Inflationary scenarios with primordial magnetic fields and gravitational waves

上田, 和茂

出版情報：Kyushu University, 2022, 博士（理学）, 課程博士
バージョン：
権利関係：

Doctral Thesis

Inflationary scenarios with primordial magnetic fields and gravitational waves

Dissertation Submitted to Kyushu University

for the Degree of Doctor of Science

Kazushige Ueda

Quantum field theory in curved spacetime, Cosmology, and Astrophysics/Astronomy,

Department of Physics, Graduate School of Science, Kyushu University,

744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan

March 2023

Abstract

In the standard framework of the early universe, quantum states of graviton generated during inflation become squeezed states. Since such gravitons tend to enhance their observables, it is expected that gravitons may be detectable by observations in the future. However, if they interact with environmental matter fields, such as primordial magnetic fields generated during inflation, the gravitons may lose their quantum coherence. Thus, it is important to investigate whether the primordial magnetic field affects the squeezed state of graviton or not. As a first step for analyzing the decoherence of the gravitons, we assume two models of the magnetic field. One of the models assumes the case of minimal coupling between gravitons and photons (Model-1). In this case, the primordial magnetic field decays with the negative square of the scale factor, $\propto a^{-2}$, during inflation. Through the analysis of this model, it turns out that the gravitons are robust against the decoherence caused by the cosmological magnetic fields. We also find that the conversion rate of gravitons into photons is at a few percent at most. The other model assumes magnetic fields sustained by a gauge kinetic coupling. In this model, the primordial magnetic field decays with the negative power of the scale factor, $\propto a^{-1}$, during inflation. Not only gravitons as excitations of PGWs, but also photons as excitations of electromagnetic fields are highly squeezed in this model. They become entangled with each other through graviton to photon conversion and vice versa. We derive the reduced density matrix for the gravitons and calculate their entanglement entropy. It turns out that the state of the gravitons is not a squeezed state but a mixed state. These results may provide some hints for future observations of primordial gravitational waves and their quantum nature.

Acknowledgments

K. U. would like to express my sincere gratitude to my supervisor Prof. Sugumi Kanno for valuable advice and careful reading of this manuscript. K. U. is particularly grateful to Prof. Jiro Soda and Ann Nakato in Kobe university for their continuous encouragement, helpful discussions, and nice collaborations which enables me to complete this thesis. K. U. was supported by the Japan Society for the Promotion of Science (JSPS) KAKENHI Grant Number 20J22946, which enables K. U. to discuss many researchers in RIKEN Dr. Naritaka Oshita Dr. Akira Dohi and so on. The discussion with the above researchers greatly encouraged me to proceed my study. K. U. also want to thank the member of a theoretical astrophysics group at Kyushu University for giving me kind encouragement. In particular, daily conversations with my colleagues, Yuki Sugiyama and Daisuke Miki were very helpful. Prof. Kazuhiro Yamamoto, Prof. Akira Matsumura, Prof. Yui Kuramochi, and Prof. Ken'ichiro Nakazato gave K. U. a lot of advises and comments on the research. K. U. thank Prof. Hiroshi Suzuki, Prof. Koji Tsumura for their valuable comments and careful reading of this manuscript. All research activities during K. U.'s Ph.D. program are supported by staff in the administrative office (Science). At the end of the acknowledgments, K. U. thanks his family for their ungrudging encouragement and support.

Contents

Part I

Introduction

The inflation theory, which assumes that there was a rapid expansion in the early universe, was introduced in the early 1980s as a mechanism to solve problems in the big bang cosmology, such as the horizon problem, the flatness problem, and so on. According to the inflation theory, the origin of the large-scale structure of the present universe is thought to be quantum fluctuations of the inflaton field that caused inflation. The prediction is supported by precise observations of temperature fluctuations in the cosmic microwave background and the distribution of galaxies. Thus, inflation theory is successful in the sense of phenomenology. However, it has not been proved whether the primordial scalar fluctuations in the early universe have a quantum origin or not. Inflation theory predicts the existence of primordial gravitational waves originating from quantum fluctuations of the spacetime as well as scalar fluctuations that seeded the formation of the structure of the universe. After the discovery of gravitational waves from a binary black hole system in 2015, the detection of primordial gravitational waves has become the most important research topic. If future observations find primordial gravitational waves, it will strongly support the inflation theory. In particular, if we can observe the quantum properties of primordial gravitational waves, it will imply the discovery of gravitons. These motivation encouraging scientist to establish several future experimental projects to detect primordial gravitational waves [1, 2]. Theoretically, it is believed that the quantum states of gravitons become squeezed due to particle production during inflation. Therefore, if we can find evidence of the squeezed state of gravitons, it would be evidence of the quantum nature of primordial gravitational waves. Several ideas have been proposed to detect non-classical primordial gravitational waves using squeezed states. For instance, authors of [3] used the Hanbury Brown-Twiss interferometer to predict the intensity-intensity correlation to distinguish non-classical particles from classical particles. Gravitons that have undergone inflation are thought to remain squeezed state until now if there are no environmental influences. However, if the graviton interacts with the matter field during inflation, the squeezed state may not be maintained. Therefore, considering the influence of the environment of the Graviton during inflation is an important issue to understand the quantum nature of primordial gravitational waves.

The cosmic magnetic field is an important factor to consider in the environment around the graviton during inflation. It is known that there are magnetic fields of the order of micro-Gauss scale in the Galaxy. Several observations imply that there is a small cosmic background magnetic field between galaxies. In fact, it is difficult to explain the data of some of the TeV gamma-ray observations without the presence of cosmic magnetic fields in the very early time of the universe. The size of the current background magnetic field ranges from about 10^{-17} Gauss to 10^{-9} Gauss, and its coherence length is said to exceed mega-parsecs. If the coherence length is greater than mega-parsecs, astronomical scale, the origin of the magnetic field with such a coherence length should be the inflation. In other words, the origin of the current background magnetic field is considered to be the primordial magnetic field generated during inflation. If a primordial magnetic field is produced during inflation, it would cause the conversion of gravitons to photons and vice versa [4].

We, therefore, study the squeezing process of gravitons in the presence of a primordial magnetic field that decays slowly during inflation, to see whether gravitons can remain squeezed to this day or not. As a result, we found that the primordial magnetic field produces the maximum quantum entanglement between gravitons and photons, and the quantum state of gravitons is not a squeezed (pure) state but a mixed state of gravitons and photons. This can be interpreted as the quantum entanglement between gravitons and photons partially destroying the squeezed state of gravitons. These results may provide some hints to future observations of primordial gravitational waves and their quantum nature.

This thesis consists of three parts, Part 1: Introduction, Part 2: the analysis of the magnetic field model-1, Part 3: the analysis of the magnetic field model-2, and Part 4: conclusion.

In this part, we briefly review fundamental topics of cosmology somehow related to the main argument of the thesis and observational clue of the existence of the cosmic magnetic field.

1 the inflationary scenario

Cosmological principles insist that there is no specific place in the universe. That is the assumption that our universe is spatially homogeneous and isotropic when we see it on a large scale. This assumption is supported by the observational evidence,

1. The galaxy distribution on large scale is almost homogeneous.

2. The cosmic microwave background is almost isotropic.

There are three kinds of the geometry of the space which satisfy the cosmological principle,

$$d\ell^2 = \begin{cases} d\chi^2 + \left(\frac{\sin\sqrt{K}\chi}{\sqrt{K}}\right)^2 (d\theta^2 + \sin^2\theta d\phi^2) & (K > 0) \\ d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2) & (K = 0) \\ d\chi^2 + \left(\frac{\sinh\sqrt{-K}\chi}{\sqrt{-K}}\right)^2 \chi^2(d\theta^2 + \sin^2\theta d\phi^2) & (K < 0) \end{cases}$$
$$= d\chi^2 + f_K(\chi)^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1.1}$$

where $d\ell^2$ is a part of the background metric:

$$ds^2 = -dt^2 + a(t)^2 d\ell^2, \tag{1.2}$$

and K is the spatial curvature, Note that $d\ell^2\,(K > 0)$ corresponds to the three dimensional spherical space, $d\ell^2\,(K = 0)$ is flat space, and $d\ell^2\,(K < 0)$ corresponds to the three dimensional hyperbolic space. The time evolution of the scale factor a is written by the Einstein equation,

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{1.3}$$

where $T_{\mu\nu}$ is the energy-momentum tensor. In the case of a perfect fluid, it takes the form of

$$T^\mu_\nu = \begin{pmatrix} -\rho & 0 \\ 0 & P\,\delta^i_j \end{pmatrix}, \tag{1.4}$$

where ρ is the density and P is the pressure. $(0,0)$ component and (i,j) yields

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 + \frac{K}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho, \tag{1.5}$$

$$\left(\frac{1}{a}\frac{d^2a}{dt^2}\right) + \frac{1}{2}\left(\frac{1}{a}\frac{da}{dt}\right)^2 + \frac{K}{2a^2} - \frac{\Lambda}{2} = -4\pi GP, \tag{1.6}$$

respectively. By taking into account these relations, one can derive the acceleration of the cosmic expansion,

$$\frac{1}{a}\frac{d^2a}{dt^2} = \frac{\Lambda}{3} - \frac{4\pi G}{3}(\rho + 3P). \tag{1.7}$$

Observations of the supernovae indicate the acceleration of the cosmic expansion takes some positive value [5, 6]. Nobel Prize in Physics in 2011 was for the discovery of accelerating cosmic expansion. If the cosmological principle stands in every scale and every time, the wealth structure of the current universe, galaxies, stars, filament structure, and so on for instance, cannot be generated by the gravitational time evolution. Thus, there should be some kind of initial density fluctuations. If there are density fluctuations in the early universe, they can behave as a seed of the structure, grow up due to the gravitational interaction, and finally construct the current large-scale structure. Though, the origin of the initial density fluctuations, or how it generated still remain shrouded in mystery. Inflationary scenario gives one candidate of its answer. Then scenario introduces the initial density fluctuation as a quantum fluctuation of the field which triggered the inflation. In this thesis, we investigate the nature of the quantum state of the graviton which would be produced at the inflation era to get some hint of the quantumness of the initial density fluctuations. Inflation theory is introduced to solve several problems of the big bang cosmology in 1980s. Big bang theory well explain the current observational universe, but it doesn't well explain how our universe began. According to the big bang theory, the time evolution of the scale factor is given by $a(t) \propto t^\alpha$. In the case of $0 < \alpha < 1$, the universe undergoes the decelerate expansion. In radiation dominated era, we have $\alpha = 1/2$, and we have $\alpha = 2/3$ in matter dominated era. The horizon scale is in proportional to $t^{1-\alpha}$. Thus, the horizon scale can be small as much as we want. But it cannot explain the observation of the CMB temperature fluctuation which is very tiny, $\Delta T/T \sim 10^{-5}$. It is called horizon problem.

As is shown in the freedman equation,

$$\begin{aligned} H^2 = \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}(\rho_m + \rho_r) - \frac{K}{a^2} + \frac{\Lambda}{3} \\ &= H_0^2 \left(\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \frac{\Omega_K}{a^2} + \Omega_\Lambda\right), \end{aligned} \tag{1.8}$$

the energy density of the radiation decreases in the fourth power of the scale factor, and the energy density of the matter decreases in the third power of the scale factor. Meanwhile, the curvature term decreases in the square of the scale factor. When we assume the reheating temperature is $T_{\text{reheat}} \sim 10^{14}$GeV, and the redshift is about $z \sim 10^{27}$, the ratio of the contribution of the curvature term in the Friedman equation is very small than the present value,

$$\frac{\Omega_K/a^2}{\Omega_r/a^4} \propto a^2 \sim 10^{-54}. \tag{1.9}$$

Even now, more than 10 billion years later from the beginning of the universe, the contribution of the curvature term is less than the other energy density. It indicates that the curvature radius is significantly smaller than the horizon scale. In other words, the universe is unnaturally flat. It is called flatness problem. These problems of the big bang theory stem from the assumption that the universe keeps its decelerated expansion. With this assumption, the particle horizon can take infinitely small at the beginning. The inflation scenario solves the problem by assuming the accelerated expansion at the beginning of the universe. The time of the accelerated expansion is represented by the e-folding number $N(t)$ which is defined as

$$N(t) \equiv \log \frac{a_f}{a(t)} = \int_t^{t_f} H dt, \tag{1.10}$$

where H is the Hubble parameter defined as $H = \dot{a}/a$, and a_f denotes the scale factor at the end of inflation.

2 time development of the scale factor

In FIG 1, the time evolution of the typical physical scale is depicted. In the inflation era, the value of the vertical axis in FIG 1 is proportional to

$$\log[\text{horizon}] = \log\left[\frac{c}{H}\right] = \text{const}, \tag{2.1}$$

while the horizontal axis is proportional to

$$\log[a] = \log[e^{Ht}] = Ht. \tag{2.2}$$

In radiation dominated era, the value of the vertical axis in FIG1 is proportional to

$$\log[\text{horizon}] = \log[ct] = \log[t] + \text{const}, \tag{2.3}$$

while the horizontal axis is proportional to

$$\log[a] = \log[t^{1/2}] = \frac{1}{2}\log[t]. \tag{2.4}$$

In the matter-dominated era, the value of the vertical axis in FIG1 is written as the same form as 2.3, while the horizontal axis is proportional to

$$\log[a] = \log[t^{2/3}] = \frac{2}{3}\log[t]. \tag{2.5}$$

After taking account of these relations, one can easily draw a graph like FIG. 1 that describe the time evolution of the physical scale. Simple estimation gives some condition $N > 60$ to solve the horizon problem with the inflationary scenario. But the detail of the mechanism of how inflation begins is not determined yet, there are many candidates like slow role inflation, Chaotic inflation, and so on.

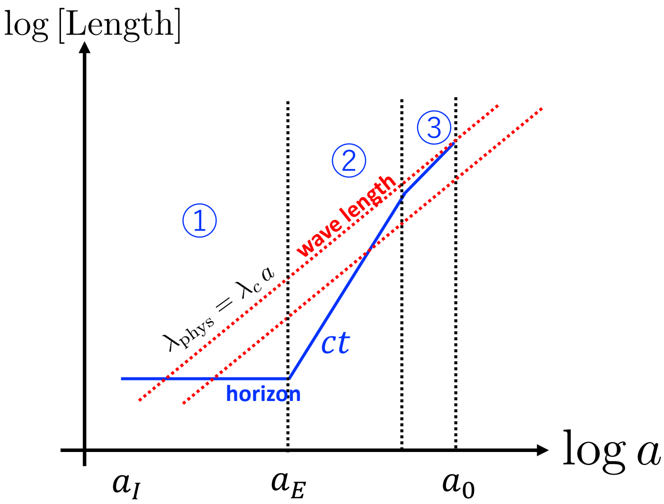

FIG 1. Outline of the time evolution of the physical scale in the inflation era (region 1), radiation-dominated era(region 2), and matter-dominated era(region 3). The blue solid line denotes the horizon scale, and the red dotted line denotes the physical scale of the wavelength.

Inflation theory has claimed that the origin of the large-scale structure of the universe and temperature fluctuations in the cosmic microwave background radiations is quantum fluctuations. Remarkably, the inflation theory also predicts the existence of primordial gravitational waves stemming from the quantum fluctuations of the spacetime (relic gravitons). After the discovery of gravitational waves from a black hole binary system [7], the detection of primordial gravitational waves has been the most important research objective [1, 2].

The notable nature of primordial gravitational waves is their quantum origin. If the relic gravitons were found, it would strongly support the inflationary universe. The finding of the relic gravitons would also give a hint of quantum gravity. Hence, it is extremely important to explore the quantum nature of the primordial gravitational waves.

In the next section, we review the foundation of cosmological perturbation.

3 cosmological perturbation

The background metric of the expanding universe is written as

$$ds^2 = a^2(\eta)\left[-d\eta^2 + \delta_{ij}dx^i dx^j\right], \tag{3.1}$$

where $a(\eta)$ is the scale factor, and η is the conformal time. The components of the connection are summarized as

$$\Gamma^0_{00} = \frac{a(\eta)}{a'(\eta)} \ , \ \ \Gamma^0_{ij} = \frac{a'(\eta)}{a(\eta)}\delta_{ij} \ , \ \ \Gamma^i_{0j} = \frac{a'(\eta)}{a(\eta)}\delta^i_j, \tag{3.2}$$

$$R_{00} = -3\left(\frac{a'(\eta)}{a(\eta)}\right)' \ , \ \ R_{ij} = \left[\frac{a''(\eta)}{a(\eta)} + \left(\frac{a'(\eta)}{a(\eta)}\right)^2\right]\delta_{ij}, \tag{3.3}$$

$$R = \frac{6}{a^2(\eta)}\frac{a''(\eta)}{a(\eta)} \ , \ \ G^0_0 = -\frac{3}{a^2(\eta)}\left(\frac{a'(\eta)}{a(\eta)}\right)^2, \tag{3.4}$$

$$G^i_j = -\frac{2}{a^2(\eta)}\left[\frac{a''(\eta)}{a(\eta)} - \frac{1}{2}\left(\frac{a'(\eta)}{a(\eta)}\right)^2\right]\delta^i_j, \tag{3.5}$$

where prime $'$ denotes the derivative with respect to the conformal time. When the matter is perfect fluid, the energy-momentum tensor is expressed as

$$T^\mu_\nu = (\rho + p)u^\mu u_\nu + p\delta^\mu_\nu, \tag{3.6}$$

where u^μ is four velocity defined as $u^\mu \equiv dx^\mu/d\tau$. The Einstein equation, $G^\mu_\nu = 8\pi G T^\mu_\nu$, is described as

$$H^2 = \frac{8\pi G}{3}\rho a^2 \tag{3.7}$$

$$2H' + H^2 = -8\pi G p a^2. \tag{3.8}$$

When the matter is a scalar field, we have

$$T^\mu_\nu = \partial^\mu\phi\partial_\nu\phi - \delta^\mu_\nu\left[\frac{1}{2}\partial_\alpha\phi \ \partial^\phi_\alpha + V(\phi)\right], \tag{3.9}$$

and thus, the density and pressure is given as

$$\rho = \frac{1}{2a^2}\phi'^{\,2} + V(\phi), \tag{3.10}$$

$$p = \frac{1}{2a^2}\phi'^{\,2} - V(\phi) \tag{3.11}$$

The background spacetime with the perturbation is described as

$$ds^2 = a^2(\eta)\left[-d\eta^2 + \delta_{ij}dx^i dx^j + a^2(\eta)\; h_{\mu\nu}(\eta, x^i)\; dx^\mu dx^\nu\right]. \tag{3.12}$$

Based on the symmetry of the metric tensor, ten components are distinguished as follows

$$h_{00} = -2A, \tag{3.13}$$

$$h_{0i} = B_{;i} + V_i \tag{3.14}$$

$$h_{ij} = 2(E_{;i;j} - \psi\delta_{ij}) + \chi_{i;j} + \chi_{j;i} + h_{ij}^{\mathrm{TT}}. \tag{3.15}$$

There are four components of the scalar perturbation $-2A$, $B_{;i}$, $2(E_{;i;j} - \psi\delta_{ij})$, the vector perturbation V_i, $\chi_{i;j} + \chi_{j;i}$, and the tensor perturbation h_{ij}^{TT}. The degree of freedom is restricted by constraints due to the gauge condition;

$$V_i^{;i} = \chi_i^{;i} = 0, \tag{3.16}$$

$$h_{ij}^{;j\mathrm{TT}} = h_i^{i\mathrm{TT}} = 0. \tag{3.17}$$

Each component is completely decoupled from the other,

$$\int d^4x\, a^4\, B_{;i}\, V^i = -\int d^4x\, a^4 B\, V^i_{;i} = 0 \tag{3.18}$$

$$\int d^4x\, a^4\, (E_{;ij} - \psi\,\delta_{ij})\, h^{ij}_{\mathrm{TT}} = 0. \tag{3.19}$$

The vector component decays due to cosmic expansion. As is described later, the tensor perturbation follows

$$h''^{\;\mathrm{TT}}_{ij} + 2H'_{ij} - \delta^2 h_{ij}^{\mathrm{TT}} = 0. \tag{3.20}$$

The background metric is rewritten as

$$ds^2 = a^2(\eta)\left[-(1+2A)d\eta^2 + 2B_{;i}dx^i d\eta + \{(1-2\psi)\delta_{ij}2E_{;ij}\}dx^i dx^j\right], \tag{3.21}$$

which remains the gauge degree of freedom $\xi^\mu = (\xi^0, \xi^{;i})$ corresponding to the coordinate transformation: $\overline{x}^\mu = x^\mu - \xi^\mu$. When we choose

$$\xi = \overline{E}, \tag{3.22}$$

$$\xi^0 = xi' - \overline{B}. \tag{3.23}$$

corresponding to the gauge transformation

$$\delta E = \overline{E} - E = \xi \tag{3.24}$$

$$\delta B = \overline{B} - B = \xi^0 + \xi', \tag{3.25}$$

that is equivalent to

$$E = 0\ , \quad B = 0. \tag{3.26}$$

Then, the metric is reduced to

$$ds^2 = a^2(\eta)\left[-(1+2A)d\eta^2 + (1-2\Psi)\delta_{ij}dx^i dx^j\right], \tag{3.27}$$

which is called longitudinal gauge or Newtonian gauge. At this time, gauge invariant quantities are $\Phi = A$ and $\Psi = \psi$.

4 Quantum field theory in curved spacetime

Generally, the action of the U(1) gauge field is written as

$$S_A = -\frac{1}{4\mu_0}\int d^4x\, F^{\mu\nu}\, F_{\mu\nu}. \tag{4.1}$$

When we take a variation of the gauge field $A_\mu \to A_\mu + \delta A_\mu$, we have

$$\begin{aligned} \delta_A S_A =& -\frac{1}{4\mu_0}\int d^4x\, 2\cdot(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu)\, F^{\mu\nu} && (4.2)\\ =& \frac{1}{\mu_0}\int d^4x\, \partial_\nu\, \delta A_\nu\, F^{\mu\nu} && (4.3)\\ =& -\frac{1}{\mu_0}\int d^4x\, \delta A_\mu\, \partial_\nu F^{\mu\nu} + (\text{surface term}). && (4.4)\end{aligned}$$

Thus, we have the Maxwell equation with the tensor form

$$\partial_\nu\, F^{\mu\nu} + \mu_0 j^\mu = 0, \tag{4.5}$$

and the Bianchi identity,

$$\partial_\rho\, F_{\mu\nu} + \partial_\mu\, F_{\nu\rho} + \partial_\nu\, F_{\rho\mu} = 0. \tag{4.6}$$

4.1 Covariant Electromagnetism

The Maxwell equation is described by the forms:

$$\nabla\cdot\boldsymbol{E} = \rho\ ,\quad \nabla\times\boldsymbol{B} - \frac{\partial\boldsymbol{E}}{\partial t} = \boldsymbol{J}\ , \tag{4.7}$$

where $\boldsymbol{E}(t,\boldsymbol{x})$ denotes the electric field, $\boldsymbol{B}(t,\boldsymbol{x})$ denotes the magnetic field, $\rho(t,\boldsymbol{x})$ denotes the electric charge density, and $\boldsymbol{J}(t,\boldsymbol{x})$ denotes the electric current. Here we introduce the scalar potential and the vector potential as

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial\boldsymbol{A}}{\partial t}\ ,\quad \boldsymbol{B} = \nabla\times\boldsymbol{A}, \tag{4.8}$$

respectively. This definition results

$$\nabla\cdot\boldsymbol{B} = 0\ ,\quad \nabla\times\boldsymbol{E} + \frac{\partial\boldsymbol{B}}{\partial t} = 0. \tag{4.9}$$

Then Eq. (4.7) is shown as

$$\Delta\phi + \tfrac{\partial}{\partial t}\nabla \cdot \boldsymbol{A} = -\rho \tag{4.10}$$

$$\Delta\boldsymbol{A} - \tfrac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla\left(\tfrac{\partial\phi}{\partial t} + \nabla \cdot \boldsymbol{A}\right) \quad = -\boldsymbol{J} \tag{4.11}$$

The Lorentz gauge condition:

$$\frac{\partial\phi}{\partial t} + \nabla \cdot \boldsymbol{A} = 0 \tag{4.12}$$

simplifies the above relation as

$$\frac{\partial^2\phi}{\partial t^2} - \Delta\phi = \rho\ , \quad \frac{\partial^2\boldsymbol{A}}{\partial t^2} - \Delta\boldsymbol{A} = \boldsymbol{J} \tag{4.13}$$

Now we introduce the electromagnetic four-potential $(A^\mu) = (\phi, \boldsymbol{A})$ and four-vector$(J^\mu) = (\rho, \boldsymbol{J})$ to unify the Maxwell equations as

$$\Box A^\mu - \partial^\mu\partial_\nu A^\nu = -J^\mu. \tag{4.14}$$

The charge conservation law is expressed as

$$\partial_\mu j^\mu = 0. \tag{4.15}$$

The gauge transformation is expressed as

$$A_\mu(x) \to \tilde{A}_\mu(x) = A_\mu(x) + \partial_\mu\theta(x), \tag{4.16}$$

and the Lorentz gauge condition is described as

$$\partial_\mu A^\mu = 0. \tag{4.17}$$

The Lorentz gauge condition is a scalar equation, and thus it is relativistically invariant. Under the Lorentz gauge, the equation of motion is given as

$$\Box A^{\mu} = -J^{\mu}. \tag{4.18}$$

Here, we introduce the electromagnetic field tensor as $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$. Explicitly, we have

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}, \quad F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}. \tag{4.19}$$

Note the $F_{\mu\nu}$ is invariