Studies on discrete differential geometry and non-steady state nucleation in terms of the elliptic theta functions

重富, 尚太

https://hdl.handle.net/2324/6787426

出版情報：Kyushu University, 2022, 博士（機能数理学），課程博士
バージョン：

Doctoral Dissertation

Studies on discrete differential geometry and non-steady state nucleation in terms of the elliptic theta functions

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February 15, 2023

Contents

1 Introduction **5**

2 Explicit formulas for isoperimetric deformations of smooth and discrete elasticae **9**

2.1 Isoperimetric deformations of planar curves . . . 9
2.1.1 Continuous deformation of smooth curves . . . 9
2.1.2 Continuous deformation of discrete curves . . . 10
2.1.3 Discrete deformation of discrete curves . . . 11
2.2 Explicit formulas in terms of τ function . . . 12
2.2.1 Continuous deformation of smooth curves . . . 12
2.2.2 Continuous deformation of discrete curves . . . 12
2.2.3 Discrete deformation of discrete curves . . . 14
2.3 Explicit Solutions . . . 14
2.3.1 Continuous deformation of elastica . . . 15
2.3.2 Continuous deformation of discrete elastica . . . 24
2.3.3 Discrete deformation of discrete elastica . . . 27
2.4 Connection between the discrete curves given in 2.3.2, 2.3.3 and the discrete elastica . . . 29

3 Explicit formulas of space curves with constant torsion and discrete space curves with constant torsion angle **33**

3.1 The Frenet-Serret formula for space curves . . . 33
3.2 Explicit formulas of space curves with constant torsion and discrete space curves with constant torsion angle . . . 34
3.2.1 Explicit formula of the Frenet frame . . . 37
3.2.2 Sufficient conditions for curve closure for the curve given in Theorem 9 . . . 37
3.3 The discrete Frenet-Serret formula for discrete space curves . . . 38
3.4 Explicit formulas of closed discrete space curves with constant torsion angle 39
3.4.1 Explicit formula of the discrete Frenet frame . . . 42

3.4.2 Sufficient conditions for curve closure for the curve given in Theorem 11 . . . 43
3.5 Proof of Theorems 8-11 . . . 43
3.6 Proof of Propositions 1-4 . . . 47

4 An explicit construction of Kaleidocycles 56
4.1 An explicit formula of closed discrete space curve with constant torsion angle 57
4.1.1 Discrete space curves . . . 57
4.1.2 An explicit formula in terms of the elliptic theta functions . . . 58
4.1.3 Sufficient conditions for curve closure . . . 62
4.1.4 Time period and drift of γ . . . 63
4.1.5 Explicit formula of the discrete Frenet frame . . . 65
4.2 Deformation of the curve that preserves segment length and torsion angle . 67
4.2.1 Bilinear equations satisfied by the τ function . . . 67
4.2.2 Isoperimetric deformation of the curve by the semi-discrete mKdV equation . . . 69
4.2.3 Isoperimetric deformation of the curve by the semi-discrete potential mKdV equation and the semi-discrete sine-Gordon equation . . . 70
4.2.4 The semi-discrete potential mKdV equation and the semi-discrete sine-Gordon equation for the potential function . . . 72
4.2.5 Semi-discrete mKdV equation for signed curvature angle . . . 75
4.3 Derivation of bilinear equations . . . 76
4.3.1 Bilinearization of the semi-discrete potential mKdV equation (4.113) 76
4.3.2 Bilinearization of the semi-discrete sine-Gordon equation (4.115) . . 78
4.3.3 Bilinearization of the semi-discrete mKdV equation (4.123) . . . 78
4.4 Proof of Proposition 11 . . . 79
4.5 Construction of solutions to bilinear equations in terms of the elliptic theta functions . . . 80
4.5.1 Proof of Theorem 12 . . . 81
4.5.2 Proof of Proposition 8 . . . 85
4.5.3 Proof of Proposition 9 . . . 85
4.5.4 Proof of Proposition 10 . . . 86
4.5.5 Construction of solutions to the semi-discrete potential mKdV equation and the semi-discrete sine-Gordon equation . . . 87
4.5.6 Construction of a solution to the semi-discrete mKdV equation . . . 90
4.6 Proof of Proposition 12 . . . 91
4.7 Proof of Proposition 13 . . . 92
4.8 Proof that C, Γ only contributes to rigid transformations . . . 94

5 Exploration of non-trivial relations for the non-steady state nucleation rate: usefulness of the elliptic theta functions for its experimental estimations **95**

5.1 Non-steady state nucleation rate 95
5.2 Relations between nucleation rate and the elliptic theta functions 100
5.3 Non-trivial relations connecting the non-steady state nucleation rates at different times 100
5.4 A difference equation for the non-steady state nucleation rate 103
5.5 Discussion 106
5.6 Conclusions 106

Bibliography **106**

Acknowledgments

I would like to thank my supervisor, Prof. Kenji Kajiwara. His guidance was a great help in promoting research. I also thank Prof. Wolfgang K. Schief, Prof. Shizuo Kaji and Prof. Nozomu Matsuura for their helpful comments and discussions.

Chapter 1

Introduction

The theta functions are known to appear in various areas of mathematics, and, interestingly, the solutions to integrable systems are expressed in terms of the theta functions. In this connection, the theta functions are also used in the study of geometric objects associated with integrable systems. It is also known that integrable systems can be discretized preserving the structure of solutions, and the methods of integrable systems are often applied in the field of discrete differential geometry, which considers the discretization of various concepts that appear in differential geometry. Thus, the theta functions have also been used to construct explicit formulas for discrete curves and surfaces. On the other hand, it is well known that the theta functions appear in physics fields such as fluid mechanics and statistical mechanics, but in the last part of this thesis, we would like to focus on a phenomenon called non-steady state nucleation, in which the properties of the theta functions have not been utilized much so far.

In this thesis, we exploit various properties of the theta functions to construct explicit formulas for some curves and discrete curves, and to derive equations that can be used in experimental studies of non-steady state nucleation. The former is a part of the stream of studies of deformations of curves based on the methods of integrable systems, and was originally inspired by Hashimoto [1] and deepened by Goldstein-Petrich and Lamb [2, 3]. Subsequently, deformations of discrete curves and discrete surfaces based on semi-discrete or discrete integrable systems, has been studied in [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. The connection between nucleation theory and the theta functions was suggested by the work of Kashchiev [16] and Shneidman [17], and the results have since been used in various experimental studies [18, 19, 20, 21, 22, 23, 24, 25].
Among them, particular attention is paid to the construction of explicit formulas for deformations of elastic curves, also known as Elastica, and their discrete analogues, and for constant torsion curves and their discrete analogues, in terms of the elliptic theta functions. As an application of the latter, we also construct an explicit formula of the Kaleidocycle.

In Chapter 2, we consider deformation of elastica, which is a well-known class of planar curves that describes the shape of thin elastic rods [26, 27, 28]. Mathematically, it is characterized by the differential equation for its curvature

$$\kappa_{xx} + \frac{1}{2}\kappa^3 + \lambda\kappa = 0, \quad \lambda \in \mathbb{R}, \tag{1.1}$$

where $\kappa = \kappa(x)$ and x is the arc length. On the other hand, it is well known that the modified KdV (mKdV) equation

$$\kappa_{xxx} + \frac{3}{2}\kappa^2\kappa_x + \kappa_t = 0. \tag{1.2}$$

describes an isoperimetric deformation of planar curves [4, 29]. The explicit formulas for the isoperimetric deformations of curves have been constructed in terms of the τ functions [4]. Since the traveling wave solutions to the modified KdV equation satisfy the equation for elastica (1.1), it is possible to construct explicit formula for the isoperimetric deformation of elastica. In this chapter, we construct an explicit formula for the isoperimetric deformation of elastica described by the modified KdV equation. Thanks to its integrability, it is also possible to construct a discrete version of elastica characterized by the integrable discrete analogue of the modified KdV equation [30, 31, 32]. We also construct the explicit formulas to both continuous and discrete deformations of the discrete elastica described by the semi-discrete potential mKdV equation and the discrete potential mKdV equation, respectively. The formulas can be regarded as the extended versions of the results by Mumford [33] and Matsuura [34].

Space curves of constant torsion and their discrete analogues have been studied, for example, in [7, 35, 36, 37, 38]. This curve often appears in discussions of the relationship between surfaces and integrable systems, and in particular, the correspondence with surfaces of constant negative Gaussian curvature and their discrete analogues is discussed in [14], and explicit formulas for such surfaces in terms of the Riemann theta functions are given. In the second part of this thesis, by utilizing the explicit formulas for smooth and discrete space curves by τ functions [6], the explicit formulas for two types of curves with constant torsion are first constructed in terms of the elliptic theta functions. Numerical experiments show that the difference between these curves is whether they can be closed or not. For the curve that can be closed, the conditions for it to be a closed curve are explicitly obtained. Then, their discrete analogues are also constructed in terms of the elliptic theta functions, and the closure conditions of the curves are also written down explicitly.

Kaleidocycle is a linkage mechanism consisting of tetrahedra and hinges connecting them. It is known that this mechanism can be made from origami and that it has multiple conserved quantities. In particular, numerical experiments have shown that special Kaleidocycles, called Möbius Kaleidocycles, have one-dimensional degree of freedom regardless of the number of tetrahedra which consists a Kaleidocycle and has been patented [39]. There is also a known work on modeling the Kaleidocycle as deformations of discrete space curves by integrable equations [40], which can be seen to be a subject of mathematical, physical, and engineering interest. In the third part of this thesis, an explicit formula for the Kaleidocycle is constructed by applying and extending some of the results of the previous chapter. The deformation of Kaleidocycle corresponds to the continuous deformation of discrete curves such that the segment length and the torsion angle are preserved. In this chapter, three types of such continuous deformation equations for the curve appear, and correspondingly, three types of differential-difference equations for the potential function. Continuous deformations of discrete space curves have been investigated by many researchers, for example, in [6, 7, 8, 9, 40], but there have been no reports of these equations being satisfied simultaneously. The results of this thesis include a proof that the explicit formula we constructed is a solution of the simultaneous equations.

In the last part of this thesis, we derive non-trivial relations for the non-steady state nucleation rate and discuss their applications by using the properties of the elliptic theta functions. Nucleation is a non-equilibrium process in which a different thermodynamic phase precipitates at a very localized region in response to supersaturation of a certain chemical species in the original phase. This physical phenomenon is universal and has been described by the classical nucleation theory, which is based on equilibrium thermodynamics. The non-steady state nucleation rate $J_k(t)$ ($\mathrm{m}^{-3}\cdot\ \mathrm{s}^{-1}$), where t (s) is time, is a physical quantity characterizing nucleation, meaning the flux of molecular clusters (nuclei) that grow beyond a critical cluster size $x = x_k$ per unit time and unit volume, when x is the cluster size. An explicit expression of the non-steady state nucleation rate as an infinite series was given in [16], and it has already been reported in [17] that the infinite series can also be represented by an elliptic theta function. Although the relation between the non-steady state nucleation rate derived from the classical nucleation theory and the elliptic theta functions has been pointed out in [16, 17], and cited in many subsequent studies [41, 42, 43], there have been no studies that take advantage of rich mathematical properties of the elliptic theta functions.

On the other hand, various difficulties exist in experimental studies based on the classical nucleation theory. For example, **(i)** the timescale of the non-steady state nucleation processes is generally very long in the low-temperature region, and it requires a large amount of effort and cost to estimate the time evolution of $J_k(t)$: the number of crystal nuclei is measured at many times over a long period of 200 hours in [23]. In other cases, **(ii)** nucleation is already in progress at the start of the measurement, and the time evolution of $J_k(t)$ before that point is unknown in [21]. In Chapter 5, we show that rich mathematical properties of the elliptic theta functions may help overcome these difficulties.

This thesis is organized as follows. In Chapter 2, based on [44], we present frameworks of the integrable deformation theory of smooth and discrete plane curves and construct explicit formulas for isoperimetric deformations of smooth and discrete elasticae in terms of the elliptic theta functions. In Chapter 3, we utilise the explicit formula for space curves by τ functions [6] and construct explicit formulas for curves with constant torsion in terms of the elliptic theta functions. There are two types of curves to be constructed, and the condition for one of them to be a closed curve can also be obtained explicitly. We also construct two explicit formulas for discrete space curves with constant torsion angle, which are discrete analogues of space curves with constant torsion; as in the case of smooth curve, we also explicitly derive the conditions under which the curve close. In Chapter 4, one of the results of the previous chapter is extended to isoperimetric deformations of closed discrete curves with constant torsion angle. The semi-discrete mKdV equation, the semi-discrete potential mKdV equation and the semi-discrete sine-Gordon equation appear as deformation equations. This result can be regarded as an explicit formula for Kaleidocycles. This is based on the results of [45]. In Chapter 5, based on [46], by taking advantage of the elliptic theta functions, we derive non-trivial relations satisfied by the non-steady state nucleation rate and discuss its applications.

Chapter 2

Explicit formulas for isoperimetric deformations of smooth and discrete elasticae

In this chapter, based on [44], we present frameworks of the integrable deformation theory of smooth and discrete plane curves and construct explicit formulas for isoperimetric deformations of smooth and discrete elasticae in terms of the elliptic theta functions.

2.1 Isoperimetric deformations of planar curves

We first summarize the formulations of three kinds of planar curve deformations [4, 5, 40].

2.1.1 Continuous deformation of smooth curves

Let $\gamma(x) \in \mathbb{R}^2$ be an arc length parameterized curve in Euclidean plane $\mathbb{R}^2$, and x be the arc length. By definition, $|\gamma_x| = 1$. Then the Frenet equation of γ is

$$\gamma_{xx} = \begin{pmatrix} 0 & -\kappa \\ \kappa & 0 \end{pmatrix} \gamma_x. \tag{2.1}$$

and the function κ is the curvature of γ. We consider the following deformation, where t is the deformation parameter[28, 29],

$$\frac{\partial}{\partial t}\gamma_x = \begin{pmatrix} 0 & \kappa_{xx} + \frac{1}{2}\kappa^3 \\ -\kappa_{xx} - \frac{1}{2}\kappa^3 & 0 \end{pmatrix} \gamma_x. \tag{2.2}$$

Then the potential function $\theta(x,t)$ defined by $\kappa = \theta_x$ satisfies the potential mKdV equation

$$\theta_{xxx} + \frac{1}{2}(\theta_x)^3 + \theta_t = 0, \tag{2.3}$$

so that κ satisfies the mKdV equation

$$\kappa_{xxx} + \frac{3}{2}\kappa^2\kappa_x + \kappa_t = 0. \tag{2.4}$$

The function θ is also called the angle function, as γ_x can be expressed as

$$\gamma_x = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}. \tag{2.5}$$

Remark 1. The traveling wave solutions of the mKdV equation satisfy the differential equation of elastica.

$$\kappa_{xx} + \frac{1}{2}\kappa^3 + \lambda\kappa = 0, \quad \lambda \in \mathbb{R}, \tag{2.6}$$

Thus, in this case, γ represents the isoperimetric deformation of elastica.

2.1.2 Continuous deformation of discrete curves

Let $\gamma_n \in \mathbb{R}^2$ $(n \in \mathbb{Z})$ be a discrete plane curve with constant segment length

$$|\gamma_{n+1} - \gamma_n| = a, \tag{2.7}$$

where $a \in \mathbb{R}_{>0}$ is a constant. We introduce the angle function Θ_n of a discrete curve γ_n by

$$\frac{\gamma_{n+1} - \gamma_n}{a} = \begin{pmatrix} \cos\Theta_n \\ \sin\Theta_n \end{pmatrix}. \tag{2.8}$$

A discrete curve γ_n satisfy

$$\frac{\gamma_{n+1} - \gamma_n}{a} = R\left(K_n\right)\frac{\gamma_n - \gamma_{n-1}}{a}. \tag{2.9}$$

for $K_n = \Theta_n - \Theta_{n-1}$, where $R\left(K_n\right)$ denotes the rotation matrix given by

$$R\left(K_n\right) = \begin{pmatrix} \cos K_n & -\sin K_n \\ \sin K_n & \cos K_n \end{pmatrix}. \tag{2.10}$$

Now consider the following continuous deformation of discrete curve [40].

$$\frac{d\gamma_n}{dt} = R\left(W_n\right)\frac{\gamma_{n+1} - \gamma_n}{a}. \tag{2.11}$$

Compatibility of the system (2.9) and (2.11) implies $W_n = -W_{n-1} - K_n$, so that the potential function θ_n is introduced by

$$K_n = \frac{\theta_{n+1} - \theta_{n-1}}{2}, \quad W_n = \frac{\theta_n - \theta_{n+1}}{2}, \tag{2.12}$$

and it follows that θ_n satisfies the semi-discrete potential mKdV equation [40]

$$\frac{d}{dt}\left(\theta_{n+1}+\theta_n\right)=\frac{4}{a}\sin\left(\frac{\theta_{n+1}-\theta_n}{2}\right). \tag{2.13}$$

Note that the angle function Θ_n can be expressed as

$$\Theta_n=\frac{\theta_{n+1}+\theta_n}{2}. \tag{2.14}$$

2.1.3 Discrete deformation of discrete curves

Next, we consider the following discrete deformation of discrete curve [4, 5]

$$\left|\gamma_{n+1}^m-\gamma_n^m\right|=a, \tag{2.15}$$

$$\frac{\gamma_{n+1}^m-\gamma_n^m}{a}=R\left(K_n^m\right)\frac{\gamma_n^m-\gamma_{n-1}^m}{a}, \tag{2.16}$$

$$\frac{\gamma_n^{m+1}-\gamma_n^m}{b}=R\left(W_n^m\right)\frac{\gamma_n^m-\gamma_{n-1}^m}{a}, \tag{2.17}$$

where $m\in\mathbb{Z}$ is a discrete deformation parameter. Compatibility of the system (2.16) and (2.17) implies the existence of the potential function θ_n^m defined by

$$K_n^m=\frac{\theta_{n+1}^m-\theta_{n-1}^m}{2},\quad W_n^m=\frac{\theta_n^{m+1}-\theta_{n+1}^m}{2}. \tag{2.18}$$

and it follows that θ_n^m satisfies the discrete potential mKdV equation

$$\tan\left(\frac{\theta_{n+1}^{m+1}-\theta_n^m}{4}\right)=\frac{b+a}{b-a}\tan\left(\frac{\theta_n^{m+1}-\theta_{n+1}^m}{4}\right). \tag{2.19}$$

Note that the angle function Θ_n^m can be expressed as

$$\Theta_n^m=\frac{\theta_{n+1}^m+\theta_n^m}{2}. \tag{2.20}$$

Remark 2. The discrete elastica was introduced in [30] as a discrete analogue of elastica. When the discrete curvature $\kappa_n=\frac{2}{a}\tan\frac{K_n}{2}$ of a discrete plane curve satisfies

$$\kappa_{n+1}+\kappa_{n-1}=\frac{c_0\kappa_n}{1+\frac{a^2}{4}\kappa_n}, \tag{2.21}$$

(where c_0 is a constant) the discrete plane curve is called a discrete elastica. In section 2.4, we will see that the discrete curves constructed in this chapter are discrete elastica.

2.2 Explicit formulas in terms of τ function

We summarize the explicit formulas of isoperimetric deformations of planar curves in terms of the τ functions [4].

2.2.1 Continuous deformation of smooth curves

Let $\tau = \tau(x, t; y)$ be a complex function depending on three continuous variables x, t and y, which satisfy the following system of bilinear equations of Hirota type,

$$\frac{1}{2} D_x D_y \, \tau \cdot \tau = -\left(\tau^*\right)^2, \tag{2.22}$$

$$D_x^2 \, \tau \cdot \tau^* = 0, \tag{2.23}$$

$$(D_x^3 + D_t)\tau \cdot \tau^* = 0, \tag{2.24}$$

where * denotes the complex conjugate. Here, D_x, D_y and D_t are Hirota's bilinear differential operators defined as follows [47].

$$D_x^m D_t^n f(x,t) \cdot g(x,t) = \left.\frac{\partial^m}{\partial y^m} \frac{\partial^n}{\partial s^n} f(x+y, t+s) g(x-y, t-s)\right|_{y=0, s=0}, \tag{2.25}$$
$$m, n = 0, 1, 2, 3, \ldots$$

Let τ be a solution to (2.22)-(2.24). Define a real function $\theta(x, t; y)$ and an $\mathbb{R}^2$ valued function $\gamma(x, t; y)$ by

$$\theta(x, t; y) = \frac{2}{\sqrt{-1}} \log \frac{\tau}{\tau^*}, \tag{2.26}$$

$$\gamma(x, t; y) = \begin{pmatrix} -\frac{1}{2} \left(\log \tau\tau^*\right)_y \\ \frac{1}{2\sqrt{-1}} \left(\log \frac{\tau}{\tau^*}\right)_y \end{pmatrix}. \tag{2.27}$$

Then for any $x, t, y \in \mathbb{R}$, the functions θ and γ satisfy (2.1)-(2.3) and (2.5) [4].

2.2.2 Continuous deformation of discrete curves

Let $\tau_n = \tau_n(t; y)$ be a complex function depending on the discrete variable n and two continuous variables t and y, which satisfies the following system of bilinear equations

$$\frac{1}{2} D_t D_y \, \tau_n \cdot \tau_n = -\left(\tau_n^*\right)^2, \tag{2.28}$$

$$D_y \tau_{n+1} \cdot \tau_n = -a \tau_{n+1}^* \tau_n^*, \tag{2.29}$$

$$D_t \tau_{n+1} \cdot \tau_n^* = -\frac{1}{a} \tau_{n+1}^* \tau_n. \tag{2.30}$$

Theorem 1. Let τ_n be a solution to (2.28)-(2.30). Define a real function $\theta_n(t;y)$ and an $\mathbb{R}^2$ valued function $\gamma_n(t;y)$ by

$$\theta_n(t;y) = \frac{2}{\sqrt{-1}} \log \frac{\tau_n}{\tau_n^*}, \tag{2.31}$$

$$\gamma_n(t;y) = \begin{pmatrix} -\frac{1}{2} \left(\log \tau_n \tau_n^*\right)_y \\ \frac{1}{2\sqrt{-1}} \left(\log \frac{\tau_n}{\tau_n^*}\right)_y \end{pmatrix}. \tag{2.32}$$

Then for any t, $y \in \mathbb{R}$ and $n \in \mathbb{Z}$, the functions θ_n and γ_n satisfy (2.8), (2.9), (2.11) and (2.13).

Since this formula differs from the result presented in [12], which considered the explicit formula for the continuous deformation of discrete plane curves, a brief proof is given.

Proof. Express $\gamma_n = {}^t(X_n, Y_n)$. From (2.29) we have

$$\left(\log \frac{\tau_{n+1}}{\tau_n}\right)_y = -a \frac{\tau_{n+1}^* \tau_n^*}{\tau_{n+1}\tau_n}, \tag{2.33}$$

Adding (2.33) and its complex conjugate, we obtain by using (2.31) and (2.32)

$$\frac{X_{n+1} - X_n}{a} = \cos \Theta_n, \quad \Theta_n = \frac{\theta_{n+1} + \theta_n}{2}. \tag{2.34}$$

Subtracting the complex conjugate of (2.33) from (2.33), we have

$$\frac{Y_{n+1} - Y_n}{a} = \sin \Theta_n. \tag{2.35}$$

Therefore we obtain

$$\frac{\gamma_{n+1} - \gamma_n}{a} = \begin{pmatrix} \cos \Theta_n \\ \sin \Theta_n \end{pmatrix}, \tag{2.36}$$

which gives (2.8). It is easy to see that (2.9) follows from (2.36). In order to show (2.11), it is convenient to identify $\mathbb{R}^2$ as $\mathbb{C}$. Then by using (2.12) and (2.36), we see that (2.11) is rewritten as

$$\frac{d\gamma_n}{dt} = e^{\sqrt{-1}W_n} \frac{\gamma_{n+1} - \gamma_n}{a} = e^{\sqrt{-1}\theta_n}. \tag{2.37}$$

Noticing that

$$\gamma_n = X_n + \sqrt{-1} Y_n = -\left(\log \tau_n^*\right)_y, \tag{2.38}$$

the left hand side of (2.37) can be rewritten by using (2.28) as

$$\frac{d\gamma_n}{dt} = -\left(\log \tau_n^*\right)_{ty} = -\frac{\frac{1}{2} D_t D_y \tau_n^* \cdot \tau_n^*}{\tau_n^* \tau_n^*} = e^{\sqrt{-1}\theta_n}, \tag{2.39}$$

which implies (2.37) and thus (2.11). Finally, by using (2.30) and (2.31), we see that

$$\begin{aligned}\frac{d}{dt}\left(\theta_{n+1}+\theta_n\right) &= \frac{2}{\sqrt{-1}}\frac{d}{dt}\left(\log\left(\frac{\tau_{n+1}}{\tau_n^*}\right)-\log\left(\frac{\tau_{n+1}^*}{\tau_n}\right)\right)\\ &= \frac{2}{\sqrt{-1}}\left(\frac{D_t\tau_{n+1}\cdot\tau_n^*}{\tau_{n+1}\tau_n^*}-\frac{D_t\tau_{n+1}^*\cdot\tau_n}{\tau_{n+1}^*\tau_n}\right)\\ &= \frac{1}{a}\frac{2}{\sqrt{-1}}\left(\frac{\tau_{n+1}\tau_n^*}{\tau_{n+1}^*\tau_n}-\frac{\tau_{n+1}^*\tau_n}{\tau_{n+1}\tau_n^*}\right)\\ &= \frac{4}{a}\sin\left(\frac{\theta_{n+1}-\theta_n}{2}\right),\end{aligned}$$

which is the semi-discrete potential mKdV equation. □

2.2.3 Discrete deformation of discrete curves

Let $\tau_n^m = \tau_n^m(y)$ be a complex function depending on two discrete variables n, m and one continuous variable y, which satisfies the following bilinear equations

$$D_y\tau_{n+1}^m\cdot\tau_n^m = -a\tau_{n+1}^{m}{}^*\tau_n^{m*}, \tag{2.40}$$

$$D_y\tau_n^{m+1}\cdot\tau_n^m = -b\tau_n^{m+1*}\tau_n^{m*}, \tag{2.41}$$

$$b\tau_n^{m+1*}\tau_{n+1}^m - a\tau_{n+1}^{m}{}^*\tau_n^{m+1} + c\tau_{n+1}^{m+1*}\tau_n^m = 0, \tag{2.42}$$

Let τ_n^m be a solution to (2.40)-(2.42). Define a real function $\theta_n^m(y)$ and an $\mathbb{R}^2$ valued function $\gamma_n^m(y)$ by

$$\theta_n^m(y) = \frac{2}{\sqrt{-1}}\log\frac{\tau_n^m}{\tau_n^{m*}}, \tag{2.43}$$

$$\gamma_n^m(y) = \begin{pmatrix} -\frac{1}{2}\left(\log\tau_n^m\tau_n^{m*}\right)_y \\ \frac{1}{2\sqrt{-1}}\left(\log\frac{\tau_n^m}{\tau_n^{m*}}\right)_y \end{pmatrix}. \tag{2.44}$$

Then for any $y\in\mathbb{R}$ and n, $m\in\mathbb{Z}$, the functions θ_n^m and γ_n^m satisfy (2.15)-(2.17) and (2.19) [4].

2.3 Explicit Solutions

In this section, we construct the solutions to the bilinear equations in terms of the elliptic theta functions. Let $\vartheta_i(v) = \vartheta_i(v; w)$ $(v, w\in\mathbb{C}$, $\mathrm{Im}(w) > 0$, $i = 1, 2, 3, 4)$ are the elliptic theta functions defined as follows [48, 49].

$$\vartheta_1(v,w) := -\sum_{n=-\infty}^{\infty} e^{\pi i w\left(n+\frac{1}{2}\right)^2+2\pi i\left(n+\frac{1}{2}\right)\left(v+\frac{1}{2}\right)}$$
$$= 2\sum_{n=0}^{\infty}(-1)^n e^{\pi i w\left(n+\frac{1}{2}\right)^2}\sin\left((2n+1)\pi v\right), \tag{2.45}$$

$$\vartheta_2(v,w) := \sum_{n=-\infty}^{\infty} e^{\pi i w\left(n+\frac{1}{2}\right)^2+2\pi i\left(n+\frac{1}{2}\right)v}$$
$$= 2\sum_{n=0}^{\infty} e^{\pi i w\left(n+\frac{1}{2}\right)^2}\cos\left((2n+1)\pi v\right), \tag{2.46}$$

$$\vartheta_3(v,w) := \sum_{n=-\infty}^{\infty} e^{\pi i w n^2+2\pi i n v}$$
$$= 1+2\sum_{n=1}^{\infty} e^{\pi i w n^2}\cos\left(2n\pi v\right), \tag{2.47}$$

$$\vartheta_4(v,w) := \sum_{n=-\infty}^{\infty} e^{\pi i w n^2+2\pi i n\left(v+\frac{1}{2}\right)}$$
$$= 1+2\sum_{n=1}^{\infty}(-1)^n e^{\pi i w n^2}\cos\left(2n\pi v\right). \tag{2.48}$$

It follows immediately from the definition that ϑ_1 is an odd function for v and that $\vartheta_2, \vartheta_3, \vartheta_4$ are even functions for v. Then define constants $w_1, w_2 \in \mathbb{C}$ as

$$w_1 = \sqrt{-1}r, \quad w_2 = \frac{1}{2} + \sqrt{-1}r, \quad r \in \mathbb{R}_{>0}. \tag{2.49}$$

2.3.1 Continuous deformation of elastica

1. Let $w = w_1$. Define constants h_1, c_1, Q_1 and q_1 as follows

$$h_1 = \frac{\vartheta_3''(0)}{\vartheta_3(0)} + \frac{\vartheta_1'(0)^2}{\vartheta_4(0)^2}\frac{\vartheta_2(0)^2}{\vartheta_3(0)^2}, \quad c_1 = \frac{\vartheta_4(0)^2}{\vartheta_1'(0)^2},$$
$$Q_1 = -\frac{1}{4}\left(\frac{\vartheta_2''(0)}{\vartheta_2(0)} + \frac{\vartheta_3''(0)}{\vartheta_3(0)}\right), \tag{2.50}$$
$$q_1 = 3\frac{\vartheta_3''(0)}{\vartheta_3(0)} + 3\frac{\vartheta_2''(0)}{\vartheta_2(0)} - \frac{\vartheta_1'''(0)}{\vartheta_1'(0)} - 3\frac{\vartheta_4''(0)}{\vartheta_4(0)}.$$

2. Let $w = w_2$. Define constants h_2, c_2, Q_2 and q_2 as follows

$$
\begin{aligned}
h_2 &= \frac{\vartheta_3''(0)}{\vartheta_3(0)} - \frac{\vartheta_1'(0)^2}{\vartheta_2(0)^2}\frac{\vartheta_4(0)^2}{\vartheta_3(0)^2}, \quad c_2 = \frac{\vartheta_2(0)^2}{\vartheta_1'(0)^2}, \\
Q_2 &= -\frac{1}{4}\left(\frac{\vartheta_3''(0)}{\vartheta_3(0)} + \frac{\vartheta_4''(0)}{\vartheta_4(0)}\right), \\
q_2 &= 3\frac{\vartheta_3''(0)}{\vartheta_3(0)} + 3\frac{\vartheta_4''(0)}{\vartheta_4(0)} - \frac{\vartheta_1'''(0)}{\vartheta_1'(0)} - 3\frac{\vartheta_2''(0)}{\vartheta_2(0)}.
\end{aligned}
\tag{2.51}
$$

Theorem 2. Let $w = w_1$. Consider the τ function

$$
\begin{aligned}
\tau &= \exp\left\{\frac{\pi i}{2}\left(z_{11} - \frac{w}{4}\right) - h_1 c_1 xy + Q_1 p^2 x^2\right\} \\
&\quad \times \vartheta_3\left(z_{11}\right), \\
z_{11} &= px + q_1 p^3 t + \frac{c_1}{p} y - \frac{1}{2} + \frac{w}{4},
\end{aligned}
\tag{2.52}
$$

where $p \in \mathbb{R} \setminus \{0\}$ is a parameter. Then, (2.52) and its complex conjugate satisfy the bilinear equations (2.22)-(2.24).

Theorem 3. Let $w = w_2$. Consider the τ function

$$
\begin{aligned}
\tau &= \exp\left(\frac{\pi i}{4} - h_2 c_2 xy + Q_2 p^2 x^2\right)\vartheta_3\left(z_{12}\right), \\
z_{12} &= px + q_2 p^3 t + \frac{c_2}{p} y + \frac{1}{4}.
\end{aligned}
\tag{2.53}
$$

Then, (2.53) and its complex conjugate satisfy the bilinear equations (2.22)-(2.24).

To prove Theorem 2, we first prove the following Lemmas.

Lemma 1. Let $\omega = \omega_1$, then for any $z \in \mathbb{C}$

$$
\vartheta_k(z)^* = \vartheta_k(z^*), \quad k = 1, 2, 3, 4, \tag{2.54}
$$

holds.

Lemma 2. For any $X \in \mathbb{C}$,

$$
D_X \vartheta_3(X) \cdot \vartheta_2(X) = \frac{\vartheta_1'(0)\vartheta_4(0)}{\vartheta_2(0)\vartheta_3(0)}\vartheta_1(X)\vartheta_4(X), \tag{2.55}
$$

$$
{D_X}^2 \vartheta_3(X) \cdot \vartheta_2(X) = \left(\frac{\vartheta_2''(0)}{\vartheta_2(0)} + \frac{\vartheta_3''(0)}{\vartheta_3(0)}\right)\vartheta_2(X)\vartheta_3(X), \tag{2.56}
$$

$$D_X{}^3\vartheta_3(X)\cdot\vartheta_2(X) = \left(\frac{\vartheta_1'''(0)\vartheta_4(0)}{\vartheta_2(0)\vartheta_3(0)} + 3\frac{\vartheta_1'(0)\vartheta_4''(0)}{\vartheta_2(0)\vartheta_3(0)}\right)\vartheta_1(X)\vartheta_4(X), \tag{2.57}$$

$$\frac{1}{2}D_X{}^2\vartheta_3(X)\cdot\vartheta_3(X) = \left(\frac{\vartheta_3''(0)}{\vartheta_3(0)} + \frac{\vartheta_1'(0)^2}{\vartheta_4(0)^2}\frac{\vartheta_2(0)^2}{\vartheta_3(0)^2}\right)\vartheta_3(X)^2 - \frac{\vartheta_1'(0)^2}{\vartheta_4(0)^2}\vartheta_2(X)^2, \tag{2.58}$$

hold.

Proof (proof of Lemma 1)**.** Consider the following theta function with characteristics.

$$\vartheta_{a,b}(z) := \sum_{n\in\mathbb{Z}} \exp\left(2\pi i(n+a)(z+b) + \pi i(n+a)^2\omega\right), \quad a,b\in\left\{0,\frac{1}{2}\right\}. \tag{2.59}$$

Then,

$$\begin{aligned} &\vartheta_1(z) = -\vartheta_{\frac{1}{2},\frac{1}{2}}(z), \quad \vartheta_2(z) = \vartheta_{\frac{1}{2},0}(z), \\ &\vartheta_3(z) = \vartheta_{0,0}(z), \quad \vartheta_4(z) = \vartheta_{0,\frac{1}{2}}(z). \end{aligned} \tag{2.60}$$

Since $\omega^* = -\omega$,

$$\begin{aligned} \vartheta_{a,b}(z)^* &= \sum_{n\in\mathbb{Z}} \exp\left(-2\pi i(n+a)(z^*+b) - \pi i(n+a)^2\omega^*\right) \\ &= \sum_{n\in\mathbb{Z}} \exp\left(2\pi i(-n-a)(z^*+b) + \pi i(n+a)^2\omega\right) \\ &= \sum_{n\in\mathbb{Z}} \exp\left(2\pi i(n+a)(z^*+b) + \pi i(n+a)^2\omega\right) \\ &= \vartheta_{a,b}(z^*), \end{aligned} \tag{2.61}$$

□

From Lemma 1, we see that the constants h_1, c_1, q_1, Q_1 defined in (2.50) are real numbers.

Proof (proof of Lemma 2)**.** For any $x, y, u, v \in \mathbb{C}$ and

$$\begin{aligned} x_1 &= \frac{1}{2}(x+y+u+v), \quad y_1 = \frac{1}{2}(x+y-u-v), \\ u_1 &= \frac{1}{2}(x-y+u-v), \quad v_1 = \frac{1}{2}(x-y-u+v), \end{aligned} \tag{2.62}$$

the elliptic theta functions satisfy the following identities [48, 49].

$$\begin{aligned} 2\vartheta_3(x_1)\vartheta_3(y_1)\vartheta_2(u_1)\vartheta_2(v_1) =& \vartheta_3(x)\vartheta_3(y)\vartheta_2(u)\vartheta_2(v) + \vartheta_4(x)\vartheta_4(y)\vartheta_1(u)\vartheta_1(v) \\ &+\vartheta_2(x)\vartheta_2(y)\vartheta_3(u)\vartheta_3(v) + \vartheta_1(x)\vartheta_1(y)\vartheta_4(u)\vartheta_4(v), \end{aligned} \tag{2.63}$$

$$
\begin{aligned}
2\vartheta_4(x_1)\vartheta_4(y_1)\vartheta_1(u_1)\vartheta_1(v_1) =&\vartheta_3(x)\vartheta_3(y)\vartheta_2(u)\vartheta_2(v) + \vartheta_4(x)\vartheta_4(y)\vartheta_1(u)\vartheta_1(v) \\
&-\vartheta_2(x)\vartheta_2(y)\vartheta_3(u)\vartheta_3(v) - \vartheta_1(x)\vartheta_1(y)\vartheta_4(u)\vartheta_4(v),
\end{aligned} \tag{2.64}
$$

Putting $x = u = X, y = v = U$ in (2.63) and (2.64) yields

$$
\begin{aligned}
&\vartheta_3(X+U)\vartheta_3(0)\vartheta_2(X-U)\vartheta_2(0) \\
&= \vartheta_3(X)\vartheta_2(X)\vartheta_3(U)\vartheta_2(U) + \vartheta_4(X)\vartheta_1(X)\vartheta_4(U)\vartheta_1(U).
\end{aligned} \tag{2.65}
$$

After differentiating (2.65) by U and setting $U = 0$, we obtain

$$
\vartheta_2(0)\vartheta_3(0)\left(\vartheta_3'(X)\vartheta_2(X) - \vartheta_3(X)\vartheta_2'(X)\right) = \vartheta_1'(0)\vartheta_4(0)\vartheta_1(X)\vartheta_4(X), \tag{2.66}
$$

which yields

$$
D_X\vartheta_3(X)\cdot\vartheta_2(X) = \frac{\vartheta_1'(0)\vartheta_4(0)}{\vartheta_2(0)\vartheta_3(0)}\vartheta_1(X)\vartheta_4(X). \tag{2.67}
$$

Similarly, differentiating both sides of (2.65) by U twice and three times, and setting $U = 0$, we obtain

$$
{D_X}^2\vartheta_3(X)\cdot\vartheta_2(X) = \left(\frac{\vartheta_2''(0)}{\vartheta_2(0)} + \frac{\vartheta_3''(0)}{\vartheta_3(0)}\right)\vartheta_2(X)\vartheta_3(X), \tag{2.68}
$$

and

$$
{D_X}^3\vartheta_3(X)\cdot\vartheta_2(X) = \left(\frac{\vartheta_1'''(0)\vartheta_4(0)}{\vartheta_2(0)\vartheta_3(0)} + 3\frac{\vartheta_1'(0)\vartheta_4''(0)}{\vartheta_2(0)\vartheta_3(0)}\right)\vartheta_1(X)\vartheta_4(X), \tag{2.69}
$$

respectively. Also, we have the following identities [48, 49].

$$
\begin{aligned}
2\vartheta_3(x_1)\vartheta_3(y_1)\vartheta_3(u_1)\vartheta_3(v_1) =&\vartheta_3(x)\vartheta_3(y)\vartheta_3(u)\vartheta_3(v) + \vartheta_4(x)\vartheta_4(y)\vartheta_4(u)\vartheta_4(v) \\
&+\vartheta_2(x)\vartheta_2(y)\vartheta_2(u)\vartheta_2(v) + \vartheta_1(x)\vartheta_1(y)\vartheta_1(u)\vartheta_1(v),
\end{aligned} \tag{2.70}
$$

$$
\begin{aligned}
2\vartheta_1(x_1)\vartheta_1(y_1)\vartheta_1(u_1)\vartheta_1(v_1) =&\vartheta_3(x)\vartheta_3(y)\vartheta_3(u)\vartheta_3(v) - \vartheta_4(x)\vartheta_4(y)\vartheta_4(u)\vartheta_4(v) \\
&-\vartheta_2(x)\vartheta_2(y)\vartheta_2(u)\vartheta_2(v) + \vartheta_1(x)\vartheta_1(y)\vartheta_1(u)\vartheta_1(v).
\end{aligned} \tag{2.71}
$$

Putting $x = y = X, u = v = U$ in (2.70) and (2.71) yields

$$
\vartheta_3(X+U)\vartheta_3(X-U)\vartheta_3(0)^2 = \vartheta_3(X)^2\vartheta_3(U)^2 + \vartheta_1(X)^2\vartheta_1(U)^2. \tag{2.72}
$$

After differentiating (2.72) by U twice and setting $U = 0$, we obtain

$$
2\vartheta_3(0)^2\left(\vartheta_3''(X)\vartheta_3(X) - \vartheta_3'(X)^2\right) = 2\vartheta_3''(0)\vartheta_3(0)\vartheta_3(X)^2 + 2\vartheta_1'(0)^2\vartheta_1(X)^2. \tag{2.73}
$$

Also, putting $x = y = X, u = v = 0$ in the following identity,

$$\begin{aligned} 2\vartheta_4(x_1)\vartheta_4(y_1)\vartheta_1(u_1)\vartheta_1(v_1) =& \vartheta_3(x)\vartheta_3(y)\vartheta_2(u)\vartheta_2(v) + \vartheta_4(x)\vartheta_4(y)\vartheta_1(u)\vartheta_1(v) \\ & -\vartheta_2(x)\vartheta_2(y)\vartheta_3(u)\vartheta_3(v) - \vartheta_1(x)\vartheta_1(y)\vartheta_4(u)\vartheta_4(v), \end{aligned} \tag{2.74}$$

we have

$$\vartheta_3(X)^2\vartheta_2(0)^2 - \vartheta_2(X)^2\vartheta_3(0)^2 = \vartheta_1(X)^2\vartheta_4(0)^2. \tag{2.75}$$

Substituting (2.75) into (2.73) yields

$$\frac{1}{2}D_X{}^2\vartheta_3(X)\cdot\vartheta_3(X) = \left(\frac{\vartheta_3''(0)}{\vartheta_3(0)} + \frac{\vartheta_1'(0)^2}{\vartheta_4(0)^2}\frac{\vartheta_2(0)^2}{\vartheta_3(0)^2}\right)\vartheta_3(X)^2 - \frac{\vartheta_1'(0)^2}{\vartheta_4(0)^2}\vartheta_2(X)^2. \tag{2.76}$$

□

Proof (Proof of Theorem 2)**.** Denote $Y = x + t + y + \frac{\omega}{4}$. Hirota's bilinear differential operators satisfy the following identities [47].

$$D_x{}^2(FG)\cdot(FH) = (D_x{}^2F\cdot F)GH + F^2(D_x{}^2G\cdot H), \tag{2.77}$$

$$D_x{}^2(e^{Q_1x^2})\cdot(e^{Q_1x^2}) = 4Q_1e^{2Q_1x^2}. \tag{2.78}$$

We see from (2.77), (2.78) that (2.56) is rewritten as

$$D_x{}^2\left(e^{Q_1x^2}\vartheta_3(Y)\right)\cdot\left(e^{Q_1x^2}\vartheta_2(Y)\right) = 0, \tag{2.79}$$

Also from

$$D_x{}^3(FG)\cdot(FH) = (D_x{}^3G\cdot H)F^2 + 3(D_xG\cdot H)(D_x{}^2F\cdot F), \tag{2.80}$$

and (2.78), we see that

$$\begin{aligned} &\left(D_x{}^3 + q_1D_t\right)\left(e^{Q_1x^2}\vartheta_3(Y)\right)\cdot\left(e^{Q_1x^2}\vartheta_2(Y)\right) \\ &= e^{2Q_1x^2}\left\{D_Y{}^3\vartheta_3(Y)\cdot\vartheta_2(Y) + (12Q_1 + q_1)D_Y\vartheta_3(Y)\cdot\vartheta_2(Y)\right\}, \end{aligned} \tag{2.81}$$

holds. From (2.55) and (2.57), we see that

$$\left(D_x{}^3 + q_1D_t\right)\left(e^{Q_1x^2}\vartheta_3(Y)\right)\cdot\left(e^{Q_1x^2}\vartheta_2(Y)\right) = 0, \tag{2.82}$$

holds, where q_1 is given by

$$q_1 = 3\frac{\vartheta_3''(0)}{\vartheta_3(0)} + 3\frac{\vartheta_2''(0)}{\vartheta_2(0)} - \frac{\vartheta_1'''(0)}{\vartheta_1'(0)} - 3\frac{\vartheta_4''(0)}{\vartheta_4(0)}, \tag{2.83}$$

Also from

$$\frac{\partial^2}{\partial x\partial y}\log F = \frac{D_xD_yF\cdot F}{2F^2}, \tag{2.84}$$

we see that

$$\frac{\partial^2}{\partial x\partial y}\log e^{-h_1xy}F=-h_1+\frac{D_xD_yF\cdot F}{2F^2}=\frac{D_xD_ye^{-h_1xy}F\cdot e^{-h_1xy}F}{2e^{-2h_1xy}F^2}, \tag{2.85}$$

holds. From (2.85) and (2.58), we see that

$$\begin{aligned}&\frac{1}{2}D_xD_y\vartheta_3(Y)\cdot\vartheta_3(Y)\\=&\frac{1}{2}{D_Y}^2\vartheta_3(Y)\cdot\vartheta_3(Y)=\left(\frac{\vartheta_3''(0)}{\vartheta_3(0)}+\frac{\vartheta_1'(0)^2}{\vartheta_4(0)^2}\frac{\vartheta_2(0)^2}{\vartheta_3(0)^2}\right)\vartheta_3(Y)^2-\frac{\vartheta_1'(0)^2}{\vartheta_4(0)^2}\vartheta_2(Y)^2,\end{aligned} \tag{2.86}$$

and

$$\frac{1}{2}D_xD_y\left(e^{-h_1xy}\vartheta_3(Y)\right)\cdot\left(e^{-h_1xy}\vartheta_3(Y)\right)=-e^{-2h_1xy}\frac{\vartheta_1'(0)^2}{\vartheta_4(0)^2}\vartheta_2(Y)^2, \tag{2.87}$$

hold. From the above calculations, we see that (2.52) and its complex conjugate satisfy the bilinear equations (2.22)-(2.24). □

Next, we prove the following two Lemmas to prove Theorem 3.

Lemma 3. Let $\omega=\omega_2$, then for any $z\in\mathbb{C}$

$$\begin{aligned}\vartheta_1(z)^*&=\exp\left(-\frac{\pi i}{4}\right)\vartheta_1(z^*),\\\vartheta_2(z)^*&=\exp\left(-\frac{\pi i}{4}\right)\vartheta_2(z^*),\\\vartheta_3(z)^*&=\vartheta_4(z^*),\\\vartheta_4(z)^*&=\vartheta_3(z^*),\end{aligned} \tag{2.88}$$

hold.

Lemma 4. For any $X\in\mathbb{C}$, the following identities hold.

$$D_X\vartheta_3(X)\cdot\vartheta_4(X)=-\frac{\vartheta_1'(0)\vartheta_2(0)}{\vartheta_3(0)\vartheta_4(0)}\vartheta_1(X)\vartheta_2(X), \tag{2.89}$$

$${D_X}^2\vartheta_3(X)\cdot\vartheta_4(X)=\left(\frac{\vartheta_3''(0)}{\vartheta_3(0)}+\frac{\vartheta_4''(0)}{\vartheta_4(0)}\right)\vartheta_3(X)\vartheta_4(X), \tag{2.90}$$

$${D_X}^3\vartheta_3(X)\cdot\vartheta_4(X)=-\left(\frac{\vartheta_1'''(0)\vartheta_2(0)}{\vartheta_3(0)\vartheta_4(0)}+3\frac{\vartheta_1'(0)\vartheta_2''(0)}{\vartheta_3(0)\vartheta_4(0)}\right)\vartheta_1(X)\vartheta_2(X), \tag{2.91}$$

$$\frac{1}{2}{D_X}^2\vartheta_3(X)\cdot\vartheta_3(X)=\left(\frac{\vartheta_3''(0)}{\vartheta_3(0)}-\frac{\vartheta_1'(0)^2}{\vartheta_2(0)^2}\frac{\vartheta_4(0)^2}{\vartheta_3(0)^2}\right)\vartheta_3(X)^2+\frac{\vartheta_1'(0)^2}{\vartheta_2(0)^2}\vartheta_4(X)^2, \tag{2.92}$$

Proof (proof of Lemma 3)**.** Let $\omega = \omega_2$, then $\omega^* = 1 - \omega$ and

$$\begin{aligned}
-\vartheta_1(z)^* =& \sum_{n\in\mathbb{Z}} \exp\left(-2\pi i\left(n+\frac{1}{2}\right)\left(z^*+\frac{1}{2}\right) - \pi i\left(n+\frac{1}{2}\right)^2 \omega^*\right) \\
=& \sum_{n\in\mathbb{Z}} \exp\left(2\pi i\left(-n-\frac{1}{2}\right)\left(z^*+\frac{1}{2}\right) - \pi i\left(n+\frac{1}{2}\right)^2 (1-\omega)\right) \\
=& \sum_{n\in\mathbb{Z}} \exp\left(2\pi i\left(-n-\frac{1}{2}\right)\left(z^*+\frac{1}{2}\right) + \pi i\left(n+\frac{1}{2}\right)^2 \omega - \pi i\left(n+\frac{1}{2}\right)^2\right),
\end{aligned} \tag{2.93}$$

hold. Since

$$\exp\left(-\pi i\left(n+\frac{1}{2}\right)^2\right) = \exp\left(-\pi i\left(n^2+n\right) - \frac{\pi i}{4}\right) = \exp\left(-\frac{\pi i}{4}\right), \tag{2.94}$$

holds, we have

$$\begin{aligned}
-\vartheta_1(z)^* =& \exp\left(-\frac{\pi i}{4}\right)\sum_{n\in\mathbb{Z}} \exp\left(2\pi i\left(-n-\frac{1}{2}\right)\left(z^*+\frac{1}{2}\right) + \pi i\left(n+\frac{1}{2}\right)^2 \omega\right) \\
=& \exp\left(-\frac{\pi i}{4}\right)\sum_{n'\in\mathbb{Z}} \exp\left(2\pi i\left(n'+\frac{1}{2}\right)\left(z^*+\frac{1}{2}\right) + \pi i\left(n'+\frac{1}{2}\right)^2 \omega\right) \\
=& -\exp\left(-\frac{\pi i}{4}\right)\vartheta_1(z^*), \quad (n' = -n-1).
\end{aligned} \tag{2.95}$$

Also, from $\vartheta_1\left(z-\frac{1}{2}\right) = -\vartheta_2(z)$

$$-\vartheta_2(z)^* = \vartheta_1\left(z-\frac{1}{2}\right)^* = \exp\left(-\frac{\pi i}{4}\right)\vartheta_1\left(z^*-\frac{1}{2}\right) = -\exp\left(-\frac{\pi i}{4}\right)\vartheta_2(z^*), \tag{2.96}$$

holds. We also see that

$$\begin{aligned}
\vartheta_3(z)^* =& \sum_{n\in\mathbb{Z}} \exp\left(-2\pi i z^* - \pi i n^2 \omega^*\right) \\
=& \sum_{n\in\mathbb{Z}} \exp\left(-2\pi i n z^* + \pi i n^2 \omega - \pi i n^2\right), \\
=& \sum_{n\in\mathbb{Z}} \exp\left(-2\pi i n\left(z^*+\frac{1}{2}\right) + \pi i n^2 \omega\right) \\
=& \sum_{n\in\mathbb{Z}} \exp\left(2\pi i n\left(z^*+\frac{1}{2}\right) + \pi i n^2 \omega\right) = \vartheta_4(z^*),
\end{aligned} \tag{2.97}$$

holds. □

From Lemma 3, we see that the constants h_2, c_2, q_2, Q_2 defined in (2.51) are real numbers.

Proof (proof of Lemma 4)**.** The elliptic theta functions satisfy the following identities [48, 49],

$$\begin{aligned}2\vartheta_3(x_1)\vartheta_3(y_1)\vartheta_4(u_1)\vartheta_4(v_1) =&\vartheta_3(x)\vartheta_3(y)\vartheta_4(u)\vartheta_4(v) + \vartheta_4(x)\vartheta_4(y)\vartheta_3(u)\vartheta_3(v)\\ &-\vartheta_2(x)\vartheta_2(y)\vartheta_1(u)\vartheta_1(v) - \vartheta_1(x)\vartheta_1(y)\vartheta_2(u)\vartheta_2(v),\end{aligned} \tag{2.98}$$

$$\begin{aligned}-2\vartheta_2(x_1)\vartheta_2(y_1)\vartheta_1(u_1)\vartheta_1(v_1) =&\vartheta_3(x)\vartheta_3(y)\vartheta_4(u)\vartheta_4(v) - \vartheta_4(x)\vartheta_4(y)\vartheta_3(u)\vartheta_3(v)\\ &-\vartheta_2(x)\vartheta_2(y)\vartheta_1(u)\vartheta_1(v) + \vartheta_1(x)\vartheta_1(y)\vartheta_2(u)\vartheta_2(v),\end{aligned} \tag{2.99}$$

Putting $x = u = X, y = v = U$ in (2.98) and (2.99) yields

$$\begin{aligned}&\vartheta_3(X+U)\vartheta_3(0)\vartheta_4(X-U)\vartheta_4(0)\\ &= \vartheta_3(X)\vartheta_4(X)\vartheta_3(U)\vartheta_4(U) - \vartheta_1(X)\vartheta_2(X)\vartheta_1(U)\vartheta_2(U).\end{aligned} \tag{2.100}$$

After differentiating (2.100) by U and setting $U = 0$, we obtain

$$\vartheta_3(0)\vartheta_4(0)\left(\vartheta_3'(X)\vartheta_4(X) - \vartheta_3(X)\vartheta_4'(X)\right) = -\vartheta_1'(0)\vartheta_2(0)\vartheta_1(X)\vartheta_2(X), \tag{2.101}$$

From (2.101) we get

$$D_X\vartheta_3(X)\cdot\vartheta_4(X) = -\frac{\vartheta_1'(0)\vartheta_2(0)}{\vartheta_3(0)\vartheta_4(0)}\vartheta_1(X)\vartheta_2(X). \tag{2.102}$$

Similarly, differentiating both sides of (2.100) by U twice and three times, and setting $U = 0$, we obtain

$$D_X{}^2\vartheta_3(X)\cdot\vartheta_4(X) = \left(\frac{\vartheta_3''(0)}{\vartheta_3(0)} + \frac{\vartheta_4''(0)}{\vartheta_4(0)}\right)\vartheta_3(X)\vartheta_4(X), \tag{2.103}$$

$$D_X{}^3\vartheta_3(X)\cdot\vartheta_4(X) = -\left(\frac{\vartheta_1'''(0)\vartheta_2(0)}{\vartheta_3(0)\vartheta_4(0)} + 3\frac{\vartheta_1'(0)\vartheta_2''(0)}{\vartheta_3(0)\vartheta_4(0)}\right)\vartheta_1(X)\vartheta_2(X). \tag{2.104}$$

Also, we have the following identity [48, 49],

$$\begin{aligned}-2\vartheta_2(x_1)\vartheta_2(y_1)\vartheta_1(u_1)\vartheta_1(v_1) =&\vartheta_3(x)\vartheta_3(y)\vartheta_4(u)\vartheta_4(v) - \vartheta_4(x)\vartheta_4(y)\vartheta_3(u)\vartheta_3(v)\\ &-\vartheta_2(x)\vartheta_2(y)\vartheta_1(u)\vartheta_1(v) + \vartheta_1(x)\vartheta_1(y)\vartheta_2(u)\vartheta_2(v),\end{aligned} \tag{2.105}$$

Putting $x = y = X, u = v = 0$ into (2.105) yields

$$\vartheta_4(X)^2\vartheta_3(0)^2 - \vartheta_3(X)^2\vartheta_4(0)^2 = \vartheta_1(X)^2\vartheta_2(0)^2, \tag{2.106}$$

Substituting (2.106) into (2.73) yields

$$\frac{1}{2}{D_X}^2\vartheta_3(X)\cdot\vartheta_3(X)=\left(\frac{\vartheta_3''(0)}{\vartheta_3(0)}-\frac{\vartheta_1'(0)^2}{\vartheta_2(0)^2}\frac{\vartheta_4(0)^2}{\vartheta_3(0)^2}\right)\vartheta_3(X)^2+\frac{\vartheta_1'(0)^2}{\vartheta_2(0)^2}\vartheta_4(X)^2. \tag{2.107}$$

□

Proof (proof of Theorem 3)**.** Denote $Y=x+t+y$. From (2.90),

$${D_x}^2\left(e^{Q_2x^2}\vartheta_3(Y)\right)\cdot\left(e^{Q_2x^2}\vartheta_4(Y)\right)=0, \tag{2.108}$$

holds. Also, we see that

$$\begin{aligned}&\left({D_x}^3+q_2D_t\right)\left(e^{Q_2x^2}\vartheta_3(Y)\right)\cdot\left(e^{Q_2x^2}\vartheta_4(Y)\right)\\&=e^{2Q_2x^2}\left\{{D_Y}^3\vartheta_3(Y)\cdot\vartheta_4(Y)+(12Q_2+q_2)D_Y\vartheta_3(Y)\cdot\vartheta_4(Y)\right\},\end{aligned} \tag{2.109}$$

holds. From (2.109), (2.89) and (2.91), we see that

$$\left({D_x}^3+q_2D_t\right)\left(e^{Q_2x^2}\vartheta_3(Y)\right)\cdot\left(e^{Q_2x^2}\vartheta_4(Y)\right)=0, \tag{2.110}$$

holds, where q_2 is given by

$$q_2=3\frac{\vartheta_3''(0)}{\vartheta_3(0)}+3\frac{\vartheta_4''(0)}{\vartheta_4(0)}-\frac{\vartheta_1'''(0)}{\vartheta_1'(0)}-3\frac{\vartheta_2''(0)}{\vartheta_2(0)}, \tag{2.111}$$

Also from (2.92),

$$\begin{aligned}&\frac{1}{2}D_xD_y\vartheta_3(Y)\cdot\vartheta_3(Y)\\=&\frac{1}{2}{D_Y}^2\vartheta_3(Y)\cdot\vartheta_3(Y)=\left(\frac{\vartheta_3''(0)}{\vartheta_3(0)}-\frac{\vartheta_1'(0)^2}{\vartheta_2(0)^2}\frac{\vartheta_4(0)^2}{\vartheta_3(0)^2}\right)\vartheta_3(Y)^2+\frac{\vartheta_1'(0)^2}{\vartheta_2(0)^2}\vartheta_4(Y)^2,\end{aligned} \tag{2.112}$$

holds and thus

$$\frac{1}{2}D_xD_y\left(e^{-h_2xy}\vartheta_3(Y)\right)\cdot\left(e^{-h_2xy}\vartheta_3(Y)\right)=e^{-2h_2xy}\frac{\vartheta_1'(0)^2}{\vartheta_2(0)^2}\vartheta_4(Y)^2, \tag{2.113}$$

From the above calculations, we see that (2.53) and its complex conjugate satisfy the bilinear equations (2.22)-(2.24). □

2.3.2 Continuous deformation of discrete elastica

1. Let $w = w_1$. Define parameters ϵ_1, δ_1 and a_1 as follows

$$\begin{aligned} \epsilon_1(\alpha) &= \frac{\vartheta_1'(\alpha)\vartheta_4(0)}{\vartheta_1'(0)\vartheta_1(\alpha)}, \quad \delta_1(\alpha) = \frac{\vartheta_4(0)\vartheta_4'(\alpha)}{\vartheta_1'(0)\vartheta_4(\alpha)}, \\ a_1(\alpha) &= \frac{1}{p}\frac{\vartheta_1(\alpha)}{\vartheta_4(\alpha)}, \end{aligned} \tag{2.114}$$

where $\alpha \in \mathbb{R}\backslash(1/2)\mathbb{Z}$ is a parameter.

2. Let $w = w_2$. Define parameters ϵ_2, δ_2 and a_2 as follows

$$\begin{aligned} \epsilon_2(\alpha) &= \frac{\vartheta_1'(\alpha)\vartheta_2(0)}{\vartheta_1'(0)\vartheta_1(\alpha)}, \quad \delta_2(\alpha) = \frac{\vartheta_2(0)\vartheta_2'(\alpha)}{\vartheta_1'(0)\vartheta_2(\alpha)}, \\ a_2(\alpha) &= \frac{1}{p}\frac{\vartheta_1(\alpha)}{\vartheta_2(\alpha)}. \end{aligned} \tag{2.115}$$

Theorem 4. Let $w = w_1$. Consider the τ function

$$\begin{aligned} \tau_n &= \exp\left[\frac{\pi i}{2}\left(z_{21} - \frac{w}{4}\right) - h_1 c_1 t y\right] \\ &\times \exp\left[-n\left(\epsilon_1(\alpha)pt + \delta_1(\alpha)\frac{y}{p}\right)\right]\vartheta_3\left(z_{21}\right), \\ z_{21} &= \alpha n + \frac{\vartheta_4(0)}{\vartheta_1'(0)}\left(pt + \frac{y}{p}\right) - \frac{1}{2} + \frac{w}{4}, \end{aligned} \tag{2.116}$$

Then, (2.116) and its complex conjugate satisfy the bilinear equations (2.28)-(2.30) with $a = a_1(\alpha)$.

Theorem 5. Let $w = w_2$. Consider the τ function

$$\begin{aligned} \tau_n &= \exp\left[\frac{\pi i}{4} - h_2 c_2 t y - n\left(\epsilon_2(\alpha)pt + \delta_2(\alpha)\frac{y}{p}\right)\right] \\ &\times \vartheta_3\left(z_{22}\right), \\ z_{22} &= \alpha n + \frac{\vartheta_2(0)}{\vartheta_1'(0)}\left(pt + \frac{y}{p}\right) + \frac{1}{4}. \end{aligned} \tag{2.117}$$

Then, (2.117) and its complex conjugate satisfy the bilinear equations (2.28)-(2.30) with $a = a_2(\alpha)$.

To prove Theorem 4, we first show the following Lemma

Lemma 5. For any $X, \alpha \in \mathbb{C}$, the following identities hold.

$$\begin{aligned} &D_X \vartheta_3(X+\alpha)\cdot\vartheta_3(X) \\ &= \frac{\vartheta_4'(\alpha)}{\vartheta_4(\alpha)}\vartheta_3(X+\alpha)\vartheta_3(X) - \frac{\vartheta_1'(0)\vartheta_1(\alpha)}{\vartheta_4(0)\vartheta_4(\alpha)}\vartheta_2(X+\alpha)\vartheta_2(X), \end{aligned} \tag{2.118}$$

$$\begin{aligned} &D_X \vartheta_3(X+\alpha)\cdot\vartheta_2(X) \\ &= \frac{\vartheta_1'(\alpha)}{\vartheta_1(\alpha)}\vartheta_3(X+\alpha)\vartheta_2(X) - \frac{\vartheta_1'(0)\vartheta_4(\alpha)}{\vartheta_1(\alpha)\vartheta_4(0)}\vartheta_2(X+\alpha)\vartheta_3(X), \end{aligned} \tag{2.119}$$

Proof (proof of Lemma 5)**.** The elliptic theta functions satisfy the following identities

$$\begin{aligned} 2\vartheta_3(x_1)\vartheta_3(y_1)\vartheta_4(u_1)\vartheta_4(v_1) =&\vartheta_3(x)\vartheta_3(y)\vartheta_4(u)\vartheta_4(v) + \vartheta_4(x)\vartheta_4(y)\vartheta_3(u)\vartheta_3(v) \\ &-\vartheta_2(x)\vartheta_2(y)\vartheta_1(u)\vartheta_1(v) - \vartheta_1(x)\vartheta_1(y)\vartheta_2(u)\vartheta_2(v), \end{aligned} \tag{2.120}$$

$$\begin{aligned} -2\vartheta_3(x_1)\vartheta_3(y_1)\vartheta_4(u_1)\vartheta_4(v_1) =&\vartheta_3(x)\vartheta_3(y)\vartheta_4(u)\vartheta_4(v) - \vartheta_4(x)\vartheta_4(y)\vartheta_3(u)\vartheta_3(v) \\ &-\vartheta_2(x)\vartheta_2(y)\vartheta_1(u)\vartheta_1(v) + \vartheta_1(x)\vartheta_1(y)\vartheta_2(u)\vartheta_2(v), \end{aligned} \tag{2.121}$$

Putting $x = X, y = X+\alpha, u = U, v = U+\alpha$ in (2.120) and (2.121) yields

$$\begin{aligned} &\vartheta_3(X+\alpha+U)\vartheta_3(X-U)\vartheta_4(\alpha)\vartheta_4(0) \\ &= \vartheta_3(X)\vartheta_3(X+\alpha)\vartheta_4(U)\vartheta_4(U+\alpha) - \vartheta_2(X)\vartheta_2(X+\alpha)\vartheta_1(U)\vartheta_1(U+\alpha), \end{aligned} \tag{2.122}$$

After differentiating (2.122) by U and setting $U = 0$, we obtain

$$\begin{aligned} &\left(\vartheta_3'(X+\alpha)\vartheta_3(X) - \vartheta_3(X+\alpha)\vartheta_3'(X)\right)\vartheta_4(\alpha)\vartheta_4(0) \\ &= \vartheta_3(X)\vartheta_3(X+\alpha)\vartheta_4(0)\vartheta_4'(\alpha) - \vartheta_2(X)\vartheta_2(X+\alpha)\vartheta_1'(0)\vartheta_1(\alpha). \end{aligned} \tag{2.123}$$

Also, we have the following identities

$$\begin{aligned} 2\vartheta_3(x_1)\vartheta_3(y_1)\vartheta_4(u_1)\vartheta_4(v_1) =&\vartheta_3(x)\vartheta_3(y)\vartheta_4(u)\vartheta_4(v) + \vartheta_4(x)\vartheta_4(y)\vartheta_3(u)\vartheta_3(v) \\ &-\vartheta_2(x)\vartheta_2(y)\vartheta_1(u)\vartheta_1(v) - \vartheta_1(x)\vartheta_1(y)\vartheta_2(u)\vartheta_2(v), \end{aligned} \tag{2.124}$$

$$\begin{aligned} -2\vartheta_3(x_1)\vartheta_3(y_1)\vartheta_4(u_1)\vartheta_4(v_1) =&\vartheta_3(x)\vartheta_3(y)\vartheta_4(u)\vartheta_4(v) - \vartheta_4(x)\vartheta_4(y)\vartheta_3(u)\vartheta_3(v) \\ &-\vartheta_2(x)\vartheta_2(y)\vartheta_1(u)\vartheta_1(v) + \vartheta_1(x)\vartheta_1(y)\vartheta_2(u)\vartheta_2(v), \end{aligned} \tag{2.125}$$

Putting $x = X+\alpha, y = -U, u = -X, v = U+\alpha$ in (2.124) and (2.125) yields

$$\begin{aligned} &\vartheta_1(\alpha)\vartheta_4(0)\vartheta_3(X+\alpha+U)\vartheta_2(X-U) \\ &= \vartheta_3(X+\alpha)\vartheta_2(X)\vartheta_1(U+\alpha)\vartheta_4(U) - \vartheta_2(X+\alpha)\vartheta_3(X)\vartheta_1(U)\vartheta_4(U+\alpha). \end{aligned} \tag{2.126}$$

After differentiating (2.126) by U and setting $U = 0$, we obtain

$$\begin{aligned} &\left(\vartheta_3'(X+\alpha)\vartheta_2(X) - \vartheta_3(X+\alpha)\vartheta_2'(X)\right)\vartheta_1(\alpha)\vartheta_4(0) \\ &= \vartheta_3(X+\alpha)\vartheta_2(X)\vartheta_1'(\alpha)\vartheta_4(0) - \vartheta_2(X+\alpha)\vartheta_3(X)\vartheta_1'(0)\vartheta_4(\alpha). \end{aligned} \tag{2.127}$$

□

Proof (proof of Theorem 4)**.** Denote $Y = \alpha n + t + ly + \frac{\omega}{4}$. From (2.58),

$$\begin{aligned}&\frac{1}{2}D_tD_y\vartheta_3(Y)\cdot\vartheta_3(Y)\\ =&l\frac{1}{2}{D_Y}^2\vartheta_3(Y)\cdot\vartheta_3(Y) = l\left(\frac{\vartheta_3''(0)}{\vartheta_3(0)}+\frac{\vartheta_1'(0)^2}{\vartheta_4(0)^2}\frac{\vartheta_2(0)^2}{\vartheta_3(0)^2}\right)\vartheta_3(Y)^2 - l\frac{\vartheta_1'(0)^2}{\vartheta_4(0)^2}\vartheta_2(Y)^2,\end{aligned} \tag{2.128}$$

holds. Also from Hirota's bilinear differential operator's properties, we see that

$$D_x\left(e^{a(n+1)x}F_{n+1}\right)\cdot\left(e^{anx}G_n\right) = e^{a(2n+1)x}\left(aF_{n+1}G_n + D_xF_{n+1}\cdot G_n\right), \tag{2.129}$$

$$D_x\left(e^{a(n+1)x}F_{n+1}\right)\cdot\left(e^{anx}F_n\right) = e^{a(2n+1)x}\left(aF_{n+1}F_n + D_xF_{n+1}\cdot F_n\right), \tag{2.130}$$

hold for any $a \in \mathbb{R}$. From the above calculations, we see that (2.116) and its complex conjugate satisfy the bilinear equations (2.28)-(2.30) with $a = a_1(\alpha)$. □

Next, we show the following Lemma to prove Theorem 5.

Lemma 6. For any $X, \alpha \in \mathbb{C}$, the following identities hold.

$$\begin{aligned}&D_X\vartheta_3(X+\alpha)\cdot\vartheta_3(X)\\ =&\frac{\vartheta_2'(\alpha)}{\vartheta_2(\alpha)}\vartheta_3(X+\alpha)\vartheta_3(X) + \frac{\vartheta_1'(0)\vartheta_1(\alpha)}{\vartheta_2(0)\vartheta_2(\alpha)}\vartheta_4(X+\alpha)\vartheta_4(X),\end{aligned} \tag{2.131}$$

$$\begin{aligned}&D_X\vartheta_3(X+\alpha)\cdot\vartheta_4(X)\\ =&\frac{\vartheta_1'(\alpha)}{\vartheta_1(\alpha)}\vartheta_3(X+\alpha)\vartheta_4(X) - \frac{\vartheta_1'(0)\vartheta_2(\alpha)}{\vartheta_1(\alpha)\vartheta_2(0)}\vartheta_4(X+\alpha)\vartheta_3(X).\end{aligned} \tag{2.132}$$

Proof (proof of Lemma 6)**.** The elliptic theta functions satisfy the following identities

$$\begin{aligned}2\vartheta_3(x_1)\vartheta_3(y_1)\vartheta_2(u_1)\vartheta_2(v_1) =&\vartheta_3(x)\vartheta_3(y)\vartheta_2(u)\vartheta_2(v) + \vartheta_4(x)\vartheta_4(y)\vartheta_1(u)\vartheta_1(v)\\ &+\vartheta_2(x)\vartheta_2(y)\vartheta_3(u)\vartheta_3(v) + \vartheta_1(x)\vartheta_1(y)\vartheta_4(u)\vartheta_4(v),\end{aligned} \tag{2.133}$$

$$\begin{aligned}2\vartheta_4(x_1)\vartheta_4(y_1)\vartheta_1(u_1)\vartheta_1(v_1) =&\vartheta_3(x)\vartheta_3(y)\vartheta_2(u)\vartheta_2(v) + \vartheta_4(x)\vartheta_4(y)\vartheta_1(u)\vartheta_1(v)\\ &-\vartheta_2(x)\vartheta_2(y)\vartheta_3(u)\vartheta_3(v) - \vartheta_1(x)\vartheta_1(y)\vartheta_4(u)\vartheta_4(v),\end{aligned} \tag{2.134}$$

Putting $x = X, y = X+\alpha, u = U, v = U+\alpha$ in (2.133) and (2.134) yields

$$\begin{aligned}&\vartheta_3(X+\alpha+U)\vartheta_3(X-U)\vartheta_2(\alpha)\vartheta_2(0)\\ &= \vartheta_3(X)\vartheta_3(X+\alpha)\vartheta_2(U)\vartheta_2(U+\alpha) + \vartheta_4(X)\vartheta_4(X+\alpha)\vartheta_1(U)\vartheta_1(U+\alpha),\end{aligned} \tag{2.135}$$

After differentiating (2.135) by U and setting $U = 0$, we obtain

$$\begin{aligned}&\left(\vartheta_3'(X+\alpha)\vartheta_3(X) - \vartheta_3(X+\alpha)\vartheta_3'(X)\right)\vartheta_2(\alpha)\vartheta_2(0)\\ &= \vartheta_3(X)\vartheta_3(X+\alpha)\vartheta_2(0)\vartheta_2'(\alpha) + \vartheta_4(X)\vartheta_4(X+\alpha)\vartheta_1'(0)\vartheta_1(\alpha).\end{aligned} \tag{2.136}$$

Also, we have the following identities

$$\begin{aligned}2\vartheta_1(x_1)\vartheta_2(y_1)\vartheta_4(u_1)\vartheta_3(v_1) =&\vartheta_3(x)\vartheta_4(y)\vartheta_2(u)\vartheta_1(v)+\vartheta_4(x)\vartheta_3(y)\vartheta_1(u)\vartheta_2(v)\\&+\vartheta_2(x)\vartheta_1(y)\vartheta_3(u)\vartheta_4(v)+\vartheta_1(x)\vartheta_2(y)\vartheta_4(u)\vartheta_3(v),\end{aligned} \tag{2.137}$$

$$\begin{aligned}-2\vartheta_2(x_1)\vartheta_1(y_1)\vartheta_3(u_1)\vartheta_4(v_1) =&\vartheta_3(x)\vartheta_4(y)\vartheta_2(u)\vartheta_1(v)+\vartheta_4(x)\vartheta_3(y)\vartheta_1(u)\vartheta_2(v)\\&-\vartheta_2(x)\vartheta_1(y)\vartheta_3(u)\vartheta_4(v)-\vartheta_1(x)\vartheta_2(y)\vartheta_4(u)\vartheta_3(v),\end{aligned} \tag{2.138}$$

Putting $x = X + \alpha, y = -X, u = -U, v = U + \alpha$ in (2.137) and (2.138) yields

$$\begin{aligned}&\vartheta_1(\alpha)\vartheta_2(0)\vartheta_4(X-U)\vartheta_3(X+\alpha+U)\\&=\vartheta_3(X+\alpha)\vartheta_4(X)\vartheta_2(U)\vartheta_1(U+\alpha)+\vartheta_4(X+\alpha)\vartheta_3(X)\vartheta_1(U)\vartheta_2(U+\alpha),\end{aligned} \tag{2.139}$$

After differentiating (2.139) by U and setting $U = 0$ we obtain

$$\begin{aligned}&\left(\vartheta_3'(X+\alpha)\vartheta_4(X)-\vartheta_3(X+\alpha)\vartheta_4'(X)\right)\vartheta_1(\alpha)\vartheta_2(0)\\&=\vartheta_3(X+\alpha)\vartheta_4(X)\vartheta_1'(\alpha)\vartheta_2(0)-\vartheta_4(X+\alpha)\vartheta_3(X)\vartheta_1'(0)\vartheta_2(\alpha).\end{aligned} \tag{2.140}$$

□

Proof (proof of Theorem 5)**.** Denote $Y = \alpha n + t + ly$. From (2.92),

$$\begin{aligned}&\frac{1}{2}D_tD_y\vartheta_3(Y)\cdot\vartheta_3(Y)\\=&l\frac{1}{2}{D_Y}^2\vartheta_3(Y)\cdot\vartheta_3(Y)=l\left(\frac{\vartheta_3''(0)}{\vartheta_3(0)}-\frac{\vartheta_1'(0)^2}{\vartheta_2(0)^2}\frac{\vartheta_4(0)^2}{\vartheta_3(0)^2}\right)\vartheta_3(Y)^2+l\frac{\vartheta_1'(0)^2}{\vartheta_2(0)^2}\vartheta_4(Y)^2,\end{aligned} \tag{2.141}$$

holds. Then, we see that (2.117) and its complex conjugate satisfy the bilinear equations (2.28)-(2.30) with $a = a_2(\alpha)$.□

2.3.3 Discrete deformation of discrete elastica

1. Let $w = w_1$. Define the parameter C_1 as follows

$$C_1(\alpha,\beta)=\frac{1}{p}\frac{\vartheta_4(0)\vartheta_1(\alpha-\beta)}{\vartheta_4(\alpha)\vartheta_4(\beta)}, \tag{2.142}$$

   where $\beta \in \mathbb{R}\backslash(1/2)\mathbb{Z}$ is a parameter.

2. Let $w = w_2$. Define the parameter C_2 as follows

$$C_2(\alpha,\beta)=\frac{1}{p}\frac{\vartheta_2(0)\vartheta_1(\alpha-\beta)}{\vartheta_2(\alpha)\vartheta_2(\beta)}. \tag{2.143}$$

Theorem 6. Let $w = w_1$. Consider the τ function

$$\begin{aligned}\tau_n^m &= \exp\left[\frac{\pi i}{2}\left(z_{31} - \frac{w}{4}\right) - (\delta_1(\alpha)n + \delta_1(\beta)m)\,\frac{y}{p}\right] \\ &\quad \times \vartheta_3\left(z_{31}\right), \\ z_{31} &= \alpha n + \beta m + \frac{\vartheta_4(0)}{\vartheta_1'(0)}\frac{y}{p} - \frac{1}{2} + \frac{w}{4}.\end{aligned} \tag{2.144}$$

Then, (2.144) and its complex conjugate satisfy the bilinear equations (2.40)-(2.42) with $a = a_1(\beta)$, $b = a_1(\alpha)$ and $c = C_1(\alpha, \beta)$.

Theorem 7. Let $w = w_2$. Consider the τ function

$$\begin{aligned}\tau_n^m &= \exp\left[\frac{\pi i}{4} - (\delta_2(\alpha)n + \delta_2(\beta)m)\,\frac{y}{p}\right]\vartheta_3\left(z_{32}\right), \\ z_{32} &= \alpha n + \beta m + \frac{\vartheta_2(0)}{\vartheta_1'(0)}\frac{y}{p} + \frac{1}{4}.\end{aligned} \tag{2.145}$$

Then, (2.145) and its complex conjugate satisfy the bilinear equations (2.40)-(2.42) with $a = a_2(\beta)$, $b = a_2(\alpha)$ and $c = C_2(\alpha, \beta)$.

To prove Theorem 6, we first show the following Lemma

Lemma 7. For any $X, \alpha, \beta \in \mathbb{C}$, the following identity hold.

$$\begin{aligned}&-\frac{\vartheta_1(\alpha-\beta)\vartheta_4(0)}{\vartheta_4(\alpha)\vartheta_4(\beta)}\vartheta_2(X+\alpha+\beta)\vartheta_3(X) \\ &= \frac{\vartheta_1(\beta)}{\vartheta_4(\beta)}\vartheta_2(X+\beta)\vartheta_3(X+\alpha) - \frac{\vartheta_1(\alpha)}{\vartheta_4(\alpha)}\vartheta_2(X+\alpha)\vartheta_3(X+\beta).\end{aligned} \tag{2.146}$$

Proof (proof of Lemma 7)**.** The elliptic theta functions satisfy the following identities [48, 49].

$$\begin{aligned}2\vartheta_1(x_1)\vartheta_2(y_1)\vartheta_4(u_1)\vartheta_3(v_1) =&\vartheta_3(x)\vartheta_4(y)\vartheta_2(u)\vartheta_1(v) + \vartheta_4(x)\vartheta_3(y)\vartheta_1(u)\vartheta_2(v) \\ &+\vartheta_2(x)\vartheta_1(y)\vartheta_3(u)\vartheta_4(v) + \vartheta_1(x)\vartheta_2(y)\vartheta_4(u)\vartheta_3(v),\end{aligned} \tag{2.147}$$

$$\begin{aligned}-2\vartheta_4(x_1)\vartheta_3(y_1)\vartheta_1(u_1)\vartheta_2(v_1) =&\vartheta_3(x)\vartheta_4(y)\vartheta_2(u)\vartheta_1(v) - \vartheta_4(x)\vartheta_3(y)\vartheta_1(u)\vartheta_2(v) \\ &+\vartheta_2(x)\vartheta_1(y)\vartheta_3(u)\vartheta_4(v) - \vartheta_1(x)\vartheta_2(y)\vartheta_4(u)\vartheta_3(v),\end{aligned} \tag{2.148}$$

Putting $x = X + \alpha, y = -\alpha, u = -X - \beta, v = \beta$ in (2.147) and (2.148) yields

$$\begin{aligned}&\vartheta_1(\alpha-\beta)\vartheta_4(0)\vartheta_2(X+\alpha+\beta)\vartheta_3(X) \\ &= \vartheta_1(\alpha)\vartheta_4(\beta)\vartheta_3(X+\beta)\vartheta_2(X+\alpha) - \vartheta_1(\beta)\vartheta_4(\alpha)\vartheta_3(X+\alpha)\vartheta_2(X+\beta),\end{aligned} \tag{2.149}$$

Dividing both sides of (2.149) by $\vartheta_4(\alpha)\vartheta_4(\beta)$ yields (2.146). $\square$

Proof (proof of Theorem 6)**.** From (2.118), (2.130) and (2.146), we see that (2.144) and its complex conjugate satisfy the bilinear equations (2.40)-(2.42) with $a = a_1(\beta)$, $b = a_1(\alpha)$ and $c = C_1(\alpha, \beta)$. □

To prove Theorem 7, we first show the following Lemma

Lemma 8. For any $X, \alpha, \beta \in \mathbb{C}$, the following identity hold.

$$\begin{aligned}
&-\frac{\vartheta_1(\alpha-\beta)\vartheta_2(0)}{\vartheta_2(\alpha)\vartheta_2(\beta)}\vartheta_3(X)\vartheta_4(X+\alpha+\beta) \\
&=\frac{\vartheta_1(\beta)}{\vartheta_2(\beta)}\vartheta_4(X+\beta)\vartheta_3(X+\alpha)-\frac{\vartheta_1(\alpha)}{\vartheta_2(\alpha)}\vartheta_4(X+\alpha)\vartheta_3(X+\beta).
\end{aligned} \tag{2.150}$$

Proof (proof of Lemma 8)**.** The elliptic theta functions satisfy the following identities [48, 49].

$$\begin{aligned}
2\vartheta_3(x_1)\vartheta_4(y_1)\vartheta_2(u_1)\vartheta_1(v_1) =&\vartheta_3(x)\vartheta_4(y)\vartheta_2(u)\vartheta_1(v)-\vartheta_4(x)\vartheta_3(y)\vartheta_1(u)\vartheta_2(v) \\
&-\vartheta_2(x)\vartheta_1(y)\vartheta_3(u)\vartheta_4(v)+\vartheta_1(x)\vartheta_2(y)\vartheta_4(u)\vartheta_3(v),
\end{aligned} \tag{2.151}$$

$$\begin{aligned}
-2\vartheta_4(x_1)\vartheta_3(y_1)\vartheta_1(u_1)\vartheta_2(v_1) =&\vartheta_3(x)\vartheta_4(y)\vartheta_2(u)\vartheta_1(v)-\vartheta_4(x)\vartheta_3(y)\vartheta_1(u)\vartheta_2(v) \\
&+\vartheta_2(x)\vartheta_1(y)\vartheta_3(u)\vartheta_4(v)-\vartheta_1(x)\vartheta_2(y)\vartheta_4(u)\vartheta_3(v),
\end{aligned} \tag{2.152}$$

Putting $x = X+\alpha, y = X+\beta, u = \alpha, v = \beta$ in (2.151) and (2.152) yields

$$\begin{aligned}
&-\vartheta_1(\alpha-\beta)\vartheta_2(0)\vartheta_4(X+\alpha+\beta)\vartheta_3(X) \\
&=\vartheta_2(\alpha)\vartheta_1(\beta)\vartheta_4(X+\beta)\vartheta_3(X+\alpha)-\vartheta_2(\beta)\vartheta_1(\alpha)\vartheta_4(X+\alpha)\vartheta_3(X+\beta),
\end{aligned} \tag{2.153}$$

Dividing both sides of (2.153) by $\vartheta_2(\alpha)\vartheta_2(\beta)$ yields (2.150). □

Proof (proof of Theorem 7)**.** From (2.131), (2.130) and (2.150), we see that (2.145) and its complex conjugate satisfy the bilinear equations (2.40)-(2.42) with $a = a_2(\beta)$, $b = a_2(\alpha)$ and $c = C_2(\alpha, \beta)$. □

2.4 Connection between the discrete curves given in 2.3.2, 2.3.3 and the discrete elastica

Here we check that the discrete curvature of the discrete curves given by Theorems 4, 5, 6, 7 satisfy the difference equation for the discrete elastica (2.21). Denoting $t_n = \tan\frac{K_n}{2}$, (2.21) can be rewritten as

$$t_{n+1}+t_{n-1}=\frac{c_0 t_n}{1+t_n^2}, \tag{2.154}$$

According to [34], the solutions to the equations (2.154) can be written as follows

1. When $w = w_1$

$$t_n = i\frac{\vartheta_1(\alpha)\vartheta_4(\alpha)}{\vartheta_2(\alpha)\vartheta_3(\alpha)}\frac{\vartheta_1\left(\alpha n - d - \frac{\omega}{4}\right)\vartheta_4\left(\alpha n - d - \frac{\omega}{4}\right)}{\vartheta_2\left(\alpha n - d - \frac{\omega}{4}\right)\vartheta_3\left(\alpha n - d - \frac{\omega}{4}\right)}, \tag{2.155}$$

2. When $w = w_2$

$$t_n = -i\frac{\vartheta_1(\alpha)\vartheta_2(\alpha)}{\vartheta_3(\alpha)\vartheta_4(\alpha)}\frac{\vartheta_1\left(\alpha n - d\right)\vartheta_2\left(\alpha n - d\right)}{\vartheta_3\left(\alpha n - d\right)\vartheta_4\left(\alpha n - d\right)}. \tag{2.156}$$

where $d \in \mathbb{R}, \alpha \in \mathbb{R} \setminus (1/2\mathbb{Z})$ are constants. In this section, it is shown that the discrete curvatures of the curves given by Theorems 4, 5, 6, and 7 satisfy (2.21).

Proof. First, we prove the case when $w = w_1$. This corresponds to the curves given by Theorems 4 and 6. Fix the time variable t in Theorem 4. Then the discrete curvature of the curve is

$$K_n = \frac{\theta_{n+1} - \theta_{n-1}}{2} = \frac{1}{i}\log\frac{\vartheta_3(z_{21}+\alpha)\vartheta_2(z_{21}-\alpha)}{\vartheta_2(z_{21}+\alpha)\vartheta_3(z_{21}-\alpha)}. \tag{2.157}$$

Since,

$$t_n = \tan\frac{K_n}{2} = \frac{1}{i}\frac{e^{i\frac{K_n}{2}} - e^{-i\frac{K_n}{2}}}{e^{i\frac{K_n}{2}} + e^{-i\frac{K_n}{2}}}, \tag{2.158}$$

$$\begin{aligned} t_n =& \frac{1}{i}\frac{\sqrt{\frac{\vartheta_3(z_{21}+\alpha)\vartheta_2(z_{21}-\alpha)}{\vartheta_2(z_{21}+\alpha)\vartheta_3(z_{21}-\alpha)}} - \sqrt{\frac{\vartheta_2(z_{21}+\alpha)\vartheta_3(z_{21}-\alpha)}{\vartheta_3(z_{21}+\alpha)\vartheta_2(z_{21}-\alpha)}}}{\sqrt{\frac{\vartheta_3(z_{21}+\alpha)\vartheta_2(z_{21}-\alpha)}{\vartheta_2(z_{21}+\alpha)\vartheta_3(z_{21}-\alpha)}} + \sqrt{\frac{\vartheta_2(z_{21}+\alpha)\vartheta_3(z_{21}-\alpha)}{\vartheta_3(z_{21}+\alpha)\vartheta_2(z_{21}-\alpha)}}} \\ =& \frac{1}{i}\frac{\vartheta_3(z_{21}+\alpha)\vartheta_2(z_{21}-\alpha) - \vartheta_2(z_{21}+\alpha)\vartheta_3(z_{21}-\alpha)}{\vartheta_3(z_{21}+\alpha)\vartheta_2(z_{21}-\alpha) + \vartheta_2(z_{21}+\alpha)\vartheta_3(z_{21}-\alpha)}. \end{aligned} \tag{2.159}$$

holds. On the other hand, the elliptic theta functions satisfy the following identities [48, 49].

$$\begin{aligned} 2\vartheta_1(x_1)\vartheta_2(y_1)\vartheta_4(u_1)\vartheta_3(v_1) =& \vartheta_3(x)\vartheta_4(y)\vartheta_2(u)\vartheta_1(v) + \vartheta_4(x)\vartheta_3(y)\vartheta_1(u)\vartheta_2(v) \\ &+ \vartheta_2(x)\vartheta_1(y)\vartheta_3(u)\vartheta_4(v) + \vartheta_1(x)\vartheta_2(y)\vartheta_4(u)\vartheta_3(v), \end{aligned} \tag{2.160}$$

$$\begin{aligned} -2\vartheta_4(x_1)\vartheta_3(y_1)\vartheta_1(u_1)\vartheta_2(v_1) =& \vartheta_3(x)\vartheta_4(y)\vartheta_2(u)\vartheta_1(v) - \vartheta_4(x)\vartheta_3(y)\vartheta_1(u)\vartheta_2(v) \\ &+ \vartheta_2(x)\vartheta_1(y)\vartheta_3(u)\vartheta_4(v) - \vartheta_1(x)\vartheta_2(y)\vartheta_4(u)\vartheta_3(v). \end{aligned} \tag{2.161}$$

Putting $x = z_{21} + \alpha, y = v = -\alpha, u = -z_{21} + \alpha$ in (2.160) and (2.161) yields

$$\vartheta_3(z_{21}+\alpha)\vartheta_2(z_{21}-\alpha) + \vartheta_2(z_{21}+\alpha)\vartheta_3(z_{21}-\alpha) = \frac{\vartheta_4(0)\vartheta_1(2\alpha)}{\vartheta_1(\alpha)\vartheta_4(\alpha)}\vartheta_3(z_{21})\vartheta_2(z_{21}). \tag{2.162}$$

The elliptic theta functions also satisfy the following identities [48, 49].

$$\begin{aligned} 2\vartheta_4(x_1)\vartheta_4(y_1)\vartheta_1(u_1)\vartheta_1(v_1) =& \vartheta_3(x)\vartheta_3(y)\vartheta_2(u)\vartheta_2(v) + \vartheta_4(x)\vartheta_4(y)\vartheta_1(u)\vartheta_1(v) \\ &- \vartheta_2(x)\vartheta_2(y)\vartheta_3(u)\vartheta_3(v) - \vartheta_1(x)\vartheta_1(y)\vartheta_4(u)\vartheta_4(v), \end{aligned} \tag{2.163}$$

$$\begin{aligned}2\vartheta_1(x_1)\vartheta_1(y_1)\vartheta_4(u_1)\vartheta_4(v_1) =&\vartheta_3(x)\vartheta_3(y)\vartheta_2(u)\vartheta_2(v) - \vartheta_4(x)\vartheta_4(y)\vartheta_1(u)\vartheta_1(v)\\ &-\vartheta_2(x)\vartheta_2(y)\vartheta_3(u)\vartheta_3(v) + \vartheta_1(x)\vartheta_1(y)\vartheta_4(u)\vartheta_4(v),\end{aligned} \tag{2.164}$$

Putting $x = z_{21} + \alpha, y = \alpha, u = z_{21} - \alpha, v = -\alpha$ in (2.162) and (2.163) yields

$$\vartheta_3(z_{21}+\alpha)\vartheta_2(z_{21}-\alpha) - \vartheta_2(z_{21}+\alpha)\vartheta_3(z_{21}-\alpha) = \frac{\vartheta_4(0)\vartheta_1(2\alpha)}{\vartheta_2(\alpha)\vartheta_3(\alpha)}\vartheta_1(z_{21})\vartheta_4(z_{21}). \tag{2.165}$$

Dividing (2.165) by (2.162) yields

$$t_n = \frac{1}{i}\frac{\vartheta_1(\alpha)\vartheta_4(\alpha)}{\vartheta_2(\alpha)\vartheta_3(\alpha)}\frac{\vartheta_1(z_{21})\,\vartheta_4(z_{21})}{\vartheta_2(z_{21})\,\vartheta_3(z_{21})}, \tag{2.166}$$

This is equivalent to (2.155). It can also be proved by the same calculation that the discrete curve given by Theorem 6 is a discrete elastica. Next, we prove the case when $w = w_2$. This corresponds to the curves given by Theorems 5 and 7. Fix the time variable t in Theorem 5. Then the discrete curvature of the curve is

$$K_n = \frac{\theta_{n+1} - \theta_{n-1}}{2} = \frac{1}{i}\log\frac{\vartheta_3(z_{22}+\alpha)\vartheta_4(z_{22}-\alpha)}{\vartheta_4(z_{22}+\alpha)\vartheta_3(z_{22}-\alpha)}, \tag{2.167}$$

Since,

$$t_n = \tan\frac{K_n}{2} = \frac{1}{i}\frac{e^{i\frac{K_n}{2}} - e^{-i\frac{K_n}{2}}}{e^{i\frac{K_n}{2}} + e^{-i\frac{K_n}{2}}}, \tag{2.168}$$

$$\begin{aligned}t_n =&\frac{1}{i}\frac{\sqrt{\frac{\vartheta_3(z_{22}+\alpha)\vartheta_4(z_{22}-\alpha)}{\vartheta_4(z_{22}+\alpha)\vartheta_3(z_{22}-\alpha)}} - \sqrt{\frac{\vartheta_4(z_{22}+\alpha)\vartheta_3(z_{22}-\alpha)}{\vartheta_3(z_{22}+\alpha)\vartheta_4(z_{22}-\alpha)}}}{\sqrt{\frac{\vartheta_3(z_{22}+\alpha)\vartheta_4(z_{22}-\alpha)}{\vartheta_4(z_{22}+\alpha)\vartheta_3(z_{22}-\alpha)}} + \sqrt{\frac{\vartheta_4(z_{22}+\alpha)\vartheta_3(z_{22}-\alpha)}{\vartheta_3(z_{22}+\alpha)\vartheta_4(z_{22}-\alpha)}}}\\ =&\frac{1}{i}\frac{\vartheta_3(z_{22}+\alpha)\vartheta_4(z_{22}-\alpha) - \vartheta_4(z_{22}+\alpha)\vartheta_3(z_{22}-\alpha)}{\vartheta_3(z_{22}+\alpha)\vartheta_4(z_{22}-\alpha) + \vartheta_4(z_{22}+\alpha)\vartheta_3(z_{22}-\alpha)},\end{aligned} \tag{2.169}$$

holds. On the other hand, the elliptic theta functions satisfy the following identities [48, 49].

$$\begin{aligned}2\vartheta_1(x_1)\vartheta_2(y_1)\vartheta_4(u_1)\vartheta_3(v_1) =& \vartheta_3(x)\vartheta_4(y)\vartheta_2(u)\vartheta_1(v) + \vartheta_4(x)\vartheta_3(y)\vartheta_1(u)\vartheta_2(v)\\ &+ \vartheta_2(x)\vartheta_1(y)\vartheta_3(u)\vartheta_4(v) + \vartheta_1(x)\vartheta_2(y)\vartheta_4(u)\vartheta_3(v),\end{aligned} \tag{2.170}$$

$$\begin{aligned}-2\vartheta_2(x_1)\vartheta_1(y_1)\vartheta_3(u_1)\vartheta_4(v_1) =& \vartheta_3(x)\vartheta_4(y)\vartheta_2(u)\vartheta_1(v) + \vartheta_4(x)\vartheta_3(y)\vartheta_1(u)\vartheta_2(v)\\ &- \vartheta_2(x)\vartheta_1(y)\vartheta_3(u)\vartheta_4(v) - \vartheta_1(x)\vartheta_2(y)\vartheta_4(u)\vartheta_3(v),\end{aligned} \tag{2.171}$$

Putting $x = z_{22} + \alpha, y = -z_{22} + \alpha, u = v = -\alpha$ in (2.170) and (2.171) yields

$$\vartheta_3(z_{22}+\alpha)\vartheta_4(z_{22}-\alpha) + \vartheta_4(z_{22}+\alpha)\vartheta_3(z_{22}-\alpha) = \frac{\vartheta_2(0)\vartheta_1(2\alpha)}{\vartheta_1(\alpha)\vartheta_2(\alpha)}\vartheta_3(z_{22})\vartheta_4(z_{22}). \tag{2.172}$$

The elliptic theta functions also satisfy the following identities [48, 49].

$$\begin{aligned}-2\vartheta_1(x_1)\vartheta_1(y_1)\vartheta_2(u_1)\vartheta_2(v_1) =& \vartheta_3(x)\vartheta_3(y)\vartheta_4(u)\vartheta_4(v) - \vartheta_4(x)\vartheta_4(y)\vartheta_3(u)\vartheta_3(v)\\ &+ \vartheta_2(x)\vartheta_2(y)\vartheta_1(u)\vartheta_1(v) - \vartheta_1(x)\vartheta_1(y)\vartheta_2(u)\vartheta_2(v),\end{aligned} \tag{2.173}$$

$$\begin{aligned}-2\vartheta_2(x_1)\vartheta_2(y_1)\vartheta_1(u_1)\vartheta_1(v_1) = \vartheta_3(x)\vartheta_3(y)\vartheta_4(u)\vartheta_4(v) - \vartheta_4(x)\vartheta_4(y)\vartheta_3(u)\vartheta_3(v) \\ - \vartheta_2(x)\vartheta_2(y)\vartheta_1(u)\vartheta_1(v) + \vartheta_1(x)\vartheta_1(y)\vartheta_2(u)\vartheta_2(v),\end{aligned} \tag{2.174}$$

Putting $x = z_{22} + \alpha, y = \alpha, u = z_{22} - \alpha, v = -\alpha$ in (2.173) and (2.174) yields

$$\vartheta_3(z_{22}+\alpha)\vartheta_4(z_{22}-\alpha) - \vartheta_4(z_{22}+\alpha)\vartheta_3(z_{22}-\alpha) = -\frac{\vartheta_2(0)\vartheta_1(2\alpha)}{\vartheta_3(\alpha)\vartheta_4(\alpha)}\vartheta_1(z_{22})\vartheta_2(z_{22}). \tag{2.175}$$

Dividing (2.175) by (2.172) yields

$$t_n = -\frac{1}{i}\frac{\vartheta_1(\alpha)\vartheta_2(\alpha)}{\vartheta_3(\alpha)\vartheta_4(\alpha)}\frac{\vartheta_1\left(z_{22}\right)\vartheta_2\left(z_{22}\right)}{\vartheta_3\left(z_{22}\right)\vartheta_4\left(z_{22}\right)}, \tag{2.176}$$

This is equivalent to (2.156). It can also be proved by the same calculation that the discrete curve given by Theorem 7 is a discrete elastica. □

Chapter 3

Explicit formulas of space curves with constant torsion and discrete space curves with constant torsion angle

In this chapter, we utilize the explicit formula for space curves by τ functions [6] and construct explicit formulas for curves with constant torsion in terms of the elliptic theta functions. There are two types of curves to be constructed. Numerical experiments show that the difference between these curves is whether they can be closed or not. For the curve that can be closed, the conditions for it to be a closed curve are explicitly obtained. We also construct two explicit formulas for discrete space curves with constant torsion angle, which are discrete analogues of space curves with constant torsion; as in the case of smooth curve, we also explicitly derive the conditions under which the curve close.

3.1 The Frenet-Serret formula for space curves

First, we briefly review the basic formulation of space curves (see, for example, [6, 7]). Let $\gamma(x) \in \mathbb{R}^3$ be a smooth space curve parameterized by the arc-length x. We define the tangent $T(x)$, the normal $N(x)$ and the binormal $B(x)$ by

$$T(x) = \gamma'(x), \quad N(x) = \frac{\gamma''(x)}{|\gamma''(x)|}, \quad B(x) = T(x) \times N(x), \tag{3.1}$$

respectively. We also define the curvature $\kappa(x)$ and the torsion $\lambda(x)$ by

$$\kappa(x) = |\gamma''(x)|, \quad \lambda(x) = -\langle N(x), B'(x) \rangle, \tag{3.2}$$

respectively. Here $' = \partial/\partial x$, $\dot{} = \partial/\partial t$. The Frenet frame $\Phi = [T, N, B] \in \mathrm{SO}(3)$ satisfies the Frenet-Serret formula $\Phi' = \Phi L$, where

$$L = \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\lambda \\ 0 & \lambda & 0 \end{bmatrix}. \tag{3.3}$$

3.2 Explicit formulas of space curves with constant torsion and discrete space curves with constant torsion angle

Fix $w = iy, (y > 0)$. Denoting $\vartheta_j(z|w)$ as $\vartheta_j(z)$ for short, then we see that $\vartheta_j(z)^* = \vartheta_j(z^*)$ holds $(j = 1, 2, 3.4)$. Fix parameters $r, v \in \mathbb{R} \setminus (1/2\mathbb{Z})$ and define

$$d_j(x) = \frac{\vartheta_j'(x)}{\vartheta_j(x)} \quad (j = 1, 2, 3, 4). \tag{3.4}$$

Theorem 8. Define constants α_1, β_1, a_1, b_1 and c_1 as follows

$$\begin{aligned} &\alpha_1 = \sqrt{\vartheta_4(2r)\vartheta_4(0)}, \quad \beta_1 = \frac{\vartheta_1(r)\vartheta_4(r)}{\vartheta_2(r)\vartheta_3(r)}, \quad a_1 = \frac{\vartheta_4''(r)}{\vartheta_4(r)} - d_4(r)^2, \\ &b_1 = d_1(r) + d_4(r) - d_2(r) - d_3(r), \quad c_1 = \frac{\vartheta_4''(r)}{\vartheta_4(r)} - \frac{\vartheta_1''(r)}{\vartheta_1(r)} - d_4(r)^2 + d_1(r)^2. \end{aligned} \tag{3.5}$$

Noting that $r \in \mathbb{R} \setminus (1/2\mathbb{Z})$, we can check that α_1, β_1, a_1, b_1 and c_1 are real numbers. Then consider the τ functions,

$$\begin{aligned} F(x, z) &= \alpha_1 \exp\left(\frac{a_1}{c_1} xz\right) \vartheta_3\left(ix + i\frac{z}{c_1}\right), \\ f(x, z) &= \vartheta_4(r) \exp\left(\frac{a_1}{c_1} xz - id_4(r)x\right) \vartheta_3\left(ix + i\frac{z}{c_1} + r\right), \\ g(x, z) &= \vartheta_1(r) \exp\left(\frac{a_1}{c_1} xz + id_1(r)x\right) \vartheta_2\left(ix + i\frac{z}{c_1} - r\right), \\ G(x, z) &= \alpha_1 \frac{\vartheta_1'(0)}{\vartheta_4(0)} \exp\left(\frac{a_1}{c_1} xz + i(d_1(r) - d_4(r))x + \frac{\pi i}{2}\right) \vartheta_2\left(ix + i\frac{z}{c_1}\right), \\ H(x, z) &= \frac{\beta_1}{\alpha_1 b_1} \vartheta_2(0)\vartheta_3(0) \exp\left(\frac{a_1}{c_1} xz + i(d_1(r) + d_4(r))x - \frac{\pi i}{2}\right) \\ &\quad \times \vartheta_2\left(ix + i\frac{z}{c_1} - 2r\right). \end{aligned} \tag{3.6}$$

Then it follows that $F \in \mathbb{R}$, $f, g, G, H \in \mathbb{C}$, and that

$$F^2 = ff^* + gg^*, \tag{3.7}$$

$$D_x f \cdot F = -g^* G, \tag{3.8}$$

$$D_x g \cdot F = f^* G, \tag{3.9}$$

$$D_x H \cdot F = f^* g, \tag{3.10}$$

$$\frac{1}{2} D_z D_x F \cdot F = g g^*. \tag{3.11}$$

Putting γ as

$$\gamma(x,z) = \begin{pmatrix} \dfrac{H + H^*}{F} \\ \dfrac{1}{i}\dfrac{H - H^*}{F} \\ x - 2\dfrac{\partial \log F}{\partial z} \end{pmatrix}, \tag{3.12}$$

then γ is a space curve with constant torsion. The torsion λ can be written explicitly as follows

$$\lambda = \frac{1}{2i}\frac{\partial}{\partial x}\left(\log \frac{G}{G^*}\right) = d_1(r) - d_4(r). \tag{3.13}$$

Also, the curvature κ can be written explicitly as

$$\kappa = 2\frac{|G|}{F}. \tag{3.14}$$

Theorem 9. Define constants α_2, β_2, a_2, b_2 and c_2 as follows

$$\begin{aligned} &\alpha_2 = \sqrt{\vartheta_3(2r)\vartheta_3(0)}, \quad \beta_2 = \frac{\vartheta_1(r)\vartheta_3(r)}{\vartheta_2(r)\vartheta_4(r)}, \quad a_2 = \frac{\vartheta_3''(r)}{\vartheta_3(r)} - d_3(r)^2, \\ &b_2 = d_1(r) + d_4(r) + d_2(r) + d_3(r), \quad c_2 = \frac{\vartheta_3''(r)}{\vartheta_3(r)} - \frac{\vartheta_1''(r)}{\vartheta_1(r)} - d_3(r)^2 + d_1(r)^2. \end{aligned} \tag{3.15}$$

Noting that $r \in \mathbb{R} \setminus (1/2\mathbb{Z})$, we can check that α_2, β_2, a_2, b_2 and c_2 are real numbers. Then consider the τ functions,

$$\begin{aligned}
F(x,z) &= \alpha_2 \exp\left(\frac{a_2}{c_2}xz\right)\vartheta_2\left(ix+i\frac{z}{c_2}\right),\\
f(x,z) &= \vartheta_3(r)\exp\left(\frac{a_2}{c_2}xz - id_3(r)x\right)\vartheta_2\left(ix+i\frac{z}{c_2}+r\right),\\
g(x,z) &= \vartheta_1(r)\exp\left(\frac{a_2}{c_2}xz + id_1(r)x\right)\vartheta_4\left(ix+i\frac{z}{c_2}-r\right),\\
G(x,z) &= \alpha_2\frac{\vartheta_1'(0)}{\vartheta_3(0)}\exp\left(\frac{a_2}{c_2}xz + i(d_1(r)-d_3(r))x+\frac{\pi i}{2}\right)\vartheta_4\left(ix+i\frac{z}{c_2}\right),\\
H(x,z) &= \frac{\beta_2}{\alpha_2 b_2}\vartheta_2(0)\vartheta_4(0)\exp\left(\frac{a_2}{c_2}xz + i(d_1(r)+d_3(r))x+\frac{\pi i}{2}\right)\\
&\quad\times\vartheta_4\left(ix+i\frac{z}{c_2}-2r\right).
\end{aligned} \tag{3.16}$$

Then it follows that $F \in \mathbb{R}$, $f, g, G, H \in \mathbb{C}$, and that

$$F^2 = ff^* + gg^*, \tag{3.17}$$

$$D_x f \cdot F = -g^* G, \tag{3.18}$$

$$D_x g \cdot F = f^* G, \tag{3.19}$$

$$D_x H \cdot F = f^* g, \tag{3.20}$$

$$\frac{1}{2} D_z D_x F \cdot F = gg^*. \tag{3.21}$$

Putting γ as

$$\gamma(x,z) = \begin{pmatrix} \dfrac{H+H^*}{F} \\ \dfrac{1}{i}\dfrac{H-H^*}{F} \\ x - 2\dfrac{\partial \log F}{\partial z} \end{pmatrix}, \tag{3.22}$$

then γ is a space curve with constant torsion. The torsion λ can be written explicitly as follows

$$\lambda = \frac{1}{2i}\frac{\partial}{\partial x}\left(\log\frac{G}{G^*}\right) = d_1(r) - d_3(r). \tag{3.23}$$

Also, the curvature κ can be written explicitly as

$$\kappa = 2\frac{|G|}{F}. \tag{3.24}$$

3.2.1 Explicit formula of the Frenet frame

For the space curve given in the Theorem 8 (resp. 9), the explicit formula of the Frenet frame defined in section 3.1 can be constructed by using τ functions [6]. With the τ functions defined in (3.6) (resp. (3.16)), define three vectors T, N, $B \in S^2$ as follows.

$$T = \frac{1}{ff^* + gg^*} \begin{pmatrix} f^* g + f g^* \\ \frac{1}{i}\left(f^* g - f g^*\right) \\ f f^* - g g^* \end{pmatrix}, \tag{3.25}$$

$$N = \frac{1}{ff^* + gg^*} \begin{pmatrix} \left\{(f^*)^2 - (g^*)^2\right\}\nu + \left\{(f)^2 - (g)^2\right\}\nu^* \\ \frac{1}{i}\left[\left\{(f^*)^2 + (g^*)^2\right\}\nu - \left\{(f)^2 + (g)^2\right\}\nu^*\right] \\ -2\left(f^* g^* \nu + f g \nu^*\right) \end{pmatrix}, \tag{3.26}$$

$$B = \frac{1}{ff^* + gg^*} \begin{pmatrix} \frac{1}{i}\left[-\left\{(f^*)^2 - (g^*)^2\right\}\nu + \left\{(f)^2 - (g)^2\right\}\nu^*\right] \\ \left\{(f^*)^2 + (g^*)^2\right\}\nu + \left\{(f)^2 + (g)^2\right\}\nu^* \\ \frac{2}{i}\left(f^* g^* \nu - f g \nu^*\right) \end{pmatrix}. \tag{3.27}$$

where ν is defined by

$$\nu = \frac{1}{2}\frac{G}{|G|}. \tag{3.28}$$

From (3.7)-(3.11) (resp. (3.17)-(3.21)), it follows that

$$\gamma^{'} = T, \quad N = \frac{T^{'}}{|T^{'}|}, \quad B = N \times T. \tag{3.29}$$

where γ is the vector defined in (3.12) (resp. (3.22)). Also from (3.7)-(3.11) (resp. (3.17)-(3.21)), we see that the following relations hold.

$$\kappa = |T^{'}| = 2\frac{|G|}{F}, \quad \lambda = -\langle B^{'}, N\rangle = \frac{1}{2i}\left(\log\frac{G}{G^*}\right)^{'}. \tag{3.30}$$

where κ is the function defined in (3.14) (resp. (3.24)) and λ is the constant defined by (3.13) (resp. (3.23)).

3.2.2 Sufficient conditions for curve closure for the curve given in Theorem 9

We now consider the closure condition for the curve given in Theorem 9. For any complex number z and any integer n, the following hold [48, 49].

$$\begin{aligned} \vartheta_2\left(z + nw\right) &= \exp\{-\pi i\left(2nz + n^2 w\right)\}\vartheta_2\left(z\right), \\ \vartheta_4\left(z + nw\right) &= (-1)^n \exp\{-\pi i\left(2nz + n^2 w\right)\}\vartheta_4\left(z\right). \end{aligned} \tag{3.31}$$

For γ to be closed, it is sufficient if there exists $s \in \mathbb{R}$ such that

$$\begin{aligned} & x+s-2\frac{\partial}{\partial z}\log F(x+s,z) = x-2\frac{\partial}{\partial z}\log F(x,z), \\ & \frac{H(x+s,z)}{F(x+s,z)} = \frac{H(x,z)}{F(x,z)}. \end{aligned} \tag{3.32}$$

From (3.31), we see that by setting $s = my, m \in \mathbb{N}$, this condition can be rewritten as the conditions for r, y as follows.

$$\begin{aligned} & y(c_2 - 2a_2) - 4\pi = 0, \\ & (4r+1)\pi + y(d_1 + d_3) + 2\pi\frac{n}{m} = 0, \quad n \in \mathbb{Z}, \quad m \in \mathbb{N}. \end{aligned} \tag{3.33}$$

3.3 The discrete Frenet-Serret formula for discrete space curves

In this section, we briefly review the basic formulation of discrete space curves (see, for example, [7, 6]). Let $\gamma_n \in \mathbb{R}^3$ be a discrete space curve with

$$|\gamma_{n+1} - \gamma_n| = \epsilon, \tag{3.34}$$

where ϵ is a constant. We introduce the *discrete Frenet frame* $\Phi_n = [T_n, N_n, B_n] \in \mathrm{SO}(3)$ by

$$T_n = \frac{\gamma_{n+1} - \gamma_n}{\epsilon}, \quad B_n = \frac{T_{n-1} \times T_n}{|T_{n-1} \times T_n|}, \quad N_n = B_n \times T_n. \tag{3.35}$$

Then it follows that the discrete Frenet frame satisfies the *discrete Frenet-Serret formula*

$$\Phi_{n+1} = \Phi_n L_n, \quad L_n = M_1(-\lambda_{n+1}) M_3(\kappa_{n+1}), \tag{3.36}$$

where

$$M_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad M_3(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.37}$$

and ν_n, κ_n are defined by

$$\begin{aligned} & \langle T_{n-1}, T_n \rangle = \cos\kappa_n, \quad \langle B_n, B_{n-1} \rangle = \cos\lambda_n, \quad \langle B_n, N_{n-1} \rangle = \sin\lambda_n, \\ & -\pi \le \lambda_n < \pi, \quad 0 < \kappa_n < \pi. \end{aligned} \tag{3.38}$$

(3.38) means that κ_n can be regarded as a measure of the angle between T_n and T_{n-1}. We call κ the curvature angle. Similarly, (3.38) means that λ_n can be regarded as a measure of the angle between B_n and B_{n-1}. We call λ the torsion angle.

3.4 Explicit formulas of closed discrete space curves with constant torsion angle

Fix parameters $r, v \in \mathbb{R} \setminus (1/2\mathbb{Z})$ and define

$$\Delta_j = d_j\left(-\frac{1}{2}iv + r\right) - d_j\left(\frac{1}{2}iv + r\right), \quad R_j = \frac{\vartheta_j\left(-\frac{1}{2}iv + r\right)}{\vartheta_j\left(\frac{1}{2}iv + r\right)} \quad (j = 1, 2, 3, 4). \tag{3.39}$$

Then

$$\Delta_j^* = -\Delta_j, \quad R_j R_j^* = 1. \tag{3.40}$$

hold.

Theorem 10. Define

$$\begin{aligned} u_1 &= \vartheta_1\left(-\frac{1}{2}iv + r\right)\vartheta_4\left(-\frac{1}{2}iv + r\right)\vartheta_1\left(\frac{1}{2}iv + r\right)\vartheta_4\left(\frac{1}{2}iv + r\right), \\ s_1 &= \vartheta_4(2r)\vartheta_1(iv). \end{aligned} \tag{3.41}$$

Then $u_1^* = u_1, s_1^* = -s_1$ hold. Consider the τ functions

$$\begin{aligned} F_n(z) &= \alpha_1 \exp\left(\frac{n\Delta_4}{\Delta_4 - \Delta_1}z\right)\vartheta_3\left(iv\left(n - \frac{1}{2}\right) + \frac{z}{\Delta_4 - \Delta_1}\right), \\ f_n(z) &= \vartheta_4\left(-\frac{1}{2}iv + r\right)R_4^n \exp\left(\frac{\left(n + \frac{1}{2}\right)\Delta_4}{\Delta_4 - \Delta_1}z\right)\vartheta_3\left(ivn + \frac{z}{\Delta_4 - \Delta_1} + r\right), \\ g_n(z) &= \vartheta_1\left(\frac{1}{2}iv + r\right)R_1^{-n} \exp\left(\frac{\left(n + \frac{1}{2}\right)\Delta_4}{\Delta_4 - \Delta_1}z\right)\vartheta_2\left(ivn + \frac{z}{\Delta_4 - \Delta_1} - r\right), \\ G_n(z) &= \alpha_1 \frac{\vartheta_1(iv)}{\vartheta_4(0)}R_1^{-n}R_4^n \exp\left(\frac{n\Delta_4}{\Delta_4 - \Delta_1}z\right)\vartheta_2\left(iv\left(n - \frac{1}{2}\right) + \frac{z}{\Delta_4 - \Delta_1}\right), \\ H_n(z) &= \frac{u_1}{\alpha_1 s_1}R_1^{-n}R_4^{-n} \exp\left(\frac{n\Delta_4}{\Delta_4 - \Delta_1}z\right)\vartheta_2\left(iv\left(n - \frac{1}{2}\right) + \frac{z}{\Delta_4 - \Delta_1} - 2r\right). \end{aligned} \tag{3.42}$$

Then it follows that $F \in \mathbb{R}$, $f, g, G, H \in \mathbb{C}$, and that

$$F_{n+1}F_n = f_n f_n^* + g_n g_n^*, \tag{3.43}$$

$$F_{n+1}f_{n-1} = \frac{\vartheta_4(iv)}{\vartheta_4(0)}F_n f_n + G_n g_n^*, \tag{3.44}$$

$$F_{n+1}g_{n-1} = \frac{\vartheta_4(iv)}{\vartheta_4(0)}F_n g_n - G_n f_n^*, \tag{3.45}$$

$$H_{n+1}F_n - H_n F_{n+1} = f_n^* g_n, \tag{3.46}$$

$$D_z F_{n+1} \cdot F_n = g_n g_n^*. \tag{3.47}$$

Putting γ as

$$\gamma_n(z) = \begin{pmatrix} \dfrac{H_n + H_n^*}{F_n} \\ \dfrac{1}{i}\dfrac{H_n - H_n^*}{F_n} \\ n - 2\dfrac{\partial \log F_n}{\partial z} \end{pmatrix}, \tag{3.48}$$

then γ is a discrete space curve with constant segment length $|\gamma_{n+1} - \gamma_n| = 1$ [6]. One may check that the torsion angle λ is also constant.

$$\lambda_n = \frac{1}{2i} \log \left(\frac{G_n^* G_{n-1}}{G_n G_{n-1}^*} \right) = \frac{1}{i} \log \left(R_1 R_4^{-1} \right). \tag{3.49}$$

Also, the curvature angle κ can be written explicitly as

$$\kappa_n = 2 \arctan \left(\frac{\vartheta_4(0)}{\vartheta_4(iv)} \frac{|G_n|}{F_n} \right). \tag{3.50}$$

Remark 3. From (3.43)-(3.45), we can check that the following bilinear equations hold [6].

$$\frac{\vartheta_4(iv)}{\vartheta_4(0)} F_n F_n = f_n f_{n-1}^* + g_n g_{n-1}^* = f_n^* f_{n-1} + g_n^* g_{n-1}, \tag{3.51}$$

$$g_n f_{n-1} - f_n g_{n-1} = G_n F_n, \tag{3.52}$$

$$F_{n+1} F_{n-1} = \frac{\vartheta_4(iv)^2}{\vartheta_4(0)^2} F_n F_n + G_n G_n^*. \tag{3.53}$$

Theorem 11. Define

$$\begin{aligned} u_2 &= \vartheta_1 \left(-\frac{1}{2} iv + r \right) \vartheta_3 \left(-\frac{1}{2} iv + r \right) \vartheta_1 \left(\frac{1}{2} iv + r \right) \vartheta_3 \left(\frac{1}{2} iv + r \right), \\ s_2 &= \vartheta_3(2r) \vartheta_1(iv). \end{aligned} \tag{3.54}$$

Then $u_2^* = u_2, s_2^* = -s_2$ hold. Consider the τ functions

$$\begin{aligned}
F_n(z) &= \alpha_2 \exp\left(\frac{n\Delta_3}{\Delta_3 - \Delta_1} z\right) \vartheta_2\left(iv\left(n - \frac{1}{2}\right) + \frac{z}{\Delta_3 - \Delta_1}\right), \\
f_n(z) &= \vartheta_3\left(-\frac{1}{2}iv + r\right) R_3^n \exp\left(\frac{\left(n + \frac{1}{2}\right)\Delta_3}{\Delta_3 - \Delta_1} z\right) \vartheta_2\left(ivn + \frac{z}{\Delta_3 - \Delta_1} + r\right), \\
g_n(z) &= \vartheta_1\left(\frac{1}{2}iv + r\right) R_1^{-n} \exp\left(\frac{\left(n + \frac{1}{2}\right)\Delta_3}{\Delta_3 - \Delta_1} z\right) \vartheta_4\left(ivn + \frac{z}{\Delta_3 - \Delta_1} - r\right), \\
G_n(z) &= \alpha_2 \frac{\vartheta_1(iv)}{\vartheta_3(0)} R_1^{-n} R_3^n \exp\left(\frac{n\Delta_3}{\Delta_3 - \Delta_1} z\right) \vartheta_4\left(iv\left(n - \frac{1}{2}\right) + \frac{z}{\Delta_3 - \Delta_1}\right), \\
H_n(z) &= \frac{u_2}{\alpha_2 s_2} R_1^{-n} R_3^{-n} \exp\left(\frac{n\Delta_3}{\Delta_3 - \Delta_1} z\right) \vartheta_4\left(iv\left(n - \frac{1}{2}\right) + \frac{z}{\Delta_3 - \Delta_1} - 2r\right).
\end{aligned} \tag{3.55}$$

Then it follows that $F \in \mathbb{R}$, $f, g, G, H \in \mathbb{C}$, and that

$$F_{n+1}F_n = f_n f_n^* + g_n g_n^*, \tag{3.56}$$

$$F_{n+1}f_{n-1} = \frac{\vartheta_3(iv)}{\vartheta_3(0)} F_n f_n + G_n g_n^*, \tag{3.57}$$

$$F_{n+1}g_{n-1} = \frac{\vartheta_3(iv)}{\vartheta_3(0)} F_n g_n - G_n f_n^*, \tag{3.58}$$

$$H_{n+1}F_n - H_n F_{n+1} = f_n^* g_n, \tag{3.59}$$

$$D_z F_{n+1} \cdot F_n = g_n g_n^*. \tag{3.60}$$

Putting γ as

$$\gamma_n(z) = \begin{pmatrix} \dfrac{H_n + H_n^*}{F_n} \\ \dfrac{1}{i}\dfrac{H_n - H_n^*}{F_n} \\ n - 2\dfrac{\partial \log F_n}{\partial z} \end{pmatrix}, \tag{3.61}$$

then γ is a discrete space curve with constant segment length $|\gamma_{n+1} - \gamma_n| = 1$ [6]. One may check that the torsion angle λ is also constant.

$$\lambda_n = \frac{1}{2i} \log\left(\frac{G_n^* G_{n-1}}{G_n G_{n-1}^*}\right) = \frac{1}{i} \log\left(R_1 R_3^{-1}\right). \tag{3.62}$$

Also, the curvature angle κ can be written explicitly as

$$\kappa_n = 2 \arctan\left(\frac{\vartheta_3(0)}{\vartheta_3(iv)} \frac{|G_n|}{F_n}\right). \tag{3.63}$$

Remark 4. From (3.56)-(3.58), we can check that the following bilinear equations hold [6].

$$\frac{\vartheta_3(iv)}{\vartheta_3(0)}F_nF_n = f_nf_{n-1}^* + g_ng_{n-1}^* = f_n^*f_{n-1} + g_n^*g_{n-1}, \tag{3.64}$$

$$g_nf_{n-1} - f_ng_{n-1} = G_nF_n, \tag{3.65}$$

$$F_{n+1}F_{n-1} = \frac{\vartheta_3(iv)^2}{\vartheta_3(0)^2}F_nF_n + G_nG_n^*. \tag{3.66}$$

3.4.1 Explicit formula of the discrete Frenet frame

For the discrete space curve given in the Theorem 10 (resp. 11), the explicit formula of the discrete Frenet frame defined in section 3.3 can be constructed by using τ functions [6]. With the τ functions defined in (3.42) (resp. (3.55)), define three vectors T, N, $B \in S^2$ as follows.

$$T_n = \frac{1}{f_nf_n^* + g_ng_n^*}\begin{pmatrix} f_n^*g_n + f_ng_n^* \\ \frac{1}{i}\left(f_n^*g_n - f_ng_n^*\right) \\ f_nf_n^* - g_ng_n^* \end{pmatrix}, \tag{3.67}$$

$$N_n = \frac{1}{f_nf_n^* + g_ng_n^*}\begin{pmatrix} \left\{(f_n^*)^2 - (g_n^*)^2\right\}\nu_n + \left\{(f_n)^2 - (g_n)^2\right\}\nu_n^* \\ \frac{1}{i}\left[\left\{(f_n^*)^2 + (g_n^*)^2\right\}\nu_n - \left\{(f_n)^2 + (g_n)^2\right\}\nu_n^*\right] \\ -2\left(f_n^*g_n^*\nu_n + f_ng_n\nu_n^*\right) \end{pmatrix}, \tag{3.68}$$

$$B_n = \frac{1}{f_nf_n^* + g_ng_n^*}\begin{pmatrix} \frac{1}{i}\left[-\left\{(f_n^*)^2 - (g_n^*)^2\right\}\nu_n + \left\{(f_n)^2 - (g_n)^2\right\}\nu_n^*\right] \\ \left\{(f_n^*)^2 + (g_n^*)^2\right\}\nu_n + \left\{(f_n)^2 + (g_n)^2\right\}\nu_n^* \\ \frac{2}{i}\left(f_n^*g_n^*\nu_n - f_ng_n\nu_n^*\right) \end{pmatrix}. \tag{3.69}$$

where ν is defined below

$$\nu_n = \frac{1}{2}\frac{G_n}{|G_n|}. \tag{3.70}$$

From (3.43)-(3.47) (resp. (3.56)-(3.60)), it follows that

$$\gamma_{n+1} - \gamma = T_n, \quad B_n = \frac{T_{n-1}\times T_n}{|T_{n-1}\times T_n|}, \quad N_n = B_n \times T_n. \tag{3.71}$$

where γ is the vector defined in (3.48) (resp. (3.61)). Also from (3.43), (3.52) and (3.53) (resp. (3.56), (3.65) and (3.66)), we see that the following relation holds

$$\langle T_n, T_{n-1}\rangle = \cos\kappa_n, \tag{3.72}$$

where κ is the function defined in (3.50) (resp. (3.63)). Similarly, we see that the following relations hold

$$\langle B_n, B_{n-1}\rangle = \cos\lambda, \quad \langle B_n, N_{n-1}\rangle = \sin\lambda. \tag{3.73}$$

where λ is the constant defined by (3.49) (resp. (3.62)).

3.4.2 Sufficient conditions for curve closure for the curve given in Theorem 11

Here, we consider the conditions for parameters v, r, y that γ is closed. For γ to be closed,

$$\frac{H_{n+k}}{F_{n+k}} = \frac{H_n}{F_n}, \tag{3.74}$$

$$n + k - 2\frac{\partial \log F_{n+k}}{\partial z} = n - 2\frac{\partial \log F_n}{\partial z}. \tag{3.75}$$

is required. One way to determine the parameters to satisfy this is to consider the pseudoperiod. First, fix the space period k, then define $m \in \mathbb{Z} \setminus \{0\}$, and set $kv = my$. Then from (4.25), we see that

$$\begin{aligned} &\frac{H_{n+k}}{F_{n+k}} = \frac{H_n}{F_n} (R_1 R_3)^{-k} \exp(4mr\pi i) (-1)^m, \\ &F_{n+k} = F_n \exp\left(k\frac{\Delta_3}{D_{31}} z\right) \exp\left[-\pi i\left\{2m\left(iv\left(n - \frac{1}{2}\right) + \frac{z}{D_{31}} + it\right) + m^2\omega\right\}\right]. \end{aligned} \tag{3.76}$$

hold. From this, (3.74) and (3.75) can be rewritten as

$$\exp\{\pi im(4r+1)\}(R_1 R_3)^{-k} = 1, \tag{3.77}$$

$$i(\Delta_3 + \Delta_1) + 4\pi\frac{v}{y} = 0, \tag{3.78}$$

respectively. By simple calculation, we see that

$$|\exp\{\pi im(4r+1)\}(R_1 R_3)^{-k}| = 1, \tag{3.79}$$

(3.77) can also be written as follows

$$\arg\left[\exp\{\pi im(4r+1)\}(R_1 R_3)^{-k}\right] = 0. \tag{3.80}$$

3.5 Proof of Theorems 8-11

Proposition 1. For any $x \in \mathbb{R}, r \in \mathbb{R} \setminus (1/2\mathbb{Z})$, the elliptic theta functions satisfy the following identities.

$$\vartheta_3(ix)^2\vartheta_4(2r)\vartheta_4(0) = \vartheta_3(ix+r)\vartheta_3(ix-r)\vartheta_4(r)^2 + \vartheta_2(ix+r)\vartheta_2(ix-r)\vartheta_1(r)^2, \tag{3.81}$$

$$D_x\vartheta_3(ix+r)\cdot\vartheta_3(ix) = i\frac{\vartheta_4'(r)}{\vartheta_4(r)}\vartheta_3(ix+r)\vartheta_3(ix) - i\frac{\vartheta_1'(0)\vartheta_1(r)}{\vartheta_4(0)\vartheta_4(r)}\vartheta_2(ix+r)\vartheta_2(ix), \tag{3.82}$$

$$D_x\vartheta_2(ix-r)\cdot\vartheta_3(ix) = -i\frac{\vartheta_1'(r)}{\vartheta_1(r)}\vartheta_2(ix-r)\vartheta_3(ix) + i\frac{\vartheta_1'(0)\vartheta_4(r)}{\vartheta_4(0)\vartheta_1(r)}\vartheta_3(ix-r)\vartheta_2(ix), \tag{3.83}$$

$$\vartheta_2(0)\vartheta_3(0)\left\{D_x\vartheta_2(ix-2r)\cdot\vartheta_3(ix)+i\left(\frac{\vartheta_1'(r)}{\vartheta_1(r)}+\frac{\vartheta_4'(r)}{\vartheta_4(r)}\right)\vartheta_2(ix-2r)\vartheta_3(ix)\right\}$$
$$=i\vartheta_2(r)\vartheta_3(r)\left(\frac{\vartheta_1'(r)}{\vartheta_1(r)}+\frac{\vartheta_4'(r)}{\vartheta_4(r)}-\frac{\vartheta_2'(r)}{\vartheta_2(r)}-\frac{\vartheta_3'(r)}{\vartheta_3(r)}\right)\vartheta_3(ix-r)\vartheta_2(ix-r), \tag{3.84}$$

$$\vartheta_4(2r)\vartheta_4(0)\left\{\frac{1}{2}D_x^2\vartheta_3(ix)\cdot\vartheta_3(ix)+\left(\frac{\vartheta_4''(r)}{\vartheta_4(r)}-\frac{\vartheta_4'(r)^2}{\vartheta_4(r)^2}\right)\vartheta_3(ix)^2\right\}$$
$$=\left(\frac{\vartheta_4''(r)}{\vartheta_4(r)}-\frac{\vartheta_4'(r)^2}{\vartheta_4(r)^2}-\frac{\vartheta_1''(r)}{\vartheta_1(r)}+\frac{\vartheta_1'(r)^2}{\vartheta_1(r)^2}\right)\vartheta_1(r)^2\vartheta_2(ix-r)\vartheta_2(ix+r). \tag{3.85}$$

Proposition 2. For any $x\in\mathbb{R}, r\in\mathbb{R}\setminus(1/2\mathbb{Z})$, the elliptic theta functions satisfy the following identities.

$$\vartheta_2(ix)^2\vartheta_3(2r)\vartheta_3(0)=\vartheta_2(ix+r)\vartheta_2(ix-r)\vartheta_3(r)^2+\vartheta_4(ix+r)\vartheta_4(ix-r)\vartheta_1(r)^2, \tag{3.86}$$

$$D_x\vartheta_2(ix+r)\cdot\vartheta_2(ix)=i\frac{\vartheta_3'(r)}{\vartheta_3(r)}\vartheta_2(ix+r)\vartheta_2(ix)+i\frac{\vartheta_1'(0)\vartheta_1(r)}{\vartheta_3(0)\vartheta_3(r)}\vartheta_4(ix+r)\vartheta_4(ix), \tag{3.87}$$

$$D_x\vartheta_4(ix-r)\cdot\vartheta_2(ix)=-i\frac{\vartheta_1'(r)}{\vartheta_1(r)}\vartheta_4(ix-r)\vartheta_2(ix)+i\frac{\vartheta_1'(0)\vartheta_3(r)}{\vartheta_3(0)\vartheta_1(r)}\vartheta_2(ix-r)\vartheta_4(ix), \tag{3.88}$$

$$\vartheta_2(0)\vartheta_4(0)\left\{D_x\vartheta_4(ix-2r)\cdot\vartheta_2(ix)+i\left(\frac{\vartheta_1'(r)}{\vartheta_1(r)}+\frac{\vartheta_3'(r)}{\vartheta_3(r)}\right)\vartheta_4(ix-2r)\vartheta_2(ix)\right\}$$
$$=-i\vartheta_2(r)\vartheta_4(r)\left(\frac{\vartheta_1'(r)}{\vartheta_1(r)}+\frac{\vartheta_4'(r)}{\vartheta_4(r)}+\frac{\vartheta_2'(r)}{\vartheta_2(r)}-\frac{\vartheta_3'(r)}{\vartheta_3(r)}\right)\vartheta_2(ix-r)\vartheta_4(ix-r), \tag{3.89}$$

$$\vartheta_3(2r)\vartheta_3(0)\left\{\frac{1}{2}D_x^2\vartheta_2(ix)\cdot\vartheta_2(ix)+\left(\frac{\vartheta_3''(r)}{\vartheta_3(r)}-\frac{\vartheta_3'(r)^2}{\vartheta_3(r)^2}\right)\vartheta_2(ix)^2\right\}$$
$$=\left(\frac{\vartheta_3''(r)}{\vartheta_3(r)}-\frac{\vartheta_3'(r)^2}{\vartheta_3(r)^2}-\frac{\vartheta_1''(r)}{\vartheta_1(r)}+\frac{\vartheta_1'(r)^2}{\vartheta_1(r)^2}\right)\vartheta_1(r)^2\vartheta_4(ix-r)\vartheta_4(ix+r). \tag{3.90}$$

Proof (Proof of Theorem 8)**.** (3.7) follows from (3.81). (3.8) follows from

$$D_x\left(e^{ax}u\right)\cdot v=e^{ax}\left(auv+D_xu\cdot v\right). \tag{3.91}$$

and (3.82). Similarly, (3.9) follows from (3.91) and (3.83). (3.10) follows from (3.91) and (3.84). Also, (3.11) follows from

$$\frac{1}{2}D_xD_z\left(e^{axz}u\right)\cdot\left(e^{axz}u\right)=e^{2axz}\left(\frac{1}{2}D_xD_zu\cdot u+au^2\right). \tag{3.92}$$

and (3.85). □

Proof (Proof of Theorem 9)**.** (3.17) follows from (3.86). (3.18) follows from (3.91) and (3.87). Similarly, (3.19) follows from (3.91) and (3.88). (3.20) follows from (3.91) and (3.89). (3.21) follows from (3.92) and (3.90). □

Proposition 3. For any $n \in \mathbb{Z}, v, r \in \mathbb{R} \setminus (1/2\mathbb{Z})$, the elliptic theta functions satisfy the following identities.

$$\begin{aligned}
&\vartheta_3\left(iv\left(n+\frac{1}{2}\right)\right)\vartheta_3\left(iv\left(n-\frac{1}{2}\right)\right)\vartheta_4(2r)\vartheta_4(0)\\
&=\vartheta_3\left(ivn+r\right)\vartheta_3\left(ivn-r\right)\vartheta_4\left(\frac{1}{2}iv+r\right)\vartheta_4\left(-\frac{1}{2}iv+r\right)\\
&+\vartheta_2\left(ivn+r\right)\vartheta_2\left(ivn-r\right)\vartheta_1\left(\frac{1}{2}iv+r\right)\vartheta_1\left(-\frac{1}{2}iv+r\right),
\end{aligned}\tag{3.93}$$

$$\begin{aligned}
&\vartheta_3\left(iv\left(n+\frac{1}{2}\right)\right)\frac{\vartheta_4\left(\frac{1}{2}iv+r\right)}{\vartheta_4\left(-\frac{1}{2}iv+r\right)}\vartheta_4\left(-\frac{1}{2}iv+r\right)\vartheta_3\left(iv\left(n-1\right)+r\right)\\
&=\frac{\vartheta_4(iv)}{\vartheta_4(0)}\vartheta_3\left(iv\left(n-\frac{1}{2}\right)\right)\vartheta_4\left(-\frac{1}{2}iv+r\right)\vartheta_3\left(ivn+r\right)\\
&+\frac{\vartheta_1(iv)}{\vartheta_4(0)}\vartheta_2\left(iv\left(n-\frac{1}{2}\right)\right)\vartheta_1\left(-\frac{1}{2}iv+r\right)\vartheta_2\left(ivn+r\right),
\end{aligned}\tag{3.94}$$

$$\begin{aligned}
&\vartheta_3\left(iv\left(n+\frac{1}{2}\right)\right)\frac{\vartheta_1\left(-\frac{1}{2}iv+r\right)}{\vartheta_1\left(\frac{1}{2}iv+r\right)}\vartheta_1\left(\frac{1}{2}iv+r\right)\vartheta_2\left(iv\left(n-1\right)-r\right)\\
&=\frac{\vartheta_4(iv)}{\vartheta_4(0)}\vartheta_3\left(iv\left(n-\frac{1}{2}\right)\right)\vartheta_1\left(\frac{1}{2}iv+r\right)\vartheta_2\left(ivn-r\right)\\
&-\frac{\vartheta_1(iv)}{\vartheta_4(0)}\vartheta_2\left(iv\left(n-\frac{1}{2}\right)\right)\vartheta_4\left(\frac{1}{2}iv+r\right)\vartheta_3\left(ivn-r\right),
\end{aligned}\tag{3.95}$$

$$\begin{aligned}
&\frac{\vartheta_1\left(-\frac{1}{2}iv+r\right)\vartheta_4\left(-\frac{1}{2}iv+r\right)\vartheta_1\left(\frac{1}{2}iv+r\right)\vartheta_4\left(\frac{1}{2}iv+r\right)}{\vartheta_4(2r)\vartheta_1(iv)}\\
&\times\left\{\frac{\vartheta_1\left(\frac{1}{2}iv+r\right)\vartheta_4\left(\frac{1}{2}iv+r\right)}{\vartheta_1\left(-\frac{1}{2}iv+r\right)\vartheta_4\left(-\frac{1}{2}iv+r\right)}\vartheta_3\left(iv\left(n-\frac{1}{2}\right)\right)\vartheta_2\left(iv\left(n+\frac{1}{2}\right)-2r\right)\right.\\
&\left.-\vartheta_3\left(iv\left(n+\frac{1}{2}\right)\right)\vartheta_2\left(iv\left(n-\frac{1}{2}\right)-2r\right)\right\}\\
&=\vartheta_4\left(\frac{1}{2}iv+r\right)\vartheta_2\left(ivn-r\right)\vartheta_1\left(\frac{1}{2}iv+r\right)\vartheta_3\left(ivn-r\right),
\end{aligned}\tag{3.96}$$

$$
\begin{aligned}
&\vartheta_4(2r)\vartheta_4(0)\left\{D_z\vartheta_3\left(iv\left(n+\frac{1}{2}\right)+z\right)\cdot\vartheta_3\left(iv\left(n-\frac{1}{2}\right)+z\right)\right.\\
&+\left(\frac{\vartheta_4'\left(-\frac{1}{2}iv+r\right)}{\vartheta_4\left(-\frac{1}{2}iv+r\right)}-\frac{\vartheta_4'\left(\frac{1}{2}iv+r\right)}{\vartheta_4\left(\frac{1}{2}iv+r\right)}\right)\\
&\left.\times\vartheta_3\left(iv\left(n+\frac{1}{2}\right)+z\right)\vartheta_3\left(iv\left(n-\frac{1}{2}\right)+z\right)\right\}\\
&=\left(\frac{\vartheta_1'\left(\frac{1}{2}iv+r\right)}{\vartheta_1\left(\frac{1}{2}iv+r\right)}-\frac{\vartheta_1'\left(-\frac{1}{2}iv+r\right)}{\vartheta_1\left(-\frac{1}{2}iv+r\right)}+\frac{\vartheta_4'\left(-\frac{1}{2}iv+r\right)}{\vartheta_4\left(-\frac{1}{2}iv+r\right)}-\frac{\vartheta_4'\left(\frac{1}{2}iv+r\right)}{\vartheta_4\left(\frac{1}{2}iv+r\right)}\right)\\
&\times\vartheta_1\left(-\frac{1}{2}iv+r\right)\vartheta_2\left(ivn+z+r\right)\vartheta_1\left(\frac{1}{2}iv+r\right)\vartheta_2\left(ivn+z-r\right).
\end{aligned}
\tag{3.97}
$$

Proposition 4. For any $n\in\mathbb{Z}, v, r\in\mathbb{R}\setminus(1/2\mathbb{Z})$, the elliptic theta functions satisfy the following identities.

$$
\begin{aligned}
&\vartheta_2\left(iv\left(n+\frac{1}{2}\right)\right)\vartheta_2\left(iv\left(n-\frac{1}{2}\right)\right)\vartheta_3(2r)\vartheta_3(0)\\
&=\vartheta_2\left(ivn+r\right)\vartheta_2\left(ivn-r\right)\vartheta_3\left(\frac{1}{2}iv+r\right)\vartheta_3\left(-\frac{1}{2}iv+r\right)\\
&+\vartheta_4\left(ivn+r\right)\vartheta_4\left(ivn-r\right)\vartheta_1\left(\frac{1}{2}iv+r\right)\vartheta_1\left(-\frac{1}{2}iv+r\right),
\end{aligned}
\tag{3.98}
$$

$$
\begin{aligned}
&\vartheta_2\left(iv\left(n+\frac{1}{2}\right)\right)\frac{\vartheta_3\left(\frac{1}{2}iv+r\right)}{\vartheta_3\left(-\frac{1}{2}iv+r\right)}\vartheta_3\left(-\frac{1}{2}iv+r\right)\vartheta_2\left(iv\left(n-1\right)+r\right)\\
&=\frac{\vartheta_3(iv)}{\vartheta_3(0)}\vartheta_2\left(iv\left(n-\frac{1}{2}\right)\right)\vartheta_3\left(-\frac{1}{2}iv+r\right)\vartheta_2\left(ivn+r\right)\\
&+\frac{\vartheta_1(iv)}{\vartheta_3(0)}\vartheta_4\left(iv\left(n-\frac{1}{2}\right)\right)\vartheta_1\left(-\frac{1}{2}iv+r\right)\vartheta_4\left(ivn+r\right),
\end{aligned}
\tag{3.99}
$$

$$
\begin{aligned}
&\vartheta_2\left(iv\left(n+\frac{1}{2}\right)\right)\frac{\vartheta_1\left(-\frac{1}{2}iv+r\right)}{\vartheta_1\left(\frac{1}{2}iv+r\right)}\vartheta_1\left(\frac{1}{2}iv+r\right)\vartheta_4\left(iv\left(n-1\right)-r\right)\\
&=\frac{\vartheta_3(iv)}{\vartheta_3(0)}\vartheta_2\left(iv\left(n-\frac{1}{2}\right)\right)\vartheta_1\left(\frac{1}{2}iv+r\right)\vartheta_4\left(ivn-r\right)\\
&-\frac{\vartheta_1(iv)}{\vartheta_3(0)}\vartheta_4\left(iv\left(n-\frac{1}{2}\right)\right)\vartheta_3\left(\frac{1}{2}iv+r\right)\vartheta_2\left(ivn-r\right),
\end{aligned}
\tag{3.100}
$$

$$
\begin{aligned}
&\frac{\vartheta_1\left(-\frac{1}{2}iv+r\right)\vartheta_3\left(-\frac{1}{2}iv+r\right)\vartheta_1\left(\frac{1}{2}iv+r\right)\vartheta_3\left(\frac{1}{2}iv+r\right)}{\vartheta_3(2r)\vartheta_1(iv)}\\
&\times\left\{\frac{\vartheta_1\left(\frac{1}{2}iv+r\right)\vartheta_3\left(\frac{1}{2}iv+r\right)}{\vartheta_1\left(-\frac{1}{2}iv+r\right)\vartheta_3\left(-\frac{1}{2}iv+r\right)}\vartheta_2\left(iv\left(n-\frac{1}{2}\right)\right)\vartheta_4\left(iv\left(n+\frac{1}{2}\right)-2r\right)\right.\\
&\left.-\vartheta_2\left(iv\left(n+\frac{1}{2}\right)\right)\vartheta_4\left(iv\left(n-\frac{1}{2}\right)-2r\right)\right\}\\
&=\vartheta_3\left(\frac{1}{2}iv+r\right)\vartheta_2\left(ivn-r\right)\vartheta_1\left(\frac{1}{2}iv+r\right)\vartheta_4\left(ivn-r\right),
\end{aligned}
\tag{3.101}
$$

$$
\begin{aligned}
&\vartheta_3(2r)\vartheta_3(0)\left\{D_z\vartheta_2\left(iv\left(n+\frac{1}{2}\right)+z\right)\cdot\vartheta_2\left(iv\left(n-\frac{1}{2}\right)+z\right)\right.\\
&+\left(\frac{\vartheta_3'\left(-\frac{1}{2}iv+r\right)}{\vartheta_3\left(-\frac{1}{2}iv+r\right)}-\frac{\vartheta_3'\left(\frac{1}{2}iv+r\right)}{\vartheta_3\left(\frac{1}{2}iv+r\right)}\right)\\
&\left.\times\vartheta_2\left(iv\left(n+\frac{1}{2}\right)+z\right)\vartheta_2\left(iv\left(n-\frac{1}{2}\right)+z\right)\right\}\\
&=\left(\frac{\vartheta_1'\left(\frac{1}{2}iv+r\right)}{\vartheta_1\left(\frac{1}{2}iv+r\right)}-\frac{\vartheta_1'\left(-\frac{1}{2}iv+r\right)}{\vartheta_1\left(-\frac{1}{2}iv+r\right)}+\frac{\vartheta_3'\left(-\frac{1}{2}iv+r\right)}{\vartheta_3\left(-\frac{1}{2}iv+r\right)}-\frac{\vartheta_3'\left(\frac{1}{2}iv+r\right)}{\vartheta_3\left(\frac{1}{2}iv+r\right)}\right)\\
&\times\vartheta_1\left(-\frac{1}{2}iv+r\right)\vartheta_4\left(ivn+z+r\right)\vartheta_1\left(\frac{1}{2}iv+r\right)\vartheta_4\left(ivn+z-r\right).
\end{aligned}
\tag{3.102}
$$

Proof (Proof of Theorem 10)**.** (3.43) follows from (3.93). (3.44) follows from (3.94). (3.45) follows from (3.95). (3.46) follows from (3.96). (3.47) follows from (3.91) and (3.97). □

Proof (Proof of Theorem 11)**.** (3.56) follows from (3.98). (3.57) follows from (3.99). (3.58) follows from (3.100). (3.59) follows from (3.101). (3.60) follows from (3.91) and (3.102). □

3.6 Proof of Propositions 1-4

Proof (Proof of Proposition 1)**.** Let

$$
\begin{aligned}
X_1&=\frac{1}{2}(X+Y+U+V), \quad Y_1=\frac{1}{2}(X+Y-U-V),\\
U_1&=\frac{1}{2}(X-Y+U-V), \quad V_1=\frac{1}{2}(X-Y-U+V).
\end{aligned}
\tag{3.103}
$$

Then for any $X, Y, U, V \in \mathbb{C}$, the elliptic theta functions satisfy the following identities [48, 49].

$$
\begin{aligned}
&\vartheta_3(X_1)\vartheta_3(Y_1)\vartheta_4(U_1)\vartheta_4(V_1)-\vartheta_2(X_1)\vartheta_2(Y_1)\vartheta_1(U_1)\vartheta_1(V_1)\\
&=\vartheta_3(X)\vartheta_3(Y)\vartheta_4(U)\vartheta_4(V)-\vartheta_2(X)\vartheta_2(Y)\vartheta_1(U)\vartheta_1(V),
\end{aligned}
\tag{3.104}
$$

$$\begin{aligned} &\vartheta_3(X_1)\vartheta_4(Y_1)\vartheta_2(U_1)\vartheta_1(V_1) - \vartheta_2(X_1)\vartheta_1(Y_1)\vartheta_3(U_1)\vartheta_4(V_1) \\ &= \vartheta_3(X)\vartheta_4(Y)\vartheta_2(U)\vartheta_1(V) - \vartheta_2(X)\vartheta_1(Y)\vartheta_3(U)\vartheta_4(V), \end{aligned} \tag{3.105}$$

$$\begin{aligned} &\vartheta_3(X_1)\vartheta_3(Y_1)\vartheta_2(U_1)\vartheta_2(V_1) + \vartheta_4(X_1)\vartheta_4(Y_1)\vartheta_1(U_1)\vartheta_1(V_1) \\ &= \vartheta_3(X)\vartheta_3(Y)\vartheta_2(U)\vartheta_2(V) + \vartheta_4(X)\vartheta_4(Y)\vartheta_1(U)\vartheta_1(V). \end{aligned} \tag{3.106}$$

Putting

$$\begin{cases} X = ix + r, \\ Y = ix - r, \\ U = r, \\ V = -r. \end{cases} \tag{3.107}$$

in (3.104) yields (3.81). After differentiating (3.104) by h with

$$\begin{cases} X = ix, \\ Y = ix + r, \\ U = -r + h, \\ V = h. \end{cases} \tag{3.108}$$

and putting $h = 0$, we obtain (3.82). After differentiating (3.105) by h with

$$\begin{cases} X = ix, \\ Y = h, \\ U = ix - r, \\ V = r + h. \end{cases} \tag{3.109}$$

and putting $h = 0$, we obtain (3.83). After differentiating (3.106) by h with

$$\begin{cases} X = ix - r, \\ Y = r + h, \\ U = ix - r, \\ V = r + h. \end{cases} \tag{3.110}$$

and putting $h = 0$, we obtain

$$\begin{aligned} &\vartheta_2(0)\vartheta_3(0)D_x\vartheta_2(ix - 2r) \cdot \vartheta_3(ix) \\ =& - i\vartheta_2(ix - r)\vartheta_3(ix - r)\left(\vartheta_2'(r)\vartheta_3(r) + \vartheta_2(r)\vartheta_3'(r)\right) \\ &- i\vartheta_1(ix - r)\vartheta_4(ix - r)\left(\vartheta_1'(r)\vartheta_4(r) + \vartheta_1(r)\vartheta_4'(r)\right). \end{aligned} \tag{3.111}$$

Also, putting

$$\begin{cases} X = ix - r, \\ Y = r, \\ U = ix - r, \\ V = r. \end{cases} \tag{3.112}$$

in (3.106) yields

$$\begin{aligned}&\vartheta_2(0)\vartheta_3(0)\vartheta_2(ix-2r)\vartheta_3(ix)\\&=\vartheta_2(ix-r)\vartheta_3(ix-r)\vartheta_2(r)\vartheta_3(r)+\vartheta_1(ix-r)\vartheta_4(ix-r)\vartheta_1(r)\vartheta_4(r).\end{aligned}\tag{3.113}$$

Substituting (3.113) into (3.111) yields (3.84). After differentiating (3.104) by h twice with

$$\begin{cases} X = ix + r, \\ Y = ix - r, \\ U = h + r, \\ V = h - r. \end{cases}\tag{3.114}$$

and putting $h = 0$, we obtain

$$\begin{aligned}&\vartheta_4(2r)\vartheta_4(0)D_x^2\vartheta_3(ix)\cdot\vartheta_3(ix)\\=&2\vartheta_3(ix+r)\vartheta_3(ix-r)\left(\vartheta_4'(r)^2-\vartheta_4''(r)\vartheta_4(r)\right)\\&+2\vartheta_2(ix+r)\vartheta_2(ix-r)\left((\vartheta_1'(r)^2-\vartheta_1''(r)\vartheta_1(r)\right).\end{aligned}\tag{3.115}$$

Also, putting

$$\begin{cases} X = ix + r, \\ Y = ix - r, \\ U = r, \\ V = -r. \end{cases}\tag{3.116}$$

in (3.104) yields

$$\begin{aligned}&\vartheta_4(2r)\vartheta_4(0)\vartheta_3(ix)\vartheta_3(ix)\\&=\vartheta_3(ix+r)\vartheta_3(ix-r)\vartheta_4(r)\vartheta_4(r)+\vartheta_2(ix+r)\vartheta_2(ix-r)\vartheta_1(r)\vartheta_1(r).\end{aligned}\tag{3.117}$$

Substituting (3.117) into (3.115) yields (3.85). □

Proof (Proof of Proposition 2)**.** The elliptic theta functions satisfy the following identities [48, 49].

$$\begin{aligned}&\vartheta_2(X_1)\vartheta_2(Y_1)\vartheta_3(U_1)\vartheta_3(V_1)-\vartheta_4(X_1)\vartheta_4(Y_1)\vartheta_1(U_1)\vartheta_1(V_1)\\&=\vartheta_2(X)\vartheta_2(Y)\vartheta_3(U)\vartheta_3(V)-\vartheta_4(X)\vartheta_4(Y)\vartheta_1(U)\vartheta_1(V),\end{aligned}\tag{3.118}$$

$$\begin{aligned}&\vartheta_4(X_1)\vartheta_3(Y_1)\vartheta_1(U_1)\vartheta_2(V_1)-\vartheta_2(X_1)\vartheta_1(Y_1)\vartheta_3(U_1)\vartheta_4(V_1)\\&=\vartheta_4(X)\vartheta_3(Y)\vartheta_1(U)\vartheta_2(V)-\vartheta_2(X)\vartheta_1(Y)\vartheta_3(U)\vartheta_4(V),\end{aligned}\tag{3.119}$$

$$\begin{aligned}&\vartheta_2(X_1)\vartheta_2(Y_1)\vartheta_4(U_1)\vartheta_4(V_1)+\vartheta_1(X_1)\vartheta_1(Y_1)\vartheta_3(U_1)\vartheta_3(V_1)\\&=\vartheta_4(X)\vartheta_4(Y)\vartheta_2(U)\vartheta_2(V)-\vartheta_3(X)\vartheta_3(Y)\vartheta_1(U)\vartheta_1(V).\end{aligned}\tag{3.120}$$

Putting

$$\begin{cases} X = ix + r, \\ Y = ix - r, \\ U = r, \\ V = -r. \end{cases} \tag{3.121}$$

in (3.118) yields (3.86). After differentiating (3.118) by h with

$$\begin{cases} X = ix, \\ Y = ix + r, \\ U = -r + h, \\ V = h. \end{cases} \tag{3.122}$$

and putting $h = 0$, we obtain (3.87). After differentiating (3.119) by h with

$$\begin{cases} X = ix - r, \\ Y = h, \\ U = -r + h, \\ V = ix. \end{cases} \tag{3.123}$$

and putting $h = 0$, we obtain (3.88). After differentiating (3.120) by h with

$$\begin{cases} X = ix - r, \\ Y = r + h, \\ U = ix - r, \\ V = r + h. \end{cases} \tag{3.124}$$

and putting $h = 0$, we obtain

$$\begin{aligned} &\vartheta_2(0)\vartheta_4(0)D_x\vartheta_4(ix - 2r) \cdot \vartheta_2(ix) \\ =& - i\vartheta_2(ix - r)\vartheta_4(ix - r)\left(\vartheta_2'(r)\vartheta_4(r) + \vartheta_2(r)\vartheta_4'(r)\right) \\ &+ i\vartheta_1(ix - r)\vartheta_3(ix - r)\left(\vartheta_1'(r)\vartheta_3(r) + \vartheta_1(r)\vartheta_3'(r)\right). \end{aligned} \tag{3.125}$$

Also, putting

$$\begin{cases} X = ix - r, \\ Y = r, \\ U = ix - r, \\ V = r. \end{cases} \tag{3.126}$$

in (3.120) yields

$$\begin{aligned} &\vartheta_2(0)\vartheta_4(0)\vartheta_4(ix - 2r)\vartheta_2(ix) \\ &= \vartheta_2(ix - r)\vartheta_4(ix - r)\vartheta_2(r)\vartheta_4(r) - \vartheta_1(ix - r)\vartheta_3(ix - r)\vartheta_1(r)\vartheta_3(r). \end{aligned} \tag{3.127}$$

Substituting (3.127) into (3.125) yields (3.89). After differentiating (3.118) by h twice with

$$\begin{cases} X = ix + r, \\ Y = ix - r, \\ U = h + r, \\ V = h - r. \end{cases} \tag{3.128}$$

and putting $h = 0$, we obtain

$$\begin{aligned} &\vartheta_3(2r)\vartheta_3(0)D_x^2\vartheta_2(ix)\cdot\vartheta_2(ix) \\ =&2\vartheta_2(ix+r)\vartheta_2(ix-r)\left(\vartheta_3'(r)^2 - \vartheta_3''(r)\vartheta_3(r)\right) \\ &+2\vartheta_4(ix+r)\vartheta_4(ix-r)\left((\vartheta_1'(r)^2 - \vartheta_1''(r)\vartheta_1(r)\right). \end{aligned} \tag{3.129}$$

Also, putting

$$\begin{cases} X = ix + r, \\ Y = ix - r, \\ U = r, \\ V = -r. \end{cases} \tag{3.130}$$

in (3.118) yields

$$\begin{aligned} &\vartheta_3(2r)\vartheta_3(0)\vartheta_2(ix)\vartheta_2(ix) \\ &= \vartheta_2(ix+r)\vartheta_2(ix-r)\vartheta_3(r)\vartheta_3(r) + \vartheta_4(ix+r)\vartheta_4(ix-r)\vartheta_1(r)\vartheta_1(r). \end{aligned} \tag{3.131}$$

Substituting (3.131) into (3.129) yields (3.90). □

Proof (Proof of Proposition 3)**.** Putting

$$\begin{cases} X = ivn + r, \\ Y = ivn - r, \\ U = \frac{1}{2}iv + r, \\ V = \frac{1}{2}iv - r. \end{cases} \tag{3.132}$$

in (3.104) yields (3.93). Putting

$$\begin{cases} X = iv\left(n - \frac{1}{2}\right), \\ Y = ivn + r, \\ U = iv, \\ V = \frac{1}{2}iv - r. \end{cases} \tag{3.133}$$

in (3.104) yields (3.94). Putting

$$\begin{cases} X = iv\left(n-\frac{1}{2}\right), \\ Y = iv, \\ U = ivn - r, \\ V = \frac{1}{2}iv + r. \end{cases} \tag{3.134}$$

in (3.105) yields (3.95). Putting

$$\begin{cases} X = ivn - r, \\ Y = -\frac{1}{2}iv + r, \\ U = ivn - r, \\ V = -\frac{1}{2}iv + r. \end{cases} \tag{3.135}$$

in (3.106) yields

$$\begin{aligned} &\frac{\vartheta_2(0)\,\vartheta_3(0)}{\vartheta_1\left(-\frac{1}{2}iv+r\right)\vartheta_4\left(-\frac{1}{2}iv+r\right)}\vartheta_3\left(iv\left(n-\frac{1}{2}\right)\right)\vartheta_2\left(iv\left(n+\frac{1}{2}\right)-2r\right) \\ &=\frac{\vartheta_2\left(-\frac{1}{2}iv+r\right)\vartheta_3\left(-\frac{1}{2}iv+r\right)}{\vartheta_1\left(-\frac{1}{2}iv+r\right)\vartheta_4\left(-\frac{1}{2}iv+r\right)} \\ &\times\vartheta_2(ivn-r)\,\vartheta_3(ivn-r)+\vartheta_1(ivn-r)\,\vartheta_4(ivn-r)\,. \end{aligned} \tag{3.136}$$

Similarly, putting

$$\begin{cases} X = ivn - r, \\ Y = \frac{1}{2}iv + r, \\ U = ivn - r, \\ V = \frac{1}{2}iv + r. \end{cases} \tag{3.137}$$

in (3.106) yields

$$\begin{aligned} &\frac{\vartheta_2(0)\,\vartheta_3(0)}{\vartheta_1\left(\frac{1}{2}iv+r\right)\vartheta_4\left(\frac{1}{2}iv+r\right)}\vartheta_3\left(iv\left(n+\frac{1}{2}\right)\right)\vartheta_2\left(iv\left(n-\frac{1}{2}\right)-2r\right) \\ &=\frac{\vartheta_2\left(\frac{1}{2}iv+r\right)\vartheta_3\left(\frac{1}{2}iv+r\right)}{\vartheta_1\left(\frac{1}{2}iv+r\right)\vartheta_4\left(\frac{1}{2}iv+r\right)}\vartheta_2(ivn-r)\,\vartheta_3(ivn-r)+\vartheta_1(ivn-r)\,\vartheta_4(ivn-r)\,, \end{aligned} \tag{3.138}$$

Subtracting (3.138) from (3.136) and substituting

$$\begin{aligned} &\vartheta_2(0)\,\vartheta_3(0)\,\vartheta_1(iv)\,\vartheta_4(2r) \\ &=\vartheta_1\left(\frac{1}{2}iv+r\right)\vartheta_4\left(\frac{1}{2}iv+r\right)\vartheta_2\left(-\frac{1}{2}iv+r\right)\vartheta_3\left(-\frac{1}{2}iv+r\right) \\ &-\vartheta_1\left(-\frac{1}{2}iv+r\right)\vartheta_4\left(-\frac{1}{2}iv+r\right)\vartheta_2\left(\frac{1}{2}iv+r\right)\vartheta_3\left(\frac{1}{2}iv+r\right), \end{aligned} \tag{3.139}$$

in the equation, we obtain (3.96). Putting

$$\begin{cases} X = ivn + z + r, \\ Y = ivn + z - r, \\ U = \frac{1}{2}iv + r + h, \\ V = \frac{1}{2}iv - r + h. \end{cases} \tag{3.140}$$

in (3.104) yields

$$\begin{aligned} &\vartheta_3\left(iv\left(n+\frac{1}{2}\right)+z+h\right)\vartheta_3\left(iv\left(n-\frac{1}{2}\right)+z-h\right)\vartheta_4(2r)\vartheta_4(0) \\ &= \vartheta_3\left(ivn+z+r\right)\vartheta_3\left(ivn+z-r\right)\vartheta_4\left(\frac{1}{2}iv+r+h\right)\vartheta_4\left(\frac{1}{2}iv-r+h\right) \\ &- \vartheta_2\left(ivn+z+r\right)\vartheta_2\left(ivn+z-r\right)\vartheta_1\left(\frac{1}{2}iv+r+h\right)\vartheta_1\left(\frac{1}{2}iv-r+h\right), \end{aligned} \tag{3.141}$$

Differentiating (3.141) with h and then putting $h = 0$, we obtain

$$\begin{aligned} &\vartheta_4(2r)\vartheta_4(0)D_z\vartheta_3\left(iv\left(n+\frac{1}{2}\right)+z\right)\cdot\vartheta_3\left(iv\left(n-\frac{1}{2}\right)+z\right) \\ &= \left(\frac{\vartheta_4'\left(\frac{1}{2}iv+r\right)}{\vartheta_4\left(\frac{1}{2}iv+r\right)} - \frac{\vartheta_4'\left(-\frac{1}{2}iv+r\right)}{\vartheta_4\left(-\frac{1}{2}iv+r\right)}\right) \\ &\times \vartheta_4\left(-\frac{1}{2}iv+r\right)\vartheta_4\left(\frac{1}{2}iv+r\right)\vartheta_3\left(ivn+z+r\right)\vartheta_3\left(ivn+z-r\right) \\ &+ \left(\frac{\vartheta_1'\left(\frac{1}{2}iv+r\right)}{\vartheta_1\left(\frac{1}{2}iv+r\right)} - \frac{\vartheta_1'\left(-\frac{1}{2}iv+r\right)}{\vartheta_1\left(-\frac{1}{2}iv+r\right)}\right) \\ &\times \vartheta_1\left(-\frac{1}{2}iv+r\right)\vartheta_1\left(\frac{1}{2}iv+r\right)\vartheta_2\left(ivn+z+r\right)\vartheta_2\left(ivn+z-r\right). \end{aligned} \tag{3.142}$$

Putting $h = 0$ in (3.141) and substituting it into (3.142), we obtain (3.97). □

Proof (Proof of Proposition 3)**.** Putting

$$\begin{cases} X = ivn + r, \\ Y = ivn - r, \\ U = \frac{1}{2}iv + r, \\ V = \frac{1}{2}iv - r. \end{cases} \tag{3.143}$$

in (3.118) yields (3.98). Putting

$$\begin{cases} X = iv\left(n - \frac{1}{2}\right), \\ Y = ivn + r, \\ U = iv, \\ V = \frac{1}{2}iv - r. \end{cases} \tag{3.144}$$

in (3.118) yields (3.99). Putting

$$\begin{cases} X = iv\left(n - \frac{1}{2}\right), \\ Y = \frac{1}{2}iv + r, \\ U = iv, \\ V = ivn - r. \end{cases} \tag{3.145}$$

in (3.119) yields (3.100). Putting

$$\begin{cases} X = ivn - r, \\ Y = -\frac{1}{2}iv + r, \\ U = ivn - r, \\ V = -\frac{1}{2}iv + r. \end{cases} \tag{3.146}$$

in (3.120) yields

$$\begin{aligned} &\frac{\vartheta_2(0)\vartheta_4(0)}{\vartheta_1\left(-\frac{1}{2}iv + r\right)\vartheta_3\left(-\frac{1}{2}iv + r\right)}\vartheta_2\left(iv\left(n - \frac{1}{2}\right)\right)\vartheta_4\left(iv\left(n + \frac{1}{2}\right) - 2r\right) \\ &= \frac{\vartheta_2\left(-\frac{1}{2}iv + r\right)\vartheta_4\left(-\frac{1}{2}iv + r\right)}{\vartheta_1\left(-\frac{1}{2}iv + r\right)\vartheta_3\left(-\frac{1}{2}iv + r\right)} \\ &\times \vartheta_2(ivn - r)\vartheta_4(ivn - r) - \vartheta_3(ivn - r)\vartheta_1(ivn - r), \end{aligned} \tag{3.147}$$

Also, putting

$$\begin{cases} X = ivn - r, \\ Y = \frac{1}{2}iv + r, \\ U = ivn - r, \\ V = \frac{1}{2}iv + r. \end{cases} \tag{3.148}$$

in (3.120) yields

$$\begin{aligned} &\frac{\vartheta_2(0)\vartheta_4(0)}{\vartheta_1\left(\frac{1}{2}iv + r\right)\vartheta_3\left(\frac{1}{2}iv + r\right)}\vartheta_2\left(iv\left(n + \frac{1}{2}\right)\right)\vartheta_4\left(iv\left(n - \frac{1}{2}\right) - 2r\right) \\ &= \frac{\vartheta_2\left(\frac{1}{2}iv + r\right)\vartheta_4\left(\frac{1}{2}iv + r\right)}{\vartheta_1\left(\frac{1}{2}iv + r\right)\vartheta_3\left(\frac{1}{2}iv + r\right)}\vartheta_2(ivn - r)\vartheta_4(ivn - r) - \vartheta_3(ivn - r)\vartheta_1(ivn - r), \end{aligned} \tag{3.149}$$

Subtracting (3.149) from (3.147) and substituting

$$\begin{aligned} &\vartheta_3(2r)\vartheta_4(0)\vartheta_2(0)\vartheta_1(iv) \\ &= \vartheta_1\left(\frac{1}{2}iv + r\right)\vartheta_3\left(\frac{1}{2}iv + r\right)\vartheta_2\left(-\frac{1}{2}iv + r\right)\vartheta_4\left(-\frac{1}{2}iv + r\right) \\ &- \vartheta_1\left(-\frac{1}{2}iv + r\right)\vartheta_3\left(-\frac{1}{2}iv + r\right)\vartheta_2\left(\frac{1}{2}iv + r\right)\vartheta_4\left(\frac{1}{2}iv + r\right), \end{aligned} \tag{3.150}$$

in the equation, we obtain (3.101). Putting

$$\begin{cases} X = ivn + z + r, \\ Y = ivn + z - r, \\ U = \frac{1}{2}iv + r + h, \\ V = \frac{1}{2}iv - r + h. \end{cases} \tag{3.151}$$

in (3.118) yields

$$\begin{aligned} &\vartheta_2\left(iv\left(n+\frac{1}{2}\right)+z+h\right)\vartheta_2\left(iv\left(n-\frac{1}{2}\right)+z-h\right)\vartheta_3(2r)\vartheta_3(0) \\ &= \vartheta_2\left(ivn+z+r\right)\vartheta_2\left(ivn+z-r\right)\vartheta_3\left(\frac{1}{2}iv+r+h\right)\vartheta_3\left(\frac{1}{2}iv-r+h\right) \\ &- \vartheta_4\left(ivn+z+r\right)\vartheta_4\left(ivn+z-r\right)\vartheta_1\left(\frac{1}{2}iv+r+h\right)\vartheta_1\left(\frac{1}{2}iv-r+h\right), \end{aligned} \tag{3.152}$$

Differentiating (3.152) with h and then putting $h = 0$, we obtain

$$\begin{aligned} &\vartheta_3(2r)\vartheta_3(0)D_z\vartheta_2\left(iv\left(n+\frac{1}{2}\right)+z\right)\cdot\vartheta_2\left(iv\left(n-\frac{1}{2}\right)+z\right) \\ &= \left(\frac{\vartheta_3'\left(\frac{1}{2}iv+r\right)}{\vartheta_3\left(\frac{1}{2}iv+r\right)} - \frac{\vartheta_3'\left(-\frac{1}{2}iv+r\right)}{\vartheta_3\left(-\frac{1}{2}iv+r\right)}\right) \\ &\times \vartheta_3\left(-\frac{1}{2}iv+r\right)\vartheta_3\left(\frac{1}{2}iv+r\right)\vartheta_2\left(ivn+z+r\right)\vartheta_2\left(ivn+z-r\right) \\ &+ \left(\frac{\vartheta_1'\left(\frac{1}{2}iv+r\right)}{\vartheta_1\left(\frac{1}{2}iv+r\right)} - \frac{\vartheta_1'\left(-\frac{1}{2}iv+r\right)}{\vartheta_1\left(-\frac{1}{2}iv+r\right)}\right) \\ &\times \vartheta_1\left(-\frac{1}{2}iv+r\right)\vartheta_1\left(\frac{1}{2}iv+r\right)\vartheta_4\left(ivn+z+r\right)\vartheta_4\left(ivn+z-r\right). \end{aligned} \tag{3.153}$$

Putting $h = 0$ in (3.152) and substituting it into (3.153), we obtain (3.102). □

Chapter 4

An explicit construction of Kaleidocycles

In this chapter, one of the results of the previous chapter is extended to consider isoperimetric deformations of closed discrete curves with constant torsion angle. The semi-discrete mKdV equation, the semi-discrete potential mKdV equation and the semi-discrete sine-Gordon equation appear as deformation equations. This result can be regarded as an explicit formula for Kaleidocycles. Kaleidocycle is a closed linkage mechanism consisting of congruent tetrahedrons, which can be made of various materials such as paper, plastic, and metal. This mechanism is interesting in that it performs turning over motion. The classical Kaleidocycle consists of six tetrahedra, but one can also consider Kaleidocycles with seven or more tetrahedra. In particular, what is called a Möbius Kaleidocycle has only one degree of freedom for deformation, which has been shown by numerical experiments, and is highly controllable. Although a study [40] have modeled the Kaleidocycle as a closed discrete curve with a constant torsion angle and pointed out that the deformation of the curve is represented by the semi-discrete potential mKdV equation or the semi-discrete sine-Gordon equation, the explicit formula for the curve itself has not been obtained. In Chapter 4, we give an explicit formula for this model in terms of the elliptic theta functions. In conjunction with this result, it was also found that the potential function of the curve also satisfies a different kind of semi-discrete potential mKdV equation, which was not pointed out in the previous study. On the other hand, the result that the deformation of the Möbius Kaleidocycle has only one degree of freedom leads to the conclusion that all the deformation equations for this mechanism should be consistent, except for the difference in the congruent transformations. Also, the potential function is a quantity that is not affected by the congruent transformation, which means that if there is more than one differential-difference equation for it, they must be simultaneously satisfied. The potential function of the curve constructed in Chapter 4 is the solution of two semi-discrete potential mKdV equations and a semi-discrete sine-Gordon equation, which is consistent with this conclusion. This is based on the results of [45].

