\alpha^4(\sigma_1, \sigma_1^{-1}g_1\sigma_1, \sigma_2, (\sigma_1\sigma_2)^{-1}g_2\sigma_1\sigma_2) + \alpha^4(\sigma_1, \sigma_1^{-1}g_1\sigma_1, \sigma_1^{-1}g_2\sigma_1, \sigma_2)$
$+\alpha^4(g_1, \sigma_1, \sigma_2, (\sigma_1\sigma_2)^{-1}g_2\sigma_1\sigma_2) - \alpha^4(g_1, \sigma_1, \sigma_1^{-1}g_2\sigma_1, \sigma_2) + \alpha^4(g_1, g_2, \sigma_1, \sigma_2)$

We call $h^n_\sigma, H^n_{\sigma_1,\sigma_2}$ the *transgression* homomorphisms, which play roles similar to the transgression homomorphisms in [Gm].

The following Theorem 5.2.6 and Corollary 5.2.7 were shown in Appendices A and B of [CKKPY]. Here we give an elementary direct proof. See also Remark 5.2.8 below for the background of the proof.

Theorem 5.2.6. *Notations being as above, we have the following equalities.*

$$\sigma.\alpha^n - \alpha^n = h^n_\sigma(d^{n+1}(\alpha^n)) + d^n(h^{n-1}_\sigma(\alpha^n)),$$

$$\sigma_1.h^n_{\sigma_2}(\alpha^{n+1}) - h^n_{\sigma_1\sigma_2}(\alpha^{n+1}) + h^n_{\sigma_1}(\alpha^{n+1}) = H^n_{\sigma_1,\sigma_2}(d^{n+2}(\alpha^{n+1})) - d^n(H^{n-1}_{\sigma_1,\sigma_2}(\alpha^{n+1})).$$

for $\alpha^n \in C^n(\Pi, M)$ *and* $\alpha^{n+1} \in C^{n+1}(\Pi, M)$ $(n \geq 1)$.

Proof. By (5.2.4), we can see

$$(5.2.6.1)\qquad \begin{aligned} s_i \circ t_k &= \begin{cases} t_k \circ s_{i+1} & (k \leq i) \\ t_{k+1} \circ s_i & (i < k), \end{cases} \\ s_{i,j} \circ t_k &= \begin{cases} t_k \circ s_{i+1,j+1} & (k \leq i) \\ t_{k+1} \circ s_{i,j+1} & (i < k \leq j) \\ t_{k+2} \circ s_{i,j} & (j < k). \end{cases} \end{aligned}$$

We note that $t_{i+1} \circ s_{i+1} = t_{i+1} \circ s_i$. By (5.2.1) and (5.2.5), we have, for any $(g_1, g_2, \ldots, g_n) \in \Pi^n$,

$$\begin{aligned} h^n_\sigma(d^{n+1}(\alpha^n))(g_1, \ldots, g_n) &= (\sigma.\alpha^n)(g_1, \ldots, g_n) \\ &\quad + \sum_{1 \leq i \leq n} (-1)^i g_1(\alpha^n \circ s_{i-1})(g_2, \ldots, g_n) \\ &\quad + \sum_{0 \leq i \leq n, 1 \leq k \leq n} (-1)^{i+k} (\alpha^n \circ t_k \circ s_i)(g_1, \ldots, g_n) \\ &\quad + (-1)^{n+n+1} \alpha^n(g_1, \ldots, g_n) \\ &\quad + \sum_{0 \leq i \leq n-1} (-1)^{i+n+1} (\alpha^n \circ s_i)(g_1, \ldots, g_{n-1}), \\ d^n(h^{n-1}_\sigma(\alpha^n))(g_1, \ldots, g_n) &= \sum_{0 \leq i \leq n-1} (-1)^i g_1(\alpha^n \circ s_i)(g_2, \ldots, g_n) \\ &\quad + \sum_{0 \leq i \leq n-1, 1 \leq k \leq n-1} (-1)^{i+k} (\alpha^n \circ s_i \circ t_k)(g_1, \ldots, g_n) \\ &\quad + \sum_{0 \leq i \leq n-1} (-1)^{i+n} (\alpha^n \circ s_i)(g_1, \ldots, g_{n-1}), \end{aligned}$$

and

$$
\begin{aligned}
&H^n_{\sigma_1,\sigma_2}(d^{n+2}(\alpha^{n+1}))(g_1,\dots,g_n)\\
&= (\sigma_1.h^n_{\sigma_2}(\alpha^{n+1}))(g_1,\dots,g_n) + \sum_{0<i\leq j\leq n} (-1)^{i+j} g_1(\alpha^{n+1}\circ s_{i-1,j-1})(g_2,\dots,g_n)\\
&-h^n_{\sigma_1\sigma_2}(\alpha^{n+1})(g_1,\dots,g_n) + \sum_{\substack{0\leq i\leq j\leq n, 1\leq k\leq n+1\\ i\neq j \text{ or } k\neq i+1}} (-1)^{i+j+k}(\alpha^{n+1}\circ t_k\circ s_{i,j})(g_1,\dots,g_n)\\
&+h^n_{\sigma_1}(\alpha^{n+1})(g_1,\dots,g_n) + \sum_{0\leq i\leq j\leq n-1} (-1)^{i+j+n+2}(\alpha^{n+1}\circ s_{i,j})(g_1,\dots,g_{n-1}),\\
&d^n(H^{n-1}_{\sigma_1,\sigma_2}(\alpha^{n+1}))(g_1,\dots,g_n)\\
&= \sum_{0\leq i\leq j\leq n-1} (-1)^{i+j} g_1.(\alpha^{n+1}\circ s_{i,j})(g_2,\dots,g_n)\\
&+ \sum_{0\leq i\leq j\leq n-1, 1\leq k\leq n-1} (-1)^{i+j+k}(\alpha^{n+1}\circ s_{i,j}\circ t_k)(g_1,\dots,g_n)\\
&+ \sum_{0\leq i\leq j\leq n-1} (-1)^{i+j+n}(\alpha^{n+1}\circ s_{i,j})(g_1,\dots,g_{n-1}).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&h^n_\sigma(d^{n+1}(\alpha^n))(g_1,\dots,g_n) + d^n(h^{n-1}_\sigma(\alpha^n))(g_1,\dots,g_n)\\
&= (\sigma.\alpha^n)(g_1,\dots,g_n) - \alpha^n(g_1,\dots,g_n)\\
&+ \sum_{0\leq i\leq n, 1\leq k\leq n} (-1)^{i+k}(\alpha^n\circ t_k\circ s_i)(g_1,\dots,g_n)\\
&+ \sum_{0\leq i\leq n-1, 1\leq k\leq n-1} (-1)^{i+k}(\alpha^n\circ s_i\circ t_k)(g_1,\dots,g_n),
\end{aligned}
$$

and

$$
\begin{aligned}
&H^n_{\sigma_1,\sigma_2}(d^{n+2}(\alpha^{n+1}))(g_1,\dots,g_n) - d^n(H^{n-1}_{\sigma_1,\sigma_2}(\alpha^{n+1}))(g_1,\dots,g_n)\\
&= \sigma_1.h^n_{\sigma_2}(\alpha^{n+1})(g_1,\dots,g_n) - h^n_{\sigma_1\sigma_2}(\alpha^{n+1})(g_1,\dots,g_n) + h^n_{\sigma_1}(\alpha^{n+1})(g_1,\dots,g_n)\\
&+ \sum_{\substack{0\leq i\leq j\leq n, 1\leq k\leq n+1\\ i\neq j \text{ or } k\neq i+1}} (-1)^{i+j+k}(\alpha^{n+1}\circ t_k\circ s_{i,j})(g_1,\dots,g_n)\\
&- \sum_{0\leq i\leq j\leq n-1, 1\leq k\leq n-1} (-1)^{i+j+k}(\alpha^{n+1}\circ s_{i,j}\circ t_k)(g_1,\dots,g_n).
\end{aligned}
$$

By (5.2.6.1), we obtain the required equalities. □

By (5.2.3), Π acts on $Z^n(\Pi, M)$ from the left. This action is described by Theorem 5.2.6 as follows.

Corollary 5.2.7. *Suppose $\alpha \in Z^n(\Pi, M)$ $(n \geq 1)$. For $\sigma \in \Pi$, we let*

$$\beta_\sigma := h^{n-1}_\sigma(\alpha).$$

Then we have

$$\sigma.\alpha = \alpha + d^n\beta_\sigma.$$

For $\sigma, \sigma' \in \Pi$, *we have*

$$\beta_{\sigma\sigma'} = \beta_\sigma + \sigma.\beta_{\sigma'} \mod B^{n-1}(\Pi, M),$$

namely, the map $\Pi \ni \sigma \mapsto \beta_\sigma$ *mod* $B^{n-1}(\Pi, M) \in C^{n-1}(\Pi, M)/B^{n-1}(\Pi, M)$ *is a* 1*-cocycle.*

Proof. Proof. The both equalities are obtained immediately from Theorem 5.2.6, since $d^{n+1}(\alpha) = 0$ by $\alpha \in Z^n(\Pi, M)$ $(n \geq 1)$. □

Remark 5.2.8 (Algebro-topological proof of Theorem 5.2.6). For $\sigma \in \Pi$, let $\sigma^\bullet$ denote the automorphism of the cochain complex $(C^\bullet(\Pi, M), d^\bullet)$ defined by $\sigma^n(\alpha) := \sigma.\alpha$ for $\alpha \in C^n(\Pi, M)$. Then Theorem 5.2.6 asserts that the family of homomorphisms $\{h^n_\sigma : C^{n+1}(\Pi, M) \to C^n(\Pi, M)\}$ gives a homotopy connecting $\sigma^\bullet$ and $\mathrm{id}_{C^\bullet(\Pi,M)}$. Actually our explicit definition (5.2.5) is obtained by making the following algebro-topological proof concrete: We may assume Π is finite by the limit argument. Let $\mathcal{E}$ be the one-object category whose morphisms are the elements of Π. We consider two functors $\mathrm{id}_\mathcal{E}, \widehat{\sigma} : \mathcal{E} \to \mathcal{E}$ defined by $\mathrm{id}_\mathcal{E}(g) := g, \widehat{\sigma}(g) := \sigma^{-1}g\sigma$ for each morphism $g \in \Pi$. Let $\mathcal{N} : \mathrm{Cat} \to \mathrm{Fct}(\Delta^{\mathrm{op}}, \mathrm{Set})$ denote the nerve functor, where Cat is the category of small categories and $\mathrm{Fct}(\Delta^{\mathrm{op}}, \mathrm{Set})$ is the category of simplicial sets. Define the natural transformation $\eta : \widehat{\sigma} \to \mathrm{id}_\mathcal{E}$ by $\eta(*) := \sigma$ ($*$ is the unique object of $\mathcal{E}$). Then η induces a corresponding funcor $h_\eta : \mathcal{E} \times \underline{1} \to \mathcal{E}$, where $\underline{n}$ denotes the category defined by the set $\{0, 1, \ldots, n\}$ and its order. Then $\mathcal{N}h_\eta : \mathcal{N}\mathcal{E} \times \mathcal{N}\underline{1} \to \mathcal{N}\mathcal{E}$ is a homotopy connecting the two simplicial maps $\mathcal{N}\widehat{\sigma}, \mathcal{N}\mathrm{id}_\mathcal{E} : \mathcal{N}\mathcal{E} \to \mathcal{N}\mathcal{E}$. Let $C_n(\mathcal{N}\mathcal{E}) = \mathbb{Z}[\mathcal{N}\mathcal{E}(\underline{n})]$ be the group of n-chains of the simplicial set $\mathcal{N}\mathcal{E}$. By [My; Proposition 5.3] and [My; Proposition 6.2], $\mathcal{N}h_\eta$ induces a homotopy $\{h^\sigma_n : C_n(\mathcal{N}\mathcal{E}) \to C_{n+1}(\mathcal{N}\mathcal{E})\}$ connecting two chain maps $(\mathcal{N}\widehat{\sigma})_\bullet, (\mathcal{N}\mathrm{id}_\mathcal{E})_\bullet : C_\bullet(\mathcal{N}\mathcal{E}) \to C_\bullet(\mathcal{N}\mathcal{E})$. For the groups of n-cochains $C^n(\mathcal{N}\mathcal{E}, M) = \mathrm{Hom}(C_n(\mathcal{N}\mathcal{E}), M)$, the homotopy $\{h^\sigma_n\}$ induces the homotopy $\{h^n_\sigma : C^{n+1}(\mathcal{N}\mathcal{E}, M) \to C^n(\mathcal{N}\mathcal{E}, M)\}$ connecting the two cochain maps $(\mathcal{N}\widehat{\sigma})^\bullet, (\mathcal{N}\mathrm{id}_\mathcal{E})^\bullet : C^\bullet(\mathcal{N}\mathcal{E}, M) \to C^\bullet(\mathcal{N}\mathcal{E}, M)$. Since $\mathcal{N}\mathcal{E}(n)$ is Π^n, we have the isomorphisms for $i \geq 0$

$$C^n(\mathcal{N}\mathcal{E}, M) \simeq \mathrm{Map}(\Pi^n, M) = C^n(\Pi, M).$$

Under the above isomorphisms, $(\mathcal{N}\widehat{\sigma})^\bullet$ and $(\mathcal{N}\mathrm{id}_\mathcal{E})^\bullet$ are identified with $\sigma^\bullet$ and $\mathrm{id}_{C^\bullet(\Pi,M)}$, respectively, and hence $\{h^n_\sigma\}$ gives a homotopy connecting $\sigma^\bullet$ and $\mathrm{id}_{C^\bullet(\Pi,M)}$.

5.3 Arithmetic prequantization bundles and arithmetic Chern–Simons 1-cocycles

Throughout the rest of this section, we fix a natural number $N > 1$and let μ_N be the group of N-th roots of unity in the field $\mathbb{C}$ of complex numbers. We fix a primitive N-th root of unity ζ_N and the isomorphism $\mathbb{Z}/N\mathbb{Z} \simeq \mu_N$; $m \mapsto \zeta_N^m$. The base number field k (in $\mathbb{C}$) is supposed to contain μ_N. Let G be a finite group and let c be a fixed 3-cocycle of G with coefficients in $\mathbb{Z}/N\mathbb{Z}$, $c \in Z^3(G, \mathbb{Z}/N\mathbb{Z})$, where G acts on $\mathbb{Z}/N\mathbb{Z}$ trivially.

We firstly develop a local theory at a finite prime. Let $\mathfrak{p}$ be a finite prime of k and let $k_\mathfrak{p}$ be the $\mathfrak{p}$-adic field. We let $\partial V_\mathfrak{p} := \mathrm{Spec}(k_\mathfrak{p})$, which play a role analogous to the boundary of a tubular neighborhood of a knot (see the dictionary of the analogies in Introduction). Let $\Pi_\mathfrak{p}$ denote the étale fundamental group of $\partial V_\mathfrak{p}$ with base point $\mathrm{Spec}(\overline{k}_\mathfrak{p})$ ($\overline{k}_\mathfrak{p}$ being an algebraic closure of $k_\mathfrak{p}$), which is the absolute Galois group $\mathrm{Gal}(\overline{k}_\mathfrak{p}/k_\mathfrak{p})$.

Let $\mathcal{F}_\mathfrak{p}$ be the set of continuous homomorphisms of $\Pi_\mathfrak{p}$ to G:

$$\mathcal{F}_\mathfrak{p} := \mathrm{Hom}_{\mathrm{cont}}(\Pi_\mathfrak{p}, G).$$

It is a finite set on which G acts from the right by

$$\mathcal{F}_\mathfrak{p} \times G \to \mathcal{F}_\mathfrak{p}; \quad (\rho_\mathfrak{p}, g) \mapsto \rho_\mathfrak{p}.g := g^{-1}\rho_\mathfrak{p} g. \tag{5.3.1}$$

Let $\mathcal{M}_\mathfrak{p}$ denote the quotient space by this action:

$$\mathcal{M}_\mathfrak{p} := \mathcal{F}_\mathfrak{p}/G.$$

Let $\mathrm{Map}(\mathcal{F}_\mathfrak{p}, \mathbb{Z}/N\mathbb{Z})$ denote the additive group consisting of maps from $\mathcal{F}_\mathfrak{p}$ to $\mathbb{Z}/N\mathbb{Z}$, on which G acts from the left by

$$(g.\psi_\mathfrak{p})(\rho_\mathfrak{p}) := \psi_\mathfrak{p}(\rho_\mathfrak{p}.g) \tag{5.3.2}$$

for $g \in G, \psi_\mathfrak{p} \in \mathrm{Map}(\mathcal{F}_\mathfrak{p}, \mathbb{Z}/N\mathbb{Z})$ and $\rho_\mathfrak{p} \in \mathcal{F}_\mathfrak{p}$. For $\rho_\mathfrak{p} \in \mathcal{F}_\mathfrak{p}$ and $\alpha \in C^n(G, \mathbb{Z}/N\mathbb{Z})$, we denote by $\alpha \circ \rho_\mathfrak{p}$ the n-cochain of $\Pi_\mathfrak{p}$ with coefficients in $\mathbb{Z}/N\mathbb{Z}$ defined by

$$(\alpha \circ \rho_\mathfrak{p})(\gamma_1, \ldots, \gamma_n) := \alpha(\rho_\mathfrak{p}(\gamma_1), \ldots, \rho_\mathfrak{p}(\gamma_n)).$$

By (5.2.2) and (5.3.1), we have

$$(g.\alpha) \circ \rho_\mathfrak{p} = \alpha \circ (\rho_\mathfrak{p}.g) \tag{5.3.3}$$

for $g \in G, \alpha \in C^n(G, \mathbb{Z}/N\mathbb{Z})$ and $\rho_\mathfrak{p} \in \mathcal{F}_\mathfrak{p}$.

Firstly, we shall construct an arithmetic analog for $\partial V_{\mathfrak{p}} := \mathrm{Spec}(k_{\mathfrak{p}})$ of the prequantization bundle, using the given 3-cocycle $c \in Z^3(G, \mathbb{Z}/N\mathbb{Z})$. The key idea for this is due to Kim ([Ki]), who uses the conjugate G-action on c and the 2nd Galois cohomology group (Brauer group) of the local field $k_{\mathfrak{p}}$.

Let $\rho_{\mathfrak{p}} \in \mathcal{F}_{\mathfrak{p}}$ and so $c \circ \rho_{\mathfrak{p}} \in Z^3(\Pi_{\mathfrak{p}}, \mathbb{Z}/N\mathbb{Z})$. Let d denote the coboundary homomorphism $C^2(\Pi_{\mathfrak{p}}, \mathbb{Z}/N\mathbb{Z}) \to C^3(\Pi_{\mathfrak{p}}, \mathbb{Z}/N\mathbb{Z})$. We define $\mathcal{L}_{\mathfrak{p}}(\rho_{\mathfrak{p}})$ by the quotient set

$$\mathcal{L}_{\mathfrak{p}}(\rho_{\mathfrak{p}}) := d^{-1}(c \circ \rho_{\mathfrak{p}})/B^2(\Pi_{\mathfrak{p}}, \mathbb{Z}/N\mathbb{Z}). \tag{5.3.4}$$

Here we note that $d^{-1}(c \circ \rho_{\mathfrak{p}})$ is non-empty, because the cohomological dimension of $\Pi_{\mathfrak{p}}$ is 2 ([NSW; Theorem 7.1.8], [S1; Chapitre II, 5.3, Proposition 15]) and so $H^3(\Pi_{\mathfrak{p}}, \mathbb{Z}/N\mathbb{Z}) = 0$. Thus $d^{-1}(c \circ \rho_{\mathfrak{p}})$ is a $Z^2(\Pi_{\mathfrak{p}}, \mathbb{Z}/N\mathbb{Z})$-torsor in the obvious manner and so $\mathcal{L}_{\mathfrak{p}}(\rho_{\mathfrak{p}})$ is an $H^2(\Pi_{\mathfrak{p}}, \mathbb{Z}/N\mathbb{Z})$-torsor by (5.3.4) and Lemma 5.1.4 (4). Since $k_{\mathfrak{p}}$ contains μ_N and so $H^2(\Pi_{\mathfrak{p}}, \mathbb{Z}/N\mathbb{Z}) = H^2(k_{\mathfrak{p}}, \mu_N)$, the theory of Brauer groups (cf. [S2; Chapitre XII]) tells us that there is the canonical isomorphism

$$\mathrm{inv}_{\mathfrak{p}} : H^2(\Pi_{\mathfrak{p}}, \mathbb{Z}/N\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}/N\mathbb{Z}$$

and hence $\mathcal{L}_{\mathfrak{p}}(\rho_{\mathfrak{p}})$ is a $\mathbb{Z}/N\mathbb{Z}$-torsor via $\mathrm{inv}_{\mathfrak{p}}$.

Let $\mathcal{L}_{\mathfrak{p}}$ be the disjoint union of $\mathcal{L}_{\mathfrak{p}}(\rho_{\mathfrak{p}})$ over all $\rho_{\mathfrak{p}} \in \mathcal{F}_{\mathfrak{p}}$:

$$\mathcal{L}_{\mathfrak{p}} := \bigsqcup_{\rho_{\mathfrak{p}} \in \mathcal{F}_{\mathfrak{p}}} \mathcal{L}_{\mathfrak{p}}(\rho_{\mathfrak{p}})$$

and consider the projection

$$\varpi_{\mathfrak{p}} : \mathcal{L}_{\mathfrak{p}} \longrightarrow \mathcal{F}_{\mathfrak{p}}; \quad \alpha_{\mathfrak{p}} \mapsto \rho_{\mathfrak{p}} \text{ if } \alpha_{\mathfrak{p}} \in \mathcal{L}_{\mathfrak{p}}(\rho_{\mathfrak{p}}).$$

Since each fiber $\varpi_{\mathfrak{p}}^{-1}(\rho_{\mathfrak{p}}) = \mathcal{L}_{\mathfrak{p}}(\rho_{\mathfrak{p}})$ is a $\mathbb{Z}/N\mathbb{Z}$-torsor, we may regard $\mathcal{L}_{\mathfrak{p}}$ as a principal $\mathbb{Z}/N\mathbb{Z}$-bundle over $\mathcal{F}_{\mathfrak{p}}$.

Let $g \in G$. Using the transgression map h_g^2 in (5.2.5), we define $h_g \in C^2(G, \mathbb{Z}/N\mathbb{Z})/B^2(G, \mathbb{Z}/N\mathbb{Z})$ by

$$h_g := h_g^2(c) \bmod B^2(G, \mathbb{Z}/N\mathbb{Z}),$$

where $h_g^2(c)$ is the 2-cochain defined explicitly by

$$h_g^2(c)(g_1, g_2) := c(g, g^{-1}g_1 g, g^{-1}g_2 g) - c(g_1, g, g^{-1}g_2 g) + c(g_1, g_2, g),$$

where $g_1, g_2 \in G$. By Corollary 1.2.7, we have

$$g.c = c + dh_g \tag{5.3.5}$$

and

$$h_{gg'} = h_g + g.h_{g'} \tag{5.3.6}$$

for $g, g' \in G$. By (5.3.3), (5.3.4) and (5.3.5), we have

$$d(\alpha + h_g \circ \rho_{\mathfrak{p}}) = c \circ \rho_{\mathfrak{p}} + (g.c - c) \circ \rho_{\mathfrak{p}} = (g.c) \circ \rho_{\mathfrak{p}} = c \circ (\rho_{\mathfrak{p}}.g)$$

for $\alpha_{\mathfrak{p}} \in \mathcal{L}_{\mathfrak{p}}(\rho_{\mathfrak{p}})$ and so we have the isomorphism of $\mathbb{Z}/N\mathbb{Z}$-torsors

$$f_{\mathfrak{p}}(g, \rho_{\mathfrak{p}}) : \mathcal{L}_{\mathfrak{p}}(\rho_{\mathfrak{p}}) \xrightarrow{\sim} \mathcal{L}_{\mathfrak{p}}(\rho_{\mathfrak{p}}.g); \;\; \alpha_{\mathfrak{p}} \mapsto \alpha_{\mathfrak{p}} + h_g \circ \rho_{\mathfrak{p}}. \tag{5.3.7}$$

By (5.3.3) and (5.3.6), we have

$$\begin{aligned} \alpha_{\mathfrak{p}} + h_{gg'} \circ \rho_{\mathfrak{p}} &= \alpha_{\mathfrak{p}} + (h_g + g.h_{g'}) \circ \rho_{\mathfrak{p}} \\ &= \alpha_{\mathfrak{p}} + h_g \circ \rho_{\mathfrak{p}} + h_{g'} \circ (\rho_{\mathfrak{p}}.g) \end{aligned}$$

for $g, g' \in G$. It means that G acts on $\mathcal{L}_{\mathfrak{p}}$ from the right by

$$\mathcal{L}_{\mathfrak{p}} \times G \to \mathcal{L}_{\mathfrak{p}}; \;\; \alpha_{\mathfrak{p}} \mapsto \alpha_{\mathfrak{p}}.g := f(g, \rho_{\mathfrak{p}})(\alpha_{\mathfrak{p}}). \tag{5.3.8}$$

By (5.3.7), (5.3.8) and the way of the $\mathbb{Z}/N\mathbb{Z}$-action on $\mathcal{L}_{\mathfrak{p}}$, we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{L}_{\mathfrak{p}} & \xrightarrow{\cdot g} & \mathcal{L}_{\mathfrak{p}} \curvearrowleft \mathbb{Z}/N\mathbb{Z} \\ \varpi_{\mathfrak{p}} \downarrow & & \downarrow \varpi_{\mathfrak{p}} \\ \mathcal{F}_{\mathfrak{p}} & \xrightarrow{\cdot g} & \mathcal{F}_{\mathfrak{p}}, \end{array}$$

namely,

$$(\alpha_{\mathfrak{p}}.m).g = (\alpha_{\mathfrak{p}}.g).m, \;\; \varpi_{\mathfrak{p}}(\alpha_{\mathfrak{p}}.g) = \varpi_{\mathfrak{p}}(\alpha_{\mathfrak{p}}).g \tag{5.3.9}$$

for $\alpha_{\mathfrak{p}} \in \mathcal{F}_{\mathfrak{p}}$, $m \in \mathbb{Z}/N\mathbb{Z}, g \in G$. So $\mathcal{L}_{\mathfrak{p}}$ is a G-equivariant principal $\mathbb{Z}/N\mathbb{Z}$-bundle over $\mathcal{F}_{\mathfrak{p}}$. Taking the quotient by the action of G, we have the principal $\mathbb{Z}/N\mathbb{Z}$-bundle $\overline{\varpi}_{\mathfrak{p}} : \overline{\mathcal{L}}_{\mathfrak{p}} \to \mathcal{M}_{\mathfrak{p}}$. We call $\varpi_{\mathfrak{p}} : \mathcal{L}_{\mathfrak{p}} \to \mathcal{F}_{\mathfrak{p}}$ or $\overline{\varpi}_{\mathfrak{p}} : \overline{\mathcal{L}}_{\mathfrak{p}} \to \mathcal{M}_{\mathfrak{p}}$ the *arithmetic prequantization* $\mathbb{Z}/N\mathbb{Z}$*-bundle* for $\partial V_{\mathfrak{p}} := \mathrm{Spec}(k_{\mathfrak{p}})$.

Let us choose a section $x_{\mathfrak{p}} \in \Gamma(\mathcal{F}_{\mathfrak{p}}, \mathcal{L}_{\mathfrak{p}})$, namely, the map

$$x_{\mathfrak{p}} : \mathcal{F}_{\mathfrak{p}} \longrightarrow \mathcal{L}_{\mathfrak{p}} \;\text{ such that } \varpi_{\mathfrak{p}} \circ x_{\mathfrak{p}} = \mathrm{id}_{\mathcal{F}_{\mathfrak{p}}}.$$

This means that we fix a "coordinate" on $\mathcal{L}_{\mathfrak{p}}$. In fact, by the trivialization at $x_{\mathfrak{p}}(\rho_{\mathfrak{p}})$ in (5.1.3), we may identify each fiber $\mathcal{L}_{\mathfrak{p}}(\rho_{\mathfrak{p}})$ over $\rho_{\mathfrak{p}}$ with $\mathbb{Z}/N\mathbb{Z}$:

$$\varphi_{x_{\mathfrak{p}}(\rho_{\mathfrak{p}})} : \mathcal{L}_{\mathfrak{p}}(\rho_{\mathfrak{p}}) \xrightarrow{\sim} \mathbb{Z}/N\mathbb{Z}; \; \alpha_{\mathfrak{p}} \mapsto \alpha_{\mathfrak{p}} - x_{\mathfrak{p}}(\rho_{\mathfrak{p}}).$$

For $g \in G$ and $\rho_{\mathfrak{p}} \in \mathcal{F}_{\mathfrak{p}}$, we let

$$\lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}(g, \rho_{\mathfrak{p}}) := f_{\mathfrak{p}}(g, \rho_{\mathfrak{p}})(x_{\mathfrak{p}}(\rho_{\mathfrak{p}})) - x_{\mathfrak{p}}(\rho_{\mathfrak{p}}.g) = x_{\mathfrak{p}}(\rho_{\mathfrak{p}}).g - x_{\mathfrak{p}}(\rho_{\mathfrak{p}}.g) \tag{5.3.10}$$

so that we have the following commutative diagram by Lemma 5.1.4 (3):

$$\begin{array}{ccc} \mathcal{L}_{\mathfrak{p}}(\rho_{\mathfrak{p}}) & \stackrel{f_{\mathfrak{p}}(g,\rho_{\mathfrak{p}})}{\longrightarrow} & \mathcal{L}_{\mathfrak{p}}(\rho_{\mathfrak{p}}.g) \\ \varphi_{x_{\mathfrak{p}}(\rho_{\mathfrak{p}})} \downarrow & & \downarrow \varphi_{x_{\mathfrak{p}}(\rho_{\mathfrak{p}}.g)} \\ \mathbb{Z}/N\mathbb{Z} & \stackrel{+\lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}(g,\rho_{\mathfrak{p}})}{\longrightarrow} & \mathbb{Z}/N\mathbb{Z}, \end{array}$$

namely, for $\alpha_{\mathfrak{p}} \in \mathcal{L}_{\mathfrak{p}}(\rho_{\mathfrak{p}})$, we have

$$\alpha_{\mathfrak{p}}.g - x_{\mathfrak{p}}(\rho_{\mathfrak{p}}.g) = (\alpha_{\mathfrak{p}} - x_{\mathfrak{p}}(\rho_{\mathfrak{p}})) + \lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}(g, \rho_{\mathfrak{p}}). \tag{5.3.11}$$

We define the map $\lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}} : G \to \mathrm{Map}(\mathcal{F}_{\mathfrak{p}}, \mathbb{Z}/N\mathbb{Z})$ by

$$\lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}(g)(\rho_{\mathfrak{p}}) := \lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}(g, \rho_{\mathfrak{p}}) \tag{5.3.12}$$

for $g \in G$ and $\rho_{\mathfrak{p}} \in \mathcal{F}_{\mathfrak{p}}$.

Theorem 5.3.13. *For $g, g' \in G$, we have*

$$\lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}(gg') = \lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}(g) + (g.\lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}})(g').$$

Namely, the map $\lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}$ is a 1-cocycle:

$$\lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}} \in Z^1(G, \mathrm{Map}(\mathcal{F}_{\mathfrak{p}}, \mathbb{Z}/N\mathbb{Z})).$$

Proof. For $g, g' \in G$ and $\rho_{\mathfrak{p}} \in \mathcal{F}_{\mathfrak{p}}$, we have

$$\begin{aligned} \lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}(gg', \rho_{\mathfrak{p}}) &= f_{\mathfrak{p}}(gg', \rho_{\mathfrak{p}})(x_{\mathfrak{p}}(\rho_{\mathfrak{p}})) - x_{\mathfrak{p}}(\rho_{\mathfrak{p}}(gg')) \text{ by (5.3.10)} \\ &= (x_{\mathfrak{p}}(\rho_{\mathfrak{p}}) + h_{gg'} \circ \rho_{\mathfrak{p}}) - x_{\mathfrak{p}}(\rho_{\mathfrak{p}}.(gg')) \text{ by (5.3.7)} \\ &= (x_{\mathfrak{p}}(\rho_{\mathfrak{p}}) + h_g \circ \rho_{\mathfrak{p}} + h_{g'} \circ (\rho_{\mathfrak{p}}.g)) - x_{\mathfrak{p}}(\rho_{\mathfrak{p}}.(gg')) \text{ by (5.3.3), (5.3.6).} \end{aligned}$$

By Lemma 5.1.4 (1), we have

$$\begin{aligned} &(x_{\mathfrak{p}}(\rho_{\mathfrak{p}}) + h_g \circ \rho_{\mathfrak{p}} + h_{g'} \circ (\rho_{\mathfrak{p}}.g)) - x_{\mathfrak{p}}(\rho_{\mathfrak{p}}.(gg')) \\ &\qquad = \{(x_{\mathfrak{p}}(\rho_{\mathfrak{p}}) + h_g \circ \rho_{\mathfrak{p}}) - x_{\mathfrak{p}}(\rho_{\mathfrak{p}}.g)\} + \{(x_{\mathfrak{p}}(\rho_{\mathfrak{p}}.g) + h_{g'} \circ (\rho_{\mathfrak{p}}.g)) - x_{\mathfrak{p}}(\rho_{\mathfrak{p}}.(gg'))\}. \end{aligned}$$

Here we see by (5.3.7), (5.3.10) that

$$\begin{aligned} (x_{\mathfrak{p}}(\rho_{\mathfrak{p}}) + h_g \circ \rho_{\mathfrak{p}}) - x_{\mathfrak{p}}(\rho_{\mathfrak{p}}.g) &= \lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}(g, \rho_{\mathfrak{p}}), \\ (x_{\mathfrak{p}}(\rho_{\mathfrak{p}}.g) + h_{g'} \circ (\rho_{\mathfrak{p}}.g)) - x_{\mathfrak{p}}(\rho_{\mathfrak{p}}.(gg')) &= \lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}(g', \rho_{\mathfrak{p}}.g). \end{aligned}$$

Combining these, we have

$$\lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}(gg', \rho_{\mathfrak{p}}) = \lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}(g, \rho_{\mathfrak{p}}) + \lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}(g', \rho_{\mathfrak{p}}.g)$$

for any $\rho_{\mathfrak{p}} \in \mathcal{F}_{\mathfrak{p}}$. By (5.3.2) and (5.3.12), we obtain the assertion. □

We call $\lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}$ the *Chern–Simons 1-cocycle* for $\partial V_{\mathfrak{p}}$ with respect to the section $x_{\mathfrak{p}}$.

For a section $x_{\mathfrak{p}} \in \Gamma(\mathcal{F}_{\mathfrak{p}}, \mathcal{L}_{\mathfrak{p}})$, we define $\mathcal{L}_{\mathfrak{p}}^{x_{\mathfrak{p}}}$ by the product (trivial) principal $\mathbb{Z}/N\mathbb{Z}$-bundle over $\mathcal{F}_{\mathfrak{p}}$:

$$\mathcal{L}_{\mathfrak{p}}^{x_{\mathfrak{p}}} := \mathcal{F}_{\mathfrak{p}} \times \mathbb{Z}/N\mathbb{Z},$$

on which G acts from the right by

$$\mathcal{L}_{\mathfrak{p}}^{x_{\mathfrak{p}}} \times G \to \mathcal{L}_{\mathfrak{p}}^{x_{\mathfrak{p}}};\ \ ((\rho_{\mathfrak{p}}, m), g) \mapsto (\rho_{\mathfrak{p}}.g, m + \lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}(g, \rho_{\mathfrak{p}})), \tag{5.3.14}$$

and so the projection

$$\varpi_{\mathfrak{p}}^{x_{\mathfrak{p}}} : \mathcal{L}_{\mathfrak{p}}^{x_{\mathfrak{p}}} \longrightarrow \mathcal{F}_{\mathfrak{p}}$$

is G-equivariant.

Proposition 5.3.15. *We have the following isomorphism of G-equivariant principal $\mathbb{Z}/N\mathbb{Z}$-bundles*

$$\Phi_{\mathfrak{p}}^{x_{\mathfrak{p}}} : \mathcal{L}_{\mathfrak{p}} \xrightarrow{\sim} \mathcal{L}_{\mathfrak{p}}^{x_{\mathfrak{p}}};\ \ \alpha_{\mathfrak{p}} \mapsto (\varpi_{\mathfrak{p}}(\alpha_{\mathfrak{p}}), \alpha_{\mathfrak{p}} - x_{\mathfrak{p}}(\varpi_{\mathfrak{p}}(\alpha_{\mathfrak{p}}))).$$

In particular, the isomorphism class of $\mathcal{L}_{\mathfrak{p}}^{x_{\mathfrak{p}}}$ is independent of the choice of a section $x_{\mathfrak{p}}$. In other words, for another section $x'_{\mathfrak{p}} \in \Gamma(\mathcal{F}_{\mathfrak{p}}, \mathcal{L}_{\mathfrak{p}})$, we have $\mathcal{L}_{\mathfrak{p}}^{x'_{\mathfrak{p}}} \simeq \mathcal{L}_{\mathfrak{p}}^{x_{\mathfrak{p}}}$ as G-equivariant principal $\mathbb{Z}/N\mathbb{Z}$-bundles.

Proof. (i) It is easy to see that $\varpi_{\mathfrak{p}}^{x_{\mathfrak{p}}} \circ \Phi_{\mathfrak{p}}^{x_{\mathfrak{p}}} = \varpi_{\mathfrak{p}}$.
(ii) For $\alpha_{\mathfrak{p}} \in \mathcal{L}_{\mathfrak{p}}$ and $m \in \mathbb{Z}/N\mathbb{Z}$, we have

$$\begin{aligned}
\Phi_{\mathfrak{p}}^{x_{\mathfrak{p}}}(\alpha_{\mathfrak{p}}.m) &= (\varpi_{\mathfrak{p}}(\alpha_{\mathfrak{p}}.m), \alpha_{\mathfrak{p}}.m - x_{\mathfrak{p}}(\varpi_{\mathfrak{p}}(\alpha_{\mathfrak{p}}.m))) \\
&= (\varpi_{\mathfrak{p}}(\alpha_{\mathfrak{p}}), \alpha_{\mathfrak{p}}.m - x_{\mathfrak{p}}(\varpi_{\mathfrak{p}}(\alpha_{\mathfrak{p}}))) \\
&= (\varpi_{\mathfrak{p}}(\alpha_{\mathfrak{p}}), (\alpha_{\mathfrak{p}} - x_{\mathfrak{p}}(\varpi_{\mathfrak{p}}(\alpha_{\mathfrak{p}}))) + m) \text{ by Lemma 5.1.4 (1)} \\
&= \Phi_{\mathfrak{p}}^{x_{\mathfrak{p}}}(\alpha_{\mathfrak{p}}).m.
\end{aligned}$$

(iii) $\Phi_{\mathfrak{p}}^{x_{\mathfrak{p}}}$ has the inverse defined by $(\Phi_{\mathfrak{p}}^{x_{\mathfrak{p}}})^{-1}((\rho_{\mathfrak{p}}, m)) := x_{\mathfrak{p}}(\rho_{\mathfrak{p}}).m$ for $(\rho_{\mathfrak{p}}, m) \in \mathcal{F}_{\mathfrak{p}} \times \mathbb{Z}/N\mathbb{Z}$.
By (i), (ii), (iii), $\Phi_{\mathfrak{p}}^{x_{\mathfrak{p}}}$ is an isomorphism of principal $\mathbb{Z}/N\mathbb{Z}$-bundles. So it suffices to show that $\Phi^{x_{\mathfrak{p}}}$ is G-equivariant. It follows from that

$$\begin{aligned}
\Phi_{\mathfrak{p}}^{x_{\mathfrak{p}}}(\alpha_{\mathfrak{p}}.g) &= (\varpi_{\mathfrak{p}}(\alpha_{\mathfrak{p}}.g), \alpha_{\mathfrak{p}}.g - x_{\mathfrak{p}}(\varpi_{\mathfrak{p}}(\alpha_{\mathfrak{p}}.g))) \\
&= (\varpi_{\mathfrak{p}}(\alpha_{\mathfrak{p}}).g, (\alpha_{\mathfrak{p}} - x_{\mathfrak{p}}(\varpi_{\mathfrak{p}}(\alpha_{\mathfrak{p}}))) + \lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}(g, \varpi_{\mathfrak{p}}(\alpha_{\mathfrak{p}}))) \\
&= \Phi_{\mathfrak{p}}^{x_{\mathfrak{p}}}(\alpha_{\mathfrak{p}}).g,
\end{aligned}$$

where the 2nd equality holds by (5.3.9), (5.3.11) and the 3rd equality follows from (5.1.14) □

Taking the quotient of $\varpi_{\mathfrak{p}}^{x_{\mathfrak{p}}} : \mathcal{L}_{\mathfrak{p}}^{x_{\mathfrak{p}}} \to \mathcal{F}_{\mathfrak{p}}$ by the action of G, we have the principal $\mathbb{Z}/N\mathbb{Z}$-bundle $\overline{\varpi}_{\mathfrak{p}}^{x_{\mathfrak{p}}} : \overline{\mathcal{L}}_{\mathfrak{p}}^{x_{\mathfrak{p}}} \to \mathcal{M}_{\mathfrak{p}}$. We call $\varpi_{\mathfrak{p}}^{x_{\mathfrak{p}}} : \mathcal{L}_{\mathfrak{p}}^{x_{\mathfrak{p}}} \to \mathcal{F}_{\mathfrak{p}}$ or $\overline{\varpi}_{\mathfrak{p}}^{x_{\mathfrak{p}}} : \overline{\mathcal{L}}_{\mathfrak{p}}^{x_{\mathfrak{p}}} \to \mathcal{M}_{\mathfrak{p}}$ the *arithmetic prequantization principal* $\mathbb{Z}/N\mathbb{Z}$*-bundle* for $\partial V_{\mathfrak{p}}$ with respect to the section $x_{\mathfrak{p}}$.

For $x_{\mathfrak{p}}, x'_{\mathfrak{p}} \in \Gamma(\mathcal{F}_{\mathfrak{p}}, \mathcal{L}_{\mathfrak{p}})$, we define the map $\delta_{\mathfrak{p}}^{x_{\mathfrak{p}}, x'_{\mathfrak{p}}} : \mathcal{F}_{\mathfrak{p}} \to \mathbb{Z}/N\mathbb{Z}$ by

$$\delta_{\mathfrak{p}}^{x_{\mathfrak{p}}, x'_{\mathfrak{p}}}(\rho_{\mathfrak{p}}) := x_{\mathfrak{p}}(\rho_{\mathfrak{p}}) - x'_{\mathfrak{p}}(\rho_{\mathfrak{p}}) \tag{5.3.16}$$

for $\rho_{\mathfrak{p}} \in \mathcal{F}_{\mathfrak{p}}$.

Lemma 5.3.17. *For* $x_{\mathfrak{p}}, x'_{\mathfrak{p}}, x''_{\mathfrak{p}} \in \Gamma(\mathcal{F}_{\mathfrak{p}}, \mathcal{L}_{\mathfrak{p}})$*, we have*

$$\delta_{\mathfrak{p}}^{x_{\mathfrak{p}}, x_{\mathfrak{p}}} = 0,\ \delta_{\mathfrak{p}}^{x'_{\mathfrak{p}}, x_{\mathfrak{p}}} = -\delta_{\mathfrak{p}}^{x_{\mathfrak{p}}, x'_{\mathfrak{p}}},\ \delta_{\mathfrak{p}}^{x_{\mathfrak{p}}, x'_{\mathfrak{p}}} + \delta_{\mathfrak{p}}^{x'_{\mathfrak{p}}, x''_{\mathfrak{p}}} = \delta_{\mathfrak{p}}^{x_{\mathfrak{p}}, x''_{\mathfrak{p}}}.$$

Proof. These equalities follow from Lemma 5.1.4 (1). □

The following proposition tells us how $\lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}$ is changed when we change the section $x_{\mathfrak{p}}$.

Proposition 5.3.18. *For* $x_{\mathfrak{p}}, x'_{\mathfrak{p}} \in \Gamma(\mathcal{F}_{\mathfrak{p}}, \mathcal{L}_{\mathfrak{p}})$*, we have*

$$\lambda_{\mathfrak{p}}^{x'_{\mathfrak{p}}}(g) - \lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}(g) = g.\delta_{\mathfrak{p}}^{x_{\mathfrak{p}}, x'_{\mathfrak{p}}} - \delta_{\mathfrak{p}}^{x_{\mathfrak{p}}, x'_{\mathfrak{p}}}$$

for any $g \in G$*. So the cohomology class* $[\lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}] \in H^1(G, \mathrm{Map}(\mathcal{F}_{\mathfrak{p}}, \mathbb{Z}/N\mathbb{Z}))$ *is independent of the choice of a section* $x_{\mathfrak{p}}$*.*

Proof. By (5.3.10) and Lemma 5.1.4 (1), (2), we have

$$\begin{aligned}
&\lambda_{\mathfrak{p}}^{x'_{\mathfrak{p}}}(g, \rho_{\mathfrak{p}}) - \lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}(g, \rho_{\mathfrak{p}}) \\
&= (f_{\mathfrak{p}}(g, \rho_{\mathfrak{p}})(x'_{\mathfrak{p}}(\rho_{\mathfrak{p}})) - x'_{\mathfrak{p}}(\rho_{\mathfrak{p}}.g)) - (f_{\mathfrak{p}}(g, \rho_{\mathfrak{p}})(x_{\mathfrak{p}}(\rho_{\mathfrak{p}})) - x_{\mathfrak{p}}(\rho_{\mathfrak{p}}.g)) \\
&= (x_{\mathfrak{p}}(\rho_{\mathfrak{p}}.g) - x'_{\mathfrak{p}}(\rho_{\mathfrak{p}}.g)) + (f_{\mathfrak{p}}(g, \rho_{\mathfrak{p}})(x'_{\mathfrak{p}}(\rho_{\mathfrak{p}})) - f_{\mathfrak{p}}(g, \rho_{\mathfrak{p}})(x_{\mathfrak{p}}(\rho_{\mathfrak{p}}))) \\
&= (x_{\mathfrak{p}}(\rho_{\mathfrak{p}}.g) - x'_{\mathfrak{p}}(\rho_{\mathfrak{p}}.g)) + (x'_{\mathfrak{p}}(\rho_{\mathfrak{p}}) - x_{\mathfrak{p}}(\rho_{\mathfrak{p}})) \\
&= (g.\delta_{\mathfrak{p}}^{x_{\mathfrak{p}}, x'_{\mathfrak{p}}})(\rho_{\mathfrak{p}}) - \delta_{\mathfrak{p}}^{x_{\mathfrak{p}}, x'_{\mathfrak{p}}}(\rho_{\mathfrak{p}}) \ \text{ by (5.1.2)}
\end{aligned}$$

for any $g \in G$ and $\rho_{\mathfrak{p}} \in \mathcal{F}_{\mathfrak{p}}$, hence the assertion. □

By Proposition 5.3.18, we denote the cohomology class $[\lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}]$ by $[\lambda_{\mathfrak{p}}]$, which we call the *arithmetic Chern–Simons 1st cohomology class* for $\partial V_{\mathfrak{p}}$. As a corollary of Proposition 5.3.18, we can make the latter statement of Proposition 5.3.15 more precise as follows.

Corollary 5.3.19. (1) *For $x_{\mathfrak{p}}, x'_{\mathfrak{p}} \in \Gamma(\mathcal{F}_{\mathfrak{p}}, \mathcal{L}_{\mathfrak{p}})$, we have the following isomorphism of G-equivariant principal $\mathbb{Z}/N\mathbb{Z}$-bundles over $\mathcal{F}_{\mathfrak{p}}$:*

$$\Phi_{\mathfrak{p}}^{x_{\mathfrak{p}},x'_{\mathfrak{p}}} : \mathcal{L}_{\mathfrak{p}}^{x_{\mathfrak{p}}} \xrightarrow{\sim} \mathcal{L}_{\mathfrak{p}}^{x'_{\mathfrak{p}}};\ \ (\rho_{\mathfrak{p}}, m) \mapsto (\rho_{\mathfrak{p}}, m + \delta_{\mathfrak{p}}^{x_{\mathfrak{p}},x'_{\mathfrak{p}}}(\rho_{\mathfrak{p}})),$$

where $\delta_{\mathfrak{p}}^{x_{\mathfrak{p}},x'_{\mathfrak{p}}} : \mathcal{F}_{\mathfrak{p}} \to \mathbb{Z}/N\mathbb{Z}$ is the map defined in (5.3.16).
(2) *For $x_{\mathfrak{p}}, x'_{\mathfrak{p}}, x''_{\mathfrak{p}} \in \Gamma(\mathcal{F}_{\mathfrak{p}}, \mathcal{L}_{\mathfrak{p}})$, we have*

$$\begin{cases} \Phi_{\mathfrak{p}}^{x_{\mathfrak{p}},x'_{\mathfrak{p}}} \circ \Phi_{\mathfrak{p}}^{x_{\mathfrak{p}}} = \Phi_{\mathfrak{p}}^{x'_{\mathfrak{p}}}, \\ \Phi_{\mathfrak{p}}^{x_{\mathfrak{p}},x_{\mathfrak{p}}} = \mathrm{id}_{\mathcal{L}_{\mathfrak{p}}^{x_{\mathfrak{p}}}},\ \Phi_{\mathfrak{p}}^{x'_{\mathfrak{p}},x_{\mathfrak{p}}} = (\Phi_{\mathfrak{p}}^{x_{\mathfrak{p}},x'_{\mathfrak{p}}})^{-1},\ \Phi_{\mathfrak{p}}^{x'_{\mathfrak{p}},x''_{\mathfrak{p}}} \circ \Phi_{\mathfrak{p}}^{x_{\mathfrak{p}},x'_{\mathfrak{p}}} = \Phi_{\mathfrak{p}}^{x_{\mathfrak{p}},x''_{\mathfrak{p}}} \end{cases}$$

Proof. (1) We easily see that $\Phi_{\mathfrak{p}}^{x_{\mathfrak{p}},x'_{\mathfrak{p}}}$ is isomorphism of principal $\mathbb{Z}/N\mathbb{Z}$-bundles and so it suffices to show that $\Phi_{\mathfrak{p}}^{x_{\mathfrak{p}},x'_{\mathfrak{p}}}$ is G-equivariant. This follows from

$$\begin{aligned} \Phi_{\mathfrak{p}}^{x_{\mathfrak{p}},x'_{\mathfrak{p}}}((\rho_{\mathfrak{p}}, m).g) &= \Phi_{\mathfrak{p}}^{x_{\mathfrak{p}},x'_{\mathfrak{p}}}((\rho_{\mathfrak{p}}.g, m + \lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}(g, \rho_{\mathfrak{p}}))) \ \text{ by (5.3.14)} \\ &= (\rho_{\mathfrak{p}}.g, m + \lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}(g, \rho_{\mathfrak{p}}) + \delta_{\mathfrak{p}}^{x_{\mathfrak{p}},x'_{\mathfrak{p}}}(\rho_{\mathfrak{p}}.g)) \\ &= (\rho_{\mathfrak{p}}.g, m + \delta_{\mathfrak{p}}^{x_{\mathfrak{p}},x'_{\mathfrak{p}}}(\rho_{\mathfrak{p}}) + \lambda_{\mathfrak{p}}^{x'_{\mathfrak{p}}}(g, \rho_{\mathfrak{p}})) \ \text{ by Proposition 5.3.18} \\ &= \Phi_{\mathfrak{p}}^{x_{\mathfrak{p}},x'_{\mathfrak{p}}}(\rho_{\mathfrak{p}}, m).g. \end{aligned}$$

(2) The first equality follows from the definitions of $\Phi_{\mathfrak{p}}^{x_{\mathfrak{p}}}, \Phi_{\mathfrak{p}}^{x_{\mathfrak{p}},x'_{\mathfrak{p}}}$. The latter equalities follow from Lemma 5.3.17. □

Let F be a field containing μ_N. Let $L_{\mathfrak{p}}$ be the F-line bundle over $\mathcal{F}_{\mathfrak{p}}$ associated to the principal $\mathbb{Z}/N\mathbb{Z}$-bundle $\mathcal{L}_{\mathfrak{p}}$ and the homomorphism $\mathbb{Z}/N\mathbb{Z} \hookrightarrow F^{\times}$; $m \mapsto \zeta_N^m$, namely,
(5.3.20)

$$\begin{aligned} L_{\mathfrak{p}} &:= \mathcal{L}_{\mathfrak{p}} \times_{\mathbb{Z}/N\mathbb{Z}} F \\ &:= (\mathcal{L}_{\mathfrak{p}} \times F)/(\alpha_{\mathfrak{p}}, z) \sim (\alpha_{\mathfrak{p}}.m, \zeta_N^{-m} z) \ \ (\alpha_{\mathfrak{p}} \in \mathcal{L}_{\mathfrak{p}}, m \in \mathbb{Z}/N\mathbb{Z}, z \in F), \end{aligned}$$

on which G acts from the right by

(5.3.21) $$L_{\mathfrak{p}} \times G \to L_{\mathfrak{p}};\ \ ([(\alpha_{\mathfrak{p}}, z)], g) \mapsto [(\alpha_{\mathfrak{p}}.g, z)].$$

The projection

$$\varpi_{\mathfrak{p},F} : L_{\mathfrak{p}} \longrightarrow \mathcal{F}_{\mathfrak{p}};\ [(\alpha_{\mathfrak{p}}, z)] \mapsto \varpi_{\mathfrak{p}}(\alpha_{\mathfrak{p}})$$

is a G-equivariant F-line bundle. We denote the fiber $\varpi_{\mathfrak{p},F}^{-1}(\rho_{\mathfrak{p}})$ over $\rho_{\mathfrak{p}}$ by $L_{\mathfrak{p}}(\rho_{\mathfrak{p}})$:

(5.3.22) $$L_{\mathfrak{p}}(\rho_{\mathfrak{p}}) := \{[(\alpha_{\mathfrak{p}}, z)] \in L_{\mathfrak{p}} \mid \varpi_{\mathfrak{p}}(\alpha_{\mathfrak{p}}) = \rho_{\mathfrak{p}}, z \in F\}$$

We have a non-canonical bijection by fixing an $\alpha_{\mathfrak{p}} \in \mathcal{L}_{\mathfrak{p}}(\rho_{\mathfrak{p}})$:

$$L_{\mathfrak{p}}(\rho_{\mathfrak{p}}) \xrightarrow{\sim} F;\ [(\alpha_{\mathfrak{p}}, z)] \mapsto z.$$

Taking the quotient by the action of G, we obtain the F-line bundles $\overline{\varpi}_{\mathfrak{p},F} : \overline{L}_{\mathfrak{p}} \to \mathcal{M}_{\mathfrak{p}}$. We call $\varpi_{\mathfrak{p},F} : L_{\mathfrak{p}} \to \mathcal{F}_{\mathfrak{p}}$ or $\overline{\varpi}_{\mathfrak{p},F} : \overline{L}_{\mathfrak{p}} \to \mathcal{M}_{\mathfrak{p}}$ the *arithmetic prequantization F-line bundle* for $\partial V_{\mathfrak{p}}$.

