An embedding of a smooth quandle into a Lie group

米村, 拳太郎

出版情報：Kyushu University, 2022, 博士（数理学）, 課程博士
バージョン：
権利関係：

KYUSHU UNIVERSITY

DISSERTATION

An embedding of a smooth quandle into a Lie group

Author:
Kentaro YONEMURA

Supervisor:
Professor Hiroyuki OCHIAI

Graduate School of Mathematics

February 2023

"Even if you think it's going to be fatal, it's not."

Yasuharu Ōyama

"The work of mathematicians is similar to that of Demon Slayer. When you prove a theorem, it is not enough to pierce the heart or cut off the head of demons, but one must go so far as to annihilate it."

Hiroyuki Ochiai

KYUSHU UNIVERSITY

Abstract

Faculty of Mathematics
Graduate School of Mathematics

Doctor of Philosophy

An embedding of a smooth quandle into a Lie group

by Kentaro YONEMURA

We propose the conjecture that both the quandle and the manifold structure of smooth quandles may be embedded in Lie groups and show that it is correct in the case of spherical quandles in this dissertation.

Acknowledgements

I would like to express my sincere gratitude to Professor Hiroyuki Ochiai, my supervisor, for his helpful comments, suggestions, and encouragement. Without his tolerance, I could not have done this thesis.

I am grateful to Professor Takefumi Nosaka for his suggestions and advises. In particular, his careful answers to the author's questions about Nosaka, 2017b were of great help.

I wish to thank Professor Takaaki Nomura for advices on not only mathematics but also how to enjoy lives. Without his class and his book Nomura, 2018, I could not have a chance to learn Lie groups.

I thank Michiko Yonemura, my younger sister, for drawing FIGURE 2.1.

Contents

Dedicated to my family

Chapter 1

Introduction

1.1 Quandles

A *quandle* is an algebraic system closely related to knot theory. Here, we would like to substitute a brief description of its history for its explanation.

In 1982, Joyce, 1982 and Matveev, 1982 defined an algebraic system called quandles and constructed the complete knot invariant called the knot quandle or the fundamental quandle of knots. This was amazing since even the isomorphism classes of knot groups, one of the important knot invariants, are not able to classify knots. However, for some time after that, not much progress was made in the study of quandles since the algebraic structure of knot quandles was difficult.

In 1999, Carter et al., 1999 constructed the quandle cocycle invariants of (classical) knots and surface knots. The invariants are remarkable in that they are much easier than knot quandles to calculate and to investigate differences. The study of this invariant and of the quandle homology that emerged in the construction of this invariant has greatly advanced the study of the algebraic structure of quandles. As a result, the relationship between existing knot invariants and quandles is also known. For example, Inoue and Kabaya, 2014 proved that the complex volume, the Chern-Simons invariants with respect to its principal $PSL(2, \mathbb{C})$ bundle, of a hyperbolic knots is presented as the quandle cocycle invariants.

Recently, Ishikawa, n.d. defined a quandle class called *smooth quandle* and constructed its fundamental theory . A smooth quandle is a differentiable manifold with smooth operation which satisfies the quandle condition. Quandle classes are often compared to group classes and smooth quandles are analogous to Lie groups.

TABLE 1.1: Group classes vs Quandle classes.

group class	quandle class
finite groups	finite quandles
topological groups	topological quandles
Lie groups	smooth quandles

1.2 Embedding quandles into groups

Embedding quandles into groups is an important problem and has a long history since it is known that the quandles embedded into groups are important to apply quandles to knot theory and to consider the algebraic structure of quandles.

Joyce, 1982, who gave a name to quandles, gave an embedding of free quandles and suggested that the theory of quandles may be regarded as the theory of conjugation of groups. Joyce presented the conjugacy quandles as the first example in his paper. It appears that Joyce considered that conjugacy quandles are fundamental examples of quandles. See Joyce, 1982, Section 4 for more details.

Eisermann, 2014 defined a *quandle covering* [1] and constructed its fundamental theory. In his paper, he pointed out that the covering theory of quandles embedded in groups is essentially the theory of central group extensions.

Bardakov, Dey, and Singh, 2017 dealt with the following problem:

Question 1.2.1 (Bardakov, Dey, and Singh, 2017, Question 3.1)**.** *For which quandles X does there exist a group G such that X embeds in the conjugation quandle* $\mathrm{Conj}\, G$*?*

They proved that Alexander quandles associated with fixed-point free involutions are embeddable in the paper. After that, they and some people found quandles satisfying Question 1.2.1.

TABLE 1.2: Quandles satisfying Question 1.2.1.

quandles	bibliography
free quandles	Joyce, 1982, Theorem 4.1
free n-quandles	Joyce, 1982, Corollary 10.3
some Alexander quandles[2]	Bardakov, Dey, and Singh, 2017
commutative quandles	Bardakov and Nasybullov, 2020, §5
latin quandles	Bardakov and Nasybullov, 2020, §5
simple quandles	Bardakov and Nasybullov, 2020, §5
core quandles	Bergman, 2021, (6.5)
some gen. Al. quandles[3]	Dhanwani, Raundal, and Singh, 2021, Prop. 3.12
twisted conjugate quandles	Akita, 2022

Recently, Akita, 2022 proved that twisted conjugate quandles are able to be embedded into a group. Question 1.2.1 has been considered for a long time and still attracts attention from researchers.

1.3 Embedding smooth quandles into Lie groups

We suggest the following conjecture:

Conjecture 1.3.1. *For any topologically connected and algebraically connected smooth quandle X, there is a Lie group G and a smooth embedding $\iota : X \to G$ that is a quandle homomorphism if we consider G to be a conjugacy quandle.*

Conjecture 1.3.1 means that there is an embedding with respect to both of the quandle structures and manifold structures into Lie groups for any smooth quandles. In this thesis, we prove the conjecture in the case of spherical quandles. See Theorem 4.0.1. We expect that the strategy of the proof can be extend to the case of the quandles defined over compact Riemann symmetric spaces[4].

Conjecture 1.3.2. *The algebraically connected and the topologically conneted smooth quandles defined in Subsection 2.3.3 satisfy Conjecture 1.3.1.*

[1] In this thesis, we do not deal with quandle covering and “covering" means a topological covering in this thesis.

[2] Alexander quandles associated with fixed-point free involutions.

[3] Generalized Alexander quandles associated with fixed-point free automorphisms.

[4] See Subsection 2.3.3 for its definition.

Conjecture 1.3.1 is a kind of Question 1.2.1, the problem dealt with in Section 1.2. The problem setting of Conjecture 1.3.1 is important since we can use not only algebraic conditions but also geometrical conditions. For example, when we consider the conjecture, we often use the universal covering $p : \tilde{G} \to G$ of a connected Lie group G. The map p has three structures: the (topological) covering structure, the quandle covering structure defined by Eisermann, 2014 and the central extension. As a side note, the central extensions of groups are important when we deal with quandles. Since there are many aspects of the quandle structure that are not well known, it is important to be able to use the relatively well-known theory of Lie groups and Lie algebras as support. In this thesis, we also use the geometrical structure to prove the main theorem, Theorem 4.0.1.

Even if Conjecture 1.3.1 has a counterexample, the conjecture is important in developing quandle theory. There are two reasons. First, the conjecture makes some smooth quandles structure clear. In quandle theory, few quandles have well-known algebraic structures. This fact has hindered the development of quandle theory. In contrast, there are many examples in Lie group theory. For instance, the structures of classical groups are well known. Hence, it allows gradual development. For example, if the fact with respect to $SL(2, \mathbb{R})$, which is isomorphic to $Spin(2, 1)$, is known, we may extend the fact in the case $Spin(n, 1)$[5]. Second, considering the conjecture is useful for its application to knot theory since Question 1.2.1 is closely related to the application to knot invariants. A similar perspective may be valid for quandle theory. We hope that Conjecture 1.3.1 will help develop the study of smooth quandles.

Some researchers are interested in only finite quandles, since finite quandles are more useful than the other quandles involving smooth quandles for applying to knot invariants. However finite groups are often viewed as discrete subgroups of Lie groups in group theory and the view point is helpful.

1.4 Our motivation

In the main theorem, Theorem 4.0.1, we prove that Conjecture 1.3.1 in the case of spherical quandles. Our motivation to prove the main theorem is to construct an example of embedding quandles which are not faithful into conjugacy quandles.

For any quandles, there is a natural quandle homomorphism from a quandle to its inner automorphism group. A quandle is *faithful* if the natural quandle homomorphism is injective. A quandle $P^n_{\mathbb{R}}$, which is a quandle defined in Chapter 4, is an example of faithful quandles.

Quandles defined over Riemannian symmetric spaces, we introduce in Subsection 2.3.3, are Joyce's model cases of quandles. In this case, the faithfulness of the quandles is related to *Cartan embedding*, a famous mathematical object in symmetric space theory. The inner automorphism group of the quandles has a Lie group structure and the natural quandle homomorphisms from the quandles to their inner automorphism group are commonly called Cartan embedding. See also Nosaka, 2017b, Example B.3.

In most cases, we wonder if a given quandle is faithful when we approach Question 1.2.1. In the case that a given quandle is not faithful, we need to find a group to embed the quandle. The quandle treated in the main theorem is not faithful. In general, few methods to find an appropriate group is known and this is why Question 1.2.1 is difficult.

[5]The Lie group $Spin(n, 1)$ is the double covering of the unit component of $SO(n, 1)$.

Spherical quandles, we treat in the main theorem, are not a faithful. However, we consider the universal covering group of the inner automorphism group of spherical quandles and prove that the covering group is the group to embed the quandle. The manifold structure of sphercial quandles is a key of the proof. We hope that many mathematicians will be interested in Conjecture 1.3.1 and solve Question 1.2.1.

1.5 Organization of this thesis

This thesis is organized as follows. In Chapter 2, the basic notation and facts on general quandles and smooth quandles are presented. In Chapter 3, we construct a lifting of a smooth group action using a covering map. The construction is introduced in Montaldi and Ortega, 2008; Montaldi and Ortega, 2009. However, we do not describe the original construction, but a reformulation of the method sketched in Montaldi and Ortega, 2008, Remark 2.2 to apply in the proof of the main theorem. In Chapter 4, we present the main theorem on Conjecture 1.3.1 in the case of spherical quandles and prove it. In Chapter 5, some application of the main theorem concisely.

Chapter 2

Quandles

In this chapter, we introduce what we need to read this thesis on quandles. See Nosaka, 2017b, Kamada, 2017 and Ishikawa, n.d. for more details.

2.1 Quandle basics

In this section, we define quandles and the algebraic connectivity of quandles.

First, we introduce the definition of quandles.

Definition 2.1.1 (Joyce, 1982; Matveev, 1982)**.** *A quandle is a set X with a binary operation $\triangleright : X \times X \to X$ satisfying the three conditions:*
(Q1) $x \triangleright x = x$ for any $x \in X$.
(Q2) The map $S_y : X \to X$ defined by $x \mapsto x \triangleright y$ is bijective for any $y \in X$.
(Q3) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ for any $x, y, z \in X$.

We denote $S_y^{-1}(x)$ as $x \triangleright^{-1} y$ for $x, y \in X$. The conditions Q1, Q2,and Q3 in Definition 2.1.1 are consistent with Reidemeister moves, operations for knot diagrams in knot theory, I, II and III respectively. See Figure 2.1.

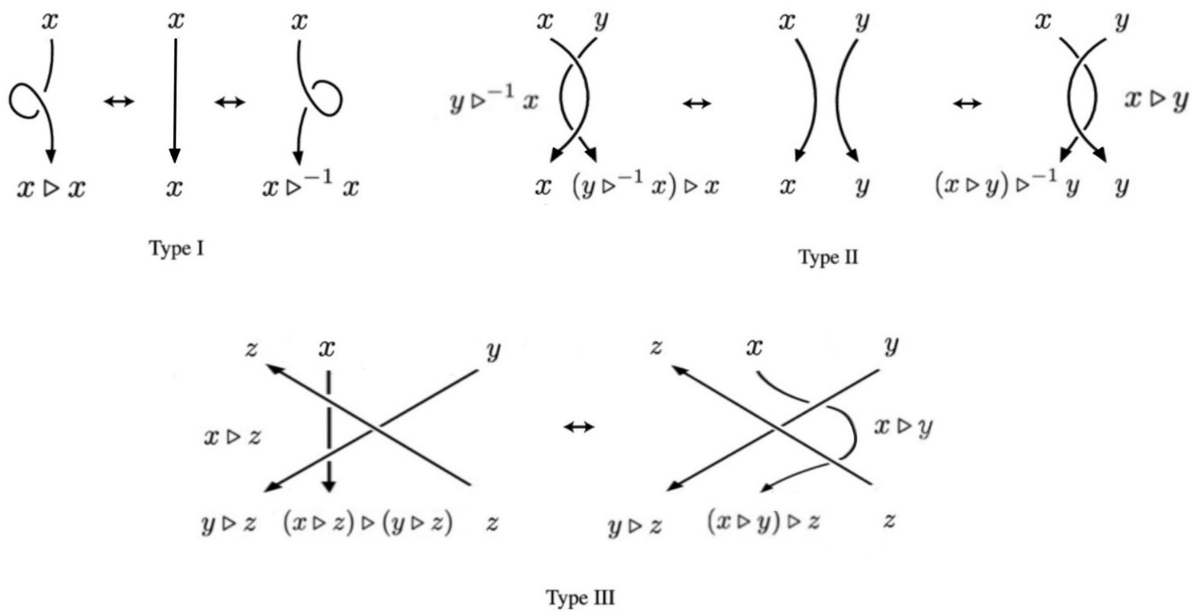

FIGURE 2.1: Geometric interpretation of the axioms in Definition 2.1.1.

Second, we define the algebraic connectivity of quandles. Suppose that X and Y are quandles. A map $f : X \to Y$ is called a homomorphism if $f(x \triangleright y) = f(x) \triangleright f(y)$

for any $x, y \in X$. Isomorphisms and automorphisms are defined similary. We denote the automorphism group of a quandle X as $\operatorname{Aut} X$. By the axiom Q3, the bijection S_y in Q2 is an automorphism. Then we call the subgroup of $\operatorname{Aut} X$ generated by $\{S_y\}_{y\in X}$ the inner automorphism group and denote it by $\operatorname{Inn} X$. The inner automorphism group $\operatorname{Inn} X$ acts on X on the right naturally. We say that X is algebraically connected if the action of $\operatorname{Inn} X$ on X is transitive.

2.2 Smooth quandles

In this section, we study the definition of smooth quandles and its inner automorphism groups. The facts dealt with in this section are those stated in Ishikawa, n.d.

Definition 2.2.1 (Ishikawa, n.d.). *A smooth quandle is a smooth manifold X with a smooth binary operation $\triangleright : X \times X \to X$ satisfying the two conditions:*

1. *The algebraic system $(X, \triangleright)$ is a quandle.*
2. *The inner automorphism $S_y : X \to X$ is a diffeomorphism for any $y \in X$.*

To state the theorem on the inner automorphism group of smooth quandles, let us prepare a subgroup of the inner automorphism of quanles. Let X be a quandle. The associated group $\operatorname{As} X$ of the quandle X generated by $\{e_x\}_{x\in X}$ with relations $e_{x\triangleright y} = e_y e_x e_y^{-1}$. There is a natural surjective group homomorphism $p : \operatorname{As} X \to \operatorname{Inn} X$. We consider the group homomorphism $\varepsilon : \operatorname{As} X \to \mathbb{Z}$ which maps every e_x to $1 \in \mathbb{Z}$, and let $\operatorname{Inn}^0 X$ be the image $p(\operatorname{Ker} \varepsilon)$.

Theorem 2.2.2 (Ishikawa, n.d.). *Suppose X is a topologically connected and algebraically connected smooth quandle. Then the inner automorpism group $\operatorname{Inn} X$ is a Lie transformation group. Furthermore, the identity component of $\operatorname{Inn} X$ is equal to $\operatorname{Inn}^0 X$ and acts on X transitively.*

By the Theorem 2.2.2, topologically connected and algebraically connected smooth quandles are homogeneous spaces.

2.3 Examples

In this section, we show some examples that we use in this thesis.

2.3.1 Conjugacy quandles

Let G be a group, and $\triangleright : G \times G \to G$ be a binary operation defined by $g \triangleright h = h^{-1}gh$. Then the algebraic system $(X, \triangleright)$ is a quandle called conjugacy quandle. We denote the quandle as $\operatorname{Conj} G$. Especially, if G is a Lie group, the quandle $\operatorname{Conj} G$ is a smooth quandle.

2.3.2 Spherical quandles

We introduce the spherical quandle defined by Azcan and Fenn, 1994. Let $\langle -, - \rangle : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ be the Euclidean inner product, and let S^n be the n sphere.

$$\left\{ \boldsymbol{x} = (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1 \right\}.$$

We define the binary operation $\triangleright : S^n \times S^n \to S^n$ as $\boldsymbol{x} \triangleright \boldsymbol{y} = 2\langle \boldsymbol{x}, \boldsymbol{y}\rangle \boldsymbol{y} - \boldsymbol{x}$ for all $\boldsymbol{x}, \boldsymbol{y} \in S^n$. Then $(S^n, \triangleright)$ is a smooth quandle and called the spherical quandle $S^n_{\mathbb{R}}$.

The inner automorphism $S_{\boldsymbol{y}}$ can be interpreted as a linear transformation which identically acts on $\boldsymbol{y}$ and $-\mathrm{Id}$ on the the subspace orthogonal of $\boldsymbol{y}$. Applying the Cartan–Dieudonné theorem [1], we get the following fact.

Proposition 2.3.1 (Nosaka, 2017a)**.** *Suppose n is strictly greater than two. If n is odd, then $\mathrm{Inn}\, S^n_{\mathbb{R}}$ is isomorphic to the orthogonal group $O(n+1)$. If n is even, then $\mathrm{Inn}\, S^n_{\mathbb{R}}$ is isomorphic to the special orthogonal group $SO(n+1)$.*

The action of inner automorphism group on the spherical quandle $S^n_{\mathbb{R}}$ is coincident with the natural action of the orthogonal group $O(n+1)$ or the special orthogonal group $SO(n+1)$ on S^n and is transitive. Hence we get the following fact.

Proposition 2.3.2. *The spherical quandle is algebraically connected.*

Remark 2.3.3. *Azcan and Fenn, 1994 defined quandles for a broader class that includes spherical quandles. We introduce their original definition here. Let K be a fieldof characteristic different from two and $\langle -, - \rangle : K^n \times K^n \to K$ be a symmetric bilinear form. K . Consider a set of the form*

$$S^n_K := \{x \in K^{n+1} | \langle x, x\rangle = 1\}.$$

We define the operation $x \triangleright y$ to be $2\langle x, y\rangle y - x \in S^n_K$. Then the pair $(S^n_K, \triangleright)$ is a quandle.

Remark 2.3.4. *Clark and Saito, 2018 defined a spherical quandle as a quandle which is different from Azcan and Fenn, 1994. However, Theorem 4.0.1, our main theorem in this thesis, provides compatibility between the two definitions. See Appendix A for more details.*

2.3.3 Quandles defined over Riemann symmetric spaces

Joyce pointed out that Loos constructed quandles over symmetric spaces in the paper Joyce, 1982. We introduce the quandle constructed in Loos, 1969.

Let X be a symmetric space, i.e., a C^∞-manifold equipped with a Riemannian metric such that each point $y \in X$ admits an isometry $s_y : X \to X$ that reverses every geodesic line $\gamma : (\mathbb{R}, 0) \to (X, y)$, meaning that $s_y \circ \gamma(t) = \gamma(-t)$. Then X has a quandle structure defined by $x \triangleright y := s_y(x)$ for all $x, y \in X$.

The spherical quandles are a kind of quandles defined by Loos: for any $\boldsymbol{y} \in S^n$, point symmetry s of S^n is coincident with the inner automorphism $S_{\boldsymbol{y}}$ of the sphercal quandle $S^n_{\mathbb{R}}$. See also Helgason, 2001. Hence the quandles defined by Loos are a kind of extension of the spherical quandles.

[1]Every orthogonal transformation can be expressed as a composition of reflections. See Cartan, 2012, Chapter I Section 10.

Chapter 3

Action of Lie groups on smooth manifolds

3.1 Preliminaries

In this thesis, the symbols in Tu, 2011 are used without refusal.

A C^∞ vector field on a smooth manifold M is a section of a tangent bundle TM of M, that is, the C^∞ map $X : M \to TM$ that makes the following diagram commutative:

$$\begin{array}{ccc} M & \xrightarrow{X} & TM \\ {\scriptstyle \mathrm{id}_M}\big\downarrow & \swarrow{\scriptstyle \text{projection}} & \\ M & & \end{array} .$$

In this thesis, we denote the Lie algebra of the C^∞ vector field on M as $\mathfrak{X}(M)$.

Since a covering map is a local diffeomorphism, we get the following fact.

Proposition 3.1.1. *Let $\tilde{M}$ and M be smooth manifolds without boundary, $p : \tilde{M} \to M$ be a smooth covering map and M be a smooth vector field on M. Then there exists a uniquely defined smooth vector field $\tilde{X}$ on $\tilde{M}$ such that*

$$dp(x)\tilde{X}(x) = X \circ p(x).$$

3.2 Smooth actions of Lie groups and Lie-Palais theorem

In this section, we discuss smooth actions of Lie groups and Lie-Palais theorem. See Wang, 2013; Gorbatsevich, Onishchik, and Vinberg, 1997 for more details.

Let G be a finite dimensional Lie group and M be a smooth manifold. Suppose G acts on M from right smoothly. The action induces a group anti-homomorphism $\tau : G \to \operatorname{Diff} M$, where $\operatorname{Diff} M$ is the group of diffeomorphisms on M. Then τ induces an infinitesimal action, that is, a Lie algebra morphism $d\tau : \mathfrak{g} \to \mathfrak{X}(M)$: for each $X \in \mathfrak{g}$, we define a vector field $X_M \in \mathfrak{X}(M)$ as

$$(X_M(x))(\xi) = \left.\frac{d}{dt}\right|_{t=0} \xi(x \cdot \exp tX),$$

where $x \in M$ and ξ is a smooth function defined in the neighborhood of point x.

In certain cases, it can constitute the reverse of the previous discussion, that is, some Lie algebra morphism can be integrated to Lie group action. One such case is Lie-Palais theorem.

Theorem 3.2.1 (Lie-Palais theorem; Palais, 1957)**.** *Let G be a connected and simply connected Lie group, $\phi : \mathfrak{g} \to \mathfrak{X}(M)$ a Lie algebra homomorphism such that each $X_M = \phi(X)$ is complete. Then there exists a unique smooth action $\tau : G \to \mathrm{Diff}(M)$ such that $d\tau = \phi$.*

3.3 Lifting Lie group actions

Suppose a connected Lie group G acts on connected manifold M smoothly, and suppose N is a covering of M. Then we can construct a smooth action of the universal covering group of G on the given cover N. We study the concrete construction of the action in this section. The construction is introduced in Montaldi and Ortega, 2008; Montaldi and Ortega, 2009. However, we do not describe the original construction, but rather a reformulation of the method sketched in Montaldi and Ortega, 2008, Remark 2.2.

Let $p : \tilde{G} \to G$ and $\pi_N : N \to M$ be covering maps of a connected Lie group G and a connected manifold M respectively. We remark that p induces a Lie algebra isomorphism $p_* : \tilde{\mathfrak{g}} \to \mathfrak{g}$, where $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$ are the Lie algebras of $\tilde{G}$ and G respectively.

Suppose G acts on M on the right. Then the action induces a unique group anti-homomorphism $\tau : G \to \mathrm{Diff}\, M$. The group anti-homomorphism also induces a Lie algebra homomorphism $d\tau : \mathfrak{g} \to \mathfrak{X}(M)$. Using Proposition 3.1.1, for any $X \in \mathfrak{g}$, an element $\tilde{X} \in \tilde{\mathfrak{g}}$ and a vector field $\tilde{X}_N \in \mathfrak{X}(N)$ satisfying

$$\begin{cases} p_*(\tilde{X}) = X \\ d\pi_N \tilde{X}_N = X_M \circ \pi_N \end{cases}$$

exist uniquely. We get a Lie algebra homomorphism

$$\tilde{\phi} : \tilde{\mathfrak{g}} \to \mathfrak{X}(N), \quad \tilde{X} \mapsto \tilde{X}_N.$$

By Lie-Palais theorem (see Theorem 3.2.1), $\tilde{\phi}$ induces a group anti-homomorphism $\tilde{\tau} : \tilde{G} \to \mathrm{Diff}(\tilde{M})$ satisfying $d\tilde{\tau} = \tilde{\phi}$. Therefore we are able to construct an action of $\tilde{G}$ on the cover N compatible with an action of G on M.

3.4 The relationship between the lifted Lie group actions and coverings

Suppose a connected Lie group G acts on a connected manifold M smoothly, and suppose N is a covering of M. As we saw in Section 3.3, we can construct a smooth action of the universal covering group $\tilde{G}$ of G on the given cover N. In this section, we see the relationship between the induced action and the covering.

Proposition 3.4.1 (Montaldi and Ortega, 2008; Montaldi and Ortega, 2009)**.** *Let $p : \tilde{G} \to G$ be a universal covering map of connected Lie group G. Then the next diagram is commutative:*

$$\begin{array}{ccc} N \times \tilde{G} & \xrightarrow{\text{action}} & N \\ {\scriptstyle \pi_N \times p}\downarrow & & \downarrow{\scriptstyle \pi_N} \\ M \times G & \xrightarrow{\text{action}} & M \end{array}.$$

Suppose $\pi : \tilde{M} \to M$ is a universal covering of M. The covering map π induces a covering morphism q_N that makes the following diagram commutative:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{q_N} & N \\ {\scriptstyle \pi}\downarrow & \swarrow {\scriptstyle \pi_N} & \\ M & & \end{array}.$$

Proposition 3.4.2. *The induced covering morphism q_N is $\tilde{G}$-equivariant, that is, for any $\tilde{x} \in \tilde{X}$ and $\tilde{g} \in \tilde{G}$, the covering morphism q_N satisfies $q_N(\tilde{x} \cdot \tilde{g}) = q_N(\tilde{x}) \cdot \tilde{g}$.*

Proof. Suppose $\tau : G \to \mathrm{Diff}(M)$ is a group anti-homomorphism corresponding to a group right action of G on M and suppose $\tilde{\tau} : \tilde{G} \to \mathrm{Diff}(\tilde{M})$ and $\tilde{\tau}_N : \tilde{G} \to \mathrm{Diff}(N)$ are group anti-homomorphisms induced by group right actions of $\tilde{G}$ on $\tilde{M}$ and N respectively. For any $\tilde{x} \in \tilde{M}$ and $\tilde{g} \in \tilde{G}$, by Proposition 3.4.1, we get

$$\pi_N \circ q_N(\tilde{x} \cdot \tilde{g}) = \pi(\tilde{x}) \cdot p(\tilde{g}) = \pi_N(q_N(\tilde{x}) \cdot \tilde{g}).$$

Therefore both $q_N \circ \tilde{\tau}(\tilde{g})$ and $\tilde{\tau}_N(\tilde{g}) \circ q_N$ are covering morphisms induced from a universal covering $\tau\left(p(\tilde{g})\right) \circ \pi : \tilde{M} \to M$ to a covering $\pi_N : N \to M$. Hence we get $q_N \circ \tilde{\tau}(\tilde{g}) = \tilde{\tau}_N(\tilde{g}) \circ q_N$ since the induced covering morphism is unique. □

Chapter 4

Embedding spherical quandles in Lie groups

In this chapter, we prove the following theorem.

Theorem 4.0.1 (Main theorem)**.** *For any positive integer n, there is a Lie group G_n and a smooth embedding $\iota_n : S^n \to G_n$ which is a quandle homomorphism if we regard S^n as a spherical quandle and G_n is a conjugacy quandle. Especially,*

$$G_n = \begin{cases} O(2) & (n = 1) \\ Spin(n+1) & (n \text{ is even}) \\ Pin(n+1) & (n \text{ is odd and } n \geq 3) \end{cases}.$$

One can easily see that spherical quandles are embedded in the pin groups in the case $n \geq 2$ since the spin group is an identity component of the pin group. Theorem 4.0.1 is already mentioned by M. Eisermann without proof. See Eisermann, 2014, Remark 3.12.

4.1 Preliminaries

Suppose $M(n, \mathbb{R})$ is the set of all n-dimensional real square matrices. We consider two Lie groups: the orthogonal group $O(n) = \{g \in M(n, \mathbb{R}) \ : \ {}^t g g = I\}$, where I is the identity matrix, and the special orthogonal group $SO(n) = \{g \in O(n) \ : \ \det g = 1\}$. Let h_n be a diagonal matrix

$$h_n = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix} \in O(n+1).$$

If n is even, the diagonal matrix h_n is also an element of $SO(n+1)$. The special orthogonal group $SO(n)$ is the unit component of orthogonal group $O(n)$. The orthogonal group $O(n)$ has two connected components and has a presentation

$$O(n) = \begin{cases} SO(n) \sqcup h_{n-1} SO(n) & (n \text{ is even}) \\ SO(n) \sqcup (-I) SO(n) & (n \text{ is odd}) \end{cases}.$$

Since $SO(n)$ is a connected Lie group, $SO(n)$ has a unique universal covering group $Spin(n)$ and a universal covering map $p : Spin(n) \to SO(n)$ [1] . The map p is a double covering and is also induced by the adjoint representation of $Spin(n+1)$.

[1]See Cartan, 2012; Chevalley, 1947.

In this thesis, we use $Pin(n)$ as a double covering group of $O(n)$. The Lie group $Spin(n)$ is a unit component of $Pin(n)$. We denote the covering map as $p : Pin(n) \to O(n)$. The covering map $p : Pin(n) \to O(n)$ induces an exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to Pin(n) \xrightarrow{p} O(n) \to 1.$$

See Atiyah, Bott, and Shapiro, 1964, §3 for more details on $Pin(n)$. We denote one of the two fibers of h_n with respect to p as $\tilde{h}_n$.

Let $\sim_n$ be an equivalence relation on the sphere defined by

$$\boldsymbol{x} \sim_n \boldsymbol{y} \Longleftrightarrow \boldsymbol{y} = \pm \boldsymbol{x} \quad (\boldsymbol{x}, \boldsymbol{y} \in S^n).$$

Then $\mathbb{R}P^n := S^n / \sim_n$ is a real projective space and the natural projection $\pi : S^n \to \mathbb{R}P^n$ is a covering space. Since the relation $\sim_n$ is a congruence relation, that is, if $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2 \in S^m$ satisfy $\boldsymbol{x}_1 \sim_n \boldsymbol{x}_2$ and $\boldsymbol{y}_1 \sim_n \boldsymbol{y}_2$, then $\boldsymbol{x}_1 \triangleright \boldsymbol{y}_1 = \boldsymbol{x}_2 \triangleright \boldsymbol{y}_2$. Hence the quandle structure of $S^m_{\mathbb{R}}$ induces a quandle structure on $\mathbb{R}P^n$. We denote the quandle over $\mathbb{R}P^n$ as $P^n_{\mathbb{R}}$. The natural action $S^n \curvearrowleft O(n+1)$ induces the action $\mathbb{R}P^n \curvearrowleft O(n+1)$ defined by

$$\pi(\boldsymbol{x}) \cdot g = \pi(\boldsymbol{x}g) \quad (\boldsymbol{x} \in S^n,\ g \in O(n+1)).$$

Remark 4.1.1. *The quandle $P^n_{\mathbb{R}}$ is introduced by Azcan and Fenn, 1994.*

4.2 In the case $n = 1$

The embedding ι_1 is presented explicitly:

$$\iota_1 : S^1 \to O(2), \quad (\cos\theta, \sin\theta) \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix}.$$

We prove ι_1 is a diffeomorphism and a quandle homomorphism if we regard S^1 as the one-dimensional spherical quandle and $O(2)$ as a conjugacy quandle. It is easy to see that the map ι_1 is a smooth embedding since the embedding ι_1 is a composite map of a diffeomorphsm

$$S^1 \to SO(2), \quad (\cos\theta, \sin\theta) \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

a natural embedding $SO(2) \to O(2)$ and a diffeomorphism[2]

$$O(2) \to O(2) \quad g \mapsto h_1 g.$$

Hence it is enough to see that ι_1 is a quandle homomorphism. For each $\theta_1, \theta_2 \in \mathbb{R}$,

$$\begin{aligned} & (\cos\theta_1, \sin\theta_1) \triangleright (\cos\theta_2, \sin\theta_2) \\ = \ & (\cos\theta_1(2\cos^2\theta_2 - 1) + 2\sin\theta_1 \sin\theta_2 \cos\theta_2, \\ & 2\cos\theta_1 \sin\theta_2 \cos\theta_2 + \sin\theta_1(2\sin^2\theta_2 - 1)) \\ = \ & (\cos\theta_1 \cos 2\theta_2 + \sin\theta_1 \sin 2\theta_2, \sin 2\theta_2 \cos\theta_1 - \cos 2\theta_2 \sin\theta_1) \\ = \ & (\cos(2\theta_2 - \theta_1), \sin(2\theta_2 - \theta_1)). \end{aligned}$$

[2] See also Hatcher, 2002, p.293.

Then

$$
\begin{aligned}
& \iota_1(\cos\theta_1, \sin\theta_1) \triangleright \iota_1(\cos\theta_2, \sin\theta_2) \\
= \; & \iota_1(\cos\theta_2, \sin\theta_2)^{-1}\iota_1(\cos\theta_1, \sin\theta_1)\iota_1(\cos\theta_2, \sin\theta_2) \\
= \; & \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ -\sin\theta_2 & -\cos\theta_2 \end{pmatrix} \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ -\sin\theta_1 & -\cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ -\sin\theta_2 & -\cos\theta_2 \end{pmatrix} \\
= \; & \begin{pmatrix} \cos(2\theta_2 - \theta_1) & -\sin(2\theta_2 - \theta_1) \\ -\sin(2\theta_2 - \theta_1) & -\cos(2\theta_2 - \theta_1) \end{pmatrix} \\
= \; & \iota_1(\cos(2\theta_2 - \theta_1), \sin(2\theta_2 - \theta_1)) \\
= \; & \iota_1((\cos\theta_1, \sin\theta_1) \triangleright (\cos\theta_2, \sin\theta_2)).
\end{aligned}
$$

4.3 Lifting orthogonal group actions on spheres

In this section, we use the discussion in Section 3.3 to construct the action of the spin group on the sphere from the action of the special orthogonal group on the real projective space.

Proposition 4.3.1. *The action* $\mathbb{R}P^n \curvearrowleft SO(n+1)$ *induces an action* $S^n \curvearrowleft Spin(n+1)$ *defined by*

$$
\boldsymbol{x} \cdot \tilde{g} = \boldsymbol{x} p(\tilde{g}) \quad (\boldsymbol{x} \in S^n,\ \tilde{g} \in Spin(n+1)). \tag{4.1}
$$

Proof. Suppose $\tilde{\tau} : Spin(n+1) \to \operatorname{Diff} S^n$ is a group antihomomorphism induced by the action defined by Equation (4.1). It is enough to show

$$
d\pi \circ d\tilde{\tau}\left(\tilde{X}_{S^n}\right) = \left(p_* \tilde{X}\right)_{\mathbb{R}P^n} \circ \pi
$$

for any $\tilde{X} \in \mathfrak{spin}(n+1)$. For any $x \in S^n$ and ξ which is a smooth function defined in the neighborhood of point x, using the fact [3] of the differential of smooth maps,

$$
\begin{aligned}
\left(d\pi \circ d\tilde{\tau}\left(\tilde{X}_{S^n}\right)(x)\right)(\xi) &= \left(d\tilde{\tau}\left(\tilde{X}_{S^n}\right)(x)\right)(\xi \circ \pi) \\
&= \left.\frac{d}{dt}\right|_{t=0} \xi \circ \pi(x \cdot \exp t\tilde{X}) \\
&= \left.\frac{d}{dt}\right|_{t=0} \xi \circ \pi(x p(\exp t\tilde{X})).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
((p_* \tilde{X})_{\mathbb{R}P^n} \circ \pi(x))(\xi) &= \left.\frac{d}{dt}\right|_{t=0} \xi(\pi(x) \cdot \exp t p_* \tilde{X}) \\
&= \left.\frac{d}{dt}\right|_{t=0} \xi \circ \pi(x \exp t p_* \tilde{X}) \\
&= \left.\frac{d}{dt}\right|_{t=0} \xi \circ \pi(x p(\exp t\tilde{X})).
\end{aligned}
$$

Here, we used the fact that the map p is an adjoint representation of $Spin(n+1)$ at the end. □

[3] See Tu, 2011, Section 8.2, equation (8.1).

4.4 An embedding quandles defined over the projective space into Lie groups

In this section, we show that a natural quandle homomorphism from the spherical quandle to its own inner automorphism group induces an embedding of the real projective space into a Lie group.

Let n be a positive integer, $\mathrm{Conj}(h_n)$ be a conjugacy class with respect to $h_n \in O(n+1)$, and inn be a map defined by

$$\mathrm{inn} : S^n \to O(n+1), \quad \boldsymbol{e}_1 g \mapsto g^{-1} h_n g. \tag{4.2}$$

Proposition 4.4.1. *The map* inn *defined by Equation (4.2) is well defined and a quandle homomorphism.*

Proof. First, we prove that the map inn is well defined. The isotropy group $\mathrm{Stab}(n, \boldsymbol{e}_1) \subset O(n+1)$ of the action $S^n \curvearrowleft O(n+1)$ with respect to $\boldsymbol{e}_1 = (1, 0, \cdots, 0) \in S^n$ is

$$\mathrm{Stab}(n, \boldsymbol{e}_1) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix} \in O(n+1) \ : \ X \in O(n) \right\} \cong O(n).$$

The isotropy group $\mathrm{Stab}(n, h) \subset O(n+1)$ of the action $\mathrm{Conj}(h_n) \curvearrowleft O(n+1)$ with respect to $h_n \in O(n+1)$ is

$$\mathrm{Stab}(n, h) = \left\{ \begin{pmatrix} a & 0 \\ 0 & X \end{pmatrix} \in O(n+1) \ : \ a \in \mathbb{R} \setminus \{0\}, \ X \in O(n) \right\}.$$

Hence the map inn is well-defined since $\mathrm{Stab}(n, \boldsymbol{e}_1)$ is a subet of $\mathrm{Stab}(n, h)$.

Second, we prove that the map inn is a quandle homomorphism. For any $\boldsymbol{x}, \boldsymbol{y} \in S^n$, there exists $A, B \in O(n+1)$ such that $\boldsymbol{x} = \boldsymbol{e}_1 A$, $\boldsymbol{y} = \boldsymbol{e}_1 B$. Since

$$\begin{aligned} \boldsymbol{x} \triangleright \boldsymbol{y} &= 2\langle \boldsymbol{x}, \boldsymbol{y} \rangle \boldsymbol{y} - \boldsymbol{x} \\ &= 2\boldsymbol{e}_1 A B^{-1} ({}^t e_1) \boldsymbol{e}_1 B - \boldsymbol{e}_1 A \\ &= \boldsymbol{e}_1 A B^{-1} (2({}^t \boldsymbol{e}_1) \boldsymbol{e}_1 - I) B \\ &= \boldsymbol{e}_1 A B^{-1} h_n B, \end{aligned}$$

then

$$\begin{aligned} \mathrm{inn}(\boldsymbol{x} \triangleright \boldsymbol{y}) &= (AB^{-1} h_n B)^{-1} h_n (AB^{-1} h_n B) \\ &= (B^{-1} h_n B)^{-1} (A^{-1} h_n A)(B^{-1} h_n B) \\ &= \mathrm{inn}(\boldsymbol{x}) \triangleright \mathrm{inn}(\boldsymbol{y}). \end{aligned}$$

Hence the map inn is a quandle homomorphism. □

Proposition 4.4.2. *The quandle homomorphism* $\mathrm{inn} : S^n_{\mathbb{R}} \to \mathrm{Conj}\, O(n+1)$ *induces a quandle isomorphism* $i_n : P^n_{\mathbb{R}} \to \mathrm{Conj}(h_n)$.

Proof. The image of the map inn is coincident with $\mathrm{Conj}(h_n)$. Since

$$\begin{aligned} \mathrm{inn}(\boldsymbol{e}_1 g_1) = \mathrm{inn}(\boldsymbol{e}_1 g_2) &\iff g_1 g_2^{-1} \in \mathrm{Stab}(n, h) \\ &\iff \boldsymbol{e}_1 g_1 \sim_n \boldsymbol{e}_1 g_2, \end{aligned}$$

the induced map

$$i_n : P^n_{\mathbb{R}} \to \mathrm{Conj}(h_n), \quad \pi(\boldsymbol{e}_1 g) \mapsto g^{-1} h_n g$$

is a quandle isomorphism. □

Remark 4.4.3. *The map i_n is a diffeomorphism.*

4.5 A covering space defined over a conjugacy class of a Lie group

First, we consider the case $n = 2m$. We construct the universal covering of the conjugacy class $\mathrm{Conj}(h_{2m})$.

Proposition 4.5.1. *Suppose the map $p : Spin(2m+1) \to SO(2m+1)$ is the universal covering of $SO(2m+1)$ and $\tilde{h}_{2m} \in Spin(2m+1)$ is a fiber of h_{2m} with respect to the covering p. Then the map*

$$\pi_h : \mathrm{Conj}(\tilde{h}_{2m}) \to \mathrm{Conj}(h_{2m}), \quad x \mapsto p(x)$$

is well-defined and the universal covering of $\mathrm{Conj}(h_{2m})$.

Proof. We denote the Lie algebra of $SO(2m+1)$ as $\mathfrak{so}(2m+1)$. The differential at the identity of the involution of $Spin(2m+1)$

$$\tilde{\Theta} : Spin(2m+1) \to Spin(2m+1), \quad \tilde{g} \mapsto \tilde{h}_{2m}^{-1}\tilde{g}\tilde{h}_{2m}$$

is coincident with the involution of $\mathfrak{so}(2m+1)$, which is isomorphic to the Lie algebra of $Spin(2m+1)$,

$$\theta : \mathfrak{so}(2m+1) \to \mathfrak{so}(2m+1), \quad X \mapsto h_{2m}Xh_{2m}(= h_{2m}^{-1}Xh_{2m} = h_{2m}Xh_{2m}^{-1}).$$

Suppose $Spin(2n+1)$ acts on $\mathrm{Conj}(\tilde{h}_{2m})$ from right by conjugation. Then the isotropy group of $\tilde{h}_{2m}$ is

$$\mathrm{Stab}(\tilde{h}_{2m}) = \{\tilde{g} \in Spin(2n+1) \mid \tilde{g}\tilde{h}_{2m} = \tilde{h}_{2m}\tilde{g}\}.$$

There exists a natural diffeomorphism

$$\mathrm{Stab}(\tilde{h}_{2m})\backslash Spin(2n+1) \to \mathrm{Conj}(\tilde{h}_{2m}), \quad \mathrm{Stab}(\tilde{h}_{2m})\tilde{g} \mapsto \tilde{g}^{-1}\tilde{h}_{2m}\tilde{g}. \tag{4.3}$$

Moreover $\mathrm{Stab}(\tilde{h}_{2m})$ is coincident with an isotropy group $\tilde{K}$ of the Cartan involution $\tilde{\Theta} : Spin(2m+1) \to Spin(2m+1)$. By Borel, 1961, Theorem 3.4, $\tilde{K} = \mathrm{Stab}(\tilde{h}_{2m})$ is topologically connected.

The conjugacy class of $O(2m+1)$

$$\mathrm{Conj}(h_{2m}) = \{g^{-1}h_{2m}g \mid g \in O(2m+1)\}$$

is also a conjugacy class of $SO(2m+1)$. The isotropy group of h_{2m} with respect to the action is

$$\left\{\begin{pmatrix} 1 & \\ & X_+ \end{pmatrix} : X_+ \in SO(2m)\right\} \sqcup \left\{\begin{pmatrix} -1 & \\ & X_- \end{pmatrix} : X_- \in O(2m) \setminus SO(2m)\right\}$$

and we denote the isotropy group as $\mathrm{Stab}(h_{2m})$. Then, there exists a natural diffeomorphism

$$\mathrm{Stab}(h_{2m})\backslash SO(2m+1) \to \mathrm{Conj}(h_{2m}), \quad \mathrm{Stab}(h_{2m})g \mapsto g^{-1}h_{2m}g. \tag{4.4}$$

Moreover the unit component of $\mathrm{Stab}(h_{2m})$ is

$$\mathrm{Stab}(h_{2m})_0 := \left\{ \begin{pmatrix} 1 & \\ & X_+ \end{pmatrix} : X_+ \in SO(2m) \right\}.$$

Based on what has been discussed so far, we construct a universal covering of $\mathrm{Stab}(h_{2m})$. According to Chevalley, 1947, p.52 Proposition 4, the map

$$\mathrm{Stab}(h_{2m})_0 \backslash SO(2m+1) \to \mathrm{Stab}(h_{2m}) \backslash SO(2m+1) \quad \mathrm{Stab}(h_{2m})_0 \tilde{g} \mapsto \mathrm{Stab}(\tilde{h}_{2m}) \tilde{g}$$

is a covering. Especially the degree of the covering is two since the cardinality of $\mathrm{Stab}(h_{2m})_0 \backslash \mathrm{Stab}(h_{2m})$ is two. Since $\tilde{K} = \mathrm{Stab}(\tilde{h}_{2m})$ is topologically connected, the Lie algebra $\mathfrak{so}(2n+1)$ is simple, and $\mathrm{Stab}(h_{2m})_0$ is closed [4], we can use Helgason, 2001, Proposition 3.6, and the map

$$\tilde{K} \backslash Spin(2m+1) \to \mathrm{Stab}(h_{2m})_0 \backslash SO(2m+1), \quad \tilde{K}\tilde{g} \mapsto \mathrm{Stab}(\tilde{h}_{2m})_0 p(\tilde{g})$$

is a universal covering. Then the composition map of them

$$\tilde{K} \backslash Spin(2m+1) \to \mathrm{Stab}(h_{2m}) \backslash SO(2m+1), \quad \tilde{K} g \mapsto \mathrm{Stab}(\tilde{h}_{2m}) g$$

is a universal covering[5]. Using the diffeomorphisms defined by Equation (4.3) and Equation (4.4), the map

$$\pi_h : \mathrm{Conj}(\tilde{h}_{2m}) \to \mathrm{Conj}(h_{2m}), \quad x \mapsto p(x)$$

is also a universal covering. □

Proposition 4.5.2. *An action* $\mathrm{Conj}(\tilde{h}_{2m}) \curvearrowleft Spin(2m+1)$ *induced by the action* $\mathrm{Conj}(h_{2m}) \curvearrowleft SO(2m+1)$ *defined by conjugation and the universal covering* $\pi_h : \mathrm{Conj}(\tilde{h}_{2m}) \to \mathrm{Conj}(h_{2m})$ *constructed in Proposition 4.5.1 is coincident with an action defined by conjugation.*

Proof. It is enough to prove that

$$d\pi_h \circ \tilde{X}_{\mathrm{Conj}(\tilde{h}_{2m})} = (p_* \tilde{X})_{\mathrm{Conj}(h_{2m})} \circ \pi_h$$

for any $\tilde{X} \in \mathfrak{spin}(2m+1)$. For any $x \in \mathrm{Conj}(\tilde{h}_{2m})$ and ξ which is a smooth function defined in the neighborhood of point $\pi_h(x)$, using the fact [6] of the differential of smooth maps, we get

$$\begin{aligned} \left(d\pi_h \circ \tilde{X}_{\mathrm{Conj}(\tilde{h}_{2m})}(x) \right)(\xi) &= \tilde{X}_{\mathrm{Conj}(\tilde{h}_{2m})}(x)(\xi \circ \pi_h) \\ &= \left.\frac{d}{dt}\right|_{t=0} \xi \circ \pi_h(x \cdot \exp t\tilde{X}) \\ &= \left.\frac{d}{dt}\right|_{t=0} \xi \circ \pi_h(\exp(-t\tilde{X}) x \exp(t\tilde{X})). \end{aligned}$$

[4] In general, each connected component of a topological spcace is closed. If a topological space has finitely many connected component, each connected components is closed and open.

[5] We applied a following fact. Let $q : X \to Y$ and $r : Y \to Z$ be covering maps and let $p = q \circ r$. Then p is a covering map if the degree of r is finite. See Munkres, 2000, §54, Exersise 4.

[6] See Tu, 2011, Section 8.2, equation (8.1).

On the other hand,

$$\begin{aligned}
((p_*\tilde{X})_{\mathrm{Conj}(h_{2m})} \circ \pi_h(x))(\xi) &= \frac{d}{dt}\Big|_{t=0} \xi(\pi_h(x) \cdot \exp tp_*\tilde{X}) \\
&= \frac{d}{dt}\Big|_{t=0} \xi(\exp(-tp_*\tilde{X})\pi_h(x)\exp(tp_*\tilde{X})) \\
&= \frac{d}{dt}\Big|_{t=0} \xi(p(\exp(-t\tilde{X}))\pi_h(x)p(\exp(t\tilde{X}))) \\
&= \frac{d}{dt}\Big|_{t=0} \xi(p(\exp(-t\tilde{X}))p(x)p(\exp(t\tilde{X}))) \\
&= \frac{d}{dt}\Big|_{t=0} \xi \circ p(\exp(-t\tilde{X})x\exp(t\tilde{X})) \\
&= \frac{d}{dt}\Big|_{t=0} \xi \circ \pi_h(\exp(-t\tilde{X})x\exp(t\tilde{X})).
\end{aligned}$$

Here, we used the fact that the map p is an adjoint representation of $Spin(2m+1)$ at the end. □

Second, we consider the case $n = 2m+1$. Since the group $Spin(n+1)$ acts on $\mathrm{Conj}(h_{2m+2}) \subset Pin(n+1)$ smoothly, we are able to do same discussion in the case $n = 2m+1$. Then we get the following fact.

Proposition 4.5.3. *The map*

$$\pi_h : \mathrm{Conj}(\tilde{h}_{2m+1}) \to \mathrm{Conj}(h_{2m+1}), \quad x \mapsto p(x)$$

is well defined and a universal covering. Moreover the action $\mathrm{Conj}(\tilde{h}_{2m+1}) \curvearrowleft Spin(2m+2)$ *induced by the action* $\mathrm{Conj}(h_{2m+1}) \curvearrowleft SO(2m+2)$ *defined by conjugation and a universal covering* $\pi_h : \mathrm{Conj}(\tilde{h}_{2m+1}) \to \mathrm{Conj}(h_{2m+1})$ *is coincident with the action defined by conjugation.*

4.6 The proof of main theorem in the case $n \geq 2$

In this section, we prove the main theorem, Theorem 4.0.1, using in the case $n \geq 2$ what we have prepared so far. Using Proposition 4.5.1 and Proposition 4.5.3, the map $\pi_h : \mathrm{Conj}(\tilde{h}_n) \to \mathrm{Conj}(h_n)$ is a universal covering. By Proposition 4.4.2 and Remark 4.4.3, the diffeomorphism i_n induces a diffeomorphism $\iota_n : S^n \xrightarrow{\sim} \mathrm{Conj}(\tilde{h}_n)$ that makes the following diagram commutative:

$$\begin{array}{ccc}
(S^n, e_1) & \xrightarrow{\iota_n} & (\mathrm{Conj}(\tilde{h}_n), \tilde{h}_n) \\
\pi \big\downarrow & & \pi_h \big\downarrow \\
(\mathbb{R}P^n, \pi(e_1)) & \xrightarrow{i_n} & (\mathrm{Conj}(h_n), h_n)
\end{array} .$$

Hence it is enough to prove that the map ι_n is a quandle homomorphism.

Proposition 4.6.1. *The map* ι_n *is a quandle homomorphism.*

Proof. For any $\boldsymbol{x}, \boldsymbol{y} \in S^n$, there exists $A, B \in SO(n+1)$ such that $\boldsymbol{x} = \boldsymbol{e}_1 A$, $\boldsymbol{y} = \boldsymbol{e}_1 B$. Then, by Proposition 3.4.2, we get

$$\begin{aligned} \iota_n(\boldsymbol{x} \triangleright \boldsymbol{y}) &= \iota_n(\boldsymbol{e}_1 A B^{-1} h_n B) \\ &= \iota_n(\boldsymbol{e}_1 A \cdot \iota_n(\boldsymbol{y})) \\ &= \iota_n(\boldsymbol{x}) \cdot \iota_n(\boldsymbol{y}) \\ &= \iota_n(\boldsymbol{y})^{-1} \iota_n(\boldsymbol{x}) \iota_n(\boldsymbol{y}) \\ &= \iota_n(\boldsymbol{x}) \triangleright \iota_n(\boldsymbol{y}). \end{aligned}$$

Therefore the diffeomorphism ι_n is a quandle homomorphism. □

Chapter 5

Application

We give some application of Theorem 4.0.1 to some knot invariants concisely.

5.1 The Chern-Simons invariant

Inoue and Kabaya, 2014 proved that the complex volume, the Chern-Simons invariants with respect to its principal $PSL(2, \mathbb{C})$ bundle, of hyperbolic knots is presented as the quandle cocycle invariants. That is, we need just two objects, a quandle coloring and a quandle 2-cocycle, to calculate the complex volume of hyperbolic knots. Inoue and Kabaya defined the parabolic quandle, which is a smooth quandle embedded in $PSL(2, \mathbb{C})$ and they proved that the parabolic quandle's quandle coloring and quandle 2-cocycle are the objects to calculate the complex volume.

Nosaka, 2019 proved that secondary characteristic classes in Dupont, 2001, closely related to the Chern-Simon class, produce cocycles of quandles. In addition, Nosaka, 2015 showed that there is a bijection between the quandle colorings and the knot group representations if there is an embedding of quandle into groups. Hence we expect that we may be able to calculate the Chern-Simons invariant with respect to principal $Spin(n+1)$ bundles with the spherical quandle $S^n_{\mathbb{R}}$ as analogy of Inoue-Kabaya's work.

5.2 The longitudinal mapping

Clark and Saito, 2018 defined a knot invariant called the longitudinal mapping. According to Clark and Saito, 2018, their invariant is a generalization of the quandle cocycle invariants. The exact definition of the knot invariant is as follows: Let K be a knot and G be a group. For each $x \in G$,

$$\mathcal{L}^x_G : \{f \in \mathrm{Hom}(\pi_1(S^3 \setminus K), G) \ : \ f(\mathfrak{m}) = x\} \to G, \quad f \mapsto f(\mathfrak{l})$$

, where $\mathfrak{m}$ is a meridian and $\mathfrak{l}$ is a longitude of K.

Since Nosaka, 2015 showed that there is a bijection between the quandle colorings and the knot group representations if there is an embedding of quandle into groups, we are able to relate $S^n_{\mathbb{R}}$-colorings to the domain of $\mathcal{L}^x_{Spin(n+1)}$ by Theorem 4.0.1. The relation makes the calculation of $\mathcal{L}^x_{Spin(n+1)}$ easier since we can use S^n, which is easier than $Spin(n+1)$, to calculate $\mathcal{L}^x_{Spin(n+1)}$.

Appendix A

The definition of the spherical quandle in dimension two

Clark and Saito, 2018 defined a family of quandles on conjugacy classes of $SU(2)$ and called them spherical quandles. Their definition of spherical quandles is different from the definition in Section 2.3.2, but we give the compatibility between the spherical quandles in the sense of Clark and Saito, 2018 and the spherical quandle $S^2_{\mathbb{R}}$.

A.1 The spherical quandle defined by Clark and Saito, 2018

In this section, we introduce the spherical quandle defined by Clark and Saito, 2018. Consider the 2-dimentional special unitary group

$$SU(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \,:\, a, b \in \mathbb{C},\ a\bar{a} + b\bar{b} = 1 \right\}$$

, where $i = \sqrt{-1}$ and $\bar{a}, \bar{b}$ are conjugation of a, b respectively. Suppose $S^2(r)$ is a conjugacy class of $SU(2)$

$$S^2(r) = \left\{ g^{-1} \begin{pmatrix} ir & 0 \\ 0 & -ir \end{pmatrix} g \,:\, g \in SU(2) \right\}$$

for any $r \in \mathbb{R}$. For any $r \in \mathbb{R} \setminus \{0\}$, the conjugacy class $S^2(r)$ is homeomorphic to the 2-dimensional sphere S^2 and is a subquandle of $\operatorname{Conj} SU(2)$. Clark and Saito, 2018 called the family of quandles $\{S^2(r)\}_{r \in (0,\pi)}$ “spherical quandles".

Remark A.1.1. *Our description is a little different from Clark and Saito, 2018's original one. However our description is equivalent to the original one since the quandle $S^2(r)$ is isomorphic to the quandle $S^2_{2\pi-2r}$ in Clark and Saito, 2018 for any $r \in (0, \pi)$. See also Clark and Saito, 2018, Lemma 4.4.*

A.2 An embedding $S^2_{\mathbb{R}}$ into SU(2)

In this section, we prove that $S^2_{\mathbb{R}}$ is isomorphic to $S^2(\frac{\pi}{2})$ and that the spherical quandles in the sense of Clark and Saito, 2018 is a kind of extension of $S^2_{\mathbb{R}}$.