4.1 An explicit formula of closed discrete space curve with constant torsion angle

In [40], a closed linkage mechanism called the Kaleidocycle is modeled as a discrete curve with constant torsion angle in $\mathbb{R}^3$, and its deformation is represented as an integrable deformation that preserves segment length and torsion angle. This correspondence is obtained by considering the hinge of the Kaleidocycle as the binormal of the space curve. It also discusses how to construct a discrete space curve from given binormals. In [7], the relationship between isoperimetric deformations of equilateral polygons on S^2 and torsion preserving deformations of discrete curves in $\mathbb{R}^3$ is discussed. Following these results, in this section, we consider constructing a discrete space curve from the given binormal. In addition, using the explicit formula for discrete space curves by the τ function given in [6], we construct an explicit formula for discrete space curves with constant torsion angle in terms of the elliptic theta functions. We also explicitly obtain the sufficient conditions satisfied by the parameters of the solution for the curve to be closed.

4.1.1 Discrete space curves

Let $\tilde{B} : \mathbb{Z} \to S^2$ be the binormal with constant torsion angle λ

$$\langle \tilde{B}_n, \tilde{B}_{n-1} \rangle = \cos\lambda, \quad 0 \le \lambda < \pi. \tag{4.1}$$

Define the tangent $T : \mathbb{Z} \to S^2$ and the normal $\tilde{N} : \mathbb{Z} \to S^2$ by

$$T_n = \frac{\tilde{B}_{n+1} \times \tilde{B}_n}{\left|\tilde{B}_{n+1} \times \tilde{B}_n\right|}, \quad \tilde{N}_n = \tilde{B}_n \times T_n. \tag{4.2}$$

Define the signed curvature angle κ by

$$\langle T_{n-1}, T_n \rangle = \cos\kappa_n, \quad -\pi < \kappa_n < \pi. \tag{4.3}$$

Then one can reconstruct the discrete space curves $\gamma : \mathbb{Z} \to \mathbb{R}^3$ from the given binormal by

$$\gamma_{n+1} = \gamma_n + T_n = \gamma_n + \frac{\tilde{B}_{n+1} \times \tilde{B}_n}{\left|\tilde{B}_{n+1} \times \tilde{B}_n\right|}. \tag{4.4}$$

where γ_0 is given. It follows that

$$|\gamma_{n+1} - \gamma_n| = 1, \tag{4.5}$$

for any $n \in \mathbb{Z}$, which means that the segment length of γ is 1. We can also check that the discrete Frenet frame $\Phi_n = [T_n, \tilde{N}_n, \tilde{B}_n] \in \mathrm{SO}(3)$ satisfies the *discrete Frenet-Serret formula*

$$\Phi_{n+1} = \Phi_n L_n, \quad L_n = M_1(-\lambda) M_3(\kappa_{n+1}), \tag{4.6}$$

where

$$M_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad M_3(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.7}$$

Remark 5. (4.1) means that λ can be regarded as the angle between $\tilde{B}_n$ and $\tilde{B}_{n-1}$, which we call the torsion angle. Also, (4.3) means that κ_n can be regarded as a signed measure of the angle between T_n and T_{n-1}, which we call this the signed curvature angle.

4.1.2 An explicit formula in terms of the elliptic theta functions

Fix $w = iy, (y > 0)$. Denoting $\vartheta_j(z|w)$ as $\vartheta_j(z)$ for short, then we see that $\vartheta_j(z)^* = \vartheta_j(z^*)$ holds $(j = 1, 2, 3.4)$. Fix parameters $r, v \in \mathbb{R} \setminus (1/2\mathbb{Z})$ and define

$$d_j(x) = \frac{\vartheta_j'(x)}{\vartheta_j(x)} \quad (j = 1, 2, 3, 4), \tag{4.8}$$

$$\alpha_2 = \sqrt{\vartheta_3(2r)\vartheta_3(0)}, \quad \beta = \frac{\vartheta_3(iv)}{\vartheta_3(0)}, \quad s_2 = \vartheta_3(2r)\vartheta_1(iv), \tag{4.9}$$

$$u_2 = \vartheta_1\left(-\frac{1}{2}iv + r\right)\vartheta_3\left(-\frac{1}{2}iv + r\right)\vartheta_1\left(\frac{1}{2}iv + r\right)\vartheta_3\left(\frac{1}{2}iv + r\right), \tag{4.10}$$

$$\Delta_j = d_j\left(-\frac{1}{2}iv + r\right) - d_j\left(\frac{1}{2}iv + r\right), \quad R_j = \frac{\vartheta_j\left(-\frac{1}{2}iv + r\right)}{\vartheta_j\left(\frac{1}{2}iv + r\right)} \quad (j = 1, 2, 3, 4). \tag{4.11}$$

Then it follows that

$$\begin{aligned} &\alpha_2^* = \alpha_2, \quad \beta^* = \beta, \quad s_2^* = -s_2, \quad u_2^* = u_2, \\ &\Delta_j^* = -\Delta_j, \quad R_j R_j^* = 1 \quad (j = 1, 2, 3, 4). \end{aligned} \tag{4.12}$$

Theorem 12. Denote $\mu_n = ivn + \frac{z}{\Delta_3-\Delta_1} + it$. Then consider the following τ functions

$$\begin{aligned}
F_n(t,z) &= \alpha_2 \exp\left(\frac{n\Delta_3}{\Delta_3-\Delta_1}z + \frac{C}{2}tz\right)\vartheta_2\left(\mu_n - \frac{1}{2}iv\right),\\
f_n(t,z) &= \vartheta_3\left(-\frac{1}{2}iv + r\right) R_3^n \exp\left(\frac{\left(n+\frac{1}{2}\right)\Delta_3}{\Delta_3-\Delta_1}z + \frac{C}{2}tz - \frac{\Gamma}{2}it\right)\vartheta_2\left(\mu_n + r\right),\\
g_n(t,z) &= \vartheta_1\left(\frac{1}{2}iv + r\right) R_1^{-n} \exp\left(\frac{\left(n+\frac{1}{2}\right)\Delta_3}{\Delta_3-\Delta_1}z + \frac{C}{2}tz + \frac{\Gamma}{2}it\right)\vartheta_4\left(\mu_n - r\right),\\
G_n(t,z) &= \alpha_2 \frac{\vartheta_1(iv)}{\vartheta_3(0)} R_1^{-n} R_3^n \exp\left(\frac{n\Delta_3}{\Delta_3-\Delta_1}z + \frac{C}{2}tz\right)\vartheta_4\left(\mu_n - \frac{1}{2}iv\right),\\
H_n(t,z) &= \frac{u_2}{\alpha_2 s_2} R_1^{-n} R_3^{-n} \exp\left(\frac{n\Delta_3}{\Delta_3-\Delta_1}z + \frac{C}{2}tz + \Gamma it\right)\vartheta_4\left(\mu_n - \frac{1}{2}iv - 2r\right),
\end{aligned} \tag{4.13}$$

where $C, \Gamma \in \mathbb{R}$ are constants. Then it follows that $F \in \mathbb{R}$, $f, g, G, H \in \mathbb{C}$, and that

$$F_{n+1}F_n = f_n f_n^* + g_n g_n^*, \tag{4.14}$$

$$F_{n+1}f_{n-1} = \beta F_n f_n + G_n g_n^*, \tag{4.15}$$

$$F_{n+1}g_{n-1} = \beta F_n g_n - G_n f_n^*, \tag{4.16}$$

$$H_{n+1}F_n - H_n F_{n+1} = f_n^* g_n, \tag{4.17}$$

$$D_z F_{n+1} \cdot F_n = g_n g_n^*, \tag{4.18}$$

hold. Putting γ as

$$\gamma_n(t,z) = \begin{pmatrix} \dfrac{H_n + H_n^*}{F_n} \\ \dfrac{1}{i}\dfrac{H_n - H_n^*}{F_n} \\ n - 2\dfrac{\partial \log F_n}{\partial z} \end{pmatrix}, \tag{4.19}$$

γ is a discrete space curve with constant segment length $|\gamma_{n+1} - \gamma_n| = 1$ and constant torsion angle λ [6]. By introducing another tau function

$$\tilde{G}_n(t,z) = -i\alpha_2 \frac{\vartheta_1(iv)}{\vartheta_3(0)} \exp\left(\frac{n\Delta_3}{\Delta_3-\Delta_1}z + \frac{C}{2}tz\right)\vartheta_4\left(\mu_n - \frac{1}{2}iv\right), \tag{4.20}$$

One can check that $\tilde{G}_n \in \mathbb{R}$ and $|G_n|^2 = \tilde{G}_n^2$ hold. By using $\tilde{G}_n$, the torsion angle λ and the signed curvature angle κ can be written explicitly as follows

$$\lambda = \frac{1}{i}\log\left(\frac{G_n^* G_{n-1}}{\tilde{G}_n \tilde{G}_{n-1}}\right) = \frac{1}{i}\log\left(R_1 R_3^{-1}\right), \tag{4.21}$$

$$
\begin{aligned}
\kappa_n(t,z) &= 2\arctan\left(\frac{\tilde{G}_n}{\beta F_n}\right) \\
&= 2\arctan\left(-i\frac{\vartheta_1(iv)\vartheta_4\left(\mu_n-\frac{1}{2}iv\right)}{\vartheta_3(iv)\vartheta_2\left(\mu_n-\frac{1}{2}iv\right)}\right).
\end{aligned} \tag{4.22}
$$

Note that the value range of arctan is taken to be $(-\pi/2, \pi/2)$.

Remark 6. C is a constant that contributes only to the parallel translation of the curve along the Z-axis, and Γ is a constant that contributes only to the rotation of the curve around the Z-axis. We will see later that the choice of these constants changes the deformation equation of the curve and the equations satisfied by the potential function. Since the explicit formula for the curve (4.19) are symmetric with respect to the X and Y axes, only rigid transformation centered on the Z-axis are considered.

We will see later that the functions defined by (4.21) and (4.22) correspond to the torsion angle and the signed curvature angle of γ, respectively. The role of parameters v, r, y, t, z, C, Γ is described in Table 4.1, and the transformation of v, r is described in Remark 8 and Remark 5. When considering a transformation of v, r, it is denoted by $\kappa_n = \kappa_n(t, z; v), \lambda = \lambda(v, r)$ to make it clear.

Table 4.1: The role of parameters v, r, y, t, z, C, Γ

Parameter names	Dependent variables affected by parameters, etc.
v	signed curvature angle and torsion angle
r	torsion angle
y	signed curvature angle, torsion angle, and time period
t, z	signed curvature angle
C, Γ	rigid transformation

Remark 7. From $\kappa_n(t+y, z) = -\kappa_n(t, z)$, we see that the time period of the curve is $2y$.

Remark 8. From $\kappa_n(t, z; -v) = -\kappa_n(-t, z; v), \lambda(-v, r) = -\lambda(v, r)$, we see that changing the sign of v can change the sign of λ and κ. Therefore, we can assume $0 \leq \lambda$.

Proposition 5. Consider the transformation $v \to v + y,\ r \to \frac{1}{2} - r$. Then, κ and λ are transformed as

$$
\kappa_n \to (-1)^{n+1}\kappa_n, \quad \lambda \to \pi + \lambda. \tag{4.23}
$$

Proof. From (4.22), we see that the signed curvature angle can be written as follows

$$
\kappa_n(t, z; v) = 2\arctan\left(-i\frac{\vartheta_1(iv)\vartheta_4\left(iv\left(n-\frac{1}{2}\right)+\frac{z}{\Delta_3-\Delta_1}+it\right)}{\vartheta_3(iv)\vartheta_2\left(iv\left(n-\frac{1}{2}\right)+\frac{z}{\Delta_3-\Delta_1}+it\right)}\right). \tag{4.24}
$$

Without loss of generality, we can assume $z = 0$. For any complex number z and any integer n, the following hold [48, 49].

$$\begin{aligned}
\vartheta_1(z+nw) &= (-1)^n \exp\{-\pi i\left(2nz+n^2w\right)\}\vartheta_1(z)\,,\\
\vartheta_2(z+nw) &= \exp\{-\pi i\left(2nz+n^2w\right)\}\vartheta_2(z)\,,\\
\vartheta_3(z+nw) &= \exp\{-\pi i\left(2nz+n^2w\right)\}\vartheta_3(z)\,,\\
\vartheta_4(z+nw) &= (-1)^n \exp\{-\pi i\left(2nz+n^2w\right)\}\vartheta_4(z)\,.
\end{aligned}\tag{4.25}$$

From (4.25), we see that

$$\begin{aligned}
&\kappa_n(t,0;v+y) = 2\arctan\left(-i\frac{\vartheta_1(iv+iy)\vartheta_4\left(iv\left(n-\frac{1}{2}\right)+it+iy\left(n-\frac{1}{2}\right)\right)}{\vartheta_3(iv+iy)\vartheta_2\left(iv\left(n-\frac{1}{2}\right)+it+iy\left(n-\frac{1}{2}\right)\right)}\right)\\
&= (-1)^{n+1}2\arctan\left(-i\frac{\vartheta_1(iv)\vartheta_4\left(iv\left(n-\frac{1}{2}\right)+it-i\frac{y}{2}\right)}{\vartheta_3(iv)\vartheta_2\left(iv\left(n-\frac{1}{2}\right)+it-i\frac{y}{2}\right)}\right)\\
&= (-1)^{n+1}\kappa_n(t,z;v)\,.
\end{aligned}\tag{4.26}$$

holds, where z is taken in such a way that the difference caused by the transformation is well absorbed (z does not affect the conditions for curve closure or torsion, which will be discussed later). Next, we check the transformation of λ. From (4.21), we see that the torsion angle can be written as follows

$$\lambda(v,r) = \frac{1}{i}\log\left(\frac{\vartheta_1\left(-\frac{1}{2}iv+r\right)}{\vartheta_1\left(\frac{1}{2}iv+r\right)}\frac{\vartheta_3\left(\frac{1}{2}iv+r\right)}{\vartheta_3\left(-\frac{1}{2}iv+r\right)}\right).\tag{4.27}$$

For any complex number z, the following hold [48, 49]

$$\begin{aligned}
\vartheta_1\left(z+\frac{1}{2}\right) &= \vartheta_2(z)\,, \quad \vartheta_2\left(z+\frac{1}{2}\right) = -\vartheta_1(z)\,,\\
\vartheta_3\left(z+\frac{1}{2}\right) &= \vartheta_4(z)\,, \quad \vartheta_4\left(z+\frac{1}{2}\right) = \vartheta_3(z)\,,
\end{aligned}\tag{4.28}$$

$$\begin{aligned}
\vartheta_1\left(z+\frac{w}{2}\right) &= i\exp\left\{-\pi i\left(z+\frac{w}{4}\right)\right\}\vartheta_4(z)\,,\\
\vartheta_2\left(z+\frac{w}{2}\right) &= \exp\left\{-\pi i\left(z+\frac{w}{4}\right)\right\}\vartheta_3(z)\,,\\
\vartheta_3\left(z+\frac{w}{2}\right) &= \exp\left\{-\pi i\left(z+\frac{w}{4}\right)\right\}\vartheta_2(z)\,,\\
\vartheta_4\left(z+\frac{w}{2}\right) &= i\exp\left\{-\pi i\left(z+\frac{w}{4}\right)\right\}\vartheta_1(z)\,.
\end{aligned}\tag{4.29}$$

We calculate the transformation of λ (4.27) using these relations. We have

$$\begin{aligned}
\lambda\left(v+y,\frac{1}{2}-r\right) &= \frac{1}{i}\log\left(\frac{\vartheta_1\left(-\frac{1}{2}iv+\frac{1}{2}-r-\frac{1}{2}iy\right)}{\vartheta_1\left(\frac{1}{2}iv+\frac{1}{2}-r+\frac{1}{2}iy\right)}\frac{\vartheta_3\left(\frac{1}{2}iv+\frac{1}{2}-r+\frac{1}{2}iy\right)}{\vartheta_3\left(-\frac{1}{2}iv+\frac{1}{2}-r-\frac{1}{2}iy\right)}\right)\\
&= \frac{1}{i}\log\left(\frac{\vartheta_2\left(-\frac{1}{2}iv-r-\frac{1}{2}iy\right)}{\vartheta_2\left(\frac{1}{2}iv-r+\frac{1}{2}iy\right)}\frac{\vartheta_4\left(\frac{1}{2}iv-r+\frac{1}{2}iy\right)}{\vartheta_4\left(-\frac{1}{2}iv-r-\frac{1}{2}iy\right)}\right)\\
&= \frac{1}{i}\log\left(i\frac{\vartheta_2\left(-\frac{1}{2}iv-r-\frac{1}{2}iy\right)}{\vartheta_3\left(\frac{1}{2}iv-r\right)}\frac{\vartheta_1\left(\frac{1}{2}iv-r\right)}{\vartheta_4\left(-\frac{1}{2}iv-r-\frac{1}{2}iy\right)}\right)\\
&= \frac{1}{i}\log\left(-\frac{\vartheta_3\left(-\frac{1}{2}iv-r\right)}{\vartheta_3\left(\frac{1}{2}iv-r\right)}\frac{\vartheta_1\left(\frac{1}{2}iv-r\right)}{\vartheta_1\left(-\frac{1}{2}iv-r\right)}\right)\\
&= \pi+\frac{1}{i}\log\left(\frac{\vartheta_1\left(-\frac{1}{2}iv+r\right)}{\vartheta_1\left(\frac{1}{2}iv+r\right)}\frac{\vartheta_3\left(\frac{1}{2}iv+r\right)}{\vartheta_3\left(-\frac{1}{2}iv+r\right)}\right)\\
&= \pi+\lambda\left(v,r\right),
\end{aligned} \tag{4.30}$$

which completes the proof of Proposition 5. □

Remark 9. From (4.14)-(4.16), we can check that the following bilinear equations hold [6].

$$\beta F_n F_n = f_n f_{n-1}^* + g_n g_{n-1}^* = f_n^* f_{n-1} + g_n^* g_{n-1}, \tag{4.31}$$

$$g_n f_{n-1} - f_n g_{n-1} = G_n F_n, \tag{4.32}$$

$$F_{n+1}F_{n-1} = \beta^2 F_n F_n + G_n G_n^*. \tag{4.33}$$

4.1.3 Sufficient conditions for curve closure

Here, we consider the following conditions for parameters v, r, y that γ is closed. For γ to be closed,

$$\frac{H_{n+k}}{F_{n+k}} = \frac{H_n}{F_n}, \tag{4.34}$$

$$n+k-2\frac{\partial \log F_{n+k}}{\partial z} = n-2\frac{\partial \log F_n}{\partial z}. \tag{4.35}$$

for some $k \in \mathbb{N}$ are required. One way to determine the parameters to satisfy (4.34) and (4.35) is to consider the pseudo-period $w = iy$. First, fix the space period k, then define $m \in \mathbb{Z} \setminus \{0\}$, and set $kv = my$. Then from (4.25), we see that

$$\begin{aligned}
&\frac{H_{n+k}}{F_{n+k}} = \frac{H_n}{F_n}\left(R_1R_3\right)^{-k}\exp\left(4mr\pi i\right)(-1)^m,\\
&F_{n+k} = F_n\exp\left(k\frac{\Delta_3}{\Delta_3-\Delta_1}z\right)\exp\left[-\pi i\left\{2m\left(\mu_n-\frac{1}{2}iv\right)+m^2w\right\}\right].
\end{aligned} \tag{4.36}$$

hold, where we denote $\mu_n = ivn + \frac{z}{\Delta_3 - \Delta_1} + it$. It follows from (4.36) that (4.34) and (4.35) can be rewritten as

$$\exp\left\{\pi im\left(4r+1\right)\right\}\left(R_1R_3\right)^{-k} = 1, \tag{4.37}$$

$$i\left(\Delta_3 + \Delta_1\right) + 4\pi\frac{v}{y} = 0, \tag{4.38}$$

respectively. (4.37) and (4.38) are sufficient conditions for γ to be closed, which are explicitly written down in terms of the elliptic theta functions. Hereafter, we assume that the parameters v, r, y are taken to satisfy the closure condition. Table 4.2 again summarizes the role of the parameters.

Table 4.2: Role of each parameter. By specifying k, m first and determining v, r, y by $kv = my$, (4.37) and (4.38), γ is a closed discrete curve with constant torsion angle.

Parameter names	Dependent variables affected by parameters, etc.	Conditions affecting parameters
v	signed curvature angle and torsion angle	closure condition
r	torsion angle	closure condition
y	signed curvature angle, torsion angle, and time period	closure condition
t, z	signed curvature angle	none
k	number of hinges	closure condition
m	integer that is even-odd with Lk	closure condition
C, Γ	rigid transformation	time period and deformation equation of γ

Remark 10. The transformation $v \to -v$ mentioned in Remark 8 corresponds to the transformation $(v, m) \to (-v, -m)$, which preserves the closure condition.

Remark 11. When v, r, y are determined to satisfy the closure condition, the transformation $(v, m) \to (v + y, m)$ can be read as $(v, m) \to (v, m - k)$.

4.1.4 Time period and drift of γ

Depending on the choices of C, Γ, the curve may drift in the Z-axis direction or rotate around the Z-axis. To prevent drift, the Z component of γ (written as γ_{nZ}) must be periodic with respect to t. By noting that C is a constant that only contributes to the drift along the Z axis and the quasiperiodic nature of the theta functions mentioned in (4.25), we can see that this condition is satisfied only if C takes special values.

Proposition 6. By putting

$$C = \frac{4\pi i}{y}\frac{1}{\Delta_3 - \Delta_1}, \tag{4.39}$$

we see that

$$\gamma_{nZ}(t+y) = \gamma_{nZ}(t). \tag{4.40}$$

hold for any n, t.

Proof.

$$\begin{aligned}
&\gamma_{nZ}(t+y) - \gamma_{nZ}(t) \\
&= -2\frac{\partial}{\partial z}\log\left\{\exp\left(\frac{C}{2}yz\right)\frac{\vartheta_2\left(iv\left(n-\frac{1}{2}\right)+\frac{z}{\Delta_3-\Delta_1}+it+iy\right)}{\vartheta_2\left(iv\left(n-\frac{1}{2}\right)+\frac{z}{\Delta_3-\Delta_1}+it\right)}\right\} \\
&= -2\frac{\partial}{\partial z}\log\exp\left\{\frac{C}{2}yz - \pi i\left(2iv\left(n-\frac{1}{2}\right)+\frac{2z}{\Delta_3-\Delta_1}+2it\right)+\pi y\right\} \\
&= -2\left(\frac{C}{2}y - \frac{2\pi i}{\Delta_3-\Delta_1}\right).
\end{aligned} \tag{4.41}$$

□

Also, since Γ is a constant that contributes only to the rotation around the Z axis, it is expected to be possible to suppress the rotation around the Z axis by defining it appropriately. Specifically, it is determined as follows

Proposition 7. We put

$$\Gamma = -\frac{\pi}{y}\left(4r+2l+1\right), \quad l \in \mathbb{Z}. \tag{4.42}$$

Then X and Y components of γ are periodic in t.

Proof. Denote $X_n = iv\left(n-\frac{1}{2}\right)+\frac{z}{\Delta_3-\Delta_1}+it$. From (4.25), we see that a sufficient condition for the X and Y components of γ to be periodic in t is that

$$\begin{aligned}
&\frac{H_n(t+y)}{F_n(t+y)}\frac{F_n(t)}{H_n(t)} = \exp\left(i\Gamma y\right)\frac{\vartheta_4\left(X_n-2r+iy\right)\vartheta_2\left(X_n\right)}{\vartheta_2\left(X_n+iy\right)\vartheta_4\left(X_n-2r\right)} \\
&= -\exp\left(i\Gamma y\right)\frac{\exp\left\{-\pi i\left(2X_n-4r+iy\right)\right\}}{\exp\left\{-\pi i\left(2X_n+iy\right)\right\}} \\
&= -\exp\left\{i\left(\Gamma y+4\pi r\right)\right\} = 1,
\end{aligned} \tag{4.43}$$

holds for any n, t. Choosing Γ as (4.42), we see that (4.43) is satisfied. □

Remark 12. One may verify that choosing l appropriately so that $|\Gamma|$ is minimized leads to the minimum length of the curve drawn by γ during one period.

4.1.5 Explicit formula of the discrete Frenet frame

Referring to the explicit formula for the discrete Frenet frame by the τ function presented in [6], we introduce a modified version of the ordinary discrete Frenet frame $\Phi_n = [T_n, \tilde{N}_n, \tilde{B}_n]$. This is the explicit formula for the discrete Frenet frame introduced in 4.1.1.

By using τ functions, define three vectors T, $\tilde{N}$, $\tilde{B} \in S^2$ as follows.

$$T_n = \frac{1}{f_n f_n^* + g_n g_n^*} \begin{pmatrix} f_n^* g_n + f_n g_n^* \\ \frac{1}{i}\left(f_n^* g_n - f_n g_n^*\right) \\ f_n f_n^* - g_n g_n^* \end{pmatrix}, \tag{4.44}$$

$$\tilde{N}_n = \frac{1}{f_n f_n^* + g_n g_n^*} \begin{pmatrix} \left\{(f_n^*)^2 - (g_n^*)^2\right\}\nu_n + \left\{(f_n)^2 - (g_n)^2\right\}\nu_n^* \\ \frac{1}{i}\left[\left\{(f_n^*)^2 + (g_n^*)^2\right\}\nu_n - \left\{(f_n)^2 + (g_n)^2\right\}\nu_n^*\right] \\ -2\left(f_n^* g_n^* \nu_n + f_n g_n \nu_n^*\right) \end{pmatrix}, \tag{4.45}$$

$$\tilde{B}_n = \frac{1}{f_n f_n^* + g_n g_n^*} \begin{pmatrix} \frac{1}{i}\left[-\left\{(f_n^*)^2 - (g_n^*)^2\right\}\nu_n + \left\{(f_n)^2 - (g_n)^2\right\}\nu_n^*\right] \\ \left\{(f_n^*)^2 + (g_n^*)^2\right\}\nu_n + \left\{(f_n)^2 + (g_n)^2\right\}\nu_n^* \\ \frac{2}{i}\left(f_n^* g_n^* \nu_n - f_n g_n \nu_n^*\right) \end{pmatrix}. \tag{4.46}$$

where ν is defined by

$$\nu_n = \frac{1}{2}\frac{G_n}{\tilde{G}_n}. \tag{4.47}$$

By using (4.15) and (4.16), we rewrite $\tilde{B}_n$ as follows

$$\tilde{B}_n = \frac{1}{2F_n \tilde{G}_n} \begin{pmatrix} \frac{1}{i}\left(g_n^* f_{n-1} - f_n g_{n-1}^* - g_n f_{n-1}^* + f_n^* g_{n-1}\right) \\ g_n^* f_{n-1} - f_n g_{n-1}^* + g_n f_{n-1}^* - f_n^* g_{n-1} \\ \frac{1}{i}\left(f_n^* f_{n-1} - f_n f_{n-1}^* - g_n^* g_{n-1} + g_n g_{n-1}^*\right) \end{pmatrix}. \tag{4.48}$$

Similarly, by using (4.15) and (4.16), we derive the following equation that will later become important when discussing the deformation equation of the curve.

$$\tan\frac{\kappa_n}{2}\tilde{N}_n = T_n - \frac{1}{2\beta F_n^2} \begin{pmatrix} g_n^* f_{n-1} + f_n g_{n-1}^* + g_n f_{n-1}^* + f_n^* g_{n-1} \\ \frac{1}{i}\left(g_n f_{n-1}^* + f_n^* g_{n-1} - g_n^* f_{n-1} - f_n g_{n-1}^*\right) \\ f_n^* f_{n-1} + f_n f_{n-1}^* - g_n^* g_{n-1} - g_n g_{n-1}^* \end{pmatrix}. \tag{4.49}$$

By direct calculation, we get the following equation

$$\tilde{B}_n \times \tilde{B}_{n-1} = \frac{G_n^* G_{n-1} - G_n G_{n-1}^*}{2i\tilde{G}_n \tilde{G}_{n-1}} \frac{1}{F_n F_{n-1}} \begin{pmatrix} f_{n-1}^* g_{n-1} + f_{n-1} g_{n-1}^* \\ \frac{1}{i}\left(f_{n-1}^* g_{n-1} - f_{n-1} g_{n-1}^*\right) \\ f_{n-1} f_{n-1}^* - g_{n-1} g_{n-1}^* \end{pmatrix} \tag{4.50}$$

$$= (\sin\lambda) T_{n-1}.$$

From this, we see that the following relations hold

$$\gamma_{n+1} - \gamma_n = T_n = \operatorname{sgn}(\lambda) \frac{\tilde{B}_{n+1} \times \tilde{B}_n}{\left|\tilde{B}_{n+1} \times \tilde{B}_n\right|}, \quad \tilde{N}_n = \tilde{B}_n \times T_n. \tag{4.51}$$

Also from (4.33), we see that the following relation holds

$$\begin{aligned} \langle T_n, T_{n-1} \rangle &= \frac{F_{n+1}F_{n-1} - 2|G_n|^2}{F_{n+1}F_{n-1}} = \frac{\beta^2 F_n^2 - |G_n|^2}{\beta^2 F_n^2 + |G_n|^2} \\ &= \frac{1 - \frac{|G_n|^2}{\beta^2 F_n^2}}{1 + \frac{|G_n|^2}{\beta^2 F_n^2}} = \frac{1 - \frac{\tilde{G}_n^2}{\beta^2 F_n^2}}{1 + \frac{\tilde{G}_n^2}{\beta^2 F_n^2}}. \end{aligned} \tag{4.52}$$

From (4.52), it follows directly that κ defined by (4.22) satisfies the following relation,

$$\langle T_n, T_{n-1} \rangle = \cos \kappa_n, \quad -\pi \leq \kappa_n \leq \pi. \tag{4.53}$$

Similarly, by direct calculation, we see that the following relations hold

$$\left\langle \tilde{B}_n, \tilde{B}_{n-1} \right\rangle = \frac{G_n^* G_{n-1} + G_n G_{n-1}^*}{2\tilde{G}_n \tilde{G}_{n-1}}, \quad \left\langle \tilde{B}_n, \tilde{N}_{n-1} \right\rangle = \frac{G_n^* G_{n-1} - G_n G_{n-1}^*}{2i\tilde{G}_n \tilde{G}_{n-1}}. \tag{4.54}$$

This shows that λ defined by (4.21) satisfies the following relation.

$$\left\langle \tilde{B}_n, \tilde{B}_{n-1} \right\rangle = \cos \lambda, \quad \left\langle \tilde{B}_n, \tilde{N}_{n-1} \right\rangle = \sin \lambda, \quad 0 \leq \lambda < \pi. \tag{4.55}$$

By putting

$$L_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \lambda & \sin \lambda \\ 0 & -\sin \lambda & \cos \lambda \end{bmatrix} \begin{bmatrix} \cos \kappa_{n+1} & -\sin \kappa_{n+1} & 0 \\ \sin \kappa_{n+1} & \cos \kappa_{n+1} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.56}$$

we see that the discrete Frenet-Serret formula $\Phi_{n+1} = \Phi_n L_n$ holds.

Remark 13. By using (4.37), we see that $\tilde{B}_0 = (-1)^m \tilde{B}_k$ holds. From this, we see that m is even-oddly equal to Lk.

Remark 14. Consider the transformation $\tilde{B}_n \to (-1)^{n+1}\tilde{B}_n$, $\tilde{N}_n \to (-1)^{n+1}\tilde{N}_n$. With this transformation, a closed curve is transferred to a closed curve. From the discrete Frenet-Serret formula, we see that

$$\left\langle \tilde{N}_n, T_{n-1} \right\rangle = -\sin \kappa_n. \tag{4.57}$$

holds. From (4.57) and (4.55), we see that this transformation leads to the following change $\kappa_n \to (-1)^{n+1}\kappa_n, \quad \lambda \to \pi + \lambda$. This indicates that this transformation corresponds to the one described in Remark 5 and Remark 11.

4.2 Deformation of the curve that preserves segment length and torsion angle

We consider the isoperimetric deformation of γ by the semi-discrete mKdV equation [7] and the isoperimetric deformations of γ by the semi-discrete potential mKdV equation and the semi-discrete sine-Gordon equation [40]. First, let us summarize the equations necessary to conduct the calculations.

4.2.1 Bilinear equations satisfied by the τ function

Define

$$\omega = \frac{\vartheta_1(2r)\,\vartheta_3(0)}{\vartheta_3(2r)\,\vartheta_1(iv)}, \tag{4.58}$$

$$\delta_1 = \frac{\vartheta_3\left(\frac{1}{2}iv+r\right)^2\vartheta_3\left(-\frac{1}{2}iv+r\right)^2}{\vartheta_3(2r)\,\vartheta_3(0)^3}, \tag{4.59}$$

$$\delta_2 = \frac{\vartheta_1\left(\frac{1}{2}iv+r\right)^2\vartheta_3\left(-\frac{1}{2}iv+r\right)^2}{\vartheta_1(iv)^2\,\vartheta_3(2r)\,\vartheta_3(0)}, \tag{4.60}$$

$$\delta_3 = \frac{-iu_2}{s_2\vartheta_3(0)\vartheta_1(iv)}\left[\frac{\vartheta_1'\left(\frac{1}{2}iv+r\right)\vartheta_3\left(-\frac{1}{2}iv+r\right)}{\vartheta_1\left(-\frac{1}{2}iv+r\right)\vartheta_3\left(\frac{1}{2}iv+r\right)}+\frac{\vartheta_1\left(\frac{1}{2}iv+r\right)\vartheta_3'\left(-\frac{1}{2}iv+r\right)}{\vartheta_1\left(-\frac{1}{2}iv+r\right)\vartheta_3\left(\frac{1}{2}iv+r\right)}\right], \tag{4.61}$$

$$\epsilon_1 = \frac{2i}{\Delta_3-\Delta_1}\frac{\vartheta_3''(0)}{\vartheta_3(0)}, \tag{4.62}$$

$$\epsilon_2 = \frac{2i}{\Delta_3-\Delta_1}\frac{\vartheta_1'(0)^2}{\vartheta_1(iv)^2}, \tag{4.63}$$

$$\beta_1 = \frac{1}{2}\left(\delta_3+\delta_3^*\right)+\frac{1}{2}i\Gamma\left(\delta_2-\delta_2^*\right), \tag{4.64}$$

$$\beta_2 = -\frac{1}{2}\left(\delta_3-\delta_3^*\right)-\frac{1}{2}i\Gamma\left(\delta_2+\delta_2^*\right). \tag{4.65}$$

Then, we see that $\omega, \beta_2 \in \sqrt{-1}\mathbb{R}, \delta_1, \epsilon_1, \epsilon_2, \beta_1 \in \mathbb{R}$, and the following Propositions hold.

Proposition 8. The following relation holds.

$$\omega = \delta_2 - \delta_2^*. \tag{4.66}$$

Proposition 9. The following identities hold.

$$2\beta_1\left(\delta_2+\delta_2^*\right)+2\omega\beta_2+\epsilon_2 = 2\delta_2\delta_3^*+2\delta_2^*\delta_3+\epsilon_2 = 0. \tag{4.67}$$

Proposition 10. The τ functions (4.13) satisfy the following bilinear equations.

$$f_n f_{n-1}^* + g_n^* g_{n-1} = \beta F_n F_n + \omega G_n G_n^*, \tag{4.68}$$

$$\omega G_n G_n^* = g_n^* g_{n-1} - g_n g_{n-1}^* = f_n f_{n-1}^* - f_n^* f_{n-1}, \tag{4.69}$$

$$H_n F_n = \delta_2 f_n^* g_{n-1} - \delta_2^* f_{n-1}^* g_n, \tag{4.70}$$

$$f_n f_{n-1}^* = \delta_1 F_n F_n + \delta_2 G_n G_n^*, \tag{4.71}$$

$$g_n^* g_{n-1} = (\beta - \delta_1) F_n F_n + (\omega - \delta_2) G_n G_n^*, \tag{4.72}$$

$$D_t H_n \cdot F_n = (\delta_3 + i\Gamma\delta_2) f_n^* g_{n-1} + (\delta_3^* - i\Gamma\delta_2^*) f_{n-1}^* g_n, \tag{4.73}$$

$$D_t D_z F_n \cdot F_n = (\epsilon_1 + C) F_n F_n + \epsilon_2 G_n G_n^*. \tag{4.74}$$

Also, for later convenience, define p_n, q_n as follows.

$$p_n = g_n^* f_{n-1} - f_n g_{n-1}^*, \tag{4.75}$$

$$q_n = g_n^* f_{n-1} + f_n g_{n-1}^*. \tag{4.76}$$

Then we have the following Proposition.

Proposition 11. The following relations hold.

$$p_n p_n^* = F_n^2 |G_n|^2 + \omega^2 |G_n|^4, \tag{4.77}$$

$$p_n q_n^* - p_n^* q_n = 2(2\delta_1 - \beta)\omega F_n^2 |G_n|^2 + 2\omega(\delta_2 + \delta_2^*)|G_n|^4, \tag{4.78}$$

$$q_n q_n^* = 4\delta_1(\beta - \delta_1) F_n^4 + \{1 + 2(\beta - 2\delta_1)(\delta_2 + \delta_2^*)\} F_n^2 |G_n|^2 - (\delta_2 + \delta_2^*)^2 |G_n|^4. \tag{4.79}$$

For later convenience, we rewrite $D_t H_n \cdot F_n$ by using (4.64), (4.65), (4.75) and (4.76) as follows

$$D_t H_n \cdot F_n = \beta_1 q_n^* + \beta_2 p_n^*. \tag{4.80}$$

Also, we rewrite (4.49) and (4.48) as follows

$$\tan\frac{\kappa_n}{2}\tilde{N}_n = T_n - \frac{1}{2\beta F_n^2}\begin{pmatrix} q_n^* + q_n \\ \frac{1}{i}(q_n^* - q_n) \\ -2(\beta - 2\delta_1) F_n^2 + 2(\delta_2 + \delta_2^*)|G_n|^2 \end{pmatrix}, \tag{4.81}$$

$$\tan\frac{\kappa_n}{2}\tilde{B}_n = \frac{1}{2\beta F_n^2}\begin{pmatrix} \frac{1}{i}(p_n - p_n^*) \\ p_n + p_n^* \\ -\frac{2}{i}\omega|G_n|^2 \end{pmatrix}. \tag{4.82}$$

4.2.2 Isoperimetric deformation of the curve by the semi-discrete mKdV equation

We first check what deformation equation the curve satisfies when C, Γ takes general values.

Theorem 13. When C, Γ takes general values, γ satisfies the following deformation equation.

$$\begin{aligned}\dot{\gamma}_n(t,z) =& 2\beta\left[\beta_1 T_n - \beta_1 \tan\frac{\kappa_n}{2}\tilde{N}_n + \frac{\beta_2}{i}\tan\frac{\kappa_n}{2}\tilde{B}_n\right]\\ &+\begin{pmatrix} 0 \\ 0 \\ 2\beta_1\left(\beta - 2\delta_1\right) - \epsilon_1 - C\end{pmatrix}.\end{aligned} \tag{4.83}$$

Proof. By differentiating (4.19) we get

$$\begin{aligned}\dot{\gamma}_n(t,z) &= \frac{1}{F_n^2}\begin{pmatrix} D_t H_n \cdot F_n + D_t H_n^* \cdot F_n \\ \frac{1}{i}\left(D_t H_n \cdot F_n - D_t H_n^* \cdot F_n\right) \\ -D_t D_z F_n \cdot F_n\end{pmatrix}\\ &= \frac{1}{F_n^2}\begin{pmatrix} \beta_1 q_n^* + \beta_2 p_n^* + \beta_1 q_n - \beta_2 p_n \\ \frac{1}{i}\left(\beta_1 q_n^* + \beta_2 p_n^* - \beta_1 q_n + \beta_2 p_n\right) \\ -\left(\epsilon_1 + C\right) F_n^2 - \epsilon_2 |G_n|^2\end{pmatrix}.\end{aligned} \tag{4.84}$$

On the other hand since

$$\begin{aligned}&\beta_1\left(T_n - \tan\frac{\kappa_n}{2}\tilde{N}_n\right) + \frac{\beta_2}{i}\tan\frac{\kappa_n}{2}\tilde{B}_n\\ &= \frac{1}{2\beta F_n^2}\begin{pmatrix} \beta_1 q_n^* + \beta_2 p_n^* + \beta_1 q_n - \beta_2 p_n \\ \frac{1}{i}\left(\beta_1 q_n^* + \beta_2 p_n^* - \beta_1 q_n + \beta_2 p_n\right) \\ -2\beta_1\left(\beta - 2\delta_1\right) F_n^2 + 2\beta_1\left(\delta_2 + \delta_2^*\right)|G_n|^2 + 2\beta_2\omega|G_n|^2\end{pmatrix}.\end{aligned} \tag{4.85}$$

holds. From (4.67), this equation can be rewritten as follows

$$\begin{aligned}&\beta_1\left(T_n - \tan\frac{\kappa_n}{2}\tilde{N}_n\right) + \frac{\beta_2}{i}\tan\frac{\kappa_n}{2}\tilde{B}_n\\ &= \frac{1}{2\beta F_n^2}\begin{pmatrix} \beta_1 q_n^* + \beta_2 p_n^* + \beta_1 q_n - \beta_2 p_n \\ \frac{1}{i}\left(\beta_1 q_n^* + \beta_2 p_n^* - \beta_1 q_n + \beta_2 p_n\right) \\ -2\beta_1\left(\beta - 2\delta_1\right) F_n^2 - \epsilon_2|G_n|^2\end{pmatrix}.\end{aligned} \tag{4.86}$$

Thus (4.83) is obtained. □

This is a deformation equation similar to the one presented in [7], and it is known that the compatibility condition of the Frenet frame yield the semi-discrete mKdV equation. We will see later that κ satisfies that equation.

4.2.3 Isoperimetric deformation of the curve by the semi-discrete potential mKdV equation and the semi-discrete sine-Gordon equation

We see that if we choose C, Γ appropriately, we can obtain the deformation equations as presented in [40]. In general, taking the C, Γ in this way causes γ to drift.

Theorem 14. By choosing C, Γ as follows

$$\begin{aligned}
&\Gamma_+ = \frac{-\left(\delta_3+\delta_3^*\right)+\left(\delta_3 e^{i\lambda}+\delta_3^* e^{-i\lambda}\right)}{i\omega},\\
&C_+ = -\epsilon_1 - i\Gamma_+\left(\omega\left(2\delta_1-\beta\right)-\frac{\delta_2+\delta_2^*}{\omega}\right)-\left(\delta_3+\delta_3^*\right)\left(2\delta_1-\beta\right)+\frac{\delta_3-\delta_3^*}{\omega},\\
&\text{or}\\
&\Gamma_- = \frac{-\left(\delta_3+\delta_3^*\right)-\left(\delta_3 e^{i\lambda}+\delta_3^* e^{-i\lambda}\right)}{i\omega},\\
&C_- = -\epsilon_1 - i\Gamma_-\left(\omega\left(2\delta_1-\beta\right)-\frac{\delta_2+\delta_2^*}{\omega}\right)-\left(\delta_3+\delta_3^*\right)\left(2\delta_1-\beta\right)+\frac{\delta_3-\delta_3^*}{\omega},
\end{aligned} \tag{4.87}$$

the following deformation equations hold.

$$\begin{aligned}
&\left\langle \dot\gamma_n, \tilde B_n \right\rangle = 0, \quad \left\langle \dot\gamma_n, \dot\gamma_n \right\rangle = \rho_+^2,\\
&\text{or}\\
&\left\langle \dot\gamma_n, \tilde B_n \right\rangle = 0, \quad \left\langle \dot\gamma_n, \dot\gamma_n \right\rangle = \rho_-^2, \quad \rho_\pm \in \mathbb{R}.
\end{aligned} \tag{4.88}$$

When choosing Γ as in (4.87), depending on the sign of the root, $\langle \dot\gamma_n, \dot\gamma_n \rangle$ also changes, and this is represented by $\rho_\pm$.

Proof. We compute the condition satisfied by C, Γ when

$$\left\langle \dot\gamma_n, \tilde B_n \right\rangle = 0, \tag{4.89}$$

$$\left\langle \dot\gamma_n, \dot\gamma_n \right\rangle = \rho_\pm^2. \tag{4.90}$$

where $\rho_\pm \in \mathbb{R}$ is a constant. From (4.48) and (4.84) we see that

$$\begin{aligned}
2iF_n^3 \tilde G_n \left\langle \dot\gamma_n, \tilde B_n \right\rangle =& \left(p_n - p_n^*\right)\left(\beta_1 q_n^* + \beta_2 p_n^* + \beta_1 q_n - \beta_2 p_n\right)\\
&+ \left(p_n + p_n^*\right)\left(\beta_1 q_n^* + \beta_2 p_n^* - \beta_1 q_n + \beta_2 p_n\right)\\
&+ 2\omega |G_n|^2 \left\{\left(\epsilon_1 + C\right) F_n^2 + \epsilon_2 |G_n|^2\right\}\\
=& 2F_n^2 |G_n|^2 \left\{2\beta_1\left(2\delta_1 - \beta\right)\omega + 2\beta_2 + \omega\left(\epsilon_1 + C\right)\right\}\\
&+ 2\omega |G_n|^4 \left\{2\beta_1\left(\delta_2 + \delta_2^*\right) + 2\omega\beta_2 + \epsilon_2\right\}.
\end{aligned} \tag{4.91}$$

holds. For (4.89) to be satisfied, it is sufficient that the following equations hold.

$$2\beta_1\left(2\delta_1-\beta\right)\omega+2\beta_2+\omega\left(\epsilon_1+C\right)=0, \tag{4.92}$$

$$2\beta_1\left(\delta_2+\delta_2^*\right)+2\omega\beta_2+\epsilon_2=0. \tag{4.93}$$

(4.93) is (4.67) itself, and it is identity. In other words, we can see that (4.92) is a necessary and sufficient condition for the realization of (4.89). By substituting (4.64) and (4.65) into (4.92), it can be rewritten in terms of C,Γ as follows.

$$i\Gamma\left(\omega\left(2\delta_1-\beta\right)-\frac{\delta_2+\delta_2^*}{\omega}\right)+\epsilon_1+C+\left(\delta_3+\delta_3^*\right)\left(2\delta_1-\beta\right)-\frac{\delta_3-\delta_3^*}{\omega}=0, \tag{4.94}$$

Also, since

$$\begin{aligned}
&F_n^4\langle\dot{\gamma}_n,\dot{\gamma}_n\rangle\\
=&\left(\beta_1q_n^*+\beta_2p_n^*+\beta_1q_n-\beta_2p_n\right)^2\\
&-\left(\beta_1q_n^*+\beta_2p_n^*-\beta_1q_n+\beta_2p_n\right)^2\\
&+\left\{\left(\epsilon_1+C\right)F_n^2+\epsilon_2|G_n|^2\right\}^2\\
=&F_n^4\left\{16\beta_1^2\delta_1\left(\beta-\delta_1\right)+\left(\epsilon_1+C\right)^2\right\}\\
&+F_n^2|G_n|^2\\
&\times\left\{4\beta_1^2\left(1+2\left(\beta-2\delta_1\right)\left(\delta_2+\delta_2^*\right)\right)-8\beta_1\beta_2\omega\left(2\delta_1-\beta\right)-4\beta_2^2+2\epsilon_2\left(\epsilon_1+C\right)\right\}\\
&+|G_n|^4\left(-4\beta_1^2\left(\delta_2+\delta_2^*\right)^2-8\beta_1\beta_2\omega\left(\delta_2+\delta_2^*\right)-4\beta_2^2\omega^2+\epsilon_2^2\right).
\end{aligned} \tag{4.95}$$

holds, we see that for (4.90) holds, it is sufficient that

$$4\beta_1^2\left(1+2\left(\beta-2\delta_1\right)\left(\delta_2+\delta_2^*\right)\right)-8\beta_1\beta_2\omega\left(2\delta_1-\beta\right)-4\beta_2^2+2\epsilon_2\left(\epsilon_1+C\right)=0, \tag{4.96}$$

$$-4\beta_1^2\left(\delta_2+\delta_2^*\right)^2-8\beta_1\beta_2\omega\left(\delta_2+\delta_2^*\right)-4\beta_2^2\omega^2+\epsilon_2^2=0. \tag{4.97}$$

hold. Since

$$\begin{aligned}
&4\beta_1^2\left(\delta_2+\delta_2^*\right)^2+8\beta_1\beta_2\omega\left(\delta_2+\delta_2^*\right)+4\beta_2^2\omega^2-\epsilon_2^2\\
=&\left\{2\beta_1\left(\delta_2+\delta_2^*\right)+2\omega\beta_2+\epsilon_2\right\}\left\{2\beta_1\left(\delta_2+\delta_2^*\right)+2\omega\beta_2-\epsilon_2\right\},
\end{aligned} \tag{4.98}$$

holds, (4.97) is identically satisfied. Therefore, (4.96) is a necessary and sufficient condition for the realization of (4.90). By using (4.67), (4.64), (4.65) and (4.94), (4.96) can be written in terms of Γ as follows.

$$i\delta_2\delta_2^*\left(\delta_2-\delta_2^*\right)\Gamma^2+2\delta_2\delta_2^*\left(\delta_3+\delta_3^*\right)\Gamma+i\left(\delta_2{\delta_3^*}^2-\delta_2^*\delta_3^2\right)=0. \tag{4.99}$$

(4.99) has the following solutions.

$$\Gamma = \frac{-\delta_2\delta_2^*\left(\delta_3+\delta_3^*\right) \pm \sqrt{\delta_2\delta_2^*\left(\delta_2\delta_3^*+\delta_2^*\delta_3\right)^2}}{i\delta_2\delta_2^*\left(\delta_2-\delta_2^*\right)}. \tag{4.100}$$

Since

$$\delta_2\delta_2^*\left(\delta_2\delta_3^*+\delta_2^*\delta_3\right)^2 = \left(\delta_2\delta_2^*\right)^2\left(\delta_3 e^{i\lambda}+\delta_3^* e^{-i\lambda}\right)^2 = \left(\delta_2\delta_2^*\right)^2\left(i\frac{\vartheta_1'(0)}{\vartheta_1(iv)}\right)^2, \tag{4.101}$$

holds, (4.100) can be rewritten as

$$\Gamma = \frac{-\left(\delta_3+\delta_3^*\right) \pm \left(\delta_3 e^{i\lambda}+\delta_3^* e^{-i\lambda}\right)}{i\omega}. \tag{4.102}$$

Thus, the deformation equations (4.88) are obtained by putting C, Γ as (4.87). □

We can also write down $\rho_\pm^2$ as follows.

Proposition 12. When C_+, Γ_+ are chosen as in (4.87),

$$\rho_+^2 = \left(i\frac{\vartheta_1'(0)}{\vartheta_1(iv)}\frac{u_2}{\vartheta_3(r)^2\vartheta_1(r)^2}\right)^2, \tag{4.103}$$

holds. Similarly, when C_-, Γ_- are chosen as in (4.87),

$$\rho_-^2 = \left(-i\frac{\vartheta_1'(0)}{\vartheta_1(iv)}\frac{u_2}{\vartheta_4(r)^2\vartheta_2(r)^2}\right)^2, \tag{4.104}$$

holds.

It is also known that when the deformation equation of γ satisfies (4.88), the potential function Θ defined by $2\kappa_n = \Theta_{n+1} - \Theta_{n-1}$ satisfies the semi-discrete potential mKdV equation or the semi-discrete sine-Gordon equation [40]. We will see later that the potential function Θ satisfies the equations.

4.2.4 The semi-discrete potential mKdV equation and the semi-discrete sine-Gordon equation for the potential function

An explicit solution in terms of the elliptic theta function

In this subsubsection, the potential function Θ is explicitly constructed in terms of the elliptic theta functions, only in this case the periodic lattice is set to $\Omega = \mathbb{Z} + w\mathbb{Z}, w = i2/y$.

Theorem 15. Let A_n, B_n be real-valued functions defined as

$$\begin{aligned} A_n(t,z) &= \vartheta_4\left(\frac{v}{y}\left(n-\frac{1}{2}\right)+\frac{t}{y}+\frac{1}{iy}\frac{z}{\Delta_3-\Delta_1}\right), \\ B_n(t,z) &= \vartheta_1\left(\frac{v}{y}\left(n-\frac{1}{2}\right)+\frac{t}{y}+\frac{1}{iy}\frac{z}{\Delta_3-\Delta_1}\right). \end{aligned} \tag{4.105}$$

Then define the function Θ as follows.

$$\Theta_n(t,z) = 4\arctan\frac{B_n}{A_n} = \frac{2}{i}\log\frac{A_n+iB_n}{A_n-iB_n}. \tag{4.106}$$

Then we see that κ defined in (4.22) satisfies

$$\kappa_n = \frac{\Theta_{n+1}-\Theta_{n-1}}{2}. \tag{4.107}$$

Proof. Without losing generality, we can put $z=0$. For any $u, v \in \mathbb{C}, w \in \mathbb{H}$, the theta functions satisfy the following identities [48, 49].

$$\frac{\vartheta_1(u;w)}{\vartheta_2(u;w)} = i\frac{\vartheta_1\left(\frac{u}{w};-\frac{1}{w}\right)}{\vartheta_4\left(\frac{u}{w};-\frac{1}{w}\right)}, \tag{4.108}$$

$$2\vartheta_1(u+v;w)\,\vartheta_4(u-v;w) = \vartheta_1\left(u;\frac{w}{2}\right)\vartheta_2\left(v;\frac{w}{2}\right)+\vartheta_2\left(u;\frac{w}{2}\right)\vartheta_1\left(v;\frac{w}{2}\right), \tag{4.109}$$

$$2\vartheta_2(u+v;w)\,\vartheta_3(u-v;w) = \vartheta_2\left(u;\frac{w}{2}\right)\vartheta_2\left(v;\frac{w}{2}\right)-\vartheta_1\left(u;\frac{w}{2}\right)\vartheta_1\left(v;\frac{w}{2}\right). \tag{4.110}$$

From this, we see that

$$\begin{aligned} \kappa_n &= 2\arctan\left(-i\frac{\vartheta_1(iv;iy)\vartheta_4\left(iv\left(n-\frac{1}{2}\right)+it;iy\right)}{\vartheta_3(iv;iy)\vartheta_2\left(iv\left(n-\frac{1}{2}\right)+it;iy\right)}\right) \\ &= 2\arctan\left(-i\frac{\frac{\vartheta_1\left(i\frac{v}{2}\left(n+\frac{1}{2}\right)+\frac{it}{2};\frac{iy}{2}\right)}{\vartheta_2\left(i\frac{v}{2}\left(n+\frac{1}{2}\right)+\frac{it}{2};\frac{iy}{2}\right)}-\frac{\vartheta_1\left(i\frac{v}{2}\left(n-\frac{3}{2}\right)+\frac{it}{2};\frac{iy}{2}\right)}{\vartheta_2\left(i\frac{v}{2}\left(n-\frac{3}{2}\right)+\frac{it}{2};\frac{iy}{2}\right)}}{1-\frac{\vartheta_1\left(i\frac{v}{2}\left(n+\frac{1}{2}\right)+\frac{it}{2};\frac{iy}{2}\right)\vartheta_1\left(i\frac{v}{2}\left(n-\frac{3}{2}\right)+\frac{it}{2};\frac{iy}{2}\right)}{\vartheta_2\left(i\frac{v}{2}\left(n+\frac{1}{2}\right)+\frac{it}{2};\frac{iy}{2}\right)\vartheta_2\left(i\frac{v}{2}\left(n-\frac{3}{2}\right)+\frac{it}{2};\frac{iy}{2}\right)}}\right) \\ &= 2\arctan\left(\frac{\frac{\vartheta_1\left(\frac{v}{y}\left(n+\frac{1}{2}\right)+\frac{t}{y};\frac{2i}{y}\right)}{\vartheta_4\left(\frac{v}{y}\left(n+\frac{1}{2}\right)+\frac{t}{y};\frac{2i}{y}\right)}-\frac{\vartheta_1\left(\frac{v}{y}\left(n-\frac{3}{2}\right)+\frac{t}{y};\frac{2i}{y}\right)}{\vartheta_4\left(\frac{v}{y}\left(n-\frac{3}{2}\right)+\frac{t}{y};\frac{2i}{y}\right)}}{1+\frac{\vartheta_1\left(\frac{v}{y}\left(n+\frac{1}{2}\right)+\frac{t}{y};\frac{2i}{y}\right)\vartheta_1\left(\frac{v}{y}\left(n-\frac{3}{2}\right)+\frac{t}{y};\frac{2i}{y}\right)}{\vartheta_4\left(\frac{v}{y}\left(n+\frac{1}{2}\right)+\frac{t}{y};\frac{2i}{y}\right)\vartheta_4\left(\frac{v}{y}\left(n-\frac{3}{2}\right)+\frac{t}{y};\frac{2i}{y}\right)}}\right) \\ &= 2\arctan\left(\frac{\vartheta_1\left(\frac{v}{y}\left(n+\frac{1}{2}\right)+\frac{t}{y};\frac{2i}{y}\right)}{\vartheta_4\left(\frac{v}{y}\left(n+\frac{1}{2}\right)+\frac{t}{y};\frac{2i}{y}\right)}\right)-2\arctan\left(\frac{\vartheta_1\left(\frac{v}{y}\left(n-\frac{3}{2}\right)+\frac{t}{y};\frac{2i}{y}\right)}{\vartheta_4\left(\frac{v}{y}\left(n-\frac{3}{2}\right)+\frac{t}{y};\frac{2i}{y}\right)}\right) \\ &= \frac{\Theta_{n+1}-\Theta_{n-1}}{2}. \end{aligned} \tag{4.111}$$

holds. □

Theorem 16. Define

$$Q_1(a) = \frac{\vartheta_1'(0)\vartheta_1(a)}{\vartheta_4(0)\vartheta_4(a)}, \quad Q_2(a) = \frac{\vartheta_1'(0)\vartheta_4(a)}{\vartheta_1(a)\vartheta_4(0)}, \\ L_1(a) = d_1(a) - d_4(a) + Q_1(a) + Q_2(a), \quad a = \frac{v}{y}. \tag{4.112}$$

Then the potential function Θ defined (4.105) and (4.106) satisfies the semi-discrete potential mKdV equation

$$\frac{d}{dt}\left(\frac{\Theta_{n+1} + \Theta_n}{2}\right) = \frac{L_1(a)}{y}\sin\left(\frac{\Theta_{n+1} - \Theta_n}{2}\right). \tag{4.113}$$

Theorem 17. Define

$$L_2(a) = d_1(a) - d_4\left(a\right) - Q_1(a) - Q_2(a), \quad a = \frac{v}{y}. \tag{4.114}$$

Then the potential function Θ defined (4.105) and (4.106) satisfies the semi-discrete sine-Gordon equation

$$\frac{d}{dt}\left(\frac{\Theta_{n+1} - \Theta_n}{2}\right) = \frac{L_2(a)}{y}\sin\left(\frac{\Theta_{n+1} + \Theta_n}{2}\right). \tag{4.115}$$

Remark 15. Consider replacing the parameter with $a = b + 1$. For any complex number z and any integer n, the following holds [48, 49].

$$\vartheta_1\left(z+n\right) = (-1)^n\vartheta_1\left(z\right), \quad \vartheta_4\left(z+n\right) = \vartheta_4\left(z\right). \tag{4.116}$$

Noting this property, we can see that

$$Q_1(b+1) = -Q_1(b), \quad Q_2(b+1) = -Q_2(b), \\ d_1(b+1) = d_1(b), \quad d_4(b+1) = d_4(b), \\ L_1(b+1) = L_2(b). \tag{4.117}$$

hold. Also, we see that

$$\begin{aligned} \Theta_n\left(t,0\right) &= 4\arctan\frac{\vartheta_1\left(a\left(n-\frac{1}{2}\right)+\frac{t}{y}\right)}{\vartheta_4\left(a\left(n-\frac{1}{2}\right)+\frac{t}{y}\right)} \\ &= 4\arctan\frac{\vartheta_1\left(b\left(n-\frac{1}{2}\right)+\frac{t}{y}+n-\frac{1}{2}\right)}{\vartheta_4\left(b\left(n-\frac{1}{2}\right)+\frac{t}{y}+n-\frac{1}{2}\right)} \\ &= (-1)^{n+1}4\arctan\frac{\vartheta_1\left(b\left(n-\frac{1}{2}\right)+\frac{t}{y}+\frac{1}{2}\right)}{\vartheta_4\left(b\left(n-\frac{1}{2}\right)+\frac{t}{y}+\frac{1}{2}\right)} \end{aligned} \tag{4.118}$$

hold. From this, we see that substituting $\Theta_n = (-1)^{n+1}\tilde{\Theta}_n$ into the semi-discrete potential mKdV equation (4.113), we obtain the semi-discrete sine-Gordon equation (4.115) as the equation satisfied by $\tilde{\Theta}_n$ is obtained. Conversely, it can also be shown that a similar transformation on the solution of the semi-discrete sine-Gordon equation (4.115) is a solution of the semi-discrete mKdV equation (4.113). This argument also holds when $z \neq 0$.

Remark 16. The transformation $a \to a+1$ described above corresponds to the transformations described in Remark 5, Remark 11, and Remark 14.

Remark 17. When the deformation equation of γ can be written as (4.89) and (4.90), it is known that the following semi-discrete potential mKdV equation

$$\frac{d}{dt}\left(\frac{\Theta_{n+1}+\Theta_n}{2}\right) = (1+\cos\lambda)\,\rho_+ \sin\left(\frac{\Theta_{n+1}-\Theta_n}{2}\right). \tag{4.119}$$

and the following semi-discrete sine-Gordon equation

$$\frac{d}{dt}\left(\frac{\Theta_{n+1}-\Theta_n}{2}\right) = (1-\cos\lambda)\,\rho_- \sin\left(\frac{\Theta_{n+1}+\Theta_n}{2}\right). \tag{4.120}$$

can be obtained from the isoperimetric condition and the compatibility condition of the Frenet frame [40]. By direct calculations, we see that the following Proposition holds.

Proposition 13.

$$\frac{L_1(v/y)}{y}\frac{1}{1+\cos\lambda} = i\frac{\vartheta_1'(0)}{\vartheta_1(iv)}\frac{u_2}{\vartheta_3(r)^2\vartheta_1(r)^2}, \tag{4.121}$$

holds. Similarly, we see that when C_- and Γ_- are chosen as in (4.87),

$$\frac{L_2(v/y)}{y}\frac{1}{1-\cos\lambda} = -i\frac{\vartheta_1'(0)}{\vartheta_1(iv)}\frac{u_2}{\vartheta_4(r)^2\vartheta_2(r)^2}, \tag{4.122}$$

holds.

Proposition 13 means that when C_+ and Γ_+ (resp. C_- and Γ_-) are chosen as in (4.87), the semi-discrete potential mKdV equation (4.113) (resp. the semi-discrete sine-Gordon equation (4.115)) coincides with the semi-discrete potential mKdV equation (4.119) (resp. (4.120)).

4.2.5 Semi-discrete mKdV equation for signed curvature angle

Theorem 18. The signed curvature angle (4.22) satisfies

$$\frac{d}{dt}\kappa_n = 2\frac{\eta_3}{\eta_4}\left(\tan\frac{\kappa_{n+1}}{2} - \tan\frac{\kappa_{n-1}}{2}\right). \tag{4.123}$$

where

$$\eta_3 = i\frac{\vartheta_1(iv)\,\vartheta_3(iv)}{\vartheta_1(2iv)\,\vartheta_3(0)}\left(d_1(iv) - d_3(iv)\right), \quad \eta_4 = \frac{\vartheta_3(0)^2}{\vartheta_3(iv)^2}. \tag{4.124}$$

Remark 18. Rewriting (4.123) in terms of the potential function Θ, we obtain

$$\frac{d}{dt}\Theta_n = 4\frac{\eta_3}{\eta_4}\tan\left(\frac{\Theta_{n+1}-\Theta_{n-1}}{4}\right), \tag{4.125}$$

This equation is also called the semi-discrete potential mKdV equation [12, 15, 50]. In other words, the potential function Θ defined by (4.105) and (4.106) is the solution of three differential-difference equations, (4.113), (4.115) and (4.125) in conjunction. It does not usually happen that these equations are simultaneously solved. For example, it can be checked by substitution that the 2 and 3-soliton solutions of the semi-discrete sine-Gordon equation constructed in [50, 51] are not solution of the semi-discrete potential mKdV equation.

Remark 19. When the deformation equation of γ can be written as (4.83), it is known that the following semi-discrete mKdV equation

$$\dot{\kappa}_n = \left(2\beta\beta_1\cos\lambda + 2\beta\frac{\beta_2}{i}\sin\lambda\right)\left(\tan\frac{\kappa_{n+1}}{2} - \tan\frac{\kappa_{n-1}}{2}\right), \tag{4.126}$$

can be obtained from the compatibility condition of the Frenet frame [7]. Noting (4.64), (4.65) and $e^{i\lambda}\delta_2 \in \mathbb{R}$, we can check that

$$2\frac{\eta_3}{\eta_4} = \left(2\beta\beta_1\cos\lambda + 2\beta\frac{\beta_2}{i}\sin\lambda\right), \tag{4.127}$$

holds regardless of the values of C, Γ.

Table 4.3 summarizes the relationships among the three deformation equations.

4.3 Derivation of bilinear equations

4.3.1 Bilinearization of the semi-discrete potential mKdV equation (4.113)

Substituting (4.106) into (4.113) and bilinearizing yields the following equation.

$$D_tA_{n+1}\cdot A_n + D_tB_{n+1}\cdot B_n = \frac{L_1}{2y}\left(\eta_1 - 1\right)\left(A_{n+1}A_n + B_{n+1}B_n\right), \tag{4.128}$$

$$D_tA_{n+1}\cdot B_n - D_tB_{n+1}\cdot A_n = \frac{L_1}{2y}\left(\eta_1 + 1\right)\left(A_{n+1}B_n - B_{n+1}A_n\right). \tag{4.129}$$

where η_1 is a real constant.

Table 4.3: Relation between the three deformation equations

Values of C, Γ	Deformation equation of the curve	Equation for potential function
$C, \Gamma \in \mathbb{R}$	$\dot{\gamma}_n = \begin{pmatrix} 0 \\ 0 \\ 2\beta_1 (\beta - 2\delta_1) - \epsilon_1 - C \end{pmatrix} + 2\beta \left(\beta_1 T_n - \beta_1 \tan \frac{\kappa_n}{2} \tilde{N}_n + \frac{\beta_2}{i} \tan \frac{\kappa_n}{2} \tilde{B}_n \right)$	$\dot{\Theta}_n = 2 \left(2\beta\beta_1 \cos \lambda + 2\beta \frac{\beta_2}{i} \sin \lambda \right) \times \tan \left(\frac{\Theta_{n+1} - \Theta_{n-1}}{4} \right)$
↓ Specialize the deformation equation by choosing rigid transformation parameters C, Γ.		
C_+, Γ_+	$\left\langle \dot{\gamma}_n, \tilde{B}_n \right\rangle = 0, \quad \langle \dot{\gamma}_n, \dot{\gamma}_n \rangle = \rho_+^2$	$\dot{\Theta}_{n+1} + \dot{\Theta}_n = 2 (1 + \cos \lambda) \rho_+ \times \sin \left(\frac{\Theta_{n+1} - \Theta_n}{2} \right)$
↕ By replacing the parameters of the solution with $v \to v + y, r \to 1/2 - r$, these two deformation equations shift to each other.		
C_-, Γ_-	$\left\langle \dot{\gamma}_n, \tilde{B}_n \right\rangle = 0, \quad \langle \dot{\gamma}_n, \dot{\gamma}_n \rangle = \rho_-^2$	$\dot{\Theta}_{n+1} - \dot{\Theta}_n = 2 (1 - \cos \lambda) \rho_- \times \sin \left(\frac{\Theta_{n+1} + \Theta_n}{2} \right)$

Proof. Let $\tau_n = A_n + iB_n$, then $\tau_n^* = A_n - iB_n$, and thus

$$\begin{aligned} \frac{d}{dt} \left(\frac{\Theta_{n+1} + \Theta_n}{2} \right) &= \frac{1}{i} \frac{d}{dt} \left(\log \frac{\tau_{n+1}}{\tau_{n+1}^*} + \log \frac{\tau_n}{\tau_n^*} \right) \\ &= \frac{1}{i} \frac{d}{dt} \left(\log \frac{\tau_{n+1}}{\tau_n^*} - \log \frac{\tau_{n+1}^*}{\tau_n} \right) \\ &= \frac{1}{i} \left(\frac{D_t \tau_{n+1} \cdot \tau_n^*}{\tau_{n+1} \tau_n^*} - \frac{D_t \tau_{n+1}^* \cdot \tau_n}{\tau_{n+1}^* \tau_n} \right). \end{aligned} \tag{4.130}$$

$$\sin \left(\frac{\Theta_{n+1} - \Theta_n}{2} \right) = \frac{1}{2i} \left(\frac{\tau_{n+1} \tau_n^*}{\tau_{n+1}^* \tau_n} - \frac{\tau_{n+1}^* \tau_n}{\tau_{n+1} \tau_n^*} \right), \tag{4.131}$$

hold. From this, it can be seen that a sufficient condition for (4.106) to be a solution of (4.113) is that

$$D_t \tau_{n+1} \cdot \tau_n^* = -\frac{L_1}{2y} \tau_{n+1}^* \tau_n + \frac{L_1}{2y} \eta_1 \tau_{n+1} \tau_n^*. \tag{4.132}$$

(η_1 is a real constant) is satisfied. Substituting $\tau_n = A_n + iB_n$ for this, we get

$$\begin{aligned} &D_t (A_{n+1} \cdot A_n + B_{n+1} \cdot B_n) - i D_t (A_{n+1} \cdot B_n - B_{n+1} \cdot A_n) \\ &= \frac{L_1}{2y} (\eta_1 - 1) (A_{n+1} A_n + B_{n+1} B_n) - i \frac{L_1}{2y} (\eta_1 + 1) (A_{n+1} B_n - B_{n+1} A_n). \end{aligned} \tag{4.133}$$

The real part of this equation yields (4.128), and the imaginary part yields (4.129). □

4.3.2 Bilinearization of the semi-discrete sine-Gordon equation (4.115)

Substituting (4.106) into (4.115) and bilinearizing yields the following equation.

$$D_tA_{n+1}\cdot A_n - D_tB_{n+1}\cdot B_n = \frac{L_2}{2y}\left(\eta_2-1\right)\left(A_{n+1}A_n - B_{n+1}B_n\right), \tag{4.134}$$

$$D_tA_{n+1}\cdot B_n + D_tB_{n+1}\cdot A_n = \frac{L_2}{2y}\left(\eta_2+1\right)\left(A_{n+1}B_n + B_{n+1}A_n\right). \tag{4.135}$$

where η_2 is a real constant.

Proof. Let $\tau_n = A_n + iB_n$, then $\tau_n^* = A_n - iB_n$ and thus

$$\begin{aligned}\frac{d}{dt}\left(\frac{\Theta_{n+1}-\Theta_n}{2}\right) &= \frac{1}{i}\frac{d}{dt}\left(\log\frac{\tau_{n+1}}{\tau_{n+1}^*} - \log\frac{\tau_n}{\tau_n^*}\right)\\ &= \frac{1}{i}\frac{d}{dt}\left(\log\frac{\tau_{n+1}}{\tau_n} - \log\frac{\tau_{n+1}^*}{\tau_n^*}\right)\\ &= \frac{1}{i}\left(\frac{D_t\tau_{n+1}\cdot\tau_n}{\tau_{n+1}\tau_n} - \frac{D_t\tau_{n+1}^*\cdot\tau_n^*}{\tau_{n+1}^*\tau_n^*}\right).\end{aligned} \tag{4.136}$$

$$\sin\left(\frac{\Theta_{n+1}+\Theta_n}{2}\right) = \frac{1}{2i}\left(\frac{\tau_{n+1}\tau_n}{\tau_{n+1}^*\tau_n^*} - \frac{\tau_{n+1}^*\tau_n^*}{\tau_{n+1}\tau_n}\right), \tag{4.137}$$

hold. From this, it can be seen that a sufficient condition for (4.106) to be a solution of (4.115) is that

$$D_t\tau_{n+1}\cdot\tau_n = -\frac{L_2}{2y}\tau_{n+1}^*\tau_n^* + \frac{L_2}{2y}\eta_2\tau_{n+1}\tau_n. \tag{4.138}$$

(η_2 is a real constant) is satisfied. Substituting $\tau_n = A_n + iB_n$ for this, we get

$$\begin{aligned}&D_t\left(A_{n+1}\cdot A_n - B_{n+1}\cdot B_n\right) + iD_t\left(A_{n+1}\cdot B_n + B_{n+1}\cdot A_n\right)\\ &= \frac{L_2}{2y}\left(\eta_2-1\right)\left(A_{n+1}A_n - B_{n+1}B_n\right) + i\frac{L_2}{2y}\left(\eta_2+1\right)\left(A_{n+1}B_n + B_{n+1}A_n\right).\end{aligned} \tag{4.139}$$

The real part of (4.139) yields (4.134), and the imaginary part yields (4.135). □

4.3.3 Bilinearization of the semi-discrete mKdV equation (4.123)

From (4.22), we observe that the solution of (4.123) can be expressed using the real functions A_n and B_n as follows.

$$\kappa_n = 2\arctan\left(\frac{B_n}{A_n}\right), \tag{4.140}$$

We obtain the following equation by substituting (4.140) into (4.123) and bilinearizing.

$$D_t B_n \cdot A_n = \eta_3 \left(B_{n+1}A_{n-1} - B_{n-1}A_{n+1}\right), \tag{4.141}$$

$$B_n^2 + A_n^2 = \eta_4 A_{n+1}A_{n-1}. \tag{4.142}$$

where η_3 and η_4 are real constants.

Proof. Noting that

$$\begin{aligned} \dot{\kappa}_n &= 2\frac{D_t B_n \cdot A_n}{B_n^2 + A_n^2}, \\ \tan\frac{\kappa_{n+1}}{2} - \tan\frac{\kappa_{n-1}}{2} &= \frac{B_{n+1}A_{n-1} - B_{n-1}A_{n+1}}{A_{n+1}A_{n-1}}. \end{aligned} \tag{4.143}$$

hold, we get (4.141) and (4.142). □

4.4 Proof of Proposition 11

Here we prove (4.77), (4.78) and (4.79).

Proof.

$$\begin{aligned} p_n p_n^* &= \left(g_n^* f_{n-1} - f_n g_{n-1}^*\right)\left(g_n f_{n-1}^* - f_n^* g_{n-1}\right) \\ &= \left(f_n^* g_{n-1}^* - g_n^* f_{n-1}^*\right)\left(f_n g_{n-1} - g_n f_{n-1}\right) \\ &\quad + \left(f_n f_{n-1}^* - f_n^* f_{n-1}\right)\left(g_n^* g_{n-1} - g_n g_{n-1}^*\right). \end{aligned} \tag{4.144}$$

Substituting (4.32) and (4.69) into (4.144) yelds (4.77).

$$\begin{aligned} & p_n q_n^* - p_n^* q_n \\ &= \left(g_n^* f_{n-1} - f_n g_{n-1}^*\right)\left(g_n f_{n-1}^* + f_n^* g_{n-1}\right) \\ &- \left(g_n f_{n-1}^* - f_n^* g_{n-1}\right)\left(g_n^* f_{n-1} + f_n g_{n-1}^*\right) \\ &= 2\left(g_n^* f_{n-1} f_n^* g_{n-1} - g_n f_{n-1}^* f_n g_{n-1}^*\right). \end{aligned} \tag{4.145}$$

On the other hand, from (4.31) and (4.69), the following holds.

$$\begin{aligned} & \beta\omega F_n^2 |G_n|^2 \\ &= \left(f_n f_{n-1}^* + g_n g_{n-1}^*\right)\left(g_n^* g_{n-1} - g_n g_{n-1}^*\right) \\ &= \left(f_n^* f_{n-1} + g_n^* g_{n-1}\right)\left(g_n^* g_{n-1} - g_n g_{n-1}^*\right) \\ &= \left(f_n f_{n-1}^* + g_n g_{n-1}^*\right)\left(f_n f_{n-1}^* - f_n^* f_{n-1}\right) \\ &= \left(f_n^* f_{n-1} + g_n^* g_{n-1}\right)\left(f_n f_{n-1}^* - f_n^* f_{n-1}\right). \end{aligned} \tag{4.146}$$

Using (4.146), we obtain

$$
\begin{aligned}
&p_n q_n^* - p_n^* q_n \\
&= f_n f_{n-1}^* f_n f_{n-1}^* - f_n^* f_{n-1} f_n^* f_{n-1} + g_n g_{n-1}^* g_n g_{n-1}^* - g_n^* g_{n-1} g_n^* g_{n-1} \\
&= \left(f_n f_{n-1}^*\right)^2 - \left(f_n^* f_{n-1}\right)^2 + \left(g_n g_{n-1}^*\right)^2 - \left(g_n^* g_{n-1}\right)^2 \\
&= \left(f_n f_{n-1}^* - f_n^* f_{n-1}\right)\left(f_n f_{n-1}^* + f_n^* f_{n-1}\right) \\
&+ \left(g_n g_{n-1}^* - g_n^* g_{n-1}\right)\left(g_n g_{n-1}^* + g_n^* g_{n-1}\right).
\end{aligned}
\tag{4.147}
$$

Substituting (4.69), (4.71) and (4.72) into (4.147) gives

$$
\begin{aligned}
&p_n q_n^* - p_n^* q_n \\
&= \omega |G_n|^2 \left(f_n f_{n-1}^* + f_n^* f_{n-1} - g_n g_{n-1}^* - g_n^* g_{n-1}\right) \\
&= 2\left(2\delta_1 - \beta\right)\omega F_n^2 |G_n|^2 + 2\omega\left(\delta_2 + \delta_2^*\right)|G_n|^4,
\end{aligned}
\tag{4.148}
$$

thus (4.78). Also from (4.32), (4.71) and (4.72), we see that

$$
\begin{aligned}
&q_n q_n^* - F_n^2 |G_n|^2 \\
&= \left(g_n^* f_{n-1} + f_n g_{n-1}^*\right)\left(g_n f_{n-1}^* + f_n^* g_{n-1}\right) \\
&- \left(g_n f_{n-1} - f_n g_{n-1}\right)\left(g_n^* f_{n-1}^* - f_n^* g_{n-1}^*\right) \\
&= \left(f_n f_{n-1}^* + f_n^* f_{n-1}\right)\left(g_n g_{n-1}^* + g_n^* g_{n-1}\right) \\
&= \left(2\delta_1 F_n^2 + \left(\delta_2 + \delta_2^*\right)|G_n|^2\right)\left(2\left(\beta - \delta_1\right)F_n^2 - \left(\delta_2 + \delta_2^*\right)|G_n|^2\right) \\
&= 4\delta_1\left(\beta - \delta_1\right)F_n^4 + 2\left(\beta - 2\delta_1\right)\left(\delta_2 + \delta_2^*\right)F_n^2|G_n|^2 - \left(\delta_2 + \delta_2^*\right)^2|G_n|^4.
\end{aligned}
\tag{4.149}
$$

holds. From (4.149), we get (4.79). □

4.5 Construction of solutions to bilinear equations in terms of the elliptic theta functions

Let

$$
\begin{aligned}
X_1 &= \frac{1}{2}(X + Y + U + V), \quad Y_1 = \frac{1}{2}(X + Y - U - V), \\
U_1 &= \frac{1}{2}(X - Y + U - V), \quad V_1 = \frac{1}{2}(X - Y - U + V),
\end{aligned}
\tag{4.150}
$$

then for any $X, Y, U, V \in \mathbb{C}$, the elliptic theta functions satisfy the following identities [48, 49].

$$
\begin{aligned}
&\vartheta_2(X_1)\vartheta_2(Y_1)\vartheta_3(U_1)\vartheta_3(V_1) - \vartheta_4(X_1)\vartheta_4(Y_1)\vartheta_1(U_1)\vartheta_1(V_1) \\
&= \vartheta_2(X)\vartheta_2(Y)\vartheta_3(U)\vartheta_3(V) - \vartheta_4(X)\vartheta_4(Y)\vartheta_1(U)\vartheta_1(V),
\end{aligned}
\tag{4.151}
$$

$$\begin{aligned}&\vartheta_4(X_1)\vartheta_3(Y_1)\vartheta_1(U_1)\vartheta_2(V_1) - \vartheta_2(X_1)\vartheta_1(Y_1)\vartheta_3(U_1)\vartheta_4(V_1)\\ &= \vartheta_4(X)\vartheta_3(Y)\vartheta_1(U)\vartheta_2(V) - \vartheta_2(X)\vartheta_1(Y)\vartheta_3(U)\vartheta_4(V),\end{aligned} \tag{4.152}$$

$$\begin{aligned}&\vartheta_2(X_1)\vartheta_2(Y_1)\vartheta_4(U_1)\vartheta_4(V_1) + \vartheta_1(X_1)\vartheta_1(Y_1)\vartheta_3(U_1)\vartheta_3(V_1)\\ &= \vartheta_4(X)\vartheta_4(Y)\vartheta_2(U)\vartheta_2(V) - \vartheta_3(X)\vartheta_3(Y)\vartheta_1(U)\vartheta_1(V).\end{aligned} \tag{4.153}$$

$$\begin{aligned}&\vartheta_3(X_1)\vartheta_3(Y_1)\vartheta_2(U_1)\vartheta_2(V_1) + \vartheta_1(X_1)\vartheta_1(Y_1)\vartheta_4(U_1)\vartheta_4(V_1)\\ &= \vartheta_3(X)\vartheta_3(Y)\vartheta_2(U)\vartheta_2(V) + \vartheta_1(X)\vartheta_1(Y)\vartheta_4(U)\vartheta_4(V).\end{aligned} \tag{4.154}$$

$$\begin{aligned}&\vartheta_4(X_1)\vartheta_4(Y_1)\vartheta_4(U_1)\vartheta_4(V_1) - \vartheta_1(X_1)\vartheta_1(Y_1)\vartheta_1(U_1)\vartheta_1(V_1)\\ &= \vartheta_4(X)\vartheta_4(Y)\vartheta_4(U)\vartheta_4(V) - \vartheta_1(X)\vartheta_1(Y)\vartheta_1(U)\vartheta_1(V).\end{aligned} \tag{4.155}$$

$$\begin{aligned}&\vartheta_4(X_1)\vartheta_4(Y_1)\vartheta_1(U_1)\vartheta_1(V_1) - \vartheta_1(X_1)\vartheta_1(Y_1)\vartheta_4(U_1)\vartheta_4(V_1)\\ &= \vartheta_4(X)\vartheta_4(Y)\vartheta_1(U)\vartheta_1(V) - \vartheta_1(X)\vartheta_1(Y)\vartheta_4(U)\vartheta_4(V).\end{aligned} \tag{4.156}$$

$$\begin{aligned}&\vartheta_4(X_1)\vartheta_4(Y_1)\vartheta_1(U_1)\vartheta_1(V_1) + \vartheta_2(X_1)\vartheta_2(Y_1)\vartheta_3(U_1)\vartheta_3(V_1)\\ &= \vartheta_3(X)\vartheta_3(Y)\vartheta_2(U)\vartheta_2(V) - \vartheta_1(X)\vartheta_1(Y)\vartheta_4(U)\vartheta_4(V).\end{aligned} \tag{4.157}$$

$$\begin{aligned}&\vartheta_3(X_1)\vartheta_3(Y_1)\vartheta_1(U_1)\vartheta_1(V_1) - \vartheta_1(X_1)\vartheta_1(Y_1)\vartheta_3(U_1)\vartheta_3(V_1)\\ &= \vartheta_3(X)\vartheta_3(Y)\vartheta_1(U)\vartheta_1(V) - \vartheta_1(X)\vartheta_1(Y)\vartheta_3(U)\vartheta_3(V).\end{aligned} \tag{4.158}$$

4.5.1 Proof of Theorem 12

Here we prove that the τ functions given in Theorem 12 satisfy (4.14)-(4.18).

Proof. Without loss of generality, we can assume $t = 0$. For proofs other than (4.18), it is also assumed that $z = 0$. Putting

$$\begin{cases} X = ivn + r,\\ Y = ivn - r,\\ U = \frac{1}{2}iv + r,\\ V = \frac{1}{2}iv - r.\end{cases} \tag{4.159}$$

in (4.151) yields

$$\begin{aligned}&\vartheta_2\left(iv\left(n+\frac{1}{2}\right)\right)\vartheta_2\left(iv\left(n-\frac{1}{2}\right)\right)\vartheta_3(2r)\vartheta_3(0)\\ &= \vartheta_2\left(ivn+r\right)\vartheta_2\left(ivn-r\right)\vartheta_3\left(\frac{1}{2}iv+r\right)\vartheta_3\left(-\frac{1}{2}iv+r\right)\\ &+ \vartheta_4\left(ivn+r\right)\vartheta_4\left(ivn-r\right)\vartheta_1\left(\frac{1}{2}iv+r\right)\vartheta_1\left(-\frac{1}{2}iv+r\right).\end{aligned} \tag{4.160}$$

(4.14) follows from (4.160). Putting

$$\begin{cases} X = iv\left(n-\frac{1}{2}\right), \\ Y = ivn + r, \\ U = iv, \\ V = \frac{1}{2}iv - r. \end{cases} \tag{4.161}$$

in (4.151) yields

$$\begin{aligned} &\vartheta_2\left(iv\left(n+\frac{1}{2}\right)\right)\frac{\vartheta_3\left(\frac{1}{2}iv+r\right)}{\vartheta_3\left(-\frac{1}{2}iv+r\right)}\vartheta_3\left(-\frac{1}{2}iv+r\right)\vartheta_2\left(iv\left(n-1\right)+r\right) \\ &= \frac{\vartheta_3(iv)}{\vartheta_3(0)}\vartheta_2\left(iv\left(n-\frac{1}{2}\right)\right)\vartheta_3\left(-\frac{1}{2}iv+r\right)\vartheta_2\left(ivn+r\right) \\ &+ \frac{\vartheta_1(iv)}{\vartheta_3(0)}\vartheta_4\left(iv\left(n-\frac{1}{2}\right)\right)\vartheta_1\left(-\frac{1}{2}iv+r\right)\vartheta_4\left(ivn+r\right). \end{aligned} \tag{4.162}$$

(4.15) follows from (4.162). Putting

$$\begin{cases} X = iv\left(n-\frac{1}{2}\right), \\ Y = \frac{1}{2}iv + r, \\ U = iv, \\ V = ivn - r. \end{cases} \tag{4.163}$$

in (4.152) yields

$$\begin{aligned} &\vartheta_2\left(iv\left(n+\frac{1}{2}\right)\right)\frac{\vartheta_1\left(-\frac{1}{2}iv+r\right)}{\vartheta_1\left(\frac{1}{2}iv+r\right)}\vartheta_1\left(\frac{1}{2}iv+r\right)\vartheta_4\left(iv\left(n-1\right)-r\right) \\ &= \frac{\vartheta_3(iv)}{\vartheta_3(0)}\vartheta_2\left(iv\left(n-\frac{1}{2}\right)\right)\vartheta_1\left(\frac{1}{2}iv+r\right)\vartheta_4\left(ivn-r\right) \\ &- \frac{\vartheta_1(iv)}{\vartheta_3(0)}\vartheta_4\left(iv\left(n-\frac{1}{2}\right)\right)\vartheta_3\left(\frac{1}{2}iv+r\right)\vartheta_2\left(ivn-r\right). \end{aligned} \tag{4.164}$$

(4.16) follows from (4.164). Putting

$$\begin{cases} X = ivn - r, \\ Y = -\frac{1}{2}iv + r, \\ U = ivn - r, \\ V = -\frac{1}{2}iv + r. \end{cases} \tag{4.165}$$

in (4.153) yields

$$\begin{aligned} &\frac{\vartheta_2\left(0\right)\vartheta_4\left(0\right)}{\vartheta_1\left(-\frac{1}{2}iv+r\right)\vartheta_3\left(-\frac{1}{2}iv+r\right)}\vartheta_2\left(iv\left(n-\frac{1}{2}\right)\right)\vartheta_4\left(iv\left(n+\frac{1}{2}\right)-2r\right) \\ &= \frac{\vartheta_2\left(-\frac{1}{2}iv+r\right)\vartheta_4\left(-\frac{1}{2}iv+r\right)}{\vartheta_1\left(-\frac{1}{2}iv+r\right)\vartheta_3\left(-\frac{1}{2}iv+r\right)} \\ &\times \vartheta_2\left(ivn-r\right)\vartheta_4\left(ivn-r\right) - \vartheta_3\left(ivn-r\right)\vartheta_1\left(ivn-r\right). \end{aligned} \tag{4.166}$$

Also, putting

$$\begin{cases} X = ivn - r, \\ Y = \frac{1}{2}iv + r, \\ U = ivn - r, \\ V = \frac{1}{2}iv + r. \end{cases} \tag{4.167}$$

in (4.153) yields

$$\begin{aligned} &\frac{\vartheta_2(0)\,\vartheta_4(0)}{\vartheta_1\left(\frac{1}{2}iv + r\right)\vartheta_3\left(\frac{1}{2}iv + r\right)}\vartheta_2\left(iv\left(n + \frac{1}{2}\right)\right)\vartheta_4\left(iv\left(n - \frac{1}{2}\right) - 2r\right) \\ &= \frac{\vartheta_2\left(\frac{1}{2}iv + r\right)\vartheta_4\left(\frac{1}{2}iv + r\right)}{\vartheta_1\left(\frac{1}{2}iv + r\right)\vartheta_3\left(\frac{1}{2}iv + r\right)}\vartheta_2(ivn - r)\,\vartheta_4(ivn - r) - \vartheta_3(ivn - r)\,\vartheta_1(ivn - r)\,. \end{aligned} \tag{4.168}$$

Subtracting (4.168) from (4.166) and substituting

$$\begin{aligned} &\vartheta_3(2r)\,\vartheta_4(0)\,\vartheta_2(0)\,\vartheta_1(iv) \\ &= \vartheta_1\left(\frac{1}{2}iv + r\right)\vartheta_3\left(\frac{1}{2}iv + r\right)\vartheta_2\left(-\frac{1}{2}iv + r\right)\vartheta_4\left(-\frac{1}{2}iv + r\right) \\ &- \vartheta_1\left(-\frac{1}{2}iv + r\right)\vartheta_3\left(-\frac{1}{2}iv + r\right)\vartheta_2\left(\frac{1}{2}iv + r\right)\vartheta_4\left(\frac{1}{2}iv + r\right), \end{aligned} \tag{4.169}$$

in the equation, we obtain

$$\begin{aligned} &\frac{\vartheta_1\left(-\frac{1}{2}iv + r\right)\vartheta_3\left(-\frac{1}{2}iv + r\right)\vartheta_1\left(\frac{1}{2}iv + r\right)\vartheta_3\left(\frac{1}{2}iv + r\right)}{\vartheta_3(2r)\vartheta_1(iv)} \\ &\times\left\{\frac{\vartheta_1\left(\frac{1}{2}iv + r\right)\vartheta_3\left(\frac{1}{2}iv + r\right)}{\vartheta_1\left(-\frac{1}{2}iv + r\right)\vartheta_3\left(-\frac{1}{2}iv + r\right)}\vartheta_2\left(iv\left(n - \frac{1}{2}\right)\right)\vartheta_4\left(iv\left(n + \frac{1}{2}\right) - 2r\right)\right. \\ &\left.-\vartheta_2\left(iv\left(n + \frac{1}{2}\right)\right)\vartheta_4\left(iv\left(n - \frac{1}{2}\right) - 2r\right)\right\} \\ &= \vartheta_3\left(\frac{1}{2}iv + r\right)\vartheta_2(ivn - r)\,\vartheta_1\left(\frac{1}{2}iv + r\right)\vartheta_4(ivn - r)\,. \end{aligned} \tag{4.170}$$

(4.17) follows from (4.170). Putting

$$\begin{cases} X = ivn + z + r, \\ Y = ivn + z - r, \\ U = \frac{1}{2}iv + r + h, \\ V = \frac{1}{2}iv - r + h. \end{cases} \tag{4.171}$$

in (4.151) yields

$$\begin{aligned}
&\vartheta_2\left(iv\left(n+\frac{1}{2}\right)+z+h\right)\vartheta_2\left(iv\left(n-\frac{1}{2}\right)+z-h\right)\vartheta_3(2r)\vartheta_3(0)\\
&=\vartheta_2\left(ivn+z+r\right)\vartheta_2\left(ivn+z-r\right)\vartheta_3\left(\frac{1}{2}iv+r+h\right)\vartheta_3\left(\frac{1}{2}iv-r+h\right)\\
&-\vartheta_4\left(ivn+z+r\right)\vartheta_4\left(ivn+z-r\right)\vartheta_1\left(\frac{1}{2}iv+r+h\right)\vartheta_1\left(\frac{1}{2}iv-r+h\right).
\end{aligned}\tag{4.172}$$

Differentiating (4.172) with h and then putting $h=0$, we obtain

$$\begin{aligned}
&\vartheta_3(2r)\vartheta_3(0)D_z\vartheta_2\left(iv\left(n+\frac{1}{2}\right)+z\right)\cdot\vartheta_2\left(iv\left(n-\frac{1}{2}\right)+z\right)\\
&=\left(\frac{\vartheta_3'\left(\frac{1}{2}iv+r\right)}{\vartheta_3\left(\frac{1}{2}iv+r\right)}-\frac{\vartheta_3'\left(-\frac{1}{2}iv+r\right)}{\vartheta_3\left(-\frac{1}{2}iv+r\right)}\right)\\
&\times\vartheta_3\left(-\frac{1}{2}iv+r\right)\vartheta_3\left(\frac{1}{2}iv+r\right)\vartheta_2\left(ivn+z+r\right)\vartheta_2\left(ivn+z-r\right)\\
&+\left(\frac{\vartheta_1'\left(\frac{1}{2}iv+r\right)}{\vartheta_1\left(\frac{1}{2}iv+r\right)}-\frac{\vartheta_1'\left(-\frac{1}{2}iv+r\right)}{\vartheta_1\left(-\frac{1}{2}iv+r\right)}\right)\\
&\times\vartheta_1\left(-\frac{1}{2}iv+r\right)\vartheta_1\left(\frac{1}{2}iv+r\right)\vartheta_4\left(ivn+z+r\right)\vartheta_4\left(ivn+z-r\right).
\end{aligned}\tag{4.173}$$

Using the equation in (4.172) with $h=0$ and substituting (4.173), we obtain

$$\begin{aligned}
&\vartheta_3(2r)\vartheta_3(0)\left\{D_z\vartheta_2\left(iv\left(n+\frac{1}{2}\right)+z\right)\cdot\vartheta_2\left(iv\left(n-\frac{1}{2}\right)+z\right)\right.\\
&+\left(\frac{\vartheta_3'\left(-\frac{1}{2}iv+r\right)}{\vartheta_3\left(-\frac{1}{2}iv+r\right)}-\frac{\vartheta_3'\left(\frac{1}{2}iv+r\right)}{\vartheta_3\left(\frac{1}{2}iv+r\right)}\right)\\
&\left.\times\vartheta_2\left(iv\left(n+\frac{1}{2}\right)+z\right)\vartheta_2\left(iv\left(n-\frac{1}{2}\right)+z\right)\right\}\\
&=\left(\frac{\vartheta_1'\left(\frac{1}{2}iv+r\right)}{\vartheta_1\left(\frac{1}{2}iv+r\right)}-\frac{\vartheta_1'\left(-\frac{1}{2}iv+r\right)}{\vartheta_1\left(-\frac{1}{2}iv+r\right)}+\frac{\vartheta_3'\left(-\frac{1}{2}iv+r\right)}{\vartheta_3\left(-\frac{1}{2}iv+r\right)}-\frac{\vartheta_3'\left(\frac{1}{2}iv+r\right)}{\vartheta_3\left(\frac{1}{2}iv+r\right)}\right)\\
&\times\vartheta_1\left(-\frac{1}{2}iv+r\right)\vartheta_4\left(ivn+z+r\right)\vartheta_1\left(\frac{1}{2}iv+r\right)\vartheta_4\left(ivn+z-r\right).
\end{aligned}\tag{4.174}$$

(4.18) follows from (4.174). □

4.5.2 Proof of Proposition 8

Proof. Putting

$$\begin{cases} X = -\frac{1}{2}iv + r, \\ Y = -\frac{1}{2}iv + r, \\ U = \frac{1}{2}iv + r, \\ V = \frac{1}{2}iv + r. \end{cases} \tag{4.175}$$

in (4.158) yields

$$\begin{aligned} &\vartheta_3\left(-\frac{1}{2}iv + r\right)^2 \vartheta_1\left(\frac{1}{2}iv + r\right)^2 - \vartheta_1\left(-\frac{1}{2}iv + r\right)^2 \vartheta_3\left(\frac{1}{2}iv + r\right)^2 \\ &= \vartheta_1\left(2r\right)\vartheta_1\left(iv\right)\vartheta_3\left(0\right)^2, \end{aligned} \tag{4.176}$$

(4.66) follows from (4.176). □

4.5.3 Proof of Proposition 9

Here we prove (4.67).

Proof. Putting

$$\begin{cases} X = \frac{1}{2}iv + r + h, \\ Y = -\frac{1}{2}iv + r - h, \\ U = \frac{1}{2}iv + r - h, \\ V = -\frac{1}{2}iv + r + h. \end{cases} \tag{4.177}$$

in (4.158) yields

$$\begin{aligned} &\vartheta_3\left(\frac{1}{2}iv + r + h\right)\vartheta_3\left(-\frac{1}{2}iv + r - h\right)\vartheta_1\left(\frac{1}{2}iv + r - h\right)\vartheta_1\left(-\frac{1}{2}iv + r + h\right) \\ &- \vartheta_1\left(\frac{1}{2}iv + r + h\right)\vartheta_1\left(-\frac{1}{2}iv + r - h\right)\vartheta_3\left(\frac{1}{2}iv + r - h\right)\vartheta_3\left(-\frac{1}{2}iv + r + h\right) \\ &= \vartheta_3\left(2r\right)\vartheta_3\left(0\right)\vartheta_1\left(iv\right)\vartheta_1\left(2h\right), \end{aligned} \tag{4.178}$$

After differentiating (4.178) by h and setting $h = 0$, we obtain (4.67). □

4.5.4 Proof of Proposition 10

Here we prove that (4.13) satisfy (4.68), (4.69), (4.70), (4.71), (4.72), (4.73) and (4.74).

Proof. Except for the proof of (4.74), it is valid assuming $z = 0$. Also, except for the proof of (4.73) and (4.74), it is valid assuming $t = 0$. Putting

$$\begin{cases} X = -\frac{1}{2}iv + r, \\ Y = \frac{1}{2}iv - r, \\ U = ivn + r, \\ V = iv\left(n-1\right) - r. \end{cases} \tag{4.179}$$

in (4.157) yields

$$\begin{aligned} &\vartheta_3\left(-\frac{1}{2}iv + r\right)^2 \vartheta_2\left(ivn + r\right)\vartheta_2\left(iv\left(n-1\right) - r\right) \\ &+ \vartheta_1\left(-\frac{1}{2}iv + r\right)^2 \vartheta_4\left(ivn + r\right)\vartheta_4\left(iv\left(n-1\right) - r\right) \\ &= \vartheta_3\left(2r\right)\vartheta_3\left(iv\right)\vartheta_2\left(iv\left(n-\frac{1}{2}\right)\right)^2 - \vartheta_1\left(2r\right)\vartheta_1\left(iv\right)\vartheta_4\left(iv\left(n-\frac{1}{2}\right)\right)^2. \end{aligned} \tag{4.180}$$

(4.68) follows from (4.180). Also, (4.69) follows from (4.31) and (4.68). Putting

$$\begin{cases} X = ivn - r + it, \\ Y = -\frac{1}{2}iv - r + h, \\ U = \frac{1}{2}iv - r + h, \\ V = iv\left(n-1\right) - r + it. \end{cases} \tag{4.181}$$

in (4.152) yields

$$\begin{aligned} &\vartheta_4\left(iv\left(n-\frac{1}{2}\right) - 2r + it + h\right)\vartheta_3\left(0\right)\vartheta_1\left(iv\right)\vartheta_2\left(iv\left(n-\frac{1}{2}\right) + it - h\right) \\ &= \vartheta_4\left(ivn - r + it\right)\vartheta_3\left(-\frac{1}{2}iv - r + h\right) \\ &\times \vartheta_1\left(\frac{1}{2}iv - r + h\right)\vartheta_2\left(iv\left(n-1\right) - r + it\right) \\ &- \vartheta_2\left(ivn - r + it\right)\vartheta_1\left(-\frac{1}{2}iv - r + h\right) \\ &\times \vartheta_3\left(\frac{1}{2}iv - r + h\right)\vartheta_4\left(iv\left(n-1\right) - r + it\right), \end{aligned} \tag{4.182}$$

After differentiating (4.182) by h and putting $h = 0$, we obtain (4.73) by rearrangement. Also, rearranging the equation with $h = 0$ in (4.182) yields (4.70). Putting

$$\begin{cases} X = \frac{1}{2}iv + r, \\ Y = -\frac{1}{2}iv - r, \\ U = iv\left(n - \frac{1}{2}\right), \\ V = -iv\left(n - \frac{1}{2}\right). \end{cases} \tag{4.183}$$

in (4.154) yields

$$\begin{aligned} &\vartheta_3(0)^2\,\vartheta_2(ivn + r)\,\vartheta_2(iv(n-1) - r) \\ &= \vartheta_3\left(\frac{1}{2}iv + r\right)^2 \vartheta_2\left(iv\left(n - \frac{1}{2}\right)\right)^2 - \vartheta_1\left(\frac{1}{2}iv + r\right)^2 \vartheta_4\left(iv\left(n - \frac{1}{2}\right)\right)^2. \end{aligned} \tag{4.184}$$

(4.71) follows from (4.184). Also, (4.72) follows from (4.68) and (4.71). Putting

$$\begin{cases} X = iv\left(n - \frac{1}{2}\right) + it + \frac{z}{\Delta_3 - \Delta_1}, \\ Y = iv\left(n - \frac{1}{2}\right) + it + \frac{z}{\Delta_3 - \Delta_1}, \\ U = h, \\ V = h. \end{cases} \tag{4.185}$$

in (4.151) yields

$$\begin{aligned} &\vartheta_2\left(iv\left(n - \frac{1}{2}\right) + it + \frac{z}{\Delta_3 - \Delta_1} + h\right)\vartheta_2\left(iv\left(n - \frac{1}{2}\right) + it + \frac{z}{\Delta_3 - \Delta_1} - h\right) \\ &= -\vartheta_4\left(iv\left(n - \frac{1}{2}\right) + it + \frac{z}{\Delta_3 - \Delta_1}\right)^2 \frac{\vartheta_1(h)^2}{\vartheta_3(0)^2} \\ &+ \vartheta_2\left(iv\left(n - \frac{1}{2}\right) + it + \frac{z}{\Delta_3 - \Delta_1}\right)^2 \frac{\vartheta_3(h)^2}{\vartheta_3(0)^2}. \end{aligned} \tag{4.186}$$

After differentiating (4.186) twice with h and putting $h = 0$, we obtain (4.74). □

4.5.5 Construction of solutions to the semi-discrete potential mKdV equation and the semi-discrete sine-Gordon equation

Here we construct solutions of (4.128) and (4.129) as well as (4.134) and (4.135) in terms of the elliptic theta functions.

Proposition 14. The real functions A and B defined by (4.105) are the solutions of (4.128) and (4.129) with

$$\eta_1 = \frac{d_1(a) + d_4(a) - Q_1(a) + Q_4(a)}{L_1}. \tag{4.187}$$

Proposition 15. The real functions A and B defined by (4.105) are the solutions of (4.134) and (4.135) with

$$\eta_2 = \frac{d_1(a) + d_4(a) + Q_1(a) - Q_4(a)}{L_2}. \tag{4.188}$$

Proof. The proof holds even assuming $z = 0$. After differentiating the equation in (4.155) with

$$\begin{cases} X = a\left(n + \frac{1}{2}\right) + \frac{t}{y}, \\ Y = a\left(n - \frac{1}{2}\right) + \frac{t}{y}, \\ U = a + h, \\ V = h. \end{cases} \tag{4.189}$$

and putting $h = 0$, we obtain

$$\begin{aligned} & D_t \vartheta_4 \left(a\left(n + \frac{1}{2}\right) + \frac{t}{y}\right) \cdot \vartheta_4 \left(a\left(n - \frac{1}{2}\right) + \frac{t}{y}\right) \\ & = \frac{1}{y}\frac{\vartheta_4'(a)}{\vartheta_4(a)} \vartheta_4 \left(a\left(n + \frac{1}{2}\right) + \frac{t}{y}\right) \vartheta_4 \left(a\left(n - \frac{1}{2}\right) + \frac{t}{y}\right) \\ & - \frac{1}{y}\frac{\vartheta_1(a)\,\vartheta_1'(0)}{\vartheta_4(a)\,\vartheta_4(0)} \vartheta_1 \left(a\left(n + \frac{1}{2}\right) + \frac{t}{y}\right) \vartheta_1 \left(a\left(n - \frac{1}{2}\right) + \frac{t}{y}\right) \\ & = \frac{1}{y} d_4(a) \vartheta_4 \left(a\left(n + \frac{1}{2}\right) + \frac{t}{y}\right) \vartheta_4 \left(a\left(n - \frac{1}{2}\right) + \frac{t}{y}\right) \\ & - \frac{1}{y} Q_1(a) \vartheta_1 \left(a\left(n + \frac{1}{2}\right) + \frac{t}{y}\right) \vartheta_1 \left(a\left(n - \frac{1}{2}\right) + \frac{t}{y}\right). \end{aligned} \tag{4.190}$$

Also, after differentiating the equation in (4.156) with

$$\begin{cases} X = a\left(n + \frac{1}{2}\right) + \frac{t}{y}, \\ Y = a\left(n - \frac{1}{2}\right) + \frac{t}{y}, \\ U = a + h, \\ V = h. \end{cases} \tag{4.191}$$

and putting $h = 0$, we obtain

$$\begin{aligned}
&D_t\vartheta_1\left(a\left(n+\frac{1}{2}\right)+\frac{t}{y}\right)\cdot\vartheta_1\left(a\left(n-\frac{1}{2}\right)+\frac{t}{y}\right)\\
&=\frac{1}{y}\frac{\vartheta_4'(a)}{\vartheta_4(a)}\vartheta_1\left(a\left(n+\frac{1}{2}\right)+\frac{t}{y}\right)\vartheta_1\left(a\left(n-\frac{1}{2}\right)+\frac{t}{y}\right)\\
&-\frac{1}{y}\frac{\vartheta_1(a)\vartheta_1'(0)}{\vartheta_4(a)\vartheta_4(0)}\vartheta_4\left(a\left(n+\frac{1}{2}\right)+\frac{t}{y}\right)\vartheta_4\left(a\left(n-\frac{1}{2}\right)+\frac{t}{y}\right)\\
&=\frac{1}{y}d_4(a)\vartheta_1\left(a\left(n+\frac{1}{2}\right)+\frac{t}{y}\right)\vartheta_1\left(a\left(n-\frac{1}{2}\right)+\frac{t}{y}\right)\\
&-\frac{1}{y}Q_1(a)\vartheta_4\left(a\left(n+\frac{1}{2}\right)+\frac{t}{y}\right)\vartheta_4\left(a\left(n-\frac{1}{2}\right)+\frac{t}{y}\right).
\end{aligned}\tag{4.192}$$

Also, after differentiating the equation in (4.156) with

$$\begin{cases} X = a\left(n+\frac{1}{2}\right)+\frac{t}{y}, \\ Y = a+h, \\ U = a\left(n-\frac{1}{2}\right)+\frac{t}{y}, \\ V = h. \end{cases}\tag{4.193}$$

and putting $h = 0$, we obtain

$$\begin{aligned}
&D_t\vartheta_1\left(a\left(n+\frac{1}{2}\right)+\frac{t}{y}\right)\cdot\vartheta_4\left(a\left(n-\frac{1}{2}\right)+\frac{t}{y}\right)\\
&=\frac{1}{y}\frac{\vartheta_1'(a)}{\vartheta_1(a)}\vartheta_1\left(a\left(n+\frac{1}{2}\right)+\frac{t}{y}\right)\vartheta_4\left(a\left(n-\frac{1}{2}\right)+\frac{t}{y}\right)\\
&-\frac{1}{y}\frac{\vartheta_4(a)\vartheta_1'(0)}{\vartheta_1(a)\vartheta_4(0)}\vartheta_4\left(a\left(n+\frac{1}{2}\right)+\frac{t}{y}\right)\vartheta_1\left(a\left(n-\frac{1}{2}\right)+\frac{t}{y}\right)\\
&=\frac{1}{y}d_1(a)\vartheta_1\left(a\left(n+\frac{1}{2}\right)+\frac{t}{y}\right)\vartheta_4\left(a\left(n-\frac{1}{2}\right)+\frac{t}{y}\right)\\
&-\frac{1}{y}Q_2(a)\vartheta_4\left(a\left(n+\frac{1}{2}\right)+\frac{t}{y}\right)\vartheta_1\left(a\left(n-\frac{1}{2}\right)+\frac{t}{y}\right).
\end{aligned}\tag{4.194}$$

Also, after differentiating the equation in (4.156) with

$$\begin{cases} X = a\left(n-\frac{1}{2}\right)+\frac{t}{y}, \\ Y = a+h, \\ U = a\left(n+\frac{1}{2}\right)+\frac{t}{y}, \\ V = h. \end{cases}\tag{4.195}$$

and putting $h = 0$, we obtain

$$\begin{aligned}
&D_t\vartheta_4\left(a\left(n+\frac{1}{2}\right)+\frac{t}{y}\right)\cdot\vartheta_1\left(a\left(n-\frac{1}{2}\right)+\frac{t}{y}\right)\\
&=\frac{1}{y}\frac{\vartheta_1'(a)}{\vartheta_1(a)}\vartheta_4\left(a\left(n+\frac{1}{2}\right)+\frac{t}{y}\right)\vartheta_1\left(a\left(n-\frac{1}{2}\right)+\frac{t}{y}\right)\\
&-\frac{1}{y}\frac{\vartheta_4(a)\,\vartheta_1'(0)}{\vartheta_1(a)\,\vartheta_4(0)}\vartheta_1\left(a\left(n+\frac{1}{2}\right)+\frac{t}{y}\right)\vartheta_4\left(a\left(n-\frac{1}{2}\right)+\frac{t}{y}\right)\\
&=\frac{1}{y}d_1(a)\vartheta_4\left(a\left(n+\frac{1}{2}\right)+\frac{t}{y}\right)\vartheta_1\left(a\left(n-\frac{1}{2}\right)+\frac{t}{y}\right)\\
&-\frac{1}{y}Q_2(a)\vartheta_1\left(a\left(n+\frac{1}{2}\right)+\frac{t}{y}\right)\vartheta_4\left(a\left(n-\frac{1}{2}\right)+\frac{t}{y}\right).
\end{aligned} \tag{4.196}$$

Adding (4.190) and (4.192) yields (4.128), and subtracting (4.194) from (4.196) yields (4.129). Also, subtracting (4.192) from (4.190) yields (4.134), and adding (4.196) and (4.194) yields (4.135). □

4.5.6 Construction of a solution to the semi-discrete mKdV equation

Here we construct the solutions of (4.141) and (4.142) in terms of the elliptic theta functions.

Proposition 16.

$$\begin{aligned}
B_n &= -i\vartheta_1(iv)\vartheta_4\left(iv\left(n-\frac{1}{2}\right)+it+\frac{z}{\Delta_3-\Delta_1}\right),\\
A_n &= \vartheta_3(iv)\vartheta_2\left(iv\left(n-\frac{1}{2}\right)+it+\frac{z}{\Delta_3-\Delta_1}\right).
\end{aligned} \tag{4.197}$$

is a solution to (4.141) and (4.142), and thus (4.140) satisfies in (4.123) with (4.124).

Proof. The proof holds assuming $z = 0$. After differentiating the equation in (4.152) with

$$\begin{cases} X = iv\left(n+\frac{1}{2}\right)+it,\\ Y = iv+h,\\ U = -iv+h,\\ V = iv\left(n-\frac{3}{2}\right)+it. \end{cases} \tag{4.198}$$

and putting $h=0$, we obtain

$$\begin{aligned}
&D_t\left\{-i\vartheta_1(iv)\vartheta_4\left(iv\left(n-\frac{1}{2}\right)+it\right)\right\}\cdot\left\{\vartheta_3(iv)\vartheta_2\left(iv\left(n-\frac{1}{2}\right)+it\right)\right\}\\
&=\frac{\vartheta_1(iv)\vartheta_3(iv)}{\vartheta_1(2iv)\vartheta_3(0)}\frac{\vartheta_1'(iv)\,\vartheta_3(iv)-\vartheta_1(iv)\,\vartheta_3'(iv)}{i\vartheta_1(iv)\,\vartheta_3(iv)}\\
&\times\left\{-i\vartheta_1(iv)\vartheta_4\left(iv\left(n-\frac{3}{2}\right)+it\right)\right\}\left\{\vartheta_3(iv)\vartheta_2\left(iv\left(n+\frac{1}{2}\right)+it\right)\right\}\\
&-\frac{\vartheta_1(iv)\vartheta_3(iv)}{\vartheta_1(2iv)\vartheta_3(0)}\frac{\vartheta_1'(iv)\,\vartheta_3(iv)-\vartheta_1(iv)\,\vartheta_3'(iv)}{i\vartheta_1(iv)\,\vartheta_3(iv)}\\
&\times\left\{-i\vartheta_1(iv)\vartheta_4\left(iv\left(n+\frac{1}{2}\right)+it\right)\right\}\left\{\vartheta_3(iv)\vartheta_2\left(iv\left(n-\frac{3}{2}\right)+it\right)\right\}.
\end{aligned}\tag{4.199}$$

After rearranging (4.199), we see that (4.197) is a solution to (4.141). Also, putting

$$\begin{cases}X=iv,\\ Y=-iv,\\ U=iv\left(n-\frac{1}{2}\right)+it,\\ V=-iv\left(n-\frac{1}{2}\right)-it.\end{cases}\tag{4.200}$$

in (4.154) yields

$$\begin{aligned}
&\frac{\vartheta_3(0)^2}{\vartheta_3(iv)^2}\vartheta_3(iv)\vartheta_2\left(iv\left(n+\frac{1}{2}\right)+it\right)\vartheta_3(iv)\vartheta_2\left(iv\left(n-\frac{3}{2}\right)+it\right)\\
&=\left\{\vartheta_3(iv)\vartheta_2\left(iv\left(n-\frac{1}{2}\right)+it\right)\right\}^2+\left\{-i\vartheta_1(iv)\vartheta_4\left(iv\left(n-\frac{1}{2}\right)+it\right)\right\}^2.
\end{aligned}\tag{4.201}$$

From (4.201), we see that (4.197) is a solution to (4.142). □

4.6 Proof of Proposition 12

Here we prove that (4.103) (resp. (4.104)) holds when C_+,Γ_+ (resp. C_-,Γ_-) are chosen as in (4.87).

Proof. From (4.9), (4.59), (4.60), (4.61) and (4.64), we see that when C_+,Γ_+ are chosen as in (4.87), the following relation holds

$$16\beta_1^2\left(\beta-\delta_1\right)\delta_1=4\left(i\frac{\vartheta_1'(0)}{\vartheta_1(iv)}\right)^2\frac{u_2^2}{\vartheta_3(2r)^2\vartheta_3(0)^6}.\tag{4.202}$$

Also, from (4.9), (4.58), (4.59), (4.60), (4.61), (4.63), (4.64), (4.92) and (4.93), we see that when C_+, Γ_+ are chosen as in (4.87), the following relation holds

$$(\epsilon_1 + C)^2 = \left(i\frac{\vartheta_1'(0)}{\vartheta_1(iv)}\right)^2 \frac{u_2^2}{\vartheta_3(2r)^2\vartheta_3(0)^6} \frac{\left(\vartheta_3(r)^4 - \vartheta_1(r)^4\right)^2}{\vartheta_3(r)^4\vartheta_1(r)^4}. \tag{4.203}$$

From (4.95), (4.202) and (4.203), we see that

$$\rho_+^2 = 16\beta_1^2\left(\beta - \delta_1\right)\delta_1 + (\epsilon_1 + C)^2 = \left(i\frac{\vartheta_1'(0)}{\vartheta_1(iv)}\right)^2 \frac{u_2^2}{\vartheta_3(r)^4\vartheta_1(r)^4}, \tag{4.204}$$

holds. (4.204) is equivalent to (4.103). From (4.9), (4.59), (4.60), (4.61) and (4.64), we see that when C_-, Γ_- are chosen as in (4.87), the following relation holds

$$16\beta_1^2\left(\beta - \delta_1\right)\delta_1 = 4\left(i\frac{\vartheta_1'(0)}{\vartheta_1(iv)}\right)^2 \frac{u_2^2}{\vartheta_3(2r)^2\vartheta_3(0)^6}. \tag{4.205}$$

Also, from (4.9), (4.58), (4.59), (4.60), (4.61), (4.63), (4.64), (4.92) and (4.93), we see that when C_-, Γ_- are chosen as in (4.87), the following relation holds

$$(\epsilon_1 + C)^2 = \left(i\frac{\vartheta_1'(0)}{\vartheta_1(iv)}\right)^2 \frac{u_2^2}{\vartheta_3(2r)^2\vartheta_3(0)^6} \frac{\left(\vartheta_4(r)^4 - \vartheta_2(r)^4\right)^2}{\vartheta_4(r)^4\vartheta_2(r)^4}. \tag{4.206}$$

From (4.95), (4.205) and (4.206), we see that

$$\rho_-^2 = 16\beta_1^2\left(\beta - \delta_1\right)\delta_1 + (\epsilon_1 + C)^2 = \left(i\frac{\vartheta_1'(0)}{\vartheta_1(iv)}\right)^2 \frac{u_2^2}{\vartheta_4(r)^4\vartheta_2(r)^4}, \tag{4.207}$$

holds. (4.207) is equivalent to (4.104). □

4.7 Proof of Proposition 13

Here we prove that (4.121) and (4.122) hold.