Let $L_{\mathfrak{p}}^{x_{\mathfrak{p}}}$ be the product F-line bundle over $\mathcal{F}_{\mathfrak{p}}$:

$$L_{\mathfrak{p}}^{x_{\mathfrak{p}}} := \mathcal{F}_{\mathfrak{p}} \times F,$$

on which G acts from the right by

$$L_{\mathfrak{p}}^{x_{\mathfrak{p}}} \times G \to L_{\mathfrak{p}}^{x_{\mathfrak{p}}};\ \ ((\rho_{\mathfrak{p}}, z), g) \mapsto (\rho_{\mathfrak{p}}.g, z\zeta_N^{\lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}(g,\rho_{\mathfrak{p}})}), \tag{5.3.23}$$

and the projection

$$\varpi_{\mathfrak{p},F}^{x_{\mathfrak{p}}} : L_{\mathfrak{p}}^{x_{\mathfrak{p}}} \longrightarrow \mathcal{F}_{\mathfrak{p}}$$

is G-equivariant. Then we have the following Proposition similar to Proposition 5.3.15 and Corollary 5.3.19.

Proposition 5.3.24. *We have the following isomorphism of G-equivariant F-line bundles over $\mathcal{F}_{\mathfrak{p}}$*

$$\Phi_{\mathfrak{p},F}^{x_{\mathfrak{p}}} : L_{\mathfrak{p}} \xrightarrow{\sim} L_{\mathfrak{p}}^{x_{\mathfrak{p}}}; [(\alpha_{\mathfrak{p}}, z)] \mapsto (\varpi_{\mathfrak{p}}(\alpha_{\mathfrak{p}}), z\zeta_N^{\alpha_{\mathfrak{p}} - x_{\mathfrak{p}}(\varpi_{\mathfrak{p}}(\alpha_{\mathfrak{p}}))}).$$

For another section $x'_{\mathfrak{p}}$, we have the following isomorphism of G-equivariant F-line bundles over $\mathcal{F}_{\mathfrak{p}}$

$$\Phi_{\mathfrak{p},F}^{x_{\mathfrak{p}},x'_{\mathfrak{p}}} : L_{\mathfrak{p}}^{x_{\mathfrak{p}}} \xrightarrow{\sim} L_{\mathfrak{p}}^{x'_{\mathfrak{p}}} : (\rho_{\mathfrak{p}}, z) \mapsto (\rho_{\mathfrak{p}}, z\zeta_N^{\delta_{\mathfrak{p}}^{x_{\mathfrak{p}},x'_{\mathfrak{p}}}(\rho_{\mathfrak{p}})}),$$

where $\delta_{\mathfrak{p}}^{x_{\mathfrak{p}},x'_{\mathfrak{p}}} : \mathcal{F}_{\mathfrak{p}} \to \mathbb{Z}/N\mathbb{Z}$ is the map in (5.3.16), and we have the equalities

$$\begin{cases} \Phi_{\mathfrak{p},F}^{x_{\mathfrak{p}},x'_{\mathfrak{p}}} \circ \Phi_{\mathfrak{p},F}^{x_{\mathfrak{p}}} = \Phi_{\mathfrak{p},F}^{x'_{\mathfrak{p}}} \\ \Phi_{\mathfrak{p},F}^{x_{\mathfrak{p}},x_{\mathfrak{p}}} = \mathrm{id}_{L_{\mathfrak{p},F}^{x_{\mathfrak{p}}}},\ \Phi_{\mathfrak{p},F}^{x'_{\mathfrak{p}},x_{\mathfrak{p}}} = (\Phi_{\mathfrak{p},F}^{x_{\mathfrak{p}},x'_{\mathfrak{p}}})^{-1},\ \Phi_{\mathfrak{p},F}^{x'_{\mathfrak{p}},x''_{\mathfrak{p}}} \circ \Phi_{\mathfrak{p},F}^{x_{\mathfrak{p}},x'_{\mathfrak{p}}} = \Phi_{\mathfrak{p},F}^{x_{\mathfrak{p}},x''_{\mathfrak{p}}} \end{cases}$$

for $x_{\mathfrak{p}}, x'_{\mathfrak{p}}, x''_{\mathfrak{p}} \in \Gamma(\mathcal{F}_{\mathfrak{p}}, \mathcal{L}_{\mathfrak{p}})$.

Proof. (i) It is easy to see that $\varpi_{\mathfrak{p},F}^{x_{\mathfrak{p}}} \circ \Phi_{\mathfrak{p},F}^{x_{\mathfrak{p}}} = \varpi_{\mathfrak{p},F}$.
(ii) For $\rho_{\mathfrak{p}} \in \mathcal{F}_{\mathfrak{p}}$, we let

$$L_{\mathfrak{p}}^{x_{\mathfrak{p}}}(\rho_{\mathfrak{p}}) := (\varpi_{\mathfrak{p},F}^{x_{\mathfrak{p}}})^{-1}(\rho_{\mathfrak{p}}) = \{(\rho_{\mathfrak{p}}, z) \mid z \in F\} \simeq F.$$

So $\Phi^{x_{\mathfrak{p}}}_{\mathfrak{p},F}$ restricted to a fiber over $\rho_{\mathfrak{p}}$

$$\Phi^{x_{\mathfrak{p}}}_{\mathfrak{p},F}|_{L_{\mathfrak{p}}(\rho_{\mathfrak{p}})} : L_{\mathfrak{p}}(\rho_{\mathfrak{p}}) \longrightarrow L^{x_{\mathfrak{p}}}_{\mathfrak{p}}(\rho_{\mathfrak{p}}); \quad [(\alpha_{\mathfrak{p}}, z)] \mapsto (\rho_{\mathfrak{p}}, z\zeta_N^{\alpha_{\mathfrak{p}} - x_{\mathfrak{p}}(\rho_{\mathfrak{p}})})$$

is F-linear.
(iv) For $g \in G$, we have

$$\begin{aligned}
\Phi^{x_{\mathfrak{p}}}_{\mathfrak{p},F}([(\alpha_{\mathfrak{p}}, z)].g) &= \Phi^{x_{\mathfrak{p}}}_{\mathfrak{p},F}([(\alpha_{\mathfrak{p}}.g, z)]) \text{ by (5.3.21)}\\
&= (\varpi_{\mathfrak{p}}(\alpha_{\mathfrak{p}}.g), z\zeta_N^{\alpha_{\mathfrak{p}}.g - x_{\mathfrak{p}}(\varpi_{\mathfrak{p}}(\alpha_{\mathfrak{p}}.g))})\\
&= (\varpi_{\mathfrak{p}}(\alpha_{\mathfrak{p}}).g, z\zeta^{(\alpha_{\mathfrak{p}} - x_{\mathfrak{p}}(\rho_{\mathfrak{p}})) + \lambda^{x_{\mathfrak{p}}}_{\mathfrak{p}}(g, \rho_{\mathfrak{p}})}) \text{ by (5.3.11)}\\
&= \Phi^{x_{\mathfrak{p}}}_{\mathfrak{p},F}([\alpha_{\mathfrak{p}}, z)]).g \text{ by (5.3.23).}
\end{aligned}$$

Hence $\Phi^{x_{\mathfrak{p}}}_{\mathfrak{p},F}$ is the isomorphism of G-equivariant F-line bundles over $\mathcal{F}_{\mathfrak{p}}$.

The proofs of the latter parts are similar to those of Corollary 5.3.19 (1), (2). □

Taking the quotient of $\varpi^{x_{\mathfrak{p}}}_{\mathfrak{p},F} : L^{x_{\mathfrak{p}}}_{\mathfrak{p}} \to \mathcal{F}_{\mathfrak{p}}$ by the action of G, we have the F-line bundle $\overline{\varpi}^{x_{\mathfrak{p}}}_{\mathfrak{p},F} : \overline{L}^{x_{\mathfrak{p}}}_{\mathfrak{p}} \to \mathcal{M}_{\mathfrak{p}}$. We call $\varpi^{x_{\mathfrak{p}}}_{\mathfrak{p},F} : L^{x_{\mathfrak{p}}}_{\mathfrak{p}} \to \mathcal{F}_{\mathfrak{p}}$ or $\overline{\varpi}^{x_{\mathfrak{p}}}_{\mathfrak{p},F} : \overline{L}^{x_{\mathfrak{p}}}_{\mathfrak{p}} \to \mathcal{M}_{\mathfrak{p}}$ the *arithmetic prequantization F-line bundle* for $\partial V_{\mathfrak{p}}$ with respect to the section $x_{\mathfrak{p}}$.

Let $S = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ be a finite set of finite primes of k and let $\partial V_S := \partial V_{\mathfrak{p}_1} \sqcup \cdots \sqcup \partial V_{\mathfrak{p}_r}$. Let $\mathcal{F}_S$ be the direct product of $\mathcal{F}_{\mathfrak{p}_i}$'s:

$$\mathcal{F}_S := \mathcal{F}_{\mathfrak{p}_1} \times \cdots \times \mathcal{F}_{\mathfrak{p}_r}.$$

It is a finite set on which G acts diagonally from the right, namely,

$$\mathcal{F}_S \times G \to \mathcal{F}_S; \quad (\rho_S, g) \mapsto \rho_S.g := (\rho_{\mathfrak{p}_1}.g, \dots, \rho_{\mathfrak{p}_r}.g) \tag{5.3.25}$$

for $\rho_S = (\rho_{\mathfrak{p}_1}, \dots, \rho_{\mathfrak{p}_r}) \in \mathcal{F}_S$ and let $\mathcal{M}_S$ denote the quotient space by this action

$$\mathcal{M}_S := \mathcal{F}_S/G.$$

Let $\mathrm{Map}(\mathcal{F}_S, \mathbb{Z}/N\mathbb{Z})$ be the additive group of maps from $\mathcal{F}_S$ to $\mathbb{Z}/N\mathbb{Z}$, on which G acts from the left by

$$(g.\psi_S)(\rho_S) := \psi_S(\rho_S.g) \tag{5.3.26}$$

for $\psi_S \in \mathrm{Map}(\mathcal{F}_S, \mathbb{Z}/N\mathbb{Z}), g \in G$ and $\rho_S \in \mathcal{F}_S$.

For $\rho_S = (\rho_{\mathfrak{p}_1}, \dots, \rho_{\mathfrak{p}_r}) \in \mathcal{F}_S$, let $\mathcal{L}_S(\rho_S)$ be the quotient space of the product $\mathcal{L}_{\mathfrak{p}_1}(\rho_{\mathfrak{p}_1}) \times \cdots \times \mathcal{L}_{\mathfrak{p}_r}(\rho_{\mathfrak{p}_r})$:

$$\mathcal{L}_S(\rho_S) := (\mathcal{L}_{\mathfrak{p}_1}(\rho_{\mathfrak{p}_1}) \times \cdots \times \mathcal{L}_{\mathfrak{p}_r}(\rho_{\mathfrak{p}_r}))/\sim, \tag{5.3.27}$$

where the equivalence relation $\sim$ is defined by

$$(5.3.28) \qquad (\alpha_{\mathfrak{p}_1}, \ldots, \alpha_{\mathfrak{p}_r}) \sim (\alpha'_{\mathfrak{p}_1}, \ldots, \alpha'_{\mathfrak{p}_r}) \Longleftrightarrow \sum_{i=1}^{r} (\alpha_{\mathfrak{p}_i} - \alpha'_{\mathfrak{p}_i}) = 0.$$

We see easily that $\mathcal{L}_S(\rho_S)$ is equipped with the simply transitive action of $\mathbb{Z}/N\mathbb{Z}$ defined by

$$\begin{array}{l} \mathcal{L}_S(\rho_S) \times \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathcal{L}_S(\rho_S); \\ ([\alpha_S], m) \mapsto [\alpha_S].m := [(\alpha_{\mathfrak{p}_1}.m, \ldots, \alpha_{\mathfrak{p}_r})] = \cdots = [(\alpha_{\mathfrak{p}_1}, \ldots, \alpha_{\mathfrak{p}_r}.m)] \end{array}$$

for $\alpha_S = (\alpha_{\mathfrak{p}_1}, \ldots, \alpha_{\mathfrak{p}_r})$ and hence $\mathcal{L}_S(\rho_S)$ is a $\mathbb{Z}/N\mathbb{Z}$-torsor.

Let $\mathcal{L}_S$ be the disjoint union of $\mathcal{L}_{\mathfrak{p}}(\rho_S)$ for $\rho_S \in \mathcal{F}_S$:

$$(5.3.29) \qquad \mathcal{L}_S := \bigsqcup_{\rho_S \in \mathcal{F}_S} \mathcal{L}_S(\rho_S),$$

on which G acts diagonally from the right by

$$(5.3.30) \qquad \mathcal{L}_S \times G \longrightarrow \mathcal{L}_S;\ ([(\alpha_{\mathfrak{p}_1}, \ldots, \alpha_{\mathfrak{p}_r})], g) \mapsto [(\alpha_{\mathfrak{p}_1}.g, \ldots, \alpha_{\mathfrak{p}_r}.g)].$$

Consider the projection

$$\varpi_S : \mathcal{L}_S \longrightarrow \mathcal{F}_S;\ [\alpha_S] = [(\alpha_{\mathfrak{p}_i})] \mapsto (\varpi_{\mathfrak{p}_i}(\alpha_{\mathfrak{p}_i})),$$

which is G-equivariant. Since each fiber $\varpi_{\mathfrak{p}}^{-1}(\rho_S) = \mathcal{L}_S(\rho_S)$ is a $\mathbb{Z}/N\mathbb{Z}$-torsor, we may regard $\varpi_S : \mathcal{L}_S \longrightarrow \mathcal{F}_S$ as a G-equivariant principal $\mathbb{Z}/N\mathbb{Z}$-bundle. Taking the quotient by the action of G, we have the principal $\mathbb{Z}/N\mathbb{Z}$-bundle $\overline{\varpi}_S : \overline{\mathcal{L}}_S \to \mathcal{M}_S$. We call $\varpi_S : \mathcal{L}_S \to \mathcal{F}_S$ or $\overline{\varpi}_S : \overline{\mathcal{L}}_S \to \mathcal{M}_S$ the *arithmetic prequantization $\mathbb{Z}/N\mathbb{Z}$-bundle* for $\partial V_S = \mathrm{Spec}(k_{\mathfrak{p}_1}) \sqcup \cdots \sqcup \mathrm{Spec}(k_{\mathfrak{p}_r})$.

Let x_S be a section of ϖ_S, $x_S \in \Gamma(\mathcal{F}_S, \mathcal{L}_S)$. By (5.3.27) and (5.3.29), it is written as $x_S = [(x_{\mathfrak{p}_1}, \ldots, x_{\mathfrak{p}_r})]$, where $x_{\mathfrak{p}_i} \in \Gamma(\mathcal{F}_{\mathfrak{p}_i}, \mathcal{L}_{\mathfrak{p}_i})$ for $1 \leq i \leq r$. For $g \in G$ and $\rho_S = (\rho_{\mathfrak{p}_i}) \in \mathcal{F}_S$, we set

$$(5.3.31) \qquad \lambda_S^{x_S}(g, \rho_S) := \lambda_{\mathfrak{p}_1}^{x_{\mathfrak{p}_1}}(g, \rho_{\mathfrak{p}_1}) + \cdots + \lambda_{\mathfrak{p}_r}^{x_{\mathfrak{p}_r}}(g, \rho_{\mathfrak{p}_r})$$

and define the map $\lambda_S^{x_S} : G \to \mathrm{Map}(\mathcal{F}_S, \mathbb{Z}/N\mathbb{Z})$ by

$$(5.3.32) \qquad \lambda_S^{x_S}(g)(\rho_S) := \lambda_S^{x_S}(g, \rho_S)$$

for $g \in G$ and $\rho_S \in \mathcal{F}_S$.

Lemma 5.3.33. (1) *Let $x'_{\mathfrak{p}_i} \in \Gamma(\mathcal{F}_{\mathfrak{p}_i}, \mathcal{L}_{\mathfrak{p}_i})$ be another section for $1 \leq i \leq r$ such that $[(x'_{\mathfrak{p}_1}, \ldots, x'_{\mathfrak{p}_r})] = x_S$. Then we have*

$$\sum_{i=1}^{r} \lambda_{\mathfrak{p}_i}^{x_{\mathfrak{p}_i}}(g, \rho_{\mathfrak{p}_i}) = \sum_{i=1}^{r} \lambda_{\mathfrak{p}_i}^{x'_{\mathfrak{p}_i}}(g, \rho_{\mathfrak{p}_i})$$

for $g \in G$ and $\rho_{\mathfrak{p}_i} \in \mathcal{F}_{\mathfrak{p}_i}$. So $\lambda_S^{x_S}(g, \rho_S)$ is independent of the choic of $x_{\mathfrak{p}_i}$'s such that $x_S = [(x_{\mathfrak{p}_1}, \ldots, x_{\mathfrak{p}_r})]$.
(2) *The map $\lambda_S^{x_S}$ is a 1-cocycle*:

$$\lambda_S^{x_S} \in Z^1(G, \mathrm{Map}(\mathcal{F}_S, \mathbb{Z}/N\mathbb{Z})).$$

Proof. (1) Since $(x_{\mathfrak{p}_1}(\rho_{\mathfrak{p}_1}), \ldots, x_{\mathfrak{p}_r}(\rho_{\mathfrak{p}_r})) \sim (x'_{\mathfrak{p}_1}(\rho_{\mathfrak{p}_1}), \ldots, x'_{\mathfrak{p}_r}(\rho_{\mathfrak{p}_r}))$, by (5.3.28), we have

$$\sum_{i=1}^{r} (x_{\mathfrak{p}_i}(\rho_{\mathfrak{p}_i}) - x'_{\mathfrak{p}_i}(\rho_{\mathfrak{p}_i})) = 0.$$

for any $\rho_{\mathfrak{p}_i} \in \mathcal{F}_{\mathfrak{p}_i}$. Therefore we have

$$\begin{aligned}
\sum_{i=1}^{r} \lambda_{\mathfrak{p}_i}^{x_{\mathfrak{p}_i}}(g, \rho_{\mathfrak{p}_i}) &= \sum_{i=1}^{r} (f_{\mathfrak{p}_i}(g, \rho_{\mathfrak{p}_i})(x_{\mathfrak{p}_i}(\rho_{\mathfrak{p}_i})) - x_{\mathfrak{p}_i}(\rho_{\mathfrak{p}_i}.g)) \quad \text{by (5.3.10)} \\
&= \sum_{i=1}^{r} ((f_{\mathfrak{p}_i}(g, \rho_{\mathfrak{p}_i})(x_{\mathfrak{p}_i}(\rho_{\mathfrak{p}_i})) - f_{\mathfrak{p}_i}(g, \rho_{\mathfrak{p}_i})(x'_{\mathfrak{p}_i}(\rho_{\mathfrak{p}_i}))) \\
&\quad + \sum_{i=1}^{r} (f_{\mathfrak{p}_i}(g, \rho_{\mathfrak{p}_i})(x'_{\mathfrak{p}_i}(\rho_{\mathfrak{p}_i})) - x'_{\mathfrak{p}_i}(\rho_{\mathfrak{p}_i}.g)) \\
&\quad + \sum_{i=1}^{r} (x'_{\mathfrak{p}_i}(\rho_{\mathfrak{p}_i}.g) - x_{\mathfrak{p}_i}(\rho_{\mathfrak{p}_i}.g)) \qquad \text{by Lemma 5.1.4 (1)} \\
&= \sum_{i=1}^{r} (f_{\mathfrak{p}_i}(g, \rho_{\mathfrak{p}_i})(x'_{\mathfrak{p}_i}(\rho_{\mathfrak{p}_i})) - x'_{\mathfrak{p}_i}(\rho_{\mathfrak{p}_i}.g)) \quad \text{by Lemma 5.1.4 (2)} \\
&= \sum_{i=1}^{r} \lambda_{\mathfrak{p}_i}^{x'_{\mathfrak{p}_i}}(g, \rho_{\mathfrak{p}_i})
\end{aligned}$$

for $g \in G$ and $\rho_{\mathfrak{p}_i} \in \mathcal{F}_{\mathfrak{p}_i}$.
(2) By Theorem 5.3.13, (5.3.26), (5.3.31) and (5.3.32), we have

$$\begin{aligned}
\lambda_S^{x_S}(gg', \rho_S) &= \sum_{i=1}^{r} \lambda_{\mathfrak{p}_i}^{x_{\mathfrak{p}_i}}(gg', \rho_{\mathfrak{p}_i}) \\
&= \sum_{i=1}^{r} \lambda_{\mathfrak{p}_i}^{x_{\mathfrak{p}_i}}(g, \rho_{\mathfrak{p}_i}) + \sum_{i=1}^{r} \lambda_{\mathfrak{p}_i}^{x_{\mathfrak{p}_i}}(g', \rho_{\mathfrak{p}_i}.g) \\
&= \lambda_S^{x_S}(g, \rho_S) + \lambda_S^{x_S}(g', \rho_S.g) \\
&= (\lambda_S^{x_S}(g) + (g.\lambda_S^{x_S})(g'))(\rho_S)
\end{aligned}$$

for $g \in G$ and $\rho_S = (\rho_{\mathfrak{p}_i}) \in \mathcal{F}_S$. Thus we obtain the assertion. $\square$

We call $\lambda_S^{x_S}$ the *arithmetic Chern–Simons 1-cocycle* for ∂V_S with respect to x_S.

Proposition 5.3.34. *Let $x'_S = [(x'_{\mathfrak{p}_1}, \ldots, x'_{\mathfrak{p}_r})] \in \Gamma(\mathcal{F}_S, \mathcal{L}_S)$ be another section of ϖ_S. We define the map $\delta_S^{x_S, x'_S} : \mathcal{F}_S \to \mathbb{Z}/N\mathbb{Z}$ by*

$$\delta_S^{x_S, x'_S}(\rho_S) := \sum_{i=1}^{r} \delta_{\mathfrak{p}_i}^{x_{\mathfrak{p}_i}, x'_{\mathfrak{p}_i}}(\rho_{\mathfrak{p}_i})$$

for $\rho_S = (\rho_{\mathfrak{p}_i}) \in \mathcal{F}_S$, where $\delta_{\mathfrak{p}_i}^{x_{\mathfrak{p}_i}, x'_{\mathfrak{p}_i}}$ is the map defined in (5.3.16). Then we have

$$\lambda_S^{x'_S}(g) - \lambda_S^{x_S}(g) = g.\delta_S^{x_S, x'_S} - \delta_S^{x_S, x'_S}$$

for $g \in G$. So the cohomology class $[\lambda_S^{x_S}] \in H^1(G, \mathrm{Map}(\mathcal{F}_S, \mathbb{Z}/N\mathbb{Z}))$ is independent of the choice of x_S.

Proof. First, note that $\delta_S^{x_S, x'_S}$ is proved to be independent of the choices of $x_{\mathfrak{p}_i}$'s in the similar manner to the proof of Lemma 5.3.33 (1). By the definition of $\delta_S^{x_S, x'_S}$, the formula follows from Proposition 5.3.18 by taking the sum over $\mathfrak{p}_i \in S$. $\square$

We denote the cohomology class $[\lambda_S^{x_S}]$ by $[\lambda_S]$, which we call the *arithmetic Chern–Simons 1st cohomology class* for ∂V_S.

Let $\mathcal{L}_S^{x_S}$ be the product principal $\mathbb{Z}/N\mathbb{Z}$-bundle over $\mathcal{F}_S$:

$$\mathcal{L}_S^{x_S} := \mathcal{F}_S \times \mathbb{Z}/N\mathbb{Z},$$

on which G acts from the right by

$$\mathcal{L}_S^{x_S} \times G \to \mathcal{L}_S^{x_S}; \ \ ((\rho_S, m), g) \mapsto (\rho_S.g, m + \lambda_S^{x_S}(g, \rho_S)).$$

Proposition 5.3.35. *We have the following isomorphism of G-equivariant principal $\mathbb{Z}/N\mathbb{Z}$-bundles over $\mathcal{F}_S$:*

$$\Phi_S^{x_S} : \mathcal{L}_S \xrightarrow{\sim} \mathcal{L}_S^{x_S}; \ [\alpha_S] = [(\alpha_{\mathfrak{p}_1}, \ldots, \alpha_{\mathfrak{p}_r})] \mapsto (\varpi_S([\alpha_S]), \sum_{i=1}^{r}(\alpha_{\mathfrak{p}_i} - x_{\mathfrak{p}_i}(\varpi_{\mathfrak{p}_i}(\alpha_{\mathfrak{p}_i})))).$$

For another section x'_S, we have the following isomorphism of G-equivariant F-line bundles over $\mathcal{F}_S$

$$\Phi_S^{x_S, x'_S} : \mathcal{L}_S^{x_S} \xrightarrow{\sim} \mathcal{L}_S^{x'_S} : (\rho_S, m) \mapsto (\rho_S, m + \delta_S^{x_S, x'_S}(\rho_S)),$$

where $\delta_S^{x_S,x'_S} : \mathcal{F}_S \to \mathbb{Z}/N\mathbb{Z}$ *is the map in Proposition 5.3.34. For* $x_S, x'_S, x''_S \in \Gamma(\mathcal{F}_S, \mathcal{L}_S)$ *we have the equalities*

$$\begin{cases} \Phi_S^{x_S,x'_S} \circ \Phi_S^{x_S} = \Phi_S^{x'_S}, \\ \Phi_S^{x_S,x_S} = \mathrm{id}_{\mathcal{L}_S^{x_S}},\ \Phi_S^{x'_S,x_S} = (\Phi_S^{x_S,x'_S})^{-1},\ \Phi_S^{x'_S,x''_S} \circ \Phi_S^{x_S,x'_S} = \Phi_S^{x_S,x''_S}. \end{cases}$$

Proof. First, suppose $[(\alpha_{\mathfrak{p}_1}, \dots, \alpha_{\mathfrak{p}_r})] = [(\alpha'_{\mathfrak{p}_1}, \dots, \alpha'_{\mathfrak{p}_r})]$. Then $\varpi_{\mathfrak{p}_i}(\alpha_{\mathfrak{p}_i}) = \varpi_{\mathfrak{p}_i}(\alpha'_{\mathfrak{p}_i})$ and $\sum_{i=1}^r (\alpha'_{\mathfrak{p}_i} - \alpha_{\mathfrak{p}_i}) = 0$ by (5.3.28). So we have

$$\begin{aligned} \sum_{i=1}^r (\alpha'_{\mathfrak{p}_i} - x_{\mathfrak{p}_i}(\varpi_{\mathfrak{p}_i}(\alpha'_{\mathfrak{p}_i}))) &= \sum_{i=1}^r \big((\alpha'_{\mathfrak{p}_i} - \alpha_{\mathfrak{p}_i}) + (\alpha_{\mathfrak{p}_i} - x_{\mathfrak{p}_i}(\varpi_{\mathfrak{p}_i}(\alpha'_{\mathfrak{p}_i})))\big) \\ &= \sum_{i=1}^r (\alpha_{\mathfrak{p}_i} - x_{\mathfrak{p}_i}(\varpi_{\mathfrak{p}_i}(\alpha_{\mathfrak{p}_i}))). \end{aligned}$$

The proofs of the assertions go well in the similar manner to those of Proposition 5.3.15 and Corollary 5.3.19, by taking the sum over $\mathfrak{p}_i \in S$. □

Taking the quotient by the action of G, we obtain the principal $\mathbb{Z}/N\mathbb{Z}$-bundle $\overline{\varpi}_S^{x_S} : \overline{\mathcal{L}}_S^{x_S} \to \mathcal{M}_S$. We call $\varpi_S^{x_S} : \mathcal{L}_S^{x_S} \to \mathcal{F}_S$ or $\overline{\varpi}_S^{x_S} : \overline{\mathcal{L}}_S^{x_S} \to \mathcal{M}_S$ the *arithmetic prequantization principal* $\mathbb{Z}/N\mathbb{Z}$*-bundle* for ∂V_S with respect to x_S.

Let L_S be the F-line bundle associated to the principal $\mathbb{Z}/N\mathbb{Z}$-bundle $\mathcal{L}_S$ over $\mathcal{F}_S$ and the homomorphism $\mathbb{Z}/N\mathbb{Z} \to F^\times; m \mapsto \zeta_N^m$:

(5.3.36)

$$\begin{aligned} L_S &:= \mathcal{L}_S \times_{\mathbb{Z}/N\mathbb{Z}} F \\ &:= (\mathcal{L}_S \times F)/([\alpha_S], z) \sim ([\alpha_S].m, \zeta_N^{-m} z) \ \ ([\alpha_S] \in \mathcal{L}_S, m \in \mathbb{Z}/N\mathbb{Z}, z \in F), \end{aligned}$$

on which G acts from the right by

$$L_S \times G \longrightarrow L_S; \ \ ([([\alpha_S], z)], g) \mapsto [([\alpha_S].g, z)]. \tag{5.3.37}$$

The projection

$$\varpi_{S,F} : L_S \longrightarrow \mathcal{F}_S; \ [([\alpha_S], z)] \mapsto \varpi_S([\alpha_S])$$

is a G-equivariant F-line bundle. We denote the fiber $\varpi_{S,F}^{-1}(\rho_S)$ over ρ_S by $L_S(\rho_S)$, which is non-canonically bijective to F by fixing $[\alpha_S] \in \mathcal{L}_S(\rho_S)$:

$$L_S(\rho_S) := \{[([\alpha_S], z)] \in L_S \,|\, \varpi_S([\alpha_S]) = \rho_S\} \overset{\sim}{\to} F; \ [([\alpha_S], z)] \mapsto z. \tag{5.3.38}$$

Taking the quotient by the action of G, we obtain the F-line bundle $\overline{\varpi}_{S,F} : \overline{L}_S \to \mathcal{M}_S$. We call $\varpi_{S,F} : L_S \to \mathcal{F}_S$ or $\overline{\varpi}_{S,F} : \overline{L}_S \to \mathcal{M}_S$ the *arithmetic prequntization* F*-line bundle* for ∂V_S.

Let $L_S^{x_S}$ be the trivial F-line bundle over $\mathcal{F}_S$:

$$L_S^{x_S} := \mathcal{F}_S \times F,$$

on which G acts from the right by

$$L_S^{x_S} \times G \to L_S^{x_S};\ ((\rho_S, z), g) \mapsto (\rho_S.g, z\zeta_N^{\lambda_S^{x_S}(g,\rho_S)}).$$

Proposition 5.3.39. *We have the following isomorphism of G-equivariant F-line bundles over $\mathcal{F}_S$:*

$$\Phi_{S,F}^{x_S} : L_S \xrightarrow{\sim} L_S^{x_S};\ [([\alpha_S], z)] \mapsto (\varpi_S([\alpha_S]), z\zeta_N^{\sum_{i=1}^r (\alpha_{\mathfrak{p}_i} - x_{\mathfrak{p}_i}(\varpi_{\mathfrak{p}_i}(\alpha_{\mathfrak{p}_i})))})$$

For another section x'_S, we the following isomorphism of G-equivariant F-line bundles over $\mathcal{F}_S$

$$\Phi_{S,F}^{x_S,x'_S} : L_S^{x_S} \xrightarrow{\sim} L_S^{x'_S} : [(\rho_S, z)] \mapsto [(\rho_S, z\zeta_N^{\delta_S^{x_S,x'_S}(\rho_S)})],$$

where $\delta_S^{x_S,x'_S} : \mathcal{F}_S \to \mathbb{Z}/N\mathbb{Z}$ is the map in Proposition 5.3.34. For $x_S, x'_S, x''_S \in \Gamma(\mathcal{F}_S, \mathcal{L}_S)$, we have the equalities

$$\begin{cases} \Phi_{S,F}^{x_S,x'_S} \circ \Phi_{S,F}^{x_S} = \Phi_{S,F}^{x'_S}, \\ \Phi_{S,F}^{x_S,x_S} = \mathrm{id}_{\mathcal{L}_S^{x_S}},\ \Phi_{S,F}^{x'_S,x_S} = (\Phi_{S,F}^{x_S,x'_S})^{-1},\ \Phi_{S,F}^{x'_S,x''_S} \circ \Phi_{S,F}^{x_S,x'_S} = \Phi_{S,F}^{x_S,x''_S}. \end{cases}$$

Proof. The assertions can be proved in the similar manner to those of the assertions in Proposition 5.3.24, by taking the sum over $\mathfrak{p}_i \in S$. □

Taking the quotient by the action of G, we obtain the line F-bundle $\overline{\varpi}_{S,F}^{x_S} : \overline{L}_S^{x_S} \to \mathcal{M}_S$. We call $\varpi_{S,F}^{x_S} : L_S^{x_S} \to \mathcal{F}_S$ or $\overline{\varpi}_{S,F}^{x_S} : \overline{L}_S^{x_S} \to \mathcal{M}_S$ the *arithmetic prequantization F-line bundle* for ∂V_S with respect to x_S.

We may also give the description of L_S in terms of the tensor product of F-line bundles. Let $p_i : \mathcal{F}_S \to \mathcal{F}_{\mathfrak{p}_i}$ be the i-th projection. Let $p_i^*(L_{\mathfrak{p}_i})$ be the F-line bundle over $\mathcal{F}_S$ induced from $L_{\mathfrak{p}_i}$ by p_i:

$$p_i^*(L_{\mathfrak{p}_i}) := \{(\rho_S, [(\alpha_{\mathfrak{p}_i}, z_i)]) \in \mathcal{F}_S \times L_{\mathfrak{p}_i} \,|\, p_i(\rho_S) = \varpi_{\mathfrak{p}_i}(\alpha_{\mathfrak{p}_i})\},$$

and let

$$p_i^*(\varpi_{\mathfrak{p}_i}) : p_i^*(L_{\mathfrak{p}_i}) \longrightarrow \mathcal{F}_S;\ (\rho_S, [(\alpha_{\mathfrak{p}_i}, z_i)]) \mapsto \rho_S$$

be the induced projection. The fiber over $\rho_S = (\rho_{\mathfrak{p}_i})$ is given by

$$\begin{aligned} p_i^*(\varpi_{\mathfrak{p}_i})^{-1}(\rho_S) &= \{\rho_S\} \times \{[(\alpha_{\mathfrak{p}_i}, z_i)] \in L_{\mathfrak{p}_i} \mid \rho_{\mathfrak{p}_i} = \varpi_{\mathfrak{p}_i}(\alpha_{\mathfrak{p}_i}), z_i \in F\} \\ &\simeq L_{\mathfrak{p}_i}(\rho_{\mathfrak{p}_i}) \\ &\simeq F, \end{aligned}$$

where $L_{\mathfrak{p}_i}(\rho_{\mathfrak{p}_i})$ is as in (5.3.22). Let $L_{\mathfrak{p}_1} \boxtimes \cdots \boxtimes L_{\mathfrak{p}_r}$ be the tensor product of $p_i^*(L_{\mathfrak{p}_i})$'s:

$$L_{\mathfrak{p}_1} \boxtimes \cdots \boxtimes L_{\mathfrak{p}_r} := p_1^*(L_{\mathfrak{p}_1}) \otimes \cdots \otimes p_r^*(L_{\mathfrak{p}_r}),$$

which is an F-line bundle over $\mathcal{F}_S$. An element of $L_{\mathfrak{p}_1} \boxtimes \cdots \boxtimes L_{\mathfrak{p}_r}$ is written by

$$(\rho_S, [(\alpha_{\mathfrak{p}_1}, z_1)] \otimes \cdots \otimes [(\alpha_{\mathfrak{p}_r}, z_r)]),$$

where $\rho_S = (\rho_{\mathfrak{p}_i}) \in \mathcal{F}_S$, $[(\alpha_{\mathfrak{p}_i}, z_i)] \in L_{\mathfrak{p}_i}(\rho_{\mathfrak{p}_i})$. Let $\varpi_S^{\boxtimes} : L_{\mathfrak{p}_1} \boxtimes \cdots \boxtimes L_{\mathfrak{p}_r} \to \mathcal{F}_S$ be the projection. For fiber over ρ_S, we have

$$(\varpi_S^{\boxtimes})^{-1}(\rho_S) \xrightarrow{\sim} F; (\rho_S, [(\alpha_{\mathfrak{p}_1}, z_1)] \otimes \cdots \otimes [(\alpha_{\mathfrak{p}_r}, z_r)]) \mapsto \prod_{i=1}^{r} z_i. \tag{5.3.40}$$

The right action of G on $L_{\mathfrak{p}_1} \boxtimes \cdots \boxtimes L_{\mathfrak{p}_r}$ is given by

(5.3.41)

$$\begin{aligned} &L_{\mathfrak{p}_1} \boxtimes \cdots \boxtimes L_{\mathfrak{p}_r} \times G \to L_{\mathfrak{p}_1} \boxtimes \cdots \boxtimes L_{\mathfrak{p}_r}; \\ &((\rho_S, [(\alpha_{\mathfrak{p}_1}, z_1)] \otimes \cdots \otimes [(\alpha_{\mathfrak{p}_r}, z_r)]), g) \mapsto (\rho_S.g, [(\alpha_{\mathfrak{p}_1}.g, z_1)] \otimes \cdots \otimes [(\alpha_{\mathfrak{p}_r}.g, z_r)]). \end{aligned}$$

The projection $\varpi_S^{\boxtimes}$ is G-equivariant.

Proposition 5.3.42. *We have the following isomorphism of G-equivariant F-line bundles over $\mathcal{F}_S$*

$$\begin{aligned} &\Phi_{S,F}^{\boxtimes} : L_{\mathfrak{p}_1} \boxtimes \cdots \boxtimes L_{\mathfrak{p}_r} \xrightarrow{\sim} L_S; \\ &(\rho_S, [(\alpha_{\mathfrak{p}_1}, z_1)] \otimes \cdots \otimes [(\alpha_{\mathfrak{p}_r}, z_r)]) \mapsto [([\alpha_S], \textstyle\prod_{i=1}^r z_i)], \end{aligned}$$

where $\rho_S = (\rho_{\mathfrak{p}_i}) \in \mathcal{F}_S$, $[(\alpha_{\mathfrak{p}_i}, z_i)] \in L_{\mathfrak{p}_i}(\rho_{\mathfrak{p}_i})$, and $\alpha_S = (\alpha_{\mathfrak{p}_1}, \ldots, \alpha_{\mathfrak{p}_r})$.

Proof. If $(\alpha_{\mathfrak{p}_i}, z_i)$ is changed to $(\alpha_{\mathfrak{p}_i}.m_i, \zeta_N^{-m_i} z_i)$ for $m_i \in \mathbb{Z}/N\mathbb{Z}$, $(\alpha_S, \prod_{j=1}^r z_j)$ is changed to $([\alpha_S].m_i, \zeta_N^{-m_i} \prod_{j=1}^r z_j) \sim ([\alpha_S], \prod_{j=1}^r z_j)$. So, by (5.3.20) and (5.3.36), $\Phi_{S,F}^{\boxtimes}$ is well-defined.
(i) It is easy to see that $\varpi_{S,F} \circ \Phi_{S,F}^{\boxtimes} = \varpi_S^{\boxtimes}$.
(ii) By (5.3.40), $\Phi_{S,F}^{\boxtimes}$ restricted to a fiber over ρ_S is F-linear.
(iii) By (5.3.30), (5.3.37) and (5.3.41), we see that $\Phi_{S,F}^{\boxtimes}$ is G-equivariant.
Therefore $\Phi_{S,F}^{\boxtimes}$ is a morphism of G-equivariant F-line bundles over $\mathcal{F}_S$. The inverse is given by

$$\begin{aligned} &(\Phi_{S,F}^{\boxtimes})^{-1} : L_S \xrightarrow{\sim} L_{\mathfrak{p}_1} \boxtimes \cdots \boxtimes L_{\mathfrak{p}_r}; \\ &([\alpha_S], z) \mapsto (\varpi_S([\alpha_S]), [(\alpha_{\mathfrak{p}_1}, z)] \otimes [(\alpha_{\mathfrak{p}_2}, 1)] \otimes \cdots \otimes [(\alpha_{\mathfrak{p}_r}, 1)]), \end{aligned}$$

Hence $\Phi_{S,F}^{\boxtimes}$ is a G-equivariant isomorphism. $\square$

5.4 Arithmetic Chern–Simons functionals

Let $\mathcal{O}_k$ be the ring of integers of k. Let $X_k := \mathrm{Spec}(\mathcal{O}_k)$ and let X_k^∞ denote the set of infinite primes of k. We set $\overline{X}_k := X_k \sqcup X_k^\infty$. Let $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ be a finite set of finite primes of k. Let $\overline{X}_S := \overline{X}_k \setminus S$. We denote by Π_S the modified étale fundamental group of $\overline{X}_S$ with geometric base point $\mathrm{Spec}(\overline{k})$ ($\overline{k}$ being a fixed algebraic closure of k), which is the Galois group of the maximal subextension k_S of $\overline{k}$ over k, unramified outside S. We assume that all maximal ideals of $\mathcal{O}_k$ dividing N are contained in S (in particular, S is non-empty).

Let $\mathcal{F}_{\overline{X}_S}$ denote the set of continuous representations of Π_S to G:

$$\mathcal{F}_{\overline{X}_S} := \mathrm{Hom}_{\mathrm{cont}}(\Pi_S, G),$$

on which G acts from the right by

$$\mathcal{F}_{\overline{X}_S} \times G \to \mathcal{F}_{\overline{X}_S}; \quad (\rho, g) \mapsto \rho.g := g^{-1}\rho g, \tag{5.4.1}$$

and let $\mathcal{M}_{\overline{X}_S}$ denote the quotient set by this action:

$$\mathcal{M}_{\overline{X}_S} := \mathcal{F}_{\overline{X}_S}/G.$$

Let $\mathrm{Map}(\mathcal{F}_{\overline{X}_S}, \mathbb{Z}/N\mathbb{Z})$ be the additive group of maps from $\mathcal{F}_{\overline{X}_S}$ to $\mathbb{Z}/N\mathbb{Z}$, on which G acts from the left by

$$(g.\psi)(\rho) := \psi(\rho.g) \tag{5.4.2}$$

for $g \in G, \psi \in \mathrm{Map}(\mathcal{F}_{\overline{X}_S}, \mathbb{Z}/N\mathbb{Z})$ and $\rho \in \mathcal{F}_{\overline{X}_S}$.

We fix an embedding $\overline{k} \hookrightarrow \overline{k}_{\mathfrak{p}_i}$, which induces the continuous homomorphism for each $1 \le i \le r$

$$\iota_{\mathfrak{p}_i} : \Pi_{\mathfrak{p}_i} \longrightarrow \Pi_S.$$

Let $\mathrm{res}_{\mathfrak{p}_i}$ and res_S denote the restriction maps (the pull-backs by $\iota_{\mathfrak{p}_i}$) defined by

$$\begin{aligned} &\mathrm{res}_{\mathfrak{p}_i} : \mathcal{F}_{\overline{X}_S} \longrightarrow \mathcal{F}_{\mathfrak{p}_i}; \quad \rho \mapsto \rho \circ \iota_{\mathfrak{p}_i}, \\ &\mathrm{res}_S := (\mathrm{res}_{\mathfrak{p}_i}) : \mathcal{F}_{\overline{X}_S} \longrightarrow \mathcal{F}_S; \quad \rho \mapsto (\rho \circ \iota_{\mathfrak{p}_i}), \end{aligned} \tag{5.4.3}$$

which are G-equivariant by (5.3.1), (5.3.25) and (5.4.1). We denote by $\mathrm{Res}_{\mathfrak{p}_i}$ and Res_S the homomorphisms on cochains defined by

(5.4.4)

$$\begin{aligned} &\mathrm{Res}_{\mathfrak{p}_i} : C^n(\Pi_S, \mathbb{Z}/N\mathbb{Z}) \longrightarrow C^n(\Pi_{\mathfrak{p}_i}, \mathbb{Z}/N\mathbb{Z}); \quad \alpha \mapsto \alpha \circ \iota_{\mathfrak{p}_i}, \\ &\mathrm{Res}_S := (\mathrm{Res}_{\mathfrak{p}_i}) : C^n(\Pi_S, \mathbb{Z}/N\mathbb{Z}) \longrightarrow \textstyle\prod_{i=1}^r C^n(\Pi_{\mathfrak{p}_i}, \mathbb{Z}/N\mathbb{Z}); \quad \alpha \mapsto (\alpha \circ \iota_{\mathfrak{p}_i}). \end{aligned}$$

Firstly, we note the following

Lemma 5.4.5. *We have*

$$H^3(\Pi_S, \mathbb{Z}/N\mathbb{Z}) = 0.$$

Proof. It suffices to show that the p-primary part $H^3(\Pi_S, \mathbb{Z}/N\mathbb{Z})(p) = 0$ for any prime number p. Since $H^3(\Pi_S, \mathbb{Z}/N\mathbb{Z})(p) = 0$ for $p \nmid N$, we may assume that $p \mid N$.
Case that $N > 2$. Then k is totally imaginary and so $\Pi_S = \Pi_{S\cup X_k^\infty}$ ($\Pi_{S\cup X_k^\infty} := \pi_1^{\text{ét}}(\text{Spec}(\mathcal{O}_k \setminus S)$ being the Galois group of the maximal extension of k unramified outside $S \cup X_k^\infty$). By our assumption on S, all primes over p are contained in S. So the cohomological p-dimension $\text{cd}_p(\Pi_S) \le 2$ by [NSW; Proposition 8.3.18]. Hence $H^3(\Pi_S, \mathbb{Z}/N\mathbb{Z})(p) = 0$.
Case that $N = 2$ and so $p = 2$. Since S does not contain any real primes of k, the cohomological 2-dimension $\text{cd}_2(\Pi_S) \le 2$ by [NSW; Theorem 10.6.7]. Hence $H^3(\Pi_S, \mathbb{Z}/2\mathbb{Z})(2) = 0$. □

Let $\rho \in \mathcal{F}_{\overline{X}_S}$ and so $c \circ \rho \in Z^3(\Pi_S, \mathbb{Z}/N\mathbb{Z})$. By Lemma 5.2.5, there is $\beta_\rho \in C^2(\Pi_S, \mathbb{Z}/N\mathbb{Z})/B^2(\Pi_S, \mathbb{Z}/N\mathbb{Z})$ such that

$$c \circ \rho = d\beta_\rho, \tag{5.4.6}$$

where $d : C^2(\Pi_S, \mathbb{Z}/N\mathbb{Z}) \to C^3(\Pi_S, \mathbb{Z}/N\mathbb{Z})$ is the coboundary homomorphism. By (5.4.3), (5.4.4) and (5.4.6), we see that

$$c \circ \text{res}_{\mathfrak{p}_i}(\rho) = d\,\text{Res}_{\mathfrak{p}_i}(\beta_\rho) \tag{5.4.7}$$

for $1 \le i \le r$. By (5.3.4), (5.3.27) and (5.4.7), we have

$$[\text{Res}_S(\beta_\rho)] \in \mathcal{L}_S(\text{res}_S(\rho)). \tag{5.4.8}$$

Let $\text{res}_S^*(\mathcal{L}_S)$ be the G-equivariant principal $\mathbb{Z}/N\mathbb{Z}$-bundle over $\mathcal{F}_{\overline{X}_S}$ induced from $\mathcal{L}_S$ by res_S:

$$\text{res}_S^*(\mathcal{L}_S) := \{(\rho, \alpha_S) \in \mathcal{F}_{\overline{X}_S} \times \mathcal{L}_S \mid \text{res}_S(\rho) = \varpi_S(\alpha_S)\}. \tag{5.4.9}$$

and let $\text{res}_S^*(\varpi_S)$ be the projection $\text{res}_S^*(\mathcal{L}_S) \to \mathcal{F}_{\overline{X}_S}$. The quotient by the action of G is the principal $\mathbb{Z}/N\mathbb{Z}$-bundle $\text{res}^*(\overline{\mathcal{L}}_S)$ over $\mathcal{M}_{\overline{X}_S}$ induced from $\overline{\mathcal{L}}_S$ by res_S. By (5.4.9), a section of $\text{res}_S^*(\varpi_S)$ is naturally identified with a map $y_S : \mathcal{F}_{\overline{X}_S} \to \mathcal{L}_S$ satisfying $\varpi_S \circ y_S = \text{res}_S$:

$$\Gamma(\mathcal{F}_{\overline{X}_S}, \text{res}_S^*(\mathcal{L}_S)) = \{y_S : \mathcal{F}_{\overline{X}_S} \to \mathcal{L}_S \mid \varpi_S \circ y_S = \text{res}_S\}, \tag{5.4.10}$$

on which G acts by $(g.y_S)(\rho) := y_S(\rho.g)$ for $\rho \in \mathcal{F}_{\overline{X}_S}, g \in G$. We denote by $\Gamma_G(\mathcal{F}_{\overline{X}_S}, \text{res}_S^*(\mathcal{L}_S))$ the set of G-equivariant sections of $\text{res}_S^*(\varpi_S)$. We define the (mod N) *arithmetic Chern–Simons functional* $CS_{\overline{X}_S} : \mathcal{F}_{\overline{X}_S} \to \mathcal{L}_S$ by

$$CS_{\overline{X}_S}(\rho) := [\text{Res}_S(\beta_\rho)] \tag{5.4.11}$$

for $\rho \in \mathcal{F}_{\overline{X}_S}$. The value $CS_{\overline{X}_S}(\rho) \in \mathcal{L}_S$ is called the *arithmetic Chern–Simons invariant* of ρ.

Lemma 5.4.12. (1) *$CS_{\overline{X}_S}(\rho)$ is independent of the choice of β_ρ.*
(2) $CS_{\overline{X}_S}$ is a G-equivariant section of $\mathrm{res}_S^*(\varpi_S)$:

$$CS_{\overline{X}_S} \in \Gamma_G(\mathcal{F}_{\overline{X}_S}, \mathrm{res}_S^*(\mathcal{L}_S)) = \Gamma(\mathcal{M}_{\overline{X}_S}, \mathrm{res}_S^*(\overline{\mathcal{L}}_S)).$$

Proof. (1) Let $\beta'_\rho \in C^2(\Pi_S, \mathbb{Z}/N\mathbb{Z})/B^2(\Pi_S, \mathbb{Z}/N\mathbb{Z})$ be another choice satisfying $c \circ \rho = d\beta'_\rho$. Then we have $\beta'_\rho = \beta_\rho + z$ for some $z \in H^2(\Pi_S, \mathbb{Z}/N\mathbb{Z})$ and so

$$\mathrm{Res}_{\mathfrak{p}_i}(\beta'_\rho) - \mathrm{Res}_{\mathfrak{p}_i}(\beta_\rho) = \mathrm{inv}_{\mathfrak{p}_i}(\mathrm{Res}_{\mathfrak{p}_i}(z)) \;\; (1 \le i \le r).$$

Noting that any primes dividing N is contained in S, Tate-Poitou exact sequence ([NSW; 8.6.10]) implies that the composite of the following maps

$$H^2(\Pi_S, \mathbb{Z}/N\mathbb{Z}) \overset{\prod_{\mathfrak{p}\in\overline{S}} \mathrm{Res}_{\mathfrak{p}}}{\longrightarrow} \prod_{\mathfrak{p}\in\overline{S}} H^2(\Pi_{\mathfrak{p}}, \mathbb{Z}/N\mathbb{Z}) \overset{\sum_{\mathfrak{p}\in\overline{S}} \mathrm{inv}_{\mathfrak{p}}}{\longrightarrow} \mathbb{Z}/N\mathbb{Z}$$

is the zero map, where $\overline{S} = S \cup X_k^\infty$. For any infinite prime $v \in X_k^\infty$, the restriction map $\Pi_v := \mathrm{Gal}(\overline{k}_v/k_v) \to \Pi_S = \mathrm{Gal}(k_S/k)$ is the trivial homomorphism, because any infinite prime is unramified in k_S/k. So $\mathrm{Res}_v : H^2(\Pi_S, \mathbb{Z}/N\mathbb{Z}) \to H^2(\Pi_v, \mathbb{Z}/N\mathbb{Z})$ is the zero map. Hence we have

$$\sum_{i=1}^r \mathrm{inv}_{\mathfrak{p}_i}(\mathrm{Res}_{\mathfrak{p}_i}(z)) = 0.$$

By (5.3.28), we obtain

$$[\mathrm{Res}_S(\beta'_\rho)] = [\mathrm{Res}_S(\beta_\rho)].$$

(2) By (5.4.8), (5.4.10) and (5.4.11), we have

$$CS_{\overline{X}_S} \in \Gamma(\mathcal{F}_{\overline{X}_S}, \mathrm{res}_S^*(\mathcal{L}_S)).$$

So it suffices to show that $CS_{\overline{X}_S}$ is G-equivariant. By (5.3.5) and (5.4.6), we have

$$d\beta_{\rho.g} = c \circ (\rho.g) = (g.c) \circ \rho = (c + dh_g) \circ \rho = d(\beta_\rho + h_g \circ \rho).$$

for $g \in G$ and $\rho \in \mathcal{F}_{\overline{X}_S}$. Therefore there is $z \in H^2(\Pi_S, \mathbb{Z}/\mathbb{Z})$ such that $\beta_{\rho.g} = \beta_\rho + h_g \circ \rho + z$ and so

$$\begin{aligned}\mathrm{Res}_S(\beta_{\rho.g}) &= \mathrm{Res}_S(\beta_\rho) + h_g \circ \mathrm{res}_S(\rho) + \mathrm{Res}_S(z)\\ &= \mathrm{Res}_S(\beta_\rho).g + \mathrm{Res}_S(z).\end{aligned}$$

By the same argument as in (1) above, we obtain

$$CS_{\overline{X}_S}(\rho.g) = [\mathrm{Res}_S(\beta_{\rho.g})] = [\mathrm{Res}_S(\beta_\rho)].g = CS_{\overline{X}_S}(\rho).g.$$

□

Let $x_S = [(x_{\mathfrak{p}_1}, \ldots, x_{\mathfrak{p}_r})] \in \Gamma(\mathcal{F}_S, \mathcal{L}_S)$ be a section and let $\mathcal{L}_S^{x_S}$ be the arithmetic prequantization principal $\mathbb{Z}/N\mathbb{Z}$-bundle over $\mathcal{F}_S$ with respect to x_S. Let $\mathrm{res}_S^*(\mathcal{L}_S^{x_S})$ be the G-equivariant principal $\mathbb{Z}/N\mathbb{Z}$-bundle over $\mathcal{F}_{\overline{X}_S}$ induced from $\mathcal{L}_S^{x_S}$ by res_S:

$$\begin{aligned} \mathrm{res}_S^*(\mathcal{L}_S^{x_S}) &= \{(\rho, (\rho_S, m)) \in \mathcal{F}_{\overline{X}_S} \times \mathcal{L}_S^{x_S} \,|\, \mathrm{res}_S(\rho) = \rho_S\} \\ &= \mathcal{F}_{\overline{X}_S} \times \mathbb{Z}/N\mathbb{Z} \end{aligned}$$

by identifying $(\rho, (\rho_S, m))$ with (ρ, m). So a section of $\mathrm{res}_S^*(\mathcal{L}_S^{x_S})$ over $\mathcal{F}_{\overline{X}_S}$ is identified with a map $\mathcal{F}_{\overline{X}_S} \to \mathbb{Z}/N\mathbb{Z}$:

$$\Gamma(\mathcal{F}_{\overline{X}_S}, \mathrm{res}_S^*(\mathcal{L}_S^{x_S})) = \mathrm{Map}(\mathcal{F}_{\overline{X}_S}, \mathbb{Z}/N\mathbb{Z}),$$

on which G acts by (5.4.2). Therefore, letting $\mathrm{Map}_G(\mathcal{F}_{\overline{X}_S}, \mathbb{Z}/N\mathbb{Z})$ denote the set of G-equivariant maps $\mathcal{F}_{\overline{X}_S} \to \mathbb{Z}/N\mathbb{Z}$, we have the identification

$$\begin{aligned} \Gamma_G(\mathcal{F}_{\overline{X}_S}, \mathrm{res}_S^*(\mathcal{L}_S^{x_S})) &= \mathrm{Map}_G(\mathcal{F}_{\overline{X}_S}, \mathbb{Z}/N\mathbb{Z}) \\ &= \{\psi : \mathcal{F}_{\overline{X}_S} \to \mathbb{Z}/N\mathbb{Z} \,|\, \psi(\rho.g) = \psi(\rho) + \lambda_S^{x_S}(g, \mathrm{res}_S(\rho)) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \rho \in \mathcal{F}_{\overline{X}_S}, g \in G\}. \end{aligned}$$

The isomorphism $\Phi_S^{x_S} : \mathcal{L}_S \overset{\sim}{\to} \mathcal{L}_S^{x_S}$ in Proposition 5.3.35 induces the isomorphism

$$\begin{array}{rcl} \Psi^{x_S} : \Gamma_G(\mathcal{F}_{\overline{X}_S}, \mathrm{res}_S^*(\mathcal{L}_S)) & \overset{\sim}{\longrightarrow} & \Gamma_G(\mathcal{F}_{\overline{X}_S}, \mathrm{res}_S^*(\mathcal{L}_S^{x_S})) = \mathrm{Map}_G(\mathcal{F}_{\overline{X}_S}, \mathbb{Z}/N\mathbb{Z}) \\ y_S & \mapsto & \Phi_S^{x_S} \circ y_S. \end{array}$$

We then define the *arithmetic Chern–Simons functional* $CS_{\overline{X}_S}^{x_S} : \mathcal{F}_{\overline{X}_S} \to \mathbb{Z}/N\mathbb{Z}$ with respect to x_S by the image of $CS_{\overline{X}_S}$ under Ψ^{x_S}:

(5.4.13) $$CS_{\overline{X}_S}^{x_S} := \Psi^{x_S}(CS_{\overline{X}_S}).$$

Theorem 5.4.14. (1) *For $\rho \in \mathcal{F}_{\overline{X}_S}$, we have*

$$CS_{\overline{X}_S}^{x_S}(\rho) = \sum_{i=1}^{r} (\mathrm{Res}_{\mathfrak{p}_i}(\beta_\rho) - x_{\mathfrak{p}_i}(\mathrm{res}_{\mathfrak{p}_i}(\rho))),$$

which is independent of the choice of β_ρ.
(2) *We have the following equality in* $C^1(G, \mathrm{Map}(\mathcal{F}_{\overline{X}_S}, \mathbb{Z}/N\mathbb{Z}))$

$$dCS_{\overline{X}_S}^{x_S} = \mathrm{res}^*(\lambda_S^{x_S}).$$

Proof. (1) This follows from the definition of $\Phi_S^{x_S}$ in Proposition 5.3.35 and (5.4.13).
(2) Since $CS_{\overline{X}_S}^{x_S} \in \mathrm{Map}_G(\mathcal{F}_{\overline{X}_S}, \mathbb{Z}/N\mathbb{Z})$, we have

$$CS_{\overline{X}_S}^{x_S}(\rho.g) = CS_{\overline{X}_S}^{x_S}(\rho) + \lambda_S^{x_S}(g, \mathrm{res}_S(\rho))$$

for $g \in G$ and $\rho \in \mathcal{F}_{\overline{X}_S}$, which means the assertion. □

Proposition 5.4.15. *Let $x'_S \in \Gamma(\mathcal{F}_S, \mathcal{L}_S)$ be another section which yields $CS_{\overline{X}_S}^{x'_S}$, and let $\delta_S^{x_S,x'_S} : \mathcal{F}_S \to \mathbb{Z}/N\mathbb{Z}$ be the map in Proposition 5.3.34. Then we have*

$$CS_{\overline{X}_S}^{x'_S}(\rho) - CS_{\overline{X}_S}^{x_S}(\rho) = \delta_S^{x_S,x'_S}(\mathrm{res}_S(\rho)).$$

Proof. By Proposition 5.4.14 (1) and Lemma 5.1.4 (1), we have

$$\begin{aligned} CS_{\overline{X}_S}^{x'_S}(\rho) - CS_{\overline{X}_S}^{x_S}(\rho) &= \sum_{i=1}^{r}(\mathrm{Res}_{\mathfrak{p}_i}(\beta_\rho) - x'_{\mathfrak{p}_i}(\mathrm{res}_{\mathfrak{p}_i}(\rho))) - \sum_{i=1}^{r}(\mathrm{Res}_{\mathfrak{p}_i}(\beta_\rho) - x_{\mathfrak{p}_i}(\mathrm{res}_{\mathfrak{p}_i}(\rho))) \\ &= \sum_{i=1}^{r}(x_{\mathfrak{p}_i}(\mathrm{res}_{\mathfrak{p}_i}(\rho)) - x'_{\mathfrak{p}_i}(\mathrm{res}_{\mathfrak{p}_i}(\rho))) \\ &= \delta_S^{x_S,x'_S}(\mathrm{res}_S(\rho)). \end{aligned}$$

□

For $x_S, x'_S \in \Gamma(\mathcal{F}_S, \mathcal{L}_S)$, the G-equivariant isomorphism $\Phi_S^{x_S,x'_S} : \mathcal{L}_S^{x_S} \overset{\sim}{\to} \mathcal{L}_S^{x'_S}$ induces the isomorphism

$$\Psi^{x_S,x'_S} : \Gamma_G(\mathcal{F}_{\overline{X}_S}, \mathrm{res}_S^*(\mathcal{L}_S^{x_S})) \overset{\sim}{\longrightarrow} \Gamma_G(\mathcal{F}_{\overline{X}_S}, \mathrm{res}_S^*(\mathcal{L}_S^{x_S}));\ \psi^{x_S} \mapsto \Phi_S^{x_S,x'_S} \circ \psi^{x_S}.$$

By Proposition 5.3.35, we have

$$\begin{cases} \Psi^{x_S,x'_S} \circ \Psi^{x_S} = \Psi^{x'_S} \\ \Psi^{x_S,x_S} = \mathrm{id}_{\Gamma_G(\mathcal{F}_{\overline{X}_S}, \mathrm{res}_S^*(\mathcal{L}_S^{x_S}))}, \Psi^{x'_S,x_S} = (\Psi^{x_S,x'_S})^{-1}, \Psi^{x'_S,x''_S} \circ \Psi^{x_S,x'_S} = \Psi^{x_S,x''_S}. \end{cases}$$

So we can define the equivalence relation $\sim$ on the disjoint union of $\Gamma_G(\mathcal{F}_{\overline{X}_S}, \mathrm{res}_S^*(\mathcal{L}_S^{x_S}))$ over $x_S \in \Gamma(\mathcal{F}_S, \mathcal{L}_S)$ by

$$\psi^{x_S} \sim \psi^{x'_S} \Longleftrightarrow \Psi^{x_S,x'_S}(\psi^{x_S}) = \psi^{x'_S}$$

for $\psi^{x_S} \in \Gamma_G(\mathcal{F}_{\overline{X}_S}, \mathrm{res}_S^*(\mathcal{L}_S^{x_S}))$ and $\psi^{x'_S} \in \Gamma_G(\mathcal{F}_{\overline{X}_S}, \mathrm{res}_S^*(\mathcal{L}_S^{x_S}))$.Since $\Phi_S^{x'_S} = \Phi_S^{x_S,x'_S} \circ \Phi_S^{x_S}$, $CS_{\overline{X}_S}^{x_S} \simeq CS_{\overline{X}_S}^{x'_S}$. Thus we have the following identification:

$$(5.4.16) \qquad \begin{array}{ccc} \Gamma_G(\mathcal{F}_{\overline{X}_S}, \mathrm{res}_S^*(\mathcal{L}_S)) & = & \bigsqcup_{x_S \in \Gamma(\mathcal{F}_S, \mathcal{L}_S)} \Gamma_G(\mathcal{F}_{\overline{X}_S}, \mathrm{res}_S^*(\mathcal{L}_S^{x_S}))/\sim; \\ \psi & \mapsto & [\Psi^{x_S}(\psi)] \end{array}$$

where $CS_{\overline{X}_S}$ and $[CS_{\overline{X}_S}^{x_S}]$ are identified.

5.5 Arithmetic quantum spaces

Following the construction of the quantum Hilbert space, we define the *arithmetic quantum space* $\mathcal{H}_S$ for ∂V_S by the space of G-equivariant sections of the arithmetic prequantization F-line bundle $\varpi_{S,F} : L_S \to \mathcal{F}_S$:

$$\mathcal{H}_S := \Gamma_G(\mathcal{F}_S, L_S) = \Gamma(\mathcal{M}_S, \overline{L}_S).$$

It is a finite dimensional F-vector space.

Let $x_S = [(x_{\mathfrak{p}_1}, \ldots, x_{\mathfrak{p}_r})] \in \Gamma(\mathcal{F}_S, \mathcal{L}_S)$ be a section and let $L_S^{x_S}$ be the arithmetic prequantization F-line bundle over $\mathcal{F}_S$ with respect to x_S and let

(5.5.1)

$$\begin{aligned} \mathcal{H}_S^{x_S} &:= \Gamma_G(\mathcal{F}_S, L_S^{x_S}) = \Gamma(\mathcal{M}_S, \overline{L}_S^{x_S}) \\ &= \{\theta : \mathcal{F}_S \to F \,|\, \theta(\rho_S.g) = \zeta_N^{\lambda_S^{x_S}(g,\rho_S)}\theta(\rho_S) \text{ for } \rho_S \in \mathcal{F}_S, g \in G\}, \end{aligned}$$

which we call the *arithmetic quantum space* for ∂V_S with respect to x_S. The isomorphism $\Phi_{S,F}^{x_S} : L_S \xrightarrow{\sim} L_S^{x_S}$ in Proposition 5.3.39 induces the isomorphism

$$\Theta^{x_S} : \mathcal{H}_S \xrightarrow{\sim} \mathcal{H}_S^{x_S};\ \theta \mapsto \Phi_{S,F}^{x_S} \circ \theta. \tag{5.5.2}$$

We call an element of $\mathcal{H}_S$ or $\mathcal{H}_S^{x_S}$ an *arithmetic theta function* (cf. Remark 5.6.4 below).

For $x_S, x'_S \in \Gamma(\mathcal{F}_S, \mathcal{L}_S)$, the isomorphism $\Phi_{S,F}^{x_S,x'_S} : L_S^{x_S} \xrightarrow{\sim} L_S^{x'_S}$ induces the isomorphism of F-vector spaces:

$$\Theta^{x_S,x'_S} : \mathcal{H}_S^{x_S} \xrightarrow{\sim} \mathcal{H}_S^{x'_S};\ \theta^{x_S} \mapsto \Phi_{S,F}^{x_S,x'_S} \circ \theta^{x_S}$$

and, by Proposition 5.3.39, we have

$$\begin{cases} \Theta^{x_S,x'_S} \circ \Theta^{x_S} = \Theta^{x'_S} \\ \Theta^{x_S,x_S} = \mathrm{id}_{\mathcal{H}_S^{x_S}}, \Theta^{x'_S,x_S} = (\Theta^{x_S,x'_S})^{-1}, \Theta^{x'_S,x''_S} \circ \Theta^{x_S,x'_S} = \Theta^{x_S,x''_S}. \end{cases}$$

So the equivalence relation $\sim$ is defined on the disjoint union of all $\mathcal{H}_S^{x_S}$ running over $x_S \in \Gamma(\mathcal{F}_S, \mathcal{L}_S)$ by

$$\theta^{x_S} \sim \theta^{x'_S} \Longleftrightarrow \Theta^{x_S,x'_S}(\theta^{x_S}) = \theta^{x'_S}$$

for $\theta^{x_S} \in \mathcal{H}_S^{x_S}$ and $\theta^{x'_S} \in \mathcal{H}_S^{x'_s}$. Then we have the following identification:

$$\mathcal{H}_S = \bigsqcup_{x_S \in \Gamma(\mathcal{F}_S, \mathcal{L}_S)} \mathcal{H}_S^{x_S} / \sim . \tag{5.3.3}$$

Remark 5.5.4. The arithmetic quantum space $\mathcal{H}_S$ is an arithmetic analog of the quantum Hilbert space $\mathcal{H}_\Sigma$ for a surface Σ in (2+1)-dimensional Chern–Simons TQFT. We recall that $\mathcal{H}_\Sigma$ is known to coincides with the space of conformal blocks ([BL]) and its dimension formula was shown by Verlinde ([Ve]). It would also be an interesting question in number theory to describe the dimension and a canonical basis of $\mathcal{H}_S$ in comparison of Verlinde's formulas.

5.6 Arithmetic Dijkgraaf–Witten partition functions

For $\rho_S \in \mathcal{F}_S$, we define the subset $\mathcal{F}_{\overline{X}_S}(\rho_S)$ of $\mathcal{F}_{\overline{X}_S}$ by

$$\mathcal{F}_{\overline{X}_S}(\rho_S) := \{\rho \in \mathcal{F}_{\overline{X}_S} \,|\, \mathrm{res}_S(\rho) = \rho_S\}.$$

We then define the *arithmetic Dijkgraaf–Witten invariant* $Z^{x_S}_{\overline{X}_S}(\rho_S)$ of ρ_S with respect to x_S by

$$Z^{x_S}_{\overline{X}_S}(\rho_S) := \frac{1}{\#G} \sum_{\rho \in \mathcal{F}_{\overline{X}_S}(\rho_S)} \zeta_N^{CS^{x_S}_{\overline{X}_S}(\rho)}. \tag{5.6.1}$$

Theorem 5.6.2. (1) $Z^{x_S}_{\overline{X}_S}(\rho_S)$ *is independent of the choice of* β_ρ.
(2) *We have*

$$Z^{x_S}_{\overline{X}_S} \in \mathcal{H}^{x_S}_S.$$

Proof. (1) This follows from Lemma 5.4.12 (1).
(2) This follows from Theorem 5.4.14 (2) and (5.6.1). □

We call $Z^{x_S}_{\overline{X}_S} \in \mathcal{H}^{x_S}_S$ the *arithmetic Dijkgraaf–Witten partition function* for $\overline{X}_S$ with respect to x_S.

The following proposition tells us how they are changed when we change x_S.

Proposition 5.6.3. *For sections* $x_S, x'_S \in \Gamma(\mathcal{F}_S, \mathcal{L}_S)$, *we have*

$$\Theta^{x_S, x'_S}(Z^{x_S}_{\overline{X}_S}) = Z^{x'_S}_{\overline{X}_S}.$$

Proof. We have

$$\begin{aligned}\Theta^{x_S,x'_S}(Z^{x_S}_{\overline{X}_S})(\rho_S) &= (\Phi^{x_S,x'_S}_{S,F}\circ Z_{\overline{X}_S})(\rho_S)\\ &= Z_{\overline{X}_S}(\rho_S)\zeta_N^{\delta_S^{x_S,x'_S}(\rho_S)} \quad \text{by Proposition 5.3.39}\\ &= \frac{1}{\#G}\sum_{\rho\in\mathcal{F}_{\overline{X}_S}(\rho_S)} \zeta_N^{CS^{x_S}_{\overline{X}_S}(\rho)+\delta_S^{x_S,x'_S}(\rho_S)} \quad \text{by (5.6.1)}\\ &= \frac{1}{\#G}\sum_{\rho\in\mathcal{F}_{\overline{X}_S}(\rho_S)} \zeta_N^{CS^{x'_S}_{\overline{X}_S}(\rho)} \quad \text{by Proposition 5.4.15}\\ &= Z^{x'_S}_{\overline{X}_S}(\rho_S)\end{aligned}$$

for $\rho_S \in \mathcal{F}_S$. So we obtain the assertion. □

By the identification (5.5.3), $Z^{x_S}_{\overline{X}_S}$ defines the element $Z_{\overline{X}_S}$ of $\mathcal{H}_S$ which is independent of the choice of x_S. We call it the *arithmetic Dijkgraaf–Witten partition function* for $\overline{X}_S$.

Remark 5.6.4. In (2+1)-dimensional Chern–Simons TQFT, an element of $\mathcal{H}_\Sigma$ for a surface Σ may be regarded as a (non-abelian) generalization of the classical theta function on the Jacobian manifold of Σ (cf. [BL]. It goes back to Weli's paper [We]. See [Mo1] for an arithmetic analog.) In this respect, it may be interesting to observe that the Dijkgraaf–Witten partition function in (5.4.1) may look like a variant of (non-abelian) Gaussian sums.

5.7 Change of the 3-cocycle c

The theory given in the section 5.3, 5.4, 5.5, and 5.6 depends on a chosen 3-cocycle c. We shall see in the following that when c is changed in the cohomology class $[c]$, objects are changed to isomorphic ones, and hence the theory depends essentially on the cohomology class $[c]$. Let $c' \in Z^3(G,\mathbb{Z}/N\mathbb{Z})$ be another 3-cocycle representing $[c]$. The objects constructed by using c' will be denoted by using $'$, for example, by $\mathcal{L}'_{\mathfrak{p}}, L'_{\mathfrak{p}}, \ldots$ etc.