Consider a matrix

$$\tilde{H}_2 = \exp \begin{pmatrix} \frac{\pi}{2} i & 0 \\ 0 & -\frac{\pi}{2} i \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in SU(2)$$

and the Lie algebra of $SU(2)$

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} ix & a \\ -\overline{a} & -ix \end{pmatrix} \;:\; x \in \mathbb{R},\, z \in \mathbb{C} \right\}.$$

The adjoint representation $\varpi' : SU(2) \to GL(\mathfrak{su}(2))$ induces a covering $\varpi : SU(2) \to SO(3)$. The map is a unique universal covering $SO(3)$ since the Lie group $SU(2)$ is connected and simply connected. Then there is a unique Lie group isomorphism $j : Spin(3) \to SU(2)$ that makes the following diagram commutative:

$$\begin{array}{ccc} (Spin(3), \tilde{h}_2) & \xrightarrow{\;j\;} & (SU(2), \tilde{H}_2)\,. \\ {\scriptstyle p}\downarrow & \swarrow {\scriptstyle \varpi} & \\ (SO(3), h_2) & & \end{array}$$

The map j induces a (smooth) quandle isomorphism between $\mathrm{Conj}(\tilde{h}_2)$ and $S^2(\frac{\pi}{2})$. Hence Theorem 4.0.1 induces a isomorphism from $S^2_{\mathbb{R}}$ to $S^2(\frac{\pi}{2})$.

Remark A.2.1. *The author gave elementary proof of the fact dealt with in this section. He also gave a concrete quandle isomorphism as follows:*

$$S^2_{\mathbb{R}} \to S^2(r), \quad (x_1, x_2, x_3) \mapsto \exp(\frac{\pi}{2} \begin{pmatrix} ix_1 & x_2 + x_3 i \\ -x_2 + x_3 i & -ix_1 \end{pmatrix}) \tag{A.1}$$

, where $\exp : \mathfrak{su}(2) \to SU(2)$ *is an exponential map. See Yonemura, 2021 for more details.*

Since $Spin(5)$ is isomorphc to $Sp(2)$, we may be able to construct a concrete embedding of $S^4_{\mathbb{R}}$ similar to Equation (A.1).

Bibliography

Akita, Toshiyuki (2022). *Embedding Alexander quandles into groups*. DOI: 10.48550/ARXIV.2210.04583. URL: https://arxiv.org/abs/2210.04583.

Atiyah, Michael F, Raoul Bott, and Arnold Shapiro (1964). "Clifford modules". In: *Topology* 3, pp. 3–38.

Azcan, Hiiseyin and Roger Fenn (1994). "Spherical representations of the link quandles". In: *Turkish J. of Mathematics* 18, pp. 102–110.

Bardakov, Valeriy and Timur Nasybullov (2020). "Embeddings of quandles into groups". In: *Journal of Algebra and its Applications* 19.07, p. 2050136.

Bardakov, Valeriy G, Pinka Dey, and Mahender Singh (2017). "Automorphism groups of quandles arising from groups". In: *Monatshefte für Mathematik* 184.4, pp. 519–530.

Bergman, George M (2021). "On core quandles of groups". In: *Communications in Algebra* 49.6, pp. 2516–2537.

Borel, Armand (1961). "Sous-groupes commutatifs et torsion des groupes de Lie compacts connexes". In: *Tohoku Mathematical Journal, Second Series* 13.2, pp. 216–240.

Cartan, Élie (2012). *The theory of spinors*. Courier Corporation.

Carter, J et al. (1999). "State-sum invariants of knotted curves and surfaces from quandle cohomology". In: *Electronic Research Announcements of the American Mathematical Society* 5.20, pp. 146–156.

Chevalley, Claude (1947). "Theory of Lie Groups I". In: *Princeton, NJ*.

Clark, W. Edwin and Masahico Saito (2018). "Longitudinal mapping knot invariant for SU(2)". In: *J. Knot Theory Ramifications* 27.11, pp. 1843014, 22. ISSN: 0218-2165. DOI: 10.1142/S0218216518430149. URL: https://doi.org/10.1142/S0218216518430149.

Dhanwani, Neeraj K, Hitesh Raundal, and Mahender Singh (2021). "Dehn quandles of groups and orientable surfaces". In: *arXiv preprint arXiv:2106.00290*.

Dupont, Johan L (2001). *Scissors Congruences, Group Homology & Characteristic Classes*. Vol. 1. World Scientific.

Eisermann, Michael (2014). "Quandle coverings and their Galois correspondence". In: *Fund. Math.* 225.1, pp. 103–168. ISSN: 0016-2736. DOI: 10.4064/fm225-1-7. URL: https://doi.org/10.4064/fm225-1-7.

Gorbatsevich, VV, Arkadij L Onishchik, and Ernest Borisovič Vinberg (1997). *Foundations of Lie theory and Lie transformation groups*. Springer.

Hatcher, Allen (2002). "Algebraic Topology". In: *http://www. math. cornell. edu/˜ hatcher/AT/ATpage. html*.

Helgason, Sigurdur (2001). *Differential geometry, Lie groups, and symmetric spaces*. AMS.

Inoue, Ayumu and Yuichi Kabaya (2014). "Quandle homology and complex volume". In: *Geometriae Dedicata* 171.1, pp. 265–292.

Ishikawa, Katsumi (n.d.). "On the classification of smooth quandles". preprint.

Joyce, David (1982). "A classifying invariant of knots, the knot quandle". In: *J. Pure Appl. Algebra* 23.1, pp. 37–65. ISSN: 0022-4049. DOI: 10.1016/0022-4049(82)90077-9. URL: https://doi.org/10.1016/0022-4049(82)90077-9.

Kamada, Seiichi (2017). *Surface-knots in 4-space*. Springer Monographs in Mathematics. An introduction. Springer, Singapore, pp. xi+212. ISBN: 978-981-10-4090-0; 978-981-10-4091-7. DOI: 10.1007/978-981-10-4091-7. URL: https://doi.org/10.1007/978-981-10-4091-7.

Loos, Ottmar (1969). *Symmetric spaces: General Theory*. Vol. 1. WA Benjamin.

Matveev, Sergei Vladimirovich (1982). "Distributive groupoids in knot theory". In: *Mat. Sb. (N.S.)* 119(161).1, pp. 78–88, 160. ISSN: 0368-8666.

Montaldi, James and Juan-Pablo Ortega (2008). "Notes on lifting group actions". MIMS Preprint. URL: http://eprints.maths.manchester.ac.uk/id/eprint/1158.

— (2009). "Symplectic group actions and covering spaces". In: *Differential Geometry and its Applications* 27.5, pp. 589–604.

Munkres, James R (2000). *Topology*. Vol. 2. Prentice Hall Upper Saddle River.

Nomura, Takaaki (2018). *Spherical Harmonics and Group Representations*. NIPPON HYORON SHA CO.,LTD.

Nosaka, Takefumi (2015). "Homotopical interpretation of link invariants from finite quandles". In: *Topology and its Applications* 193, pp. 1–30.

— (2017a). "Central extensions of groups and adjoint groups of quandles (Geometry and Analysis of Discrete Groups and Hyperbolic Spaces)". In: *RIMS Kokyuroku Bessatsu* 66, pp. 167–184.

— (2017b). *Quandles and topological pairs;Symmetry, knots, and cohomology*. SpringerBriefs in Mathematics. Symmetry, knots, and cohomology. Springer, Singapore, pp. ix+136. ISBN: 978-981-10-6792-1; 978-981-10-6793-8. DOI: 10.1007/978-981-10-6793-8. URL: https://doi.org/10.1007/978-981-10-6793-8.

— (2019). "de Rham theory and cocycles of cubical sets from smooth quandles". In: *Kodai Mathematical Journal* 42.1, pp. 111–129.

Palais, Richard S (1957). *A global formulation of the Lie theory of transportation groups*. American Mathematical Society.

Tu, Loring W. (2011). *An introduction to manifolds*. Second. Universitext. Springer, New York, pp. xviii+411. ISBN: 978-1-4419-7399-3. DOI: 10.1007/978-1-4419-7400-6. URL: https://doi.org/10.1007/978-1-4419-7400-6.

Wang, Zuoqin (2013). "Lecture 13-14: Actions of Lie groups and Lie algebra". lecture note. URL: http://staff.ustc.edu.cn/~wangzuoq/Courses/13F-Lie/Notes/Lec%2013-14.pdf.

Yonemura, Kentaro (2021). *Note on spherical quandles*. DOI: 10.48550/ARXIV.2104.04921. URL: https://arxiv.org/abs/2104.04921.