Proof. First, we calculate $L_1(v/y)/y$ and $L_2(v/y)/y$. For any $v \in \mathbb{R}$ and $y > 0$, the elliptic theta functions satisfy the following identities [48, 49].

$$\vartheta_1'\left(0, iy\right) = \pi\vartheta_2\left(0, iy\right)\vartheta_3\left(0, iy\right)\vartheta_4\left(0, iy\right), \tag{4.208}$$

$$\begin{aligned}
\vartheta_1\left(\frac{v}{y}, i\frac{2}{y}\right) &= -i\sqrt{\frac{y}{2}}\exp\left(-\pi\frac{v^2}{2y}\right)\vartheta_1\left(\frac{iv}{2}, i\frac{y}{2}\right),\\
\vartheta_2\left(\frac{v}{y}, i\frac{2}{y}\right) &= \sqrt{\frac{y}{2}}\exp\left(-\pi\frac{v^2}{2y}\right)\vartheta_4\left(\frac{iv}{2}, i\frac{y}{2}\right),\\
\vartheta_3\left(\frac{v}{y}, i\frac{2}{y}\right) &= \sqrt{\frac{y}{2}}\exp\left(-\pi\frac{v^2}{2y}\right)\vartheta_3\left(\frac{iv}{2}, i\frac{y}{2}\right),\\
\vartheta_4\left(\frac{v}{y}, i\frac{2}{y}\right) &= \sqrt{\frac{y}{2}}\exp\left(-\pi\frac{v^2}{2y}\right)\vartheta_2\left(\frac{iv}{2}, i\frac{y}{2}\right),
\end{aligned} \tag{4.209}$$

$$\begin{aligned}
\vartheta_3\left(\frac{iv}{2}, i\frac{y}{2}\right)\vartheta_4\left(\frac{iv}{2}, i\frac{y}{2}\right) &= \vartheta_4(iv, iy)\,\vartheta_4(0, iy),\\
\vartheta_1\left(\frac{iv}{2}, i\frac{y}{2}\right)\vartheta_2\left(\frac{iv}{2}, i\frac{y}{2}\right) &= \vartheta_1(iv, iy)\,\vartheta_4(0, iy),\\
-\vartheta_1\left(\frac{iv}{2}, i\frac{y}{2}\right)^2 + \vartheta_2\left(\frac{iv}{2}, i\frac{y}{2}\right)^2 &= 2\vartheta_2(iv, iy)\,\vartheta_3(0, iy).
\end{aligned} \tag{4.210}$$

From (4.112), (4.208), (4.209) and (4.210), we see that the following holds.

$$\begin{aligned}
\frac{L_1(v/y)}{y} &= \frac{1}{y}\frac{\vartheta_1'\left(\frac{v}{y}, i\frac{2}{y}\right)\vartheta_4\left(\frac{v}{y}, i\frac{2}{y}\right) - \vartheta_1\left(\frac{v}{y}, i\frac{2}{y}\right)\vartheta_4'\left(\frac{v}{y}, i\frac{2}{y}\right)}{\vartheta_1\left(\frac{v}{y}, i\frac{2}{y}\right)\vartheta_4\left(\frac{v}{y}, i\frac{2}{y}\right)}\\
&+ \frac{1}{y}\frac{\vartheta_1'\left(0, i\frac{2}{y}\right)\vartheta_1\left(\frac{v}{y}, i\frac{2}{y}\right)}{\vartheta_4\left(0, i\frac{2}{y}\right)\vartheta_4\left(\frac{v}{y}, i\frac{2}{y}\right)} + \frac{1}{y}\frac{\vartheta_1'\left(0, i\frac{2}{y}\right)\vartheta_4\left(\frac{v}{y}, i\frac{2}{y}\right)}{\vartheta_4\left(0, i\frac{2}{y}\right)\vartheta_1\left(\frac{v}{y}, i\frac{2}{y}\right)}\\
&= i\frac{\vartheta_1'(0, iy)}{\vartheta_4(0, iy)}\frac{\vartheta_4(iv, iy)}{\vartheta_1(iv, iy)} + i\frac{\vartheta_1'(0, iy)}{\vartheta_2(0, iy)}\frac{\vartheta_2(iv, iy)}{\vartheta_1(iv, iy)}\\
&= i\frac{\vartheta_1'(0, iy)}{\vartheta_1(iv, iy)}\frac{\vartheta_4(iv, iy)\,\vartheta_2(0, iy) + \vartheta_2(iv, iy)\,\vartheta_4(0, iy)}{\vartheta_4(0, iy)\,\vartheta_2(0, iy)}.
\end{aligned} \tag{4.211}$$

Similarly, from (4.114), (4.208), (4.209) and (4.210), we see that the following holds.

$$\begin{aligned}
\frac{L_2(v/y)}{y} &= \frac{1}{y}\frac{\vartheta_1'\left(\frac{v}{y}, i\frac{2}{y}\right)\vartheta_4\left(\frac{v}{y}, i\frac{2}{y}\right) - \vartheta_1\left(\frac{v}{y}, i\frac{2}{y}\right)\vartheta_4'\left(\frac{v}{y}, i\frac{2}{y}\right)}{\vartheta_1\left(\frac{v}{y}, i\frac{2}{y}\right)\vartheta_4\left(\frac{v}{y}, i\frac{2}{y}\right)}\\
&- \frac{1}{y}\frac{\vartheta_1'\left(0, i\frac{2}{y}\right)\vartheta_1\left(\frac{v}{y}, i\frac{2}{y}\right)}{\vartheta_4\left(0, i\frac{2}{y}\right)\vartheta_4\left(\frac{v}{y}, i\frac{2}{y}\right)} - \frac{1}{y}\frac{\vartheta_1'\left(0, i\frac{2}{y}\right)\vartheta_4\left(\frac{v}{y}, i\frac{2}{y}\right)}{\vartheta_4\left(0, i\frac{2}{y}\right)\vartheta_1\left(\frac{v}{y}, i\frac{2}{y}\right)}\\
&= i\frac{\vartheta_1'(0, iy)}{\vartheta_4(0, iy)}\frac{\vartheta_4(iv, iy)}{\vartheta_1(iv, iy)} - i\frac{\vartheta_1'(0, iy)}{\vartheta_2(0, iy)}\frac{\vartheta_2(iv, iy)}{\vartheta_1(iv, iy)}\\
&= i\frac{\vartheta_1'(0, iy)}{\vartheta_1(iv, iy)}\frac{\vartheta_4(iv, iy)\,\vartheta_2(0, iy) - \vartheta_2(iv, iy)\,\vartheta_4(0, iy)}{\vartheta_4(0, iy)\,\vartheta_2(0, iy)}.
\end{aligned} \tag{4.212}$$

On the other hand, from (4.10) and (4.21), we see that the following hold.

$$\begin{aligned} & 1+\cos\lambda \\ & = \frac{\left\{\vartheta_1\left(\frac{iv}{2}+r,iy\right)\vartheta_3\left(-\frac{iv}{2}+r,iy\right)+\vartheta_1\left(-\frac{iv}{2}+r,iy\right)\vartheta_3\left(\frac{iv}{2}+r,iy\right)\right\}^2}{2u_2}, \end{aligned} \tag{4.213}$$

$$\begin{aligned} & 1-\cos\lambda \\ & = -\frac{\left\{\vartheta_1\left(\frac{iv}{2}+r,iy\right)\vartheta_3\left(-\frac{iv}{2}+r,iy\right)-\vartheta_1\left(-\frac{iv}{2}+r,iy\right)\vartheta_3\left(\frac{iv}{2}+r,iy\right)\right\}^2}{2u_2}. \end{aligned} \tag{4.214}$$

From (4.211) and (4.213), we see that (4.121) holds. Similarly, from (4.212) and (4.214), we see that (4.122) holds. □

4.8 Proof that C, Γ only contributes to rigid transformations

Let $A \in \mathbb{C}, B = B(z)$.

$$\begin{aligned} & \begin{pmatrix} \cos\Gamma t & -\sin\Gamma t \\ \sin\Gamma t & \cos\Gamma t \end{pmatrix} \begin{pmatrix} A+A^* \\ \frac{1}{i}\left(A-A^*\right) \end{pmatrix} \\ & = \frac{1}{2} \begin{pmatrix} e^{i\Gamma t}+e^{-i\Gamma t} & -\frac{1}{i}\left(e^{i\Gamma t}-e^{-i\Gamma t}\right) \\ \frac{1}{i}\left(e^{i\Gamma t}-e^{-i\Gamma t}\right) & e^{i\Gamma t}+e^{-i\Gamma t} \end{pmatrix} \begin{pmatrix} A+A^* \\ \frac{1}{i}\left(A-A^*\right) \end{pmatrix} \\ & = \begin{pmatrix} e^{i\Gamma t}A+e^{-i\Gamma t}A^* \\ \frac{1}{i}\left(e^{i\Gamma t}A-e^{-i\Gamma t}A^*\right) \end{pmatrix}. \end{aligned} \tag{4.215}$$

$$2\frac{\partial \log\left(e^{\frac{C}{2}tz}B\right)}{\partial z} = Ct + 2\frac{\partial \log B}{\partial z}. \tag{4.216}$$

□

Chapter 5

Exploration of non-trivial relations for the non-steady state nucleation rate: usefulness of the elliptic theta functions for its experimental estimations

In this chapter, by taking advantage of the elliptic theta functions, we derive non-trivial relations satisfied by the non-steady state nucleation rate and discuss its applications. This result is based on [46].

5.1 Non-steady state nucleation rate

Non-steady state nucleation is a phenomenon such as that seen in the initial stage of vesiculation of carbonate water, which can be observed in daily life. This phenomenon is characterized by a quantity called the non-steady state nucleation rate $J_k(t)$. According to Section 2 in [16], $J_k(t)$ can be derived as follows. If supersaturation is constant during the nucleation process, the size distribution function $Z(x,t)$ (m^{-4}) of the cluster, the number of clusters containing molecules at size x and time t, satisfies the Zeldovich-Frenkel equation

$$\frac{\partial Z(x,t)}{\partial t} = \frac{\partial}{\partial x}\left\{D(x)Z_{\mathrm{e}}(x)\frac{\partial}{\partial x}\left(\frac{Z(x,t)}{Z_{\mathrm{e}}(x)}\right)\right\}, \tag{5.1}$$

subject to the initial and boundary conditions

$$\begin{cases} Z(x,0) = 0 & (x > 1) \\ Z(1,t) = Z_{\mathrm{e}}(1) & \\ Z(N,t) = 0 & (N > 1). \end{cases} \tag{5.2}$$

In (5.1), $D(x) = \alpha x^{\frac{2}{3}}$ (m^2/s) is the probability per unit time of a molecule joining to a cluster, where α is a frequency factor. $Z_{\mathrm{e}}(x) = N_1 \exp\left(-\Delta G(x)/k_{\mathrm{B}}T\right)$ is the equilibrium size distribution of clusters, where N_1 (m^{-4}) is the number of single molecules in a size bin when the system is in equilibrium, $\Delta G(x)$ (J) is the isothermal reversible work required for the formation of a cluster, k_{B} is the Boltzmann constant ($= 1.38 \times 10^{-23}$ J/K), and T (K) is a fixed absolute temperature. Solving (5.1) under conditions (5.2), the expression of $J_k(t)$ is obtained:

$$J_k(t) = -D(x_k) Z_{\mathrm{e}}(x_k) \frac{\partial}{\partial x} \left(\frac{Z(x,t)}{Z_{\mathrm{e}}(x)} \right)_{x=x_k}. \tag{5.3}$$

The relation between the non-steady state nucleation rate $J_k(t)$ and the steady state nucleation rate J_{st} ($\mathrm{m}^{-3} \cdot \mathrm{s}^{-1}$), calculated from the non-steady state solution and the steady-state solution of (5.1), respectively, has been already formulated by Section 4 in [16]:

$$J_k(t) = J_{\mathrm{st}} \left[1 + 2 \sum_{n=1}^{\infty} (-1)^n \exp\left(-n^2 \frac{t}{\tau} \right) \right], \tag{5.4}$$

where J_{st} is expressed as below:

$$J_{\mathrm{st}} = \frac{\sqrt{-\dfrac{1}{2k_{\mathrm{B}}T} \left(\dfrac{d^2 \Delta G(x)}{dx^2} \right)_{x=x_k}} D(x_k) Z_{\mathrm{e}}(x_k)}{\sqrt{\pi}}, \tag{5.5}$$

and τ is the non-dimensional first-order eigenvalue that characterizes the non-steady state nucleation and is expressed as below:

$$\tau = -\frac{8 k_{\mathrm{B}} T}{\pi^2 D(x_k) \left(\dfrac{d^2 \Delta G(x)}{dx^2} \right)_{x=x_k}}. \tag{5.6}$$

(5.4) is used as a reference equation in various experimental studies dealing with the non-steady state problems in nucleation kinetics on the crystallization in such as silicate glass [18, 19, 20, 21, 22, 23]. Also, the theory's validity has been verified by both theoretical and experimental studies [24, 25].

Figure 5.1 plots $J_k(t)/J_{\mathrm{st}}$ when $\tau = 1$. $J_k(t)$ increases monotonically with time, and it becomes $J_k(t) \simeq J_{\mathrm{st}}$ when $t \gtrsim \pi$.

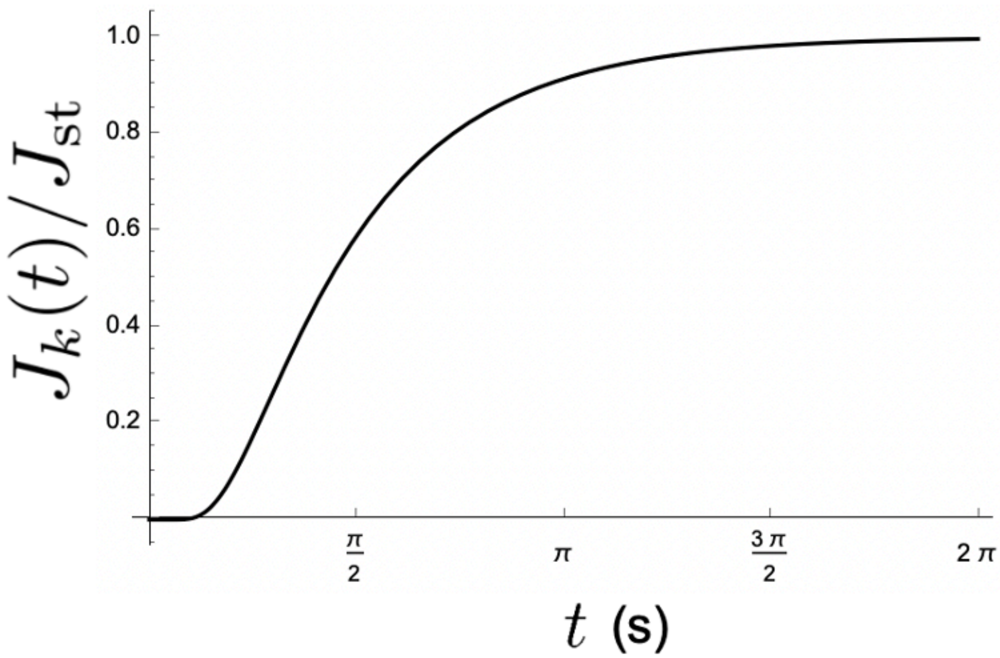

Figure 5.1: The ratio of the non-steady state nucleation rate and the steady state nucleation rate $J_k(t)/J_{\rm st}$ vs. time t (s) with the timescale of non-steady state nucleation $\tau = 1$.

To accurately determine the time evolution of $J_k(t)$, it is often necessary to closely measure the time evolution of the number of nuclei over a long time [23], which means that it is an effort- and cost-intensive process.

On the other hand, it has already been reported in [17] that the infinite series of (5.4) can also be represented by an elliptic theta function [48, 49]:

$$J_k(t) = J_{\rm st}\vartheta_4\left(0, i\frac{t}{\tau\pi}\right), \tag{5.7}$$

where note that our notation is different from that used in [17]. It is known that the elliptic theta functions satisfy several types of identities and equations [48, 49]. The following is a list of examples.

- Addition identities

$$\begin{aligned}
&2\vartheta_4(X_1,w)\vartheta_4(Y_1,w)\vartheta_3(U_1,w)\vartheta_3(V_1,w)\\
=&\vartheta_3(X,w)\vartheta_3(Y,w)\vartheta_4(U,w)\vartheta_4(V,w)\\
+&\vartheta_4(X,w)\vartheta_4(Y,w)\vartheta_3(U,w)\vartheta_3(V,w)\\
+&\vartheta_2(X,w)\vartheta_2(Y,w)\vartheta_1(U,w)\vartheta_1(V,w)\\
+&\vartheta_1(X,w)\vartheta_1(Y,w)\vartheta_2(U,w)\vartheta_2(V,w),
\end{aligned} \tag{5.8}$$

$$
\begin{aligned}
&-2\vartheta_2(X_1,w)\vartheta_2(Y_1,w)\vartheta_1(U_1,w)\vartheta_1(V_1,w)\\
=&\vartheta_3(X,w)\vartheta_3(Y,w)\vartheta_4(U,w)\vartheta_4(V,w)\\
&-\vartheta_4(X,w)\vartheta_4(Y,w)\vartheta_3(U,w)\vartheta_3(V,w)\\
&-\vartheta_2(X,w)\vartheta_2(Y,w)\vartheta_1(U,w)\vartheta_1(V,w)\\
&+\vartheta_1(X,w)\vartheta_1(Y,w)\vartheta_2(U,w)\vartheta_2(V,w),
\end{aligned}
\tag{5.9}
$$

$$
\begin{aligned}
&2\vartheta_3(X_1,w)\vartheta_3(Y_1,w)\vartheta_3(U_1,w)\vartheta_3(V_1,w)\\
=&\vartheta_3(X,w)\vartheta_3(Y,w)\vartheta_3(U,w)\vartheta_3(V,w)\\
&+\vartheta_4(X,w)\vartheta_4(Y,w)\vartheta_4(U,w)\vartheta_4(V,w)\\
&+\vartheta_2(X,w)\vartheta_2(Y,w)\vartheta_2(U,w)\vartheta_2(V,w)\\
&+\vartheta_1(X,w)\vartheta_1(Y,w)\vartheta_1(U,w)\vartheta_1(V,w).
\end{aligned}
\tag{5.10}
$$

- Heat equation

$$
\frac{\partial^2}{\partial x^2}\vartheta_i(x,w) = 4\pi i\frac{\partial}{\partial w}\vartheta_i(x,w) \quad (i=1,2,3,4). \tag{5.11}
$$

- Identity for theta constants

$$
\vartheta_1^{'}(0,w) = \pi\vartheta_2(0,w)\,\vartheta_3(0,w)\,\vartheta_4(0,w)\,. \tag{5.12}
$$

(Here, $' = \partial/\partial x$.)

- Bilinear identity connecting theta functions with w and $2w$

$$
\begin{aligned}
&2\vartheta_4(X+Y,2w)\vartheta_4(X-Y,2w)\\
=&\vartheta_3(X,w)\vartheta_4(Y,w)+\vartheta_4(X,w)\vartheta_3(Y,w),
\end{aligned}
\tag{5.13}
$$

$$
\begin{aligned}
&2\vartheta_2(X+Y,2w)\vartheta_2(X-Y,2w)\\
=&\vartheta_3(X,w)\vartheta_3(Y,w)-\vartheta_4(X,w)\vartheta_4(Y,w),
\end{aligned}
\tag{5.14}
$$

$$
\begin{aligned}
&2\vartheta_3(X+Y,2w)\vartheta_3(X-Y,2w)\\
=&\vartheta_3(X,w)\vartheta_3(Y,w)+\vartheta_4(X,w)\vartheta_4(Y,w),
\end{aligned}
\tag{5.15}
$$

$$
\begin{aligned}
&2\vartheta_2(X+Y,2w)\vartheta_3(X-Y,2w)\\
=&\vartheta_2(X,w)\vartheta_2(Y,w)-\vartheta_1(X,w)\vartheta_1(Y,w).
\end{aligned}
\tag{5.16}
$$

- Modular transformation

$$
\vartheta_4\left(\frac{X}{w},-\frac{1}{w}\right) = \sqrt{-iw}\exp\left(\frac{\pi i X^2}{w}\right)\vartheta_2(X,w)\,. \tag{5.17}
$$

Here,

$$
\begin{aligned}
X_1 &= \frac{1}{2}(X+Y+U+V), \quad Y_1 = \frac{1}{2}(X+Y-U-V), \\
U_1 &= \frac{1}{2}(X-Y+U-V), \quad V_1 = \frac{1}{2}(X-Y-U+V).
\end{aligned}
\tag{5.18}
$$

for $X, Y, U, V \in \mathbb{C}$.

Although the relation between the non-steady state nucleation rate derived from the classical nucleation theory and the elliptic theta functions (5.7) has been pointed out in previous studies [17], and cited in many subsequent studies [41, 42, 43], there have been no studies that take advantage of rich mathematical properties of the elliptic theta functions.

On the other hand, various difficulties exist in experimental studies based on the classical nucleation theory. For example, **(i)** the timescale of the non-steady state nucleation processes is generally very long in the low-temperature region, and it requires a large amount of effort and cost to estimate the time evolution of $J_k(t)$: the number of crystal nuclei is measured at many times over a long period of 200 hours in [23]. In other cases, **(ii)** nucleation is already in progress at the start of the measurement, and the time evolution of $J_k(t)$ before that point is unknown in [21]. The rich mathematical properties of the elliptic theta functions may help overcome these difficulties.

There are two objectives in Chapter 5. The first one is to derive non-trivial relations for $J_k(t)$ by accepting Kashchiev's result (5.7) as proper and applying pure mathematical operations. The second one is to solve **(i)** and **(ii)** using the rich mathematical properties of the elliptic theta functions. As the solution to **(i)**, we show that one of these result help reduce the number of data needed when estimating the time evolution of $J_k(t)$. This implies that our result help reduce the estimation cost of $J_k(t)$. As the solution to **(ii)**, we also show that it is possible to mechanically estimate when the reaction will transition to the steady state process, and how long ago the reaction started, from data on the non-steady state nucleation processes in progress.

In Section 5.2, we derived relations (Theorem 19) between the non-steady state nucleation rate and three elliptic theta functions ϑ_i $(i = 1, 2, 3)$. In Section 5.3, using relations derived in Section 5.2, we derived five non-trivial relations (Theorem 20–24) for the non-steady state nucleation rates at different times. In Section 5.4, we rewrote (5.39) in Theorem 23 as a second-order difference equation for the non-steady state nucleation rate, then discussed that its time evolution can be discretely plotted from a few measurements data of it.

5.2 Relations between nucleation rate and the elliptic theta functions

Theorem 19. Let $J_k(t)$ be a function defined by (5.7), and t be a positive real number. Then the following relations hold.

$$\vartheta_3\left(0, i\frac{t}{\tau\pi}\right) = \frac{J_k(2t)^2}{J_{\mathrm{st}}J_k(t)}, \tag{5.19}$$

$$\vartheta_2\left(0, i\frac{t}{\tau\pi}\right) = \frac{1}{J_{\mathrm{st}}}\sqrt{\frac{\tau\pi}{t}}J_k\left(\frac{\tau^2\pi^2}{t}\right), \tag{5.20}$$

$$\vartheta_1'\left(0, i\frac{t}{\tau\pi}\right) = \frac{\pi}{J_{\mathrm{st}}^3}\sqrt{\frac{\tau\pi}{t}}J_k\left(\frac{\tau^2\pi^2}{t}\right)J_k(2t)^2. \tag{5.21}$$

Proof. Putting $X = Y = 0$ and $w = it/(\pi\tau)$ into (5.13), we have

$$\vartheta_4\left(0, 2i\frac{t}{\tau\pi}\right)^2 = \vartheta_3\left(0, i\frac{t}{\tau\pi}\right)\vartheta_4\left(0, i\frac{t}{\tau\pi}\right). \tag{5.22}$$

Then by putting (5.7) into (5.22), we get (5.19). Putting $X = 0$ and $w = it/(\pi\tau)$ into (5.17), we have

$$\vartheta_2\left(0, i\frac{t}{\tau\pi}\right) = \sqrt{\frac{\tau\pi}{t}}\vartheta_4\left(0, i\frac{\tau\pi}{t}\right). \tag{5.23}$$

Then by putting (5.7) into (5.23), we get (5.20). Putting (5.19), (5.20) and (5.7) into (5.12), we have (5.21). □

5.3 Non-trivial relations connecting the non-steady state nucleation rates at different times

In this section, we use the rich mathematical properties of the elliptic theta functions listed in the introduction, (5.8)–(5.17), to relate the non-steady state nucleation rates at different times $\tau^2\pi^2/2t$, $\tau^2\pi^2/t$, t, $2t$, $4t$ each other. Each Theorem is a non-trivial relation that could not be discovered in the conventional classical nucleation theory because it is described with powers of the non-steady state nucleation rate that have no obvious direct physical meaning.

Theorem 20. Let $J_k(t)$ be a function defined by (5.7), and t be a positive real number. Then the following relation holds.

$$8\tau\frac{d}{dt}\log\frac{J_k(t)}{J_k(2t)} = \left(\frac{\tau\pi}{t}\right)^2\frac{1}{J_{\mathrm{st}}^4}J_k\left(\frac{\tau^2\pi^2}{t}\right)^4. \tag{5.24}$$

Proof. Putting $X = Y = 0$ and $U = V = x$ into (5.8), we have

$$\begin{aligned} &2\vartheta_4(x,w)\vartheta_4(-x,w)\vartheta_3(0,w)^2 \\ =&\vartheta_3(0,w)^2\vartheta_4(x,w)^2 + \vartheta_4(0,w)^2\vartheta_3(x,w)^2 \\ +&\vartheta_2(0,w)^2\vartheta_1(x,w)^2 + \vartheta_1(0,w)^2\vartheta_2(x,w)^2. \end{aligned} \tag{5.25}$$

Putting $X = Y = 0$ and $U = V = x$ into (5.9), we have

$$\begin{aligned} &-2\vartheta_2(x,w)\vartheta_2(-x,w)\vartheta_1(0,w)^2 \\ =&\vartheta_3(0,w)^2\vartheta_4(x,w)^2 - \vartheta_4(0,w)^2\vartheta_3(x,w)^2 \\ -&\vartheta_2(0,w)^2\vartheta_1(x,w)^2 + \vartheta_1(0,w)^2\vartheta_2(x,w)^2. \end{aligned} \tag{5.26}$$

Subtracting (5.26) from (5.25) and noticing that $\vartheta_1(0,w) = 0$, we obtain

$$\begin{aligned} &\vartheta_4(x,w)^2\vartheta_3(0,w)^2 \\ =&\vartheta_4(0,w)^2\vartheta_3(x,w)^2 + \vartheta_2(0,w)^2\vartheta_1(x,w)^2. \end{aligned} \tag{5.27}$$

Differentiating (5.27) twice with respect to x and putting $x = 0$, we get

$$\begin{aligned} &2\{\vartheta_4''(0,w)\vartheta_4(0,w) + \vartheta_4'(0,w)^2\}\vartheta_3(0,w)^2 \\ =&2\{\vartheta_3''(0,w)\vartheta_3(0,w) + \vartheta_3'(0,w)^2\}\vartheta_4(0,w)^2 \\ +&2\{\vartheta_1''(0,w)\vartheta_1(0,w) + \vartheta_1'(0,w)^2\}\vartheta_2(0,w)^2. \end{aligned} \tag{5.28}$$

Noticing that $\vartheta_3'(0,w) = 0$, $\vartheta_4'(0,w) = 0$ and $\vartheta_1''(0,w) = 0$, we have

$$\begin{aligned} &2\vartheta_4''(0,w)\vartheta_4(0,w)\vartheta_3(0,w)^2 \\ =&2\vartheta_3''(0,w)\vartheta_3(0,w)\vartheta_4(0,w)^2 + 2\vartheta_1'(0,w)^2\vartheta_2(0,w)^2. \end{aligned} \tag{5.29}$$

Dividing (5.29) by $2\vartheta_3(0,w)^2\vartheta_4(0,w)^2$, we have

$$\frac{\vartheta_4''(0,w)}{\vartheta_4(0,w)} = \frac{\vartheta_3''(0,w)}{\vartheta_3(0,w)} + \frac{\vartheta_1'(0,w)^2}{\vartheta_3(0,w)^2}\frac{\vartheta_2(0,w)^2}{\vartheta_4(0,w)^2}. \tag{5.30}$$

From (5.11), we see that the following equations hold

$$\begin{aligned} \vartheta_4''(0,w) &= 4\pi i\frac{\partial}{\partial w}\vartheta_4(0,w), \\ \vartheta_3''(0,w) &= 4\pi i\frac{\partial}{\partial w}\vartheta_3(0,w). \end{aligned} \tag{5.31}$$

By using (5.31), (5.30) can be rewritten as

$$\begin{aligned} &\frac{4\pi i}{\vartheta_4(0,w)}\frac{\partial}{\partial w}\vartheta_4(0,w) - \frac{4\pi i}{\vartheta_3(0,w)}\frac{\partial}{\partial w}\vartheta_3(0,w) \\ =&\frac{\vartheta_1'(0,w)^2}{\vartheta_3(0,w)^2}\frac{\vartheta_2(0,w)^2}{\vartheta_4(0,w)^2}. \end{aligned} \tag{5.32}$$

Considering the change of variable $w = it/(\pi\tau)$ and substituting (5.7) and (5.19) into (5.32), we have

$$4\tau\pi^2 \frac{d}{dt} \log \frac{J_k(t)^2}{J_k(2t)^2} = \frac{\vartheta_1'\left(0, i\dfrac{t}{\tau\pi}\right)^2}{\vartheta_3\left(0, i\dfrac{t}{\tau\pi}\right)^2} \frac{\vartheta_2\left(0, i\dfrac{t}{\tau\pi}\right)^2}{\vartheta_4\left(0, i\dfrac{t}{\tau\pi}\right)^2}. \tag{5.33}$$

Substituting (5.12) into (5.33), we get

$$8\tau\pi^2 \frac{d}{dt} \log \frac{J_k(t)}{J_k(2t)} = \pi^2 \vartheta_2\left(0, i\frac{t}{\tau\pi}\right)^4. \tag{5.34}$$

Substituting (5.20) into (5.34), we obtain (5.24). □

Theorem 21. Let $J_k(t)$ be a function defined by (5.7), and t be a positive real number. Then the following relation holds.

$$J_k(2t)^8 - J_k(t)^8 = \left(\frac{\tau\pi}{t}\right)^2 J_k\left(\frac{\tau^2\pi^2}{t}\right)^4 J_k(t)^4. \tag{5.35}$$

Proof. Putting $X = Y = U = V = 0$ and $w = it/(\tau\pi)$ into (5.10), we have

$$\vartheta_2\left(0, i\frac{t}{\tau\pi}\right)^4 = \vartheta_3\left(0, i\frac{t}{\tau\pi}\right)^4 - \vartheta_4\left(0, i\frac{t}{\tau\pi}\right)^4. \tag{5.36}$$

Substituting (5.7), (5.19) and (5.20) into (5.36), we obtain (5.35). □

Theorem 22. Let $J_k(t)$ be a function defined by (5.7), and t be a positive real number. Then the following relation holds.

$$J_k(2t)^4 - J_k(t)^4 = \frac{\tau\pi}{t} J_k\left(\frac{\tau^2\pi^2}{2t}\right)^2 J_k(t)^2. \tag{5.37}$$

Proof. Putting $X = Y = 0$ and $w = it/(\tau\pi)$ into (5.14), we have

$$\begin{aligned} &2\vartheta_4\left(0, 2i\frac{t}{\tau\pi}\right)\vartheta_4\left(0, 2i\frac{t}{\tau\pi}\right) \\ =&\vartheta_3\left(0, i\frac{t}{\tau\pi}\right)\vartheta_4\left(0, i\frac{t}{\tau\pi}\right) + \vartheta_4\left(0, i\frac{t}{\tau\pi}\right)\vartheta_3\left(0, i\frac{t}{\tau\pi}\right). \end{aligned} \tag{5.38}$$

Substituting (5.7), (5.19), and (5.20) into (5.38), we obtain (5.37). □

Theorem 23. Let $J_k(t)$ be a function defined by (5.7), and t be a positive real number. Then the following relation holds.

$$J_k(2t)^6 + J_k(t)^4 J_k(2t)^2 = 2J_k(t)^2 J_k(4t)^4. \tag{5.39}$$

Proof. Putting $X = Y = 0$ and $w = it/(\tau\pi)$ into (5.15), we have

$$\begin{aligned}&2\vartheta_3\left(0, 2i\frac{t}{\tau\pi}\right)\vartheta_3\left(0, 2i\frac{t}{\tau\pi}\right)\\ =&\vartheta_3\left(0, i\frac{t}{\tau\pi}\right)\vartheta_3\left(0, i\frac{t}{\tau\pi}\right) + \vartheta_4\left(0, i\frac{t}{\tau\pi}\right)\vartheta_4\left(0, i\frac{t}{\tau\pi}\right).\end{aligned} \tag{5.40}$$

Substituting (5.7) and (5.19) into (5.40), we obtain (5.39). □

Remark 20. Among Theorem 20–24, only Theorem 23 does not include J_{st} and τ, and is the simplest equation. In Section 5.4, this relation is rewritten as a difference equation, leading to information that may be useful in drawing the time evolution of the non-steady state nucleation rate.

Theorem 24. Let $J_k(t)$ be a function defined by (5.7), and t be a positive real number. Then the following relation holds.

$$J_k\left(\frac{\tau^2\pi^2}{2t}\right) J_k(4t)^2 = \sqrt{\frac{\tau\pi}{2t}} J_k(2t) J_k\left(\frac{\tau^2\pi^2}{t}\right)^2. \tag{5.41}$$

Proof. Putting $X = Y = 0$ and $w = it/(\tau\pi)$ into (5.16), we have

$$\begin{aligned}&2\vartheta_2\left(0, 2i\frac{t}{\tau\pi}\right)\vartheta_3\left(0, 2i\frac{t}{\tau\pi}\right)\\ =&\vartheta_2\left(0, i\frac{t}{\tau\pi}\right)\vartheta_2\left(0, i\frac{t}{\tau\pi}\right) - \vartheta_1\left(0, i\frac{t}{\tau\pi}\right)\vartheta_1\left(0, i\frac{t}{\tau\pi}\right).\end{aligned} \tag{5.42}$$

Substituting (5.7), (5.19) and (5.20) into (5.42), we obtain (5.41). □

5.4 A difference equation for the non-steady state nucleation rate

For $n \in \mathbb{Z}$ and fixed $t > 0$, we introduce a new variable $T_n = 2^n t$ and define the sequence a_n by $a_n := J_k(T_n)$. We already know that $J_k(t) > 0$ holds for any $t > 0$. From (5.37), it follows that $a_{n+1} > a_n > 0$ for any $n \in \mathbb{Z}$. It is also clear from (5.39) that a_n satisfies the following second-order difference equation:

$$a_{n+1}^6 + a_n^4 a_{n+1}^2 = 2a_n^2 a_{n+2}^4. \tag{5.43}$$

For this equation, first, we consider the case $n = 0$. Assume that the initial values a_0, a_1 are given such that $a_1 > a_0 > 0$. (5.43) is a quartic equation in a_2 and has only two real solutions as follows:

$$a_2 = \pm \left(\frac{a_1^6 + a_0^4 a_1^2}{2a_0^2} \right)^{1/4}. \tag{5.44}$$

On the other hand, since

$$\begin{aligned} & a_1^6 + a_0^4 a_1^2 - 2a_0^2 a_1^4 \\ & = a_1^2 \left(a_1^4 - 2a_0^2 a_1^2 + a_0^4 \right) = a_1^2 \left(a_1^2 - a_0^2 \right)^2 > 0. \end{aligned} \tag{5.45}$$

holds, we see that out of solutions, only

$$a_2 = \left(\frac{a_1^6 + a_0^4 a_1^2}{2a_0^2} \right)^{1/4}, \tag{5.46}$$

is a solution to (5.43) satisfying $a_2 > a_1$. This leads inductively to the following Proposition.

Proposition 17. Suppose that a_0, a_1 are given such that $a_1 > a_0 > 0$, then putting

$$a_{n+2} = \left(\frac{a_{n+1}^6 + a_n^4 a_{n+1}^2}{2a_n^2} \right)^{1/4}, \tag{5.47}$$

for each $n = 0, 1, 2, \cdots$, it follows that (5.47) is the only solution of (5.43) that satisfies $a_{n+2} > a_{n+1}$.

And conversely, for $n = -1, -2, -3 \cdots$, we can compute a_n as follows.

Proposition 18. Suppose that a_0, a_1 are given such that $a_1 > a_0 > 0$, then putting

$$a_n = \frac{\sqrt{a_{n+2}^4 - \sqrt{a_{n+2}^8 - a_{n+1}^8}}}{a_{n+1}}, \tag{5.48}$$

for each $n = -1, -2, -3, \cdots$, it follows that (5.48) is the only solution of (5.43) that satisfies $a_{n+1} > a_n > 0$.

Proof. Here we prove Proposition 18 by mathematical induction.

[I] When $n = -1$, (5.43) is a quartic equation of a_{-1} with four real solutions (any double sign):

$$a_{-1} = \pm \frac{\sqrt{a_1^4 \mp \sqrt{a_1^8 - a_0^8}}}{a_0}. \tag{5.49}$$

On the other hand, since

$$\begin{aligned} a_1^4 - \sqrt{a_1^8 - a_0^8} &> 0, \\ a_1^4 - \sqrt{a_1^8 - a_0^8} &< a_0^4, \\ a_1^4 + \sqrt{a_1^8 - a_0^8} &> a_0^4, \end{aligned} \tag{5.50}$$

follows from $a_1 > a_0 > 0$, out of (5.49)

$$a_{-1} = \frac{\sqrt{a_1^4 - \sqrt{a_1^8 - a_0^8}}}{a_0}, \tag{5.51}$$

is the only solution of (5.43) that satisfies $a_0 > a_{-1} > 0$.

[II] Assuming that the Proposition holds when $n = -k$. From the assumption, it follows that a_{-k+1}, a_{-k} satisfying $a_{-k+1} > a_{-k} > 0$ is determined. In this case, (5.43) is a quartic equation of a_{-k-1} with four real solutions (any double sign):

$$a_{-k-1} = \pm \frac{\sqrt{a_{-k+1}^4 \mp \sqrt{a_{-k+1}^8 - a_{-k}^8}}}{a_{-k}}. \tag{5.52}$$

On the other hand, since

$$\begin{aligned} a_{-k+1}^4 - \sqrt{a_{-k+1}^8 - a_{-k}^8} &> 0, \\ a_{-k+1}^4 - \sqrt{a_{-k+1}^8 - a_{-k}^8} &< a_{-k}^4, \\ a_{-k+1}^4 + \sqrt{a_{-k+1}^8 - a_{-k}^8} &> a_{-k}^4, \end{aligned} \tag{5.53}$$

follows from $a_{-k+1} > a_{-k} > 0$, out of solutions

$$a_{-k-1} = \frac{\sqrt{a_{-k+1}^4 - \sqrt{a_{-k+1}^8 - a_{-k}^8}}}{a_{-k}}, \tag{5.54}$$

is the only solution of (5.43) that satisfies $a_{-k} > a_{-k-1} > 0$. Therefore, it follows that the Proposition holds for $n = -(k+1)$. From [I] and [II], it is proved that the Proposition holds for any $n = -1, -2, -3, \cdots$. □

5.5 Discussion

Proposition 17 and Proposition 18 mean that even if parameters J_{st} and τ in $J_k(t)$ are unknown, it is possible to mechanically compute $J_k(T_n)$ for any $n \in \mathbb{Z}$ from $J_k(T_0)$ and $J_k(T_1)$. This implies that we can predict the discrete time evolution of the non-steady state nucleation rate accurately in the past and the future as long as the nuclei number density can be measured precisely enough to determine the non-steady state nucleation rate in the vicinity of given times t and $2t$ (See Fig. 5.2). This fact is expected to be very useful in the analysis of phenomena where the non-steady state process lasts a very long time before reaching the steady-state process, or if it is difficult to measure the number density of nuclei only at certain times during the non-steady state process. For example, this method can be used to restore the time evolution of $J_k(t)$ before the start of the measurement when nucleation has already progressed at that time. This result can be regarded as a solution to **(ii)**, and thus would be useful in experimental studies such as [23]. Furthermore, this method helps reduce the number of data needed when estimating the time evolution of $J_k(t)$. This result can also be regarded as a solution to **(i)**, and thus would be useful in experimental studies such as [21].

Note that when some time has passed after reaching the steady state process, supersaturation begins to decrease beyond the timescale where it can be regarded as constant, thus deviating from the applicability of Kashchiev's equation. Therefore, the results discussed above are applicable to phenomena in which supersaturation can be regarded as constant for a sufficiently long period.

5.6 Conclusions

The first objective of this chapter, to derive non-trivial relations for the non-steady state nucleation rate $J_k(t)$ by using the rich mathematical properties of the elliptic theta functions, was achieved in Section 5.2 and Section 5.3. Each relation is non-trivial that could not be discovered in the conventional classical nucleation theory. The second objective, to find solutions to two problems **(i)** it requires a large amount of effort and cost to estimate the time evolution of $J_k(t)$ under some special conditions, and **(ii)** it is impossible to measure $J_k(t)$ under some special cases, was achieved in Section 5.4 and Section 5.5 by solving the difference equation for $J_k(t)$. It is shown that our result help reduce the estimation cost of $J_k(t)$ and makes mechanical estimation of $J_k(t)$ possible from the past to the future.

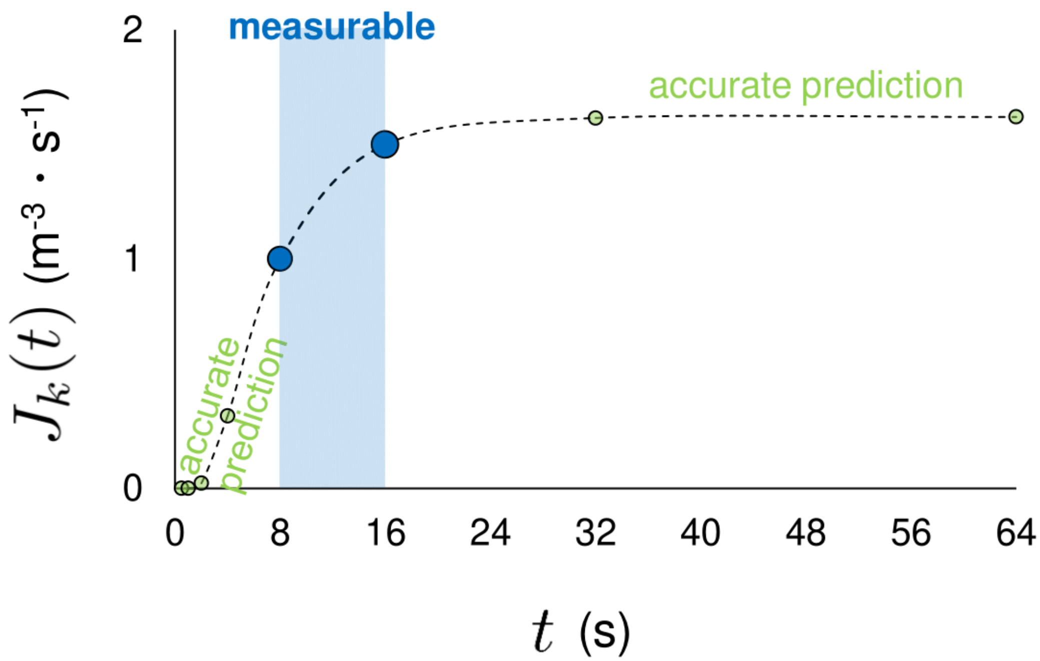

Figure 5.2: The non-steady state nucleation rate $J_k(t)$ ($\mathrm{m}^{-3}\cdot\ \mathrm{s}^{-1}$) vs. time t (s). If $J_k(t)$ can be accurately measured at only two times: $J_k(8)$=1 and $J_k(16)$=1.5 (large blue circles), $J_k(t)$ at discrete times before 8 s and after 16 s can be accurately predicted (small yellow-green circles) from Proposition 17 and Proposition 18, respectively.

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