There is $b \in C^2(G,\mathbb{Z}/N\mathbb{Z})$ such that $c' - c = db$. Then we have the isomorphism of $\mathbb{Z}/N\mathbb{Z}$-torsors for $\rho_{\mathfrak{p}} \in \mathcal{F}_{\mathfrak{p}}$:

$$\mathcal{L}_{\mathfrak{p}}(\rho_{\mathfrak{p}}) \xrightarrow{\sim} \mathcal{L}'_{\mathfrak{p}}(\rho_{\mathfrak{p}});\ \alpha_{\mathfrak{p}} \mapsto \alpha_{\mathfrak{p}} + b\circ\rho_{\mathfrak{p}},$$

which induces the following isomorphisms of arithmetic quantization bundles:

$$\text{(5.7.1)}\qquad \begin{array}{ll}\xi_{\mathfrak{p}} : \mathcal{L}_{\mathfrak{p}} \xrightarrow{\sim} \mathcal{L}'_{\mathfrak{p}}, & \xi_{\mathfrak{p},F} : L_{\mathfrak{p}} \xrightarrow{\sim} L'_{\mathfrak{p}},\\ \xi_S : \mathcal{L}_S \xrightarrow{\sim} \mathcal{L}'_S, & \xi_{S,F} : L_S \xrightarrow{\sim} L'_S.\end{array}$$

Let $x_\mathfrak{p} \in \Gamma(\mathcal{F}_\mathfrak{p}, \mathcal{L}_\mathfrak{p})$ and $x_S = [(x_{\mathfrak{p}_1}, \ldots, x_{\mathfrak{p}_r})] \in \Gamma(\mathcal{F}_S, \mathcal{L}_S)$, and let $x'_\mathfrak{p} \in \Gamma(\mathcal{F}'_\mathfrak{p}, \mathcal{L}'_\mathfrak{p})$ and $x'_S \in \Gamma(\mathcal{F}'_F, \mathcal{L}'_S)$. Denote by $\lambda'_\mathfrak{p}$ and λ'_S the arithmetic Chern–Simons 1-cocycles for $\partial V_\mathfrak{p}$ and ∂V_S with respect to $x'_\mathfrak{p}$ and x'_S, respectively. We define $\kappa_\mathfrak{p} : \mathcal{F}_\mathfrak{p} \to \mathbb{Z}/N\mathbb{Z}$ and $\kappa_S : \mathcal{F}_S \to \mathbb{Z}/N\mathbb{Z}$ by

$$\kappa_\mathfrak{p}(\rho_\mathfrak{p}) := (\xi_\mathfrak{p} \circ x_\mathfrak{p})(\rho_\mathfrak{p}) - x'_\mathfrak{p}(\rho_\mathfrak{p}), \quad \kappa_S(\rho_S) := \sum_{i=1}^{r} \kappa_{\mathfrak{p}_i}(\rho_{\mathfrak{p}_i})$$

for $\rho_\mathfrak{p} \in \mathcal{F}_\mathfrak{p}$ and $\rho_S = (\rho_{\mathfrak{p}_1}, \ldots, \rho_{\mathfrak{p}_r}) \in \mathcal{F}_S$, respectively. Then we have

$$\lambda'_\mathfrak{p}(g) - \lambda_\mathfrak{p}(g) = g.\kappa_\mathfrak{p} - \kappa_\mathfrak{p}, \quad \lambda'_S(g) - \lambda_S(g) = g.\kappa_S - \kappa_S.$$

We note that if we take $x'_\mathfrak{p} := \xi_\mathfrak{p} \circ x_\mathfrak{p}$ and $x'_S := \xi_S \circ x_S$, $\kappa_\mathfrak{p} = 0$ and so $\kappa_S = 0$. As in Corollary 5.3.19, Propositions 5.3.24, 5.3.35 and 5.3.39, using $\kappa_\mathfrak{p}$ and κ_S, we have the isomorphisms

$$\begin{array}{ll} \mathcal{L}_\mathfrak{p}^{x_\mathfrak{p}} \xrightarrow{\sim} \mathcal{L}'^{\,x'_\mathfrak{p}}_\mathfrak{p}, & L_\mathfrak{p}^{x_\mathfrak{p}} \xrightarrow{\sim} L'^{\,x'_\mathfrak{p}}_\mathfrak{p}, \\ \mathcal{L}_S^{x_S} \xrightarrow{\sim} \mathcal{L}'^{\,x'_S}_S, & L_S^{x_S} \xrightarrow{\sim} L'^{\,x'_S}_S. \end{array}$$

which are compatible with the isomorphisms in (5.1.1) via the isomorphisms $\mathcal{L}_\mathfrak{p} \simeq \mathcal{L}_\mathfrak{p}^{x_\mathfrak{p}}, L_\mathfrak{p} \simeq L_\mathfrak{p}^{x_\mathfrak{p}}, \mathcal{L}_S \simeq \mathcal{L}_S^{x_S}$ and $L_S \simeq L_S^{x_S}$ in Propositions 5.3.15, 5.3.24, 5.3.35 and 5.3.39.

The isomorphism $\xi_S : \mathcal{L}_S \overset{\sim}{\to} \mathcal{L}'_S$ induces the isomorphism

$$\Gamma_G(\mathcal{F}_{\overline{X}_S}, \mathrm{res}_S^*(\mathcal{L}_S)) \xrightarrow{\sim} \Gamma_G(\mathcal{F}_{\overline{X}_S}, \mathrm{res}_S^*(\mathcal{L}'_S))$$

which sends $CS_{\overline{X}_S}$ to $CS'_{\overline{X}_S}$, and the isomorphism $\xi_{S,F} : L_S \overset{\sim}{\to} L'_S$ induces the isomorphisms

$$\mathcal{H}_S \xrightarrow{\sim} \mathcal{H}'_S, \quad \mathcal{H}_S^{x_S} \xrightarrow{\sim} \mathcal{H}'^{\,x'_S}_S,$$

which sends $Z_{\overline{X}_S}$ to $Z'_{\overline{X}_S}$.

Remark 5.7.2. A cochain $\alpha \in C^n(G, A)$ is called *normalized* if $\alpha(g_1, \ldots, g_n) = 0$ whenever one of g_i's is 1. It is known that any cocyle is cohomologous to a normalized one, namely, any cohomology class of $H^n(G, A)$ is represented by a normalized cocycle ([NSW; Chapter I, §2, Exercise 4], [EM; Lemma 6.1]). Therefore, by the above argument, we may assume that we can take the fixed cocycle $c \in Z^3(G, \mathbb{Z}/N\mathbb{Z})$ in our theory to be normalized.

5.8 Change of number fields

Let k' be an another number field contains a primitive N-th root of unity and let $S' = \{\mathfrak{p}'_1, \ldots, \mathfrak{p}'_{r'}\}$ be a finite set of finite primes of k' such that any finite prime dividing N is contained in S'. The objects constructed by using k' and S' will be denoted by, for example, $\mathcal{L}_{\mathfrak{p}'}, L_{\mathfrak{p}'}, \mathcal{L}_{S'}, L_{S'}, \ldots$ etc, for simplicity of notations . Assume that $r = r'$ and there are isomorphisms $\xi_i : k_{\mathfrak{p}_i} \xrightarrow{\sim} k'_{\mathfrak{p}'_i}$ for $1 \leq i \leq r$. Then ξ_i's induces the following isomorphisms of arithmetic quantization bundles:

$$\xi_{\mathfrak{p}_i} : \mathcal{L}_{\mathfrak{p}_i} \xrightarrow{\sim} \mathcal{L}_{\mathfrak{p}'_i}, \quad \xi_{\mathfrak{p}_i,F} : L_{\mathfrak{p}_i} \xrightarrow{\sim} L_{\mathfrak{p}'_i}$$

$$\xi_S : \mathcal{L}_S \xrightarrow{\sim} \mathcal{L}_{S'}, \quad \xi_{S,F} : L_S \xrightarrow{\sim} L_{S'}.$$

Let $x_{\mathfrak{p}_i} \in \Gamma(\mathcal{F}_{\mathfrak{p}_i}, \mathcal{L}_{\mathfrak{p}_i})$ and $x_S = [(x_{\mathfrak{p}_1}, \ldots, x_{\mathfrak{p}_r})] \in \Gamma(\mathcal{F}_S, \mathcal{L}_S)$, and let $x_{\mathfrak{p}'_i} \in \Gamma(\mathcal{F}_{\mathfrak{p}'_i}, \mathcal{L}_{\mathfrak{p}'_i})$ and $x_{S'} = [(x_{\mathfrak{p}'_1}, \ldots, x_{\mathfrak{p}'_r})] \in \Gamma(\mathcal{F}_{S'}, \mathcal{L}_{S'})$. Then we have the isomorphisms of arithmetic prequantization bundles with respect to sections

$$\mathcal{L}_{\mathfrak{p}_i}^{x_{\mathfrak{p}_i}} \xrightarrow{\sim} \mathcal{L}_{\mathfrak{p}'_i}^{x_{\mathfrak{p}'_i}}, \quad L_{\mathfrak{p}_i}^{x_{\mathfrak{p}_i}} \xrightarrow{\sim} L_{\mathfrak{p}'_i}^{x_{\mathfrak{p}'_i}}$$

$$\mathcal{L}_S^{x_S} \xrightarrow{\sim} \mathcal{L}_{S'}^{x_{S'}}, \quad L_S^{x_S} \xrightarrow{\sim} L_{S'}^{x_{S'}}.$$

Suppose further that there is an isomorphism $\tau : k \xrightarrow{\sim} k'$ of number fields which sends $\mathfrak{p}_i$ to $\mathfrak{p}'_i$ for $1 \leq i \leq r$. so that we have the isomorphism

$$\xi : \overline{X}_S := \overline{X}_k \setminus S \xrightarrow{\sim} \overline{X}_{k'} \backslash S' =: \overline{X}_{S'}.$$

For example, let $k := \mathbb{Q}(\sqrt[3]{2}), k' := \mathbb{Q}(\sqrt[3]{2}\omega)$, $\omega := \exp(\frac{2\pi\sqrt{-1}}{3})$ and so $N = 2$. Let ξ be the isomorphism $k \xrightarrow{\sim} k'$ defined by $\xi(\sqrt[3]{2}) := \sqrt[3]{2}\omega$. Noting $2\mathcal{O}_k = (\sqrt[3]{2})^2, X^3 - 2 = (X-4)(X-7)(X-20) \bmod 31$, let $S := \{\mathfrak{p}_1 := (\sqrt[3]{2}), \mathfrak{p}_2 := (31, \sqrt[3]{2} - 4), \mathfrak{p}_2 := (31, \sqrt[3]{2} - 7), \mathfrak{p}_4 := (31, \sqrt[3]{2} - 20)\}, S' := \xi(S) = \{\mathfrak{p}'_1 := (\sqrt[3]{2}\omega), \mathfrak{p}'_2 := (31, \sqrt[3]{2}\omega - 4), \mathfrak{p}'_3 := (31, \sqrt[3]{2}\omega - 7), \mathfrak{p}'_4 := (31, \sqrt[3]{2}\omega - 20)\}$, so that we have $k_{\mathfrak{p}_1} = k'_{\mathfrak{p}'_1} = \mathbb{Q}_2$ and $k_{\mathfrak{p}_i} = k'_{\mathfrak{p}'_i} = \mathbb{Q}_{31}$ $(2 \leq i \leq 4)$. So this example satisfies the above conditions.

The isomorphism $\xi : \overline{X}_S \xrightarrow{\sim} \overline{X}_{S'}$ induces the bijection $\xi^* : \mathcal{F}_{\overline{X}_{S'}} \xrightarrow{\sim} \mathcal{F}_{\overline{X}_S}$. By the constructions in the section 5.4, 5.5, and 5.6, we have the following

Proposition 5.8.1. *The isomorphism $\xi_S : \mathcal{L}_S \xrightarrow{\sim} \mathcal{L}_{S'}$ induces the bijection*

$$\Gamma_G(\mathcal{F}_{\overline{X}_S}, \mathrm{res}_S^*(\mathcal{L}_S)) \xrightarrow{\sim} \Gamma_G(\mathcal{F}_{\overline{X}_{S'}}, \mathrm{res}_{S'}^*(\mathcal{L}_{S'}))$$

which sends $CS_{\overline{X}_S}$ to $CS_{\overline{X}_{S'}}$. The isomorphism $\xi_{S,F} : L_S \xrightarrow{\sim} L_{S'}$ induces the isomorphism

$$\mathcal{H}_S \xrightarrow{\sim} \mathcal{H}_{S'},$$

which sends $Z_{\overline{X}_S}$ to $Z_{\overline{X}_{S'}}$.

Remark 5.8.2. Proposition 5.8.1 may be regarded as an arithmetic analogue of the axiom in $(2+1)$-diemnsional TQFT, which asserts that an orientation homeomorphism $f : \Sigma \overset{\approx}{\to} \Sigma'$ between closed surfaces induces an isomorphism $\mathcal{H}_\Sigma \overset{\sim}{\to} \mathcal{H}_{\Sigma'}$ of quantum Hilbert spaces and if f extends to an orientation preserving homeomorphism $M \overset{\approx}{\to} M'$, with $\partial M = \Sigma, \partial M' = \Sigma'$, Z_M is sent to $Z_{M'}$ under the induced isomorphism $\mathcal{H}_{\partial M} \overset{\sim}{\to} \mathcal{H}_{\partial M'}$.

5.9 Disjoint union of finite sets of primes and reversing the orientation of ∂V_S

In the theory in the section 5.3, 5.4, 5.5, and 5.6, we can include the case that S is the empty set $\emptyset$ as follows.

We define $\mathcal{F}_\emptyset$ to be the space of a single point, $\mathcal{F}_\emptyset := \{*\}$. We define the arithmetic prequantization principal $\mathbb{Z}/N\mathbb{Z}$-bundle $\mathcal{L}_\emptyset$ to be $\mathbb{Z}/N\mathbb{Z}$, on which G acts trivially, so that the map $\varpi_\emptyset : \mathcal{L}_\emptyset \to \mathcal{F}_\emptyset$ is G-equivariant. So the arithmetic prequantization F-line bundle $L_\emptyset$ is defined by $\mathbb{Z}/N\mathbb{Z} \times_{\mathbb{Z}/N\mathbb{Z}} F = F$. The arithmetic Chern–Simons 1-cocycle $\lambda_\emptyset$ is defined to be 0.

Let $\tilde{\Pi}_k$ be the modified étale fundamental group of $\overline{X}_k$ defined by considering the Artin-Verdier topology on $\overline{X}_k$, which takes the real primes into account (cf. [AC], [Bi], [Zi]). It is the Galois group of the maximal extension of k unramified at all finite and infinite primes. We set

$$\mathcal{F}_{\overline{X}_k} := \mathrm{Hom}_{\mathrm{cont}}(\tilde{\Pi}_k, G).$$

Following the section 4.3, we define the mod N *arithmetic Chern–Simons invariant* $CS_{\overline{X}_k}(\rho)$ of $\rho \in \mathcal{F}_{\overline{X}_k}$ again by the image of c under the composition

$$H^3(G, \mathbb{Z}/N\mathbb{Z}) \overset{\rho^*}{\to} H^3(\tilde{\Pi}_k, \mathbb{Z}/N\mathbb{Z}) \to H^3(\overline{X}_k, \mathbb{Z}/N\mathbb{Z}) \simeq \mathbb{Z}/N\mathbb{Z},$$

where the cohomology group of $\overline{X}_k$ is the modified étale cohomology defined in the Artin-Verdier topology. Thus we have the arithmetic Chern–Simons functional $CS_{\overline{X}_k} : \mathcal{F}_{\overline{X}_k} \to \mathbb{Z}/N\mathbb{Z}$ and so we see that

$$CS_{\overline{X}_k} \in \Gamma_G(\mathcal{F}_{\overline{X}_k}, \mathrm{res}_\emptyset^*(\mathcal{L}_\emptyset)) = \mathrm{Map}(\mathcal{M}_{\overline{X}_k}, \mathbb{Z}/N\mathbb{Z}),$$

where $\mathrm{res}_\emptyset$ is the (unique) restriction map $\mathcal{F}_{\overline{X}_k} \to \mathcal{F}_\emptyset$. Then we have

$$dCS_{\overline{X}_k} = 0 = \mathrm{res}_\emptyset^*(\lambda_\emptyset).$$

The arithmetic quantum space $\mathcal{H}_\emptyset$ is defined by $\Gamma_G(\mathcal{F}_\emptyset, L_\emptyset) = F$. Following the section 4.3, we define the arithmetic Dijkgraaf–Witten invariant $Z(\overline{X}_k)$ of $\overline{X}_k$ again by

$$Z(\overline{X}_k) := \frac{1}{\#G} \sum_{\rho \in \mathcal{F}_{\overline{X}_k}} \zeta_N^{CS_{\overline{X}_k}(\rho)}$$

and the arithmetic Dijkgraaf–Witten partition function by $Z_{\overline{X}_k} : \mathcal{F}_\emptyset \to F$ by $Z_{\overline{X}_k}(*) := Z(\overline{X}_k)$ for $* \in \mathcal{F}_\emptyset$. So we have

$$Z_{\overline{X}_k} \in \mathcal{H}_\emptyset.$$

We note that when $[c]$ is trivial, $Z(\overline{X}_k)$ coincides with the (averaged) number of continuous homomorphism from $\tilde{\Pi}_k$ to G:

$$Z(\overline{X}_k) = \frac{\#\mathrm{Hom}_{\mathrm{cont}}(\tilde{\Pi}_k, G)}{\#G},$$

which is the classical invariant for the number field k.

5.10 Disjoint union of finite sets of primes and reversing the orientation of ∂V_S

Let $S_1 = \{\mathfrak{p}_1, \dots, \mathfrak{p}_{r_1}\}$ and $S_2 = \{\mathfrak{p}_{r_1+1}, \dots, \mathfrak{p}_r\}$ be disjoint sets of finite primes of k and let $S = S_1 \sqcup S_2$. We include the case where S_1 is empty, but S_2 is non-empty. (For the case where S_1 and S_2 are both empty, the following arguments are trivial.) Then we have

$$\mathcal{F}_S = \mathcal{F}_{S_1} \times \mathcal{F}_{S_2}.$$

For the arithmetic quantization principal $\mathbb{Z}/N\mathbb{Z}$-bundles, we define the map

$$\boxplus : \mathcal{L}_{S_1} \times \mathcal{L}_{S_2} \longrightarrow \mathcal{L}_S,$$

as follows. For the case that $S_1 = \emptyset$ (and so $S_2 = S$), we set

$$m \boxplus [\alpha_{S_2}] := [\alpha_{S_2}].m \tag{5.10.1}$$

for $(m, [\alpha_S]) \in \mathcal{L}_\emptyset \times \mathcal{L}_{S_2}$. For the case that $S_1 \neq \emptyset$, we set

$$[\alpha_{S_1}] \boxplus [\alpha_{S_2}] := [(\alpha_{S_1}, \alpha_{S_2})] \tag{5.10.2}$$

for $([\alpha_{S_1}], [\alpha_{S_2}]) \in \mathcal{L}_{S_1} \times \mathcal{L}_{S_2}$.

For the arithmetic quantization F-line bundles, we let $p_i^*(L_{S_i})$ be the G-equivariant F-line bundle over $\mathcal{F}_S$ induced from L_{S_i} by the projection $p_i : \mathcal{F}_S \to \mathcal{F}_{S_i}$ for $i = 1, 2$:

$$p_i^*(L_{S_i}) := \{(\rho_S, [([\alpha_{S_i}], z_i)]) \in \mathcal{F}_S \times L_{S_i} \mid \rho_{S_i} = \varpi_{S_i}([\alpha_{S_i}])\}$$

for $\rho_S = (\rho_{S_1}, \rho_{S_2})$. When $S_1 = \emptyset$, we think of $p_i^*(L_\emptyset) = F$ simply over $\mathcal{F}_\emptyset = \{*\}$. Let

$$p_i^*(\varpi_{S_i}) : p_i^*(L_{S_i}) \longrightarrow \mathcal{F}_S$$

be the projection. The fiber over $\rho_S = (\rho_{S_1}, \rho_{S_2})$ is given by

$$\begin{aligned} p_i^*(\varpi_{S_i})^{-1}(\rho_S) &= \{\rho_S\} \times \{[([\alpha_{S_i}], z_i)] \in L_{S_i} \mid \rho_{S_i} = \varpi_{S_i}([\alpha_{S_i}]), z_i \in F\} \\ &= L_{S_i}(\rho_{S_i}) \\ &\simeq F, \end{aligned}$$

where $L_{S_i}(\rho_{S_i})$ is as in (5.3.38). We set

$$L_{S_1} \boxtimes L_{S_2} := p_1^*(L_{S_1}) \otimes p_2^*(L_{S_2}),$$

which is the F-line bundle over $\mathcal{F}_S$ and whose element is written by

$$(\rho_S, [([\alpha_{S_1}], z_1)] \otimes [([\alpha_{S_2}], z_2)]),$$

where $\rho_S = (\rho_{S_1}, \rho_{S_2}) \in \mathcal{F}_S$, $[([\alpha_{S_i}], z_i)] \in L_{S_i}(\rho_{S_i})$. The right action on $L_{S_1} \boxtimes L_{S_2}$ is defined by

$$(\rho_S, [([\alpha_{S_1}], z_1)] \otimes [([\alpha_{S_2}], z_2)]).g := (\rho_S.g, [([\alpha_{S_1}].g, z_1)] \otimes [([\alpha_{S_2}].g, z_2)])$$

so that the projection $L_{S_1} \boxtimes L_{S_2} \to \mathcal{F}_S$ is G-equivariant. Then, as in Proposition 5.3.42, we have the isomorphism of G-equivariant F-line bundles over $\mathcal{F}_S$:

$$L_{S_1} \boxtimes L_{S_2} \xrightarrow{\sim} L_S;\ (\rho_S, [([\alpha_{S_1}], z_1)] \otimes [([\alpha_{S_2}], z_2)]) \mapsto [([\alpha_S], z_1 z_2)],$$

where $\alpha_S = (\alpha_{S_1}, \alpha_{S_2})$. Choose $x_{S_i} \in \Gamma(\mathcal{F}_{S_i}, \mathcal{L}_{S_i})$ and let $x_S := [(x_{S_1}, x_{S_2})] \in \Gamma(\mathcal{F}_S, \mathcal{L}_S)$. Then we see that

$$\lambda_{S_1}^{x_{S_1}}(g, \rho_{S_1}) + \lambda_{S_2}^{x_{S_2}}(g, \rho_{S_2}) = \lambda_S^{x_S}(g, \rho_S)$$

for $g \in G, \rho_S = (\rho_{S_1}, \rho_{S_2})$ and, as in the case that L_S, we have the isomorphism

$$L_{S_1}^{x_{S_1}} \boxtimes L_{S^2}^{x_{S_2}} := p_1^*(L_{S_1}^{x_{S_1}}) \otimes p_2^*(L_{S_2}^{x_{S_2}}) \xrightarrow{\sim} L_S^{x_S};\ \ ((\rho_{S_1}, \rho_{S_2}), z_1 \otimes z_2) \mapsto (\rho_S, z_1 z_2)$$

for $\rho_S = (\rho_{S_1}, \rho_{S_2})$, which is compatible with $L_{S_1} \boxtimes L_{S_2} \simeq L_S$ via Proposition 5.1.39.

Proposition 5.10.3. *For $\theta_i \in \mathcal{H}_{S_i}^{x_{S_i}}$ $(i = 1, 2)$, we define $\theta_1 \cdot \theta_2 \in \mathcal{H}_S^{x_S}$ by*

$$(\theta_1 \cdot \theta_2)(\rho_S) := \theta_1(\rho_{S_1})\theta_2(\rho_{S_2})$$

for $\rho_S = (\rho_{S_1}, \rho_{S_2})$. Then we have the following isomorphism of F-vector spaces

$$\mathcal{H}_{S_1}^{x_{S_1}} \otimes \mathcal{H}_{S_2}^{x_{S_2}} \xrightarrow{\sim} \mathcal{H}_S^{x_S};\ \theta_1 \otimes \theta_2 \mapsto \theta_1 \cdot \theta_2.$$

For $\theta_i \in \mathcal{H}_{S_i}$ $(i=1,2)$, we define $\theta_1 \boxtimes \theta_2 \in \mathcal{H}_S$ by

$$(\theta_1 \boxtimes \theta_2)(\rho_S) := p_1^*(\theta_1(\rho_{S_1})) \otimes p_2^*(\theta_2(\rho_{S_2}))$$

for $\rho_S = (\rho_{S_1}, \rho_{S_2})$. Here $p_1^*(\theta_1(\rho_{S_1})) \otimes p_2^*(\theta_2(\rho_{S_2}))$ denotes $[([\alpha_S], z_1 z_2)]$ when $\theta_i(\rho_{S_i}) = [([\alpha_{S_i}], z_i)], \alpha_S = (\alpha_{S_1}, \alpha_{S_2})$. Then we have the following isomorphism of F-vector spaces

$$\mathcal{H}_{S_1} \otimes \mathcal{H}_{S_2} \xrightarrow{\sim} \mathcal{H}_S; \ (\theta_1, \theta_2) \mapsto \theta_1 \boxtimes \theta_2.$$

The above isomorphisms are compatible via the isomorphisms $\Theta^{x_{S_i}} : \mathcal{H}_{S_i} \simeq \mathcal{H}_{S_i}^{x_{S_i}}$ $(i=1,2)$ and $\Theta^{x_S} : \mathcal{H}_S \simeq \mathcal{H}_S^{x_S}$ in (5.5.2).

Proof. We may assume by Remark 5.1.2 that the cocycle c is normalized. For $\theta \in \mathcal{H}_S^{x_S}$, set $\theta_1(\rho_{S_1}) := \theta(\rho_{S_1}, 1)$ and $\theta_2(\rho_{S_2}) := \theta(1, \rho_{S_2})$. Since c is normalized, by (5.3.7) and (5.3.10), we have $\lambda_{\mathfrak{p}}(g, 1) = 0$ for $g \in G$ and $\mathfrak{p} \in S_i$. From this, we have $\theta_i \in \mathcal{H}_{S_i}^{x_{S_i}}$. Then the map $\mathcal{H}_S^{x_S} \to \mathcal{H}_{S_1}^{x_{S_1}} \otimes \mathcal{H}_{S_2}^{x_{S_2}}$; $\theta \mapsto \theta_1 \otimes \theta_2$, gives the inverse of the former map. By the definitions, the second map is compatible with the first one via $\Theta_{S_i}^{x_{S_i}} : \mathcal{H}_{S_i} \simeq \mathcal{H}_{S_i}^{x_{S_i}}$ $(i=1,2)$ and $\Theta^{x_S} : \mathcal{H}_S \simeq \mathcal{H}_S^{x_S}$ and so we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{H}_{S_1} \otimes \mathcal{H}_{S_2} & \longrightarrow & \mathcal{H}_S \\ \Theta^{x_{S_1}} \otimes \Theta^{x_{S_2}} \wr \downarrow & & \downarrow \wr\, \Theta^{x_S} \\ \mathcal{H}_{S_1}^{x_{S_1}} \otimes \mathcal{H}_{S_2}^{x_{S_2}} & \xrightarrow{\sim} & \mathcal{H}_S^{x_S}, \end{array}$$

from which the second isomorphism follows. □

Remark 5.10.4. Proposition 5.10.3 may be regarded as an arithmetic analog of the multiplicative property that $\mathcal{H}_{\Sigma_1 \sqcup \Sigma_2} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$ for disjoint surfaces Σ_1 and Σ_2 which is one of the axioms required in $(2+1)$-dimensional TQFT (Definition 3.1.1).

For a finite prime $\mathfrak{p}$ of k, the canonical isomorphism

$$\mathrm{inv}_{\mathfrak{p}} : H^2_{\text{ét}}(\partial V_{\mathfrak{p}}, \mathbb{Z}/N\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}/N\mathbb{Z}$$

indicates that $\partial V_{\mathfrak{p}}$ is "orientable" and we choose (implicitly) the "orientation" of $\partial V_{\mathfrak{p}}$ corresponding $1 \in \mathbb{Z}/N\mathbb{Z}$. We let $\partial V_{\mathfrak{p}}^* = \partial V_{\mathfrak{p}}$ with the "opposite orientation", namely, $\mathrm{inv}_{\mathfrak{p}}([\partial V_{\mathfrak{p}}^*]) = -1$.

The arithmetic prequantization principal $\mathbb{Z}/N\mathbb{Z}$-bundle for $\partial V_{\mathfrak{p}}^*$, denoted by $\mathcal{L}_{\mathfrak{p}^*}$, is defined (formally) by $\mathcal{L}_{\mathfrak{p}}$ with the opposite action of the structure group $\mathbb{Z}/N\mathbb{Z}$, $(\alpha_{\mathfrak{p}}, m) \mapsto \alpha_{\mathfrak{p}}.(-m)$ for $\alpha_{\mathfrak{p}} \in \mathcal{L}_{\mathfrak{p}^*}$ and $m \in \mathbb{Z}/N\mathbb{Z}$. So the arithmetic prequantization F-line bundle $L_{\mathfrak{p}^*}$ for $\partial V_{\mathfrak{p}}^*$ is the dual bundle of

$L_{\mathfrak{p}}$, $L_{\mathfrak{p}^*} = L_{\mathfrak{p}}^*$. Noting $\Gamma(\mathcal{F}_{\mathfrak{p}}, \mathcal{L}_{\mathfrak{p}^*}) = \Gamma(\mathcal{F}_{\mathfrak{p}}, \mathcal{L}_{\mathfrak{p}})$, the arithmetic Chern–Simons 1-cocycle $\lambda_{\mathfrak{p}^*}^{x_{\mathfrak{p}}}$ for $\partial V_{\mathfrak{p}}^*$ is given by $-\lambda_{\mathfrak{p}}^{x_{\mathfrak{p}}}$ for $x_{\mathfrak{p}} \in \Gamma(\mathcal{F}_{\mathfrak{p}}, \mathcal{L}_{\mathfrak{p}^*})$. The actions of G on $\mathcal{L}_{\mathfrak{p}^*}^{x_{\mathfrak{p}}} = \mathcal{F}_{\mathfrak{p}} \times \mathbb{Z}/N\mathbb{Z}$ and $L_{\mathfrak{p}^*}^{x_{\mathfrak{p}}} = \mathcal{F}_{\mathfrak{p}} \times F$ are changed to those via $\lambda_{\mathfrak{p}^*}^{x_{\mathfrak{p}}}$.

For a finite set of finite primes $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$, we set $\partial V_S^* := \partial V_{\mathfrak{p}_1}^* \sqcup \cdots \sqcup \partial V_{\mathfrak{p}_r}^*$. Then the arithmetic prequantization bundles $\mathcal{L}_{S^*}, L_{S^*}, \mathcal{L}_{S^*}^{x_S}$ and $L_{S^*}^{x_S}$ ($x_S \in \Gamma(\mathcal{F}_S, \mathcal{L}_{S^*}) = \Gamma(\mathcal{F}_{\mathfrak{p}}, \mathcal{L}_S)$) are defined in the similar manner. For the arithmetic Chern–Simons 1-cocycle, we have

$$\lambda_{S^*}^{x_S} = -\lambda_S^{x_S}.$$

Let $\mathcal{H}_{S^*}^{x_S}$ be the arithmetic quantum space for ∂V_S^* with respect to x_S. Then we see that

$$\begin{aligned}\mathcal{H}_{S^*}^{x_S} &= \{\theta^* : \mathcal{F}_S \to F \mid \theta^*(\rho_S.g) = \zeta_N^{\lambda_{S^*}^{x_S}(g,\rho_S)}\theta^*(\rho_S) \text{ for } \rho_S \in \mathcal{F}_S, g \in G\}\\ &= \{\theta^* : \mathcal{F}_S \to F \mid \theta^*(\rho_S.g) = \zeta_N^{-\lambda_S(g,\rho_S)}\theta^*(\rho_S) \text{ for } \rho_S \in \mathcal{F}_S, g \in G\}\\ &= \overline{\mathcal{H}}_S^{x_S},\end{aligned}$$

where $\overline{\mathcal{H}}_S^{x_S}$ is the complex conjugate of $\mathcal{H}_S^{x_S}$. Since the pairing

$$\mathcal{H}_{S^*}^{x_S} \times \mathcal{H}_S^{x_S} \longrightarrow F;\ (\theta^*, \theta) \mapsto \sum_{\rho_S \in \mathcal{F}_S} \theta^*(\rho_S)\theta(\rho_S)$$

is a (Hermitian) perfect pairing, together with (5.3.2), we have the following

Proposition 5.10.5. *$\mathcal{H}_{S^*}^{x_S}$ and $\mathcal{H}_{S^*}$ are the dual spaces of $\mathcal{H}_S^{x_S}$ and $\mathcal{H}_S$, respectively:*

$$\mathcal{H}_{S^*}^{x_S} = (\mathcal{H}_S^{x_S})^*, \ \ \mathcal{H}_{S^*} = (\mathcal{H}_S)^*.$$

Remark 5.10.6. Proposition 5.10.5 may be regarded as an arithmetic analog of the involutory property that $\mathcal{H}_{\Sigma^*} = \mathcal{H}_{\Sigma}^*$, where $\Sigma^* = \Sigma$ with the opposite orientation, which is one of the axioms required in $(2+1)$-dimensional TQFT (Definition 3.1.1).

In the section 5.4, 5.5, and 5.6, we have chosen implicitly the orientation of $\overline{X}_S$ so that the boundary $\partial\overline{X}_S$ with induced orientation may be identified with ∂V_S. Let $\overline{X}_S^*$ denote $\overline{X}_S$ with the opposite orientation. Then, the arithmetic Chern–Simons functional and the Dijkgraaf–Witten partition function for $\overline{X}_S^*$ are given as follows:

$$CS_{\overline{X}_S^*}^{x_S} = -CS_{\overline{X}_S}^{x_S}, \ \ Z_{\overline{X}_S^*}^{x_S}(\rho_S) = \frac{1}{\#G}\sum_{\rho \in \mathcal{F}_{\overline{X}_S}} \zeta_N^{-CS_{\overline{X}_S}^{x_S}(\rho)}. \tag{5.10.7}$$

5.11 Arithmetic Chern–Simons functionals and arithmetic Dijkgraaf–Witten partition functions for V_S

For a finite prime $\mathfrak{p}$ of k, let $\mathcal{O}_{\mathfrak{p}}$ denote the ring of $\mathfrak{p}$-adic integers and we let $V_{\mathfrak{p}} := \mathrm{Spec}(\mathcal{O}_{\mathfrak{p}})$. For a non-empty finite set of finite primes $S = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_r\}$ of k, let $V_S := V_{\mathfrak{p}_1} \sqcup \cdots \sqcup V_{\mathfrak{p}_r}$, which plays a role analogous to a tubular neighborhood of a link, and so ∂V_S plays a role of the boundary of V_S. In this section, we introduce the arithmetic Chern–Simons functional and arithmetic Dijkgraaf–Witten partition function for V_S, which will be used for our gluing formula in the next section.

Let $\tilde{\Pi}_{\mathfrak{p}}$ be the étale fundamental group of $V_{\mathfrak{p}}$, namely, the Galois group of the maximal unramified extension of $k_{\mathfrak{p}}$ and we set

$$\mathcal{F}_{V_{\mathfrak{p}}} := \mathrm{Hom}_{\mathrm{cont}}(\tilde{\Pi}_{\mathfrak{p}}, G), \quad \mathcal{F}_{V_S} := \mathcal{F}_{V_{\mathfrak{p}_1}} \times \cdots \times \mathcal{F}_{V_{\mathfrak{p}_r}}.$$

Since $\tilde{\Pi}_{\mathfrak{p}} \simeq \hat{\mathbb{Z}}$ (profinite infinite cyclic group), $\mathcal{F}_{V_{\mathfrak{p}}} \simeq G$. G acts on $\mathcal{F}_{V_S}$ from the right by

$$\mathcal{F}_{V_S} \times G \to \mathcal{F}_{V_S}; \quad ((\tilde{\rho}_{\mathfrak{p}_i})_i, g) \mapsto \rho.g := (g^{-1}\tilde{\rho}_{\mathfrak{p}_i} g)_i,$$

and let $\mathcal{M}_{V_S}$ denote the quotient set by this action:

$$\mathcal{M}_{V_S} := \mathcal{F}_{V_S}/G.$$

Let $\tilde{\mathrm{res}}_{\mathfrak{p}_i} : \mathcal{F}_{V_{\mathfrak{p}_i}} \to \mathcal{F}_{\mathfrak{p}}$ and $\tilde{\mathrm{res}}_S := (\tilde{\mathrm{res}}_{\mathfrak{p}_i}) : \mathcal{F}_{V_S} \to \mathcal{F}_S$ denote the restriction maps induced by the natural continuous homomorphisms $v_{\mathfrak{p}_i} : \Pi_{\mathfrak{p}_i} \to \tilde{\Pi}_{\mathfrak{p}_i}$ $(1 \leq i \leq r)$, which are G-equivariant. We denote by $\tilde{\mathrm{Res}}_{\mathfrak{p}_i}$ and $\tilde{\mathrm{Res}}_S$ the homomorphisms on cochains given as the pull-back by $v_{\mathfrak{p}_i}$:

$$\begin{aligned}
&\tilde{\mathrm{Res}}_{\mathfrak{p}_i} : C^n(\tilde{\Pi}_{\mathfrak{p}_i}, \mathbb{Z}/N\mathbb{Z}) \longrightarrow C^n(\Pi_{\mathfrak{p}_i}, \mathbb{Z}/N\mathbb{Z}); \alpha_i \mapsto \alpha_i \circ v_{\mathfrak{p}_i},\\
&\tilde{\mathrm{Res}}_S := (\tilde{\mathrm{Res}}_{\mathfrak{p}_i}) : \prod_{i=1}^{r} C^n(\tilde{\Pi}_{\mathfrak{p}_i}, \mathbb{Z}/N\mathbb{Z}) \longrightarrow \prod_{i=1}^{r} C^n(\Pi_{\mathfrak{p}_i}, \mathbb{Z}/N\mathbb{Z}); (\alpha_i) \mapsto (\alpha_i \circ v_{\mathfrak{p}_i}).
\end{aligned}$$

For $\tilde{\rho} = (\tilde{\rho}_{\mathfrak{p}_i})_i \in \mathcal{F}_{V_S}$, $c \circ \tilde{\rho}_{\mathfrak{p}_i} \in Z^3(\tilde{\Pi}_{\mathfrak{p}_i}, \mathbb{Z}/N\mathbb{Z})$. Since $H^3(\tilde{\Pi}_{\mathfrak{p}_i}, \mathbb{Z}/N\mathbb{Z}) = 0$, there is $\tilde{\beta}_{\mathfrak{p}_i} \in C^2(\tilde{\Pi}_{\mathfrak{p}_i}, \mathbb{Z}/N\mathbb{Z})$ such that

$$c \circ \tilde{\rho}_{\mathfrak{p}_i} = d\tilde{\beta}_{\mathfrak{p}_i}.$$

We see that

$$c \circ \tilde{\mathrm{res}}_{\mathfrak{p}_i}(\tilde{\rho}_{\mathfrak{p}_i}) = d\tilde{\mathrm{Res}}_{\mathfrak{p}_i}(\tilde{\beta}_{\mathfrak{p}_i})$$

for $1 \leq i \leq r$ and we have

$$[\tilde{\mathrm{Res}}_S((\tilde{\beta}_{\mathfrak{p}_i})_i)] \in \mathcal{L}_S(\tilde{\mathrm{res}}_S(\tilde{\rho})).$$

Let $\tilde{\mathrm{res}}_S^*(\mathcal{L}_S)$ be the G-equivariant principal $\mathbb{Z}/N\mathbb{Z}$-bundle over $\mathcal{F}_{V_S}$ induced from $\mathcal{L}_S$ by $\tilde{\mathrm{res}}_S$:

$$\tilde{\mathrm{res}}_S^*(\mathcal{L}_S) := \{(\tilde{\rho}, \alpha_S) \in \mathcal{F}_{V_S} \times \mathcal{L}_S \,|\, \tilde{\mathrm{res}}_S(\tilde{\rho}) = \varpi_S(\alpha_S)\}$$

and let $\tilde{\mathrm{res}}_S^*(\varpi_S)$ be the projection $\tilde{\mathrm{res}}_S^*(\mathcal{L}_S) \to \mathcal{F}_{V_S}$. We define the *arithmetic Chern–Simons functional* $CS_{V_S} : \mathcal{F}_{V_S} \to \mathcal{L}_S$ by

$$CS_{V_S}(\tilde{\rho}) := [\tilde{\mathrm{Res}}_S((\tilde{\beta}_{\mathfrak{p}_i})_i)]$$

for $\tilde{\rho} \in \mathcal{F}_{V_S}$. The value $CS_{V_S}(\tilde{\rho})$ is called the *arithmetic Chern–Simons invariant* of $\tilde{\rho}$.

Lemma 5.11.1. (1) *$CS_{V_S}(\tilde{\rho})$ is independent of the choice of $\tilde{\beta}_{\mathfrak{p}_i}$.*
(2) *CS_{V_S} is a G-equivariant section of $\tilde{\mathrm{res}}_S^*(\varpi_S)$:*

$$CS_{V_S} \in \Gamma_G(\mathcal{F}_{V_S}, \tilde{\mathrm{res}}_S^*(\mathcal{L}_S)) = \Gamma(\mathcal{M}_{V_S}, \tilde{\mathrm{res}}_S^*(\overline{\mathcal{L}}_S)).$$

Proof. (1) This follows from the fact that the cohomological dimension of $\tilde{\Pi}_{\mathfrak{p}_i}$ is one.
(2) The proof of this lemma is almost same as Lemma 5.4.12. (2). □

For a section $x_S = [(x_{\mathfrak{p}_1}, \ldots, x_{\mathfrak{p}_r})] \in \Gamma(\mathcal{F}_S, \mathcal{L}_S)$, the isomorphism $\Phi_S^{x_S} : \mathcal{L}_S \xrightarrow{\sim} \mathcal{L}_S^{x_S}$ induces the isomorphism

$$\begin{array}{rcl} \tilde{\Psi}^{x_S} : \Gamma_G(\mathcal{F}_{V_S}, \tilde{\mathrm{res}}_S^*(\mathcal{L}_S)) & \xrightarrow{\sim} & \Gamma_G(\mathcal{F}_{V_S}, \tilde{\mathrm{res}}_S^*(\mathcal{L}_S^{x_S})) = \mathrm{Map}_G(\mathcal{F}_{V_S}, \mathbb{Z}/N\mathbb{Z}); \\ y_S & \mapsto & \Phi_S^{x_S} \circ y_S. \end{array}$$

We define the *arithmetic Chern–Simons functional* $CS_{V_S}^{x_S} : \mathcal{F}_{V_S} \to \mathbb{Z}/N\mathbb{Z}$ with respect to x_S by the image of CS_{V_S} under $\tilde{\Psi}^{x_S}$.

Proposition 5.11.2. (1) *For $\rho \in \mathcal{F}_{V_S}$, we have*

$$CS_{V_S}^{x_S}(\tilde{\rho}) = \sum_{i=1}^{r} (\tilde{\mathrm{Res}}_S(\tilde{\beta}_{\mathfrak{p}_i}) - x_{\mathfrak{p}_i}(\tilde{\mathrm{res}}_{\mathfrak{p}_i}(\tilde{\rho}_{\mathfrak{p}_i}))).$$

(2) *We have the following equality in $C^1(G, \mathrm{Map}(\mathcal{F}_{V_S}, \mathbb{Z}/N\mathbb{Z}))$*

$$dCS_{V_S} = \tilde{\mathrm{res}}^*(\lambda_S^{x_S}).$$

Proof. (1) This follows from the definition of $\tilde{\Psi}^{x_S}$.
(2) Since $CS_{V_S} \in \mathrm{Map}_G(\mathcal{F}_{V_S}, \mathbb{Z}/N\mathbb{Z})$, we have

$$CS_{V_S}^{x_S}(\tilde{\rho}.g) = CS_{V_S}^{x_S}(\tilde{\rho}) + \lambda_S^{x_S}(g, \tilde{\mathrm{res}}_S(\tilde{\rho}))$$

for $g \in G$ and $\tilde{\rho} \in \mathcal{F}_{V_S}$, which means the assertion. □

Proposition 5.11.3. *Let $x'_S \in \Gamma(\mathcal{F}_S, \mathcal{L}_S)$ be another section, which yields $CS^{x'_S}_{V_S}$ and let $\delta^{x_S,x'_S}_S : \mathcal{F}_S \to \mathbb{Z}/N\mathbb{Z}$ be the map in Proposition 5.3.34. Then we have*

$$CS^{x'_S}_{V_S}(\tilde{\rho}) - CS^{x_S}_{V_S}(\tilde{\rho}) = \delta^{x_S,x'_S}_S(\tilde{\mathrm{res}}_S(\tilde{\rho})).$$

Proof. This follows from Proposition 5.11.2. (1) and Lemma 5.1.4. □

For $\rho_S \in \mathcal{F}_S$, we define the subset $\mathcal{F}_{V_S}(\rho_S)$ of $\mathcal{F}_{V_S}$ by

$$\mathcal{F}_{V_S}(\rho_S) := \{\tilde{\rho} \in \mathcal{F}_{V_S} \,|\, \tilde{\mathrm{res}}_S(\tilde{\rho}) = \rho_S\}.$$

We then define the *arithmetic Dijkgraaf–Witten invariant* $Z_{V_S}(\rho_S)$ of ρ_S with respect to x_S by

$$Z^{x_S}_{V_S}(\rho_S) := \frac{1}{\#G} \sum_{\tilde{\rho} \in F_{V_S}(\rho_S)} \zeta_N{}^{CS^{x_S}_{V_S}(\tilde{\rho})}.$$

Theorem 5.11.4. (1) *$Z^{x_S}_{V_S}(\rho_S)$ is independent of the choice of $\tilde{\beta}_{\rho_{\mathfrak{p}_i}}$.*
(2) *We have*

$$Z^{x_S}_{V_S} \in \mathcal{H}^{x_S}_S.$$

Proof. (1) This follows from Proposition 5.11.1. (1).
(2) This follows from Proposition 5.11.2. (2). □

We call $Z^{x_S}_{V_S}$ the *arithmetic Dijkgraaf–Witten partition function* for V_S with respect to x_S.

Proposition 5.11.5. For sections $x_S, x'_S \in \Gamma(\mathcal{F}_S, \mathcal{L}_S)$ we see that

$$\Theta^{x_S,x'_S}(Z^{x_S}_{V_S}) = Z^{x'_S}_{V_S}.$$

Proof. This follows from Proposition 5.11.3. □

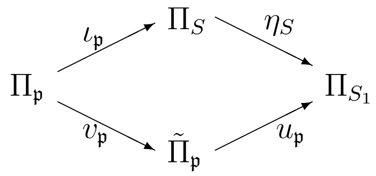
By the identification (5.5.3), $Z^{x_S}_{V_S}$ defines the element Z_{V_S} of $\mathcal{H}_S$ which is independent of the choice of x_S. We call it the *arithmetic Dijkgraaf–Witten partition function* for V_S.

In the above, the orientation of V_S is chosen so that it is compatible with that of ∂V_S as explained in the section 5.4. Let V_S^* denote V_S with opposite orientation. Then, following (5.4.7), the arithmetic Chern–Simons functional and the arithmetic Dijkgraaf–Witten partition function are given by

$$(5.11.6) \qquad CS^{x_S}_{V_S^*} = -CS^{x_S}_{V_S}, \quad Z^{x_S}_{V_S^*}(\rho_S) = \frac{1}{\#G} \sum_{\tilde{\rho} \in \mathcal{F}_{V_S}(\rho_S)} \zeta_N^{-CS^{x_S}_{\overline{X}_S}(\tilde{\rho})}.$$

5.12 Gluing formulas for arithmetic Chern–Simons invariants and gluing formulas for arithmetic Dijkgraaf–Witten partition functions

Let S_1 and S_2 be disjoint sets of finite primes of k, where S_1 may be empty and S_2 is non-empty. We assume that any prime dividing N is contained in S_2 if S_1 is empty and that any prime dividing N is contained in S_1 if S_1 is non-empty. We let $S := S_1 \sqcup S_2$. We may think of $\overline{X}_{S_1}$ as the space obtained by gluing $\overline{X}_S$ and $V_{S_2}^*$ along ∂V_{S_2}. Let $\eta_S : \Pi_S \to \Pi_{S_1}$, $\iota_{\mathfrak{p}} : \Pi_{\mathfrak{p}} \to \Pi_S$, $v_{\mathfrak{p}} : \Pi_{\mathfrak{p}} \to \tilde{\Pi}_{\mathfrak{p}}$, and $u_{\mathfrak{p}} : \tilde{\Pi}_{\mathfrak{p}} \to \Pi_{S_1}$ be the natural homomorphisms, where $\mathfrak{p} \in S_2$, so that we have $\eta_S \circ \iota_{\mathfrak{p}} = u_{\mathfrak{p}} \circ v_{\mathfrak{p}}$ for $\mathfrak{p} \in S_2$.



Let $\boxplus : \mathcal{L}_{S_1} \times \mathcal{L}_{S_2} \to \mathcal{L}_S$ be the map defined as in (5.10.1) and (5.10.2). Now we have the following decomposition formula.

Theorem 5.12.1 (*Decomposition formula*). For $\rho \in \mathrm{Hom}_{\mathrm{cont}}(\Pi_{S_1}, G)$, we have

$$CS_{\overline{X}_{S_1}}(\rho) \boxplus CS_{V_{S_2}}((\rho \circ u_{\mathfrak{p}})_{\mathfrak{p} \in S_2}) = CS_{\overline{X}_S}(\rho \circ \eta_S).$$

Proof. Case that $S_1 = \emptyset$. Although this may be well known, we give a proof for the sake of readers. By the Artin–Verdier Duality for compact support étale cohomologies ([Mi2; Chapter II. Theorem 3.1]) and modified étale cohomologies ([Bi; Theorem 5.1]), we have the following isomorphisms for a fixed $\zeta_N \in \mu_N$,

$$H^3_{\mathrm{comp}}(X_S, \mathbb{Z}/N\mathbb{Z}) \cong \mathrm{Hom}_{X_S}(\mathbb{Z}/N\mathbb{Z}, \mathbb{G}_{m,X_S})^{\sim} \cong \mu_N(k)^{\sim} \cong \mathbb{Z}/N\mathbb{Z},$$

$$H^3(\overline{X}_k, \mathbb{Z}/N\mathbb{Z}) \cong \mathrm{Hom}_{\overline{X}_k}(\mathbb{Z}/N\mathbb{Z}, \mathbb{G}_{m,\overline{X}_k})^{\sim} \cong \mu_N(k)^{\sim} \cong \mathbb{Z}/N\mathbb{Z},$$

where $\mathbb{G}_{m,X_S}$ (resp. $\mathbb{G}_{m,\overline{X}_k}$) is the sheaf of units on X_S (resp. $\overline{X}_k$) and $(-)^{\sim}$ is given by $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$. We denote the isomorphisms above by $\mathrm{inv}' : H^3_{\mathrm{comp}}(X_S, \mathbb{Z}/N\mathbb{Z}) \to \mathbb{Z}/N\mathbb{Z}$ and $\mathrm{inv} : H^3(\overline{X}_k, \mathbb{Z}/N\mathbb{Z}) \to \mathbb{Z}/N\mathbb{Z}$. Now we recall the definition of $H^3_{\mathrm{comp}}(X_S, \mathbb{Z}/N\mathbb{Z})$ ([Mi2; p.165]). We define the complex $C_{comp}(\Pi_S, \mathbb{Z}/N\mathbb{Z})$ by

$$C^n_{comp}(\Pi_S, \mathbb{Z}/N\mathbb{Z}) := C^n(\Pi_S, \mathbb{Z}/N\mathbb{Z}) \times \prod_{\mathfrak{p} \in S} C^{n-1}(\Pi_{\mathfrak{p}}, \mathbb{Z}/N\mathbb{Z}),$$

$$d(a, (b_{\mathfrak{p}})) := (da, (\mathrm{Res}_{\mathfrak{p}}(a) - db_{\mathfrak{p}})),$$

where $a \in C^n(\Pi_S, \mathbb{Z}/N\mathbb{Z})$ and $(b_{\mathfrak{p}}) \in \prod_{\mathfrak{p}\in S} C^{n-1}(\Pi_{\mathfrak{p}}, \mathbb{Z}/N\mathbb{Z})$. $H^n_{\mathrm{comp}}(X_S, \mathbb{Z}/N\mathbb{Z})$ is defined by

$$H^n_{\mathrm{comp}}(X_S, \mathbb{Z}/N\mathbb{Z}) := H^n(C^*_{comp}(\Pi_S, \mathbb{Z}/N\mathbb{Z})).$$

Then we can describe $\mathrm{inv}' : H^3_{\mathrm{comp}}(X_S, \mathbb{Z}/N\mathbb{Z}) \to \mathbb{Z}/N\mathbb{Z}$ as follows. Let $[(a, (b_{\mathfrak{p}}))] \in H^3_{\mathrm{comp}}(X_S, \mathbb{Z}/N\mathbb{Z})$. Since $da = 0$ and $H^3(\Pi_S, \mathbb{Z}/N\mathbb{Z}) = 0$, there is a cochain $b \in C^2(\Pi_S, \mathbb{Z}/N\mathbb{Z})$ such that $db = a$. Then we have

$$\mathrm{inv}'([(a, (b_{\mathfrak{p}})]) = \sum_{\mathfrak{p}\in S} \mathrm{inv}_{\mathfrak{p}}([\mathrm{Res}_{\mathfrak{p}}(b) - b_{\mathfrak{p}}]),$$

where $\mathrm{inv}_{\mathfrak{p}} : H^2(\Pi_{\mathfrak{p}}, \mathbb{Z}/N\mathbb{Z}) \to \mathbb{Z}/N\mathbb{Z}$ is the canonical isomorphism given by the theory of Brauer groups. We note that the right side of the equation above doesn't depend on the choice of b. Recall that $\tilde{\Pi}_k$ denotes the modified étale fundamental group of $\overline{X}_k$. Let $j_3 : H^3(\tilde{\Pi}_k, \mathbb{Z}/N\mathbb{Z}) \to H^3(\overline{X}_k, \mathbb{Z}/N\mathbb{Z})$ be the natural homomorphism induced by the modified Hochschild-Serre spectral sequence (Corollary 4.2.8). We describe the image of the cohomology class $[c \circ \rho] \in H^3(\tilde{\Pi}_k, \mathbb{Z}/N\mathbb{Z})$ by the composed map

$$\mathrm{inv}'^{-1} \circ \mathrm{inv} \circ j_3 : H^3(\tilde{\Pi}_k, \mathbb{Z}/N\mathbb{Z}) \to H^3_{\mathrm{comp}}(X_S, \mathbb{Z}/N\mathbb{Z}).$$

Since $c \circ (\rho \circ \eta_S) \in Z^3(\Pi_S, \mathbb{Z}/N\mathbb{Z})$ and $H^3(\Pi_S, \mathbb{Z}/N\mathbb{Z}) = 0$, there exists a cochain $\beta_{\rho\circ\eta_S} \in C^2(\Pi_S, \mathbb{Z}/N\mathbb{Z})$ such that $d\beta_{\rho\circ\eta_S} = c \circ (\rho \circ \eta_S)$. We note that $d\mathrm{Res}_{\mathfrak{p}}(\beta_{\rho\circ\eta_S}) = d(\beta_{\rho\circ\eta_S} \circ \iota_{\mathfrak{p}}) = c \circ \rho \circ u_{\mathfrak{p}} \circ v_{\mathfrak{p}}$. Since $c \circ (\rho \circ u_{\mathfrak{p}}) \in Z^3(\tilde{\Pi}_{\mathfrak{p}}, \mathbb{Z}/N\mathbb{Z})$ and $H^3(\tilde{\Pi}_{\mathfrak{p}}, \mathbb{Z}/N\mathbb{Z}) = H^2(\tilde{\Pi}_{\mathfrak{p}}, \mathbb{Z}/N\mathbb{Z}) = 0$, there exists a cochain $\tilde{\beta}_{\rho\circ u_{\mathfrak{p}}} \in C^2(\tilde{\Pi}_{\mathfrak{p}}, \mathbb{Z}/N\mathbb{Z})$ such that $d\tilde{\beta}_{\rho\circ u_{\mathfrak{p}}} = c \circ (\rho \circ u_{\mathfrak{p}})$. We set $\beta_{\rho\circ u_{\mathfrak{p}}\circ v_{\mathfrak{p}}} := \tilde{\beta}_{\rho\circ u_{\mathfrak{p}}} \circ v_{\mathfrak{p}} \in C^2(\Pi_{\mathfrak{p}}, \mathbb{Z}/N\mathbb{Z})$. So we have $d\beta_{\rho\circ u_{\mathfrak{p}}\circ v_{\mathfrak{p}}} = c \circ (\rho \circ u_{\mathfrak{p}} \circ v_{\mathfrak{p}})$. Then we obtain

$$(\mathrm{inv}'^{-1} \circ \mathrm{inv} \circ j_3)([c \circ \rho]) = [(c \circ (\rho \circ \eta_S), (\beta_{\rho\circ u_{\mathfrak{p}}}))].$$

We see that $[\mathrm{Res}_{\mathfrak{p}}(\beta_{\rho\circ\eta_S})], [\beta_{\rho\circ u_{\mathfrak{p}}\circ v_{\mathfrak{p}}}] \in \mathcal{L}_{\mathfrak{p}}(\rho \circ u_{\mathfrak{p}} \circ v_{\mathfrak{p}})$. Thus we obtain

$$\begin{aligned}
CS_{\overline{X}_k}(\rho) &= (\mathrm{inv} \circ j_3)([c \circ \rho]) \\
&= (\mathrm{inv}' \circ \mathrm{inv}'^{-1} \circ \mathrm{inv} \circ j_3)([c \circ \rho]) \\
&= \mathrm{inv}'([(c \circ (\rho \circ \eta_S), (\beta_{\rho\circ u_{\mathfrak{p}}\circ v_{\mathfrak{p}}}))]) \\
&= \sum_{\mathfrak{p}\in S} \mathrm{inv}_{\mathfrak{p}}([\mathrm{Res}_{\mathfrak{p}}(\beta_{\rho\circ\eta_S}) - \beta_{\rho\circ u_{\mathfrak{p}}\circ v_{\mathfrak{p}}}]) \\
&= CS_{\overline{X}_S}(\rho \circ \eta_S) - CS_{V_S}((\rho \circ u_{\mathfrak{p}})_{\mathfrak{p}\in S}).
\end{aligned}$$

Case that $S_1 \neq \emptyset$. Let $\beta_\rho \in C^2(\Pi_{S_1}, \mathbb{Z}/N\mathbb{Z})$ be a cochain such that $d\beta_\rho = c\circ\rho$. We have $d(\beta_\rho \circ \eta_S) = c \circ (\rho \circ \eta_S)$ and $d(\beta_\rho \circ u_\mathfrak{p}) = c \circ (\rho \circ u_\mathfrak{p})$ for $\mathfrak{p} \in S_2$. So we obtain

$$\begin{aligned} CS_{\overline{X}_{S_1}}(\rho) \boxplus CS_{V_{S_2}}((\rho \circ u_\mathfrak{p})_{\mathfrak{p}\in S_2}) &= [(\beta_\rho \circ \eta_S \circ \iota_\mathfrak{p})_{\mathfrak{p}\in S_1}] \boxplus [(\beta_\rho \circ u_\mathfrak{p} \circ v_\mathfrak{p})_{\mathfrak{p}\in S_2}] \\ &= [(\beta_\rho \circ u_\mathfrak{p} \circ v_\mathfrak{p})_{\mathfrak{p}\in S}] \\ &= [(\beta_\rho \circ \eta_S \circ \iota_\mathfrak{p})_{\mathfrak{p}\in S}] \\ &= CS_{\overline{X}_S}(\rho \circ \eta_S). \end{aligned}$$

□

Let $x_{S_i} \in \Gamma(\mathcal{F}_{S_i}, \mathcal{L}_{S_i})$ $(i = 1, 2))$ be any sections. We define the section $x_S \in \Gamma(\mathcal{F}_S, \mathcal{L}_\mathcal{S})$ by

$$x_S(\rho_{S_1}, \rho_{S_2}) := x_{S_1}(\rho_{S_1}) \boxplus x_{S_1}(\rho_{S_2}).$$

By the proof of Theorem 5.12.1, we have the following

Corollary 5.12.2. Notations being as above, we have the following equality in $\mathbb{Z}/N\mathbb{Z}$.

$$CS^{x_{S_1}}_{\overline{X}_{S_1}}(\rho) + CS^{x_{S_2}}_{V_{S_2}}((\rho \circ u_\mathfrak{p})_{\mathfrak{p}\in S_2}) = CS^{x_S}_{\overline{X}_S}(\rho \circ \eta_S).$$

We consider the situation that we obtain the space $\overline{X}_{S_1}$ by gluing $\overline{X}_S$ and $V^*_{S_2}$ along ∂V_{S_2}. We define the pairing $< \ , \ >: \mathcal{H}^{x_S}_S \times \mathcal{H}^{x_{S_2}}_{S_2{}^*} \to \mathcal{H}^{x_{S_1}}_{S_1}$ by

$$< \theta_S, \theta_{S_2{}^*} > (\rho_{S_1}) := \#G \sum_{\rho_{S_2} \in \mathcal{F}_{S_2}} \theta_S(\rho_{S_1}, \rho_{S_2}) \theta_{S_2{}^*}(\rho_{S_2}) \tag{5.12.3}$$

for $\theta_S \in \mathcal{H}^{x_S}_S, \theta_{S^*_2} \in \mathcal{H}^{x_{S_2}}_{S^*_2}$ and $\rho_{S_1} \in \mathcal{F}_{S_1}$. This induces the pairing $< \ , \ >: \mathcal{H}_S \times \mathcal{H}_{S_2{}^*} \to \mathcal{H}_{S_1}$ by (5.3.2). Now we prove the following gluing formula.

Theorem 5.12.4 *(Gluing formula)*. Notations being as above, We have the following equality

$$< Z_{\overline{X}_S}, Z_{V^*_{S_2}} > = Z_{\overline{X}_{S_1}}.$$

Proof. We show the equality

$$< Z^{x_S}_{\overline{X}_S}, Z^{x_{S_2}}_{V^*_{S_2}} > = Z^{x_{S_1}}_{\overline{X}_{S_1}}$$

for any sections $x_{S_i} \in \Gamma(\mathcal{F}_{S_i}, \mathcal{L}_{S_i})$ $(i=1,2)$. Noting (5.11.6), we have

$$
\begin{aligned}
&< Z^{x_S}_{\overline{X}_S}, Z^{x_{S_1}}_{V^*_{S_2}} > (\rho_{S_1}) \\
&= \#G \sum\nolimits_{\rho_{S_2} \in \mathcal{F}_{S_2}} \Big(\frac{1}{\#G} \sum_{\rho' \in \mathcal{F}_{\overline{X}_S}(\rho_{S_1}, \rho_{S_2})} {\zeta_N}^{CS^{x_S}_{\overline{X}_S}(\rho')} \Big) \Big(\frac{1}{\#G} \sum_{\tilde{\rho} \in \mathcal{F}_{V_{S_2}}(\rho_{S_2})} {\zeta_N}^{-CS^{x_{S_2}}_{V_{S_2}}(\tilde{\rho}')} \Big) \\
&= \sum_{\rho_{S_2} \in \mathcal{F}_{S_2}} \Big(\frac{1}{\#G} \sum_{(\rho', \tilde{\rho}) \in \mathcal{F}_{\overline{X}_S}(\rho_{S_1}, \rho_{S_2}) \times \mathcal{F}_{V_{S_2}}(\rho_{S_2})} {\zeta_N}^{CS^{x_S}_{\overline{X}_S}(\rho') - CS^{x_{S_2}}_{V_{S_2}}(\tilde{\rho})} \Big)
\end{aligned}
$$

for $\rho_{S_1} \in \mathcal{F}_{S_1}$. We define the map

$$\chi(\rho_{S_1}) : \mathcal{F}_{\overline{X}_{S_1}}(\rho_{S_1}) \to \bigsqcup_{\rho_{S_2} \in \mathcal{F}_{S_2}} \Big(\mathcal{F}_{\overline{X}_S}(\rho_{S_1}, \rho_{S_2}) \times \mathcal{F}_{V_{S_2}}(\rho_{S_2}) \Big)$$

by

$$\chi(\rho_{S_1})(\rho_1) = (\rho_1 \circ \eta_S, (\rho_1 \circ u_{\mathfrak{p}})_{\mathfrak{p} \in S_2})$$

for $\rho_1 \in \mathcal{F}_{\overline{X}_{S_1}}(\rho_{S_1})$. In order to obtain the required statement by Corollary 5.2.2, it suffices to show that $\chi(\rho_{S_1})$ is bijective. (Though this may be seen by noticing that Π_{S_1} is the push-out of the maps $\iota_{\mathfrak{p}}$ and $v_{\mathfrak{p}}$ (Π_{S_1} is the amalgamated product of Π_S and $\tilde{\Pi}_k$ along $\Pi_{\mathfrak{p}}$) for $S_2 = \{\mathfrak{p}\}$, we give here a straightforward proof.)

$\chi(\rho_{S_1})$ is injective: Suppose $\chi(\rho_{S_1})(\rho_1) = \chi(\rho_{S_1})(\rho'_1)$ for $\rho_1, \rho'_1 \in \mathcal{F}_{\overline{X}_{S_1}}(\rho_{S_1})$. Then $\rho_1 \circ \eta_S = \rho'_1 \circ \eta_S$. Since η_S is surjective, $\rho_1 = \rho'_1$.

$\chi(\rho_{S_1})$ is surjective: Let $(\rho, (\tilde{\rho}_{\mathfrak{p}})_{\mathfrak{p} \in S_2}) \in \mathcal{F}_{\overline{X}_S}(\rho_{S_1}, \rho_{S_2}) \times \mathcal{F}_{V_{S_2}}(\rho_{S_2})$. Then we have

$$\mathrm{res}_{S_1}(\rho) = \rho_{S_1}, \mathrm{res}_{S_2}(\rho) = \rho_{S_2}, \tilde{\mathrm{res}}_{S_2}((\tilde{\rho}_{\mathfrak{p}})_{\mathfrak{p} \in S_2}) = \rho_{S_2}.$$

Since $\tilde{\mathrm{res}}_{\mathfrak{p}}(\tilde{\rho}_{\mathfrak{p}})$ is unramified representation of $\Pi_{\mathfrak{p}}$ for $\mathfrak{p} \in S_2$, ρ is unramified over S_2. Therefore there is $\rho_1 \in \mathcal{F}_{\overline{X}_{S_1}}$ such that $\rho = \rho_1 \circ \eta_S$. Since we see that

$$\rho_1 \circ u_{\mathfrak{p}} \circ v_{\mathfrak{p}} = \rho_1 \circ \eta_S \circ \iota_{\mathfrak{p}} = \rho \circ \iota_{\mathfrak{p}} = \tilde{\rho}_{\mathfrak{p}} \circ v_{\mathfrak{p}}$$

for $\mathfrak{p} \in S_2$ and $v_{\mathfrak{p}}$ is surjective, we have $\rho_1 \circ u_{\mathfrak{p}} = \tilde{\rho}_{\mathfrak{p}}$ for $\mathfrak{p} \in S_2$. Hence $\chi(\rho_{S_1})(\rho_1) = (\rho, (\tilde{\rho}_{\mathfrak{p}})_{\mathfrak{p} \in S_2})$ and so $\chi(\rho_{S_1})$ is surjective. □

Remark 5.12.5 (1) In [CKKPY] and [LP], the authors used the decomposition formula (Theorem 5.12.1) in order to compute arithmetic Chern–Simons invariants $CS_{\overline{X}}(\rho)$ for various examples. In [BCGKPT] and [AC], computations of $CS_{\overline{X}}(\rho)$ have also been carried out by number theoretic methods.
(2) In [CKKPPY], arithmetic Dijkgraaf–Witten correlation functions for finite cyclic gauge groups were computed in terms of arithmetic linking numbers of primes. Their formula may be regarded as arithmetic finite analogue

of the path integral for linking numbers in abelian Chern–Simons gauge theory.
(3) In [Ki], Kim introduced arithmetic Chern–Simons functionals for the case of p-adic Lie gauge group. In [CK], an arithmetic analogue of topological BF theory was studied, and, in [CCKKPY], the authors showed an arithmetic path integral formula for the Kubota-Leopoldt p-adic L-function.
(4) A deep aspect of the 3-dimensional Chern–Simons TQFT with compact connected gauge group is a connection with 2-dimensional conformal field theory ([Ko]). For Dijkgraaf–Witten TQFT, Brylinski and McLaughlin ([BM2], [BM3]) studied the analogue for a finite gauge group of Segal–Witten reciprocity law in conformal field theory ([Seg], [BM1], [BM2]). We may find an analogous feature between central extensions of loop groups in conformal field theory and metaplectic coverings in number theory such as Segal–Witten reciprocity law and Hilbert reciprocity law ([Kub], [We], [BD]). It would be interesting to pursue this analogy in connection with (arithmetic) Chern–Simons TQFT.

References

[AC] E. Ahlqvist, M. Carlson, The cohomology of the ring of integers of a number field, arXiv:1803.08437.

[Ar] M. Artin, Grothendieck topologies, Harvard University, Cambridge, Mass, 133 pp, 1962.

[At1] M. Atiyah, Topological quantum field theories, *Inst. Hautes Etudes Sci. Publ. Math.* **68** (1988), 175–186.

[At2] M. Atiyah, The Geometry and Physics of Knots, Cambridge University Press, 1990.

[BCGKPT] F.M. Bleher, T. Chinburg, R. Greenberg, M. Kakde, G. Pappas, M. Taylor, Cup products in the étale cohomology of number fields, New York J. Math. **24** (2018), 514–542.

[BD] J.-L. Brylinski, P. Deligne, Central extensions of reductive groups by K_2, Publ. Math. de IÍHES, Tome **94**, (2001), 5-85.

[Bi] M. Bienenfeld, An etale cohomology duality theorem for number fields with a real embedding. Trans. Amer. Math. Soc. **303** (1987), no. 1, 71–96.

[BL] A. Beauville, Y. Laszlo, Conformal blocks and generalized theta functions, Commun. Math. Phys. **164** (1994), 385–419.

[BM1] J.-L. Brylinski, D.A. McLaughlin, The geometry of degree-4 characteristic classes and of line bundles on loop groups I, Duke Math.J., **75**, No.3, (1994), 603-683.

[BM2] J.-L. Brylinski, D.A. McLaughlin, The geometry of degree-4 characteristic classes and of line bundles on loop groups II, Duke Math.J., **83**, No.1, (1996), 105-139.

[BM3] J.-L. Brylinski, D.A. McLaughlin, Non-commutative reciprocity laws associated to finite groupss, Contemp. Math., **202**, Amer. Math. Soc., Providence, RI, (1997), 421-438.

[CK] M.Carlson, M.Kim, A note on abelian arithmetic BF-theory, Bull. London Math. Soc., **54**, Issue 4 (2022), 1299-1307.

[CKKPPY] H.-J. Chung, D. Kim, M. Kim, G. Pappas, J. Park, H. Yoo, Abelian arithmetic Chern–Simons theory and arithmetic linking numbers, Int. Math. Res. Not. **18** (2019), 5674–5702.

[CKKPY] H.-J. Chung, D. Kim, M. Kim, J. Park and H. Yoo, Arithmetic Chern–Simons theory II, In: p-adic Hodge theory, B. Bhatt, M. Olsson (eds) (2017 Simons Symposium on p-adic Hodge theory), Springer International Publishing, (2020), 81–128.

[DK] Y.Deng, R.Kurimaru, Arithmetic Dijkgraaf–Witten invariants for real quadratic fields, quadratic residue graphs and density formulas, preprint 2022.

[DW] R. Dijkgraaf, E. Witten, Topological gauge theories and group cohomology, Commun. Math. Phys. **129** (1990), 393–429.

[EM] S. Eilenberg, S. MacLane, Cohomology theory in abstract groups I, Annals of Math. **48** (1947), 51–78.

[Fo] R.H. Fox, Covering spaces with singularities, In: A symposium in honor of S. Lefschetz. Princeton University Press. (1957), 243–257.

[FQ] D. Freed, F. Quinn, Chern–Simons theory with finite gauge group. Comm. Math. Phys. **156** (1993), no.3, 435–472.

[Gm] K. Gomi, Extended topological quantum field theory: a toy model. (Japanese) In: Report of the 4th Kinosaki Seminar, 2007, 18 pages.

[Hi] H. Hirano, On mod 2 Dijkgraaf–Witten invariants for certain real quadratic number fileds, arXiv:1911.12964. to appear in Osaka J. Math.

[HKM] H. Hirano, J.Kim, M.Morishita, On arithmetic Dijkgraaf–Witten theory, to appear in Commun in Number Theory and Physics, **17**, no.1, (2023).

[Ki] M. Kim, Arithmetic Chern–Simons Theory I, In: Galois covers, Grothendieck-Teichmüller Theory and Dessins d'Enfants – Interactions between Geometry, Topology, Number Theory and Algebra. Springer Proceedings in Mathematics & Statistics, **330** (2020), 155-180.

[Ko] T. Kohno, Conformal field theory and topology. Translations of Mathematical Monographs, **210**. Iwanami Series in Modern Mathematics. Amer. Math. Soc. Providence, RI, 2002.

[Kub] T. Kubota, Topological coverings of SL(2) over a local field, J. Math. Soc. Japan, **19**, (1967), 114-121.

[LP] J. Lee, J. Park, Arithmetic Chern–Simons theory with real places, arXiv:1905.13610.

[My] J. P. May, Simplicial objects in algebraic topology. Reprint of 1967 original. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, II, 1992.

[Mz] B. Mazur, Notes on etale cohomology of number fields. Ann. Sci. Ecole Norm. Sup. (4) **6** (1973), 521–552.

[Mh] T. Mihara, Cohomological approach to class field theory in arithmetic topology, Canad. J. Math. **71** (2019), no. 4, 891–935.

[Mi1] J.S. Milne, Étale cohomology, volume **33** of Princeton Mathematical Series. Princeton University Press, 1980.

[Mi2] J. S. Milne, Arithmetic Duality Theorems, Perspectives in Math. Vol. **1**, Academic Press, 1986.

[Mo1] M. Morishita, Integral representations of unramified Galois groups and matrix divisors over number fields, Osaka J. Math. **32** (1995), no. 3, 565–576.

[Mo2] M. Morishita, Knots and Primes – An introduction to Arithmetic Topology. Universitext. Springer, London, 2012.

[Mo3] M. Morishita, A theory of genera for cyclic coverings of links. Proc. Japan Acad. Ser. A Math. Sci. **77**(2001), 115–118.

[MOO] H. Murakami, T. Ohtsuki, and M. Okada, Invariants of three-manifolds derived from linking matrices of framed links. Osaka J. Math. **29**(1992), 545–572.

[NSW] J. Neukirch, A. Schmidt, K. Wingberg, Cohomology of number fields. Second edition. Grundlehren der Mathematischen Wissenschaften, **323**, Springer-Verlag, Berlin, 2008.

[NU] H. Niibo, J. Ueki, Idèlic class field theory for 3-manifolds and very admissible links, Transactions of the AMS. **371**, No.12, (2019), 8467–8488.

[O] T. Ono, An introduction to algebraic number theory, The University Series in Mathematics. Plenum Press, New York, 1990.

[S1] J.-P. Serre, Cohomologie Galoisienne. Lecture Note in Math. **5**, Springer, 1965.

[S2] J.-P. Serre, Corps Locaux. Hermann, 1968.

[Seg] G. B. Segal, The definition of conformal field theory, In: Differential geometrical methods in theoretical physics (Como, 1987). Vol. **250**. NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. Dordrecht: Kluwer Acad. Publ., 1988, 165–171.

[SGA I] A. Grothendieck: Revêtements étales et groupe fondamental (Séminaire de géométrie algébrique du Bois Marie 1960–61), Documents Mathématiques (Paris) **3**, Société Mathématique de France, Paris, 2003.

[Sw] R.G. Swan, Cup products in sheaf cohomology, pure injectives, and a substitute for projective resolutions, J. Pure Appl. Algebra. **144**(1999), 169–211.

[Ta] Günter Tamme. Introduction to étale cohomology. Universitext. Springer-Verlag, Berlin, 1994. Translated from the German by Manfred Kolster.

[Tu] R. Tuler; On the linking number of a 2-bridge link. Bull. London Math. Soc. **13**(1981), 540–544.

[Ve] E. Verlinde, Fusion rules and modular transformations in 2-D conformal field theory. Nucl. phys. **B300[FS22]** (1988), 360-376.

[Wa] M. Wakui, On Dijkgraaf–Witten invariant for 3-manifolds, Osaka J. Math. **29** (1992), 675–696.

[We] A. Weil, Généralisation des fonctions abéliennes, J. Math. Pure Appl. **17** (1938), no.9, 47–87.

[Wi] E. Witten, Quantum field theory and the Jones polynomial, Commun. Math. Phys. **121** (1989), 351-399.

[Ye] D. Yetter, Topological quantum field theories associated to finite groups and crossed G-sets, J. Knot Theory and its Ramifications. **1** (1992), 1–20.

[Zi] T. Zink, Etale cohomology and duality in number fields. Appendix 2 In: Haberland, Galois cohomology of algebraic number fields, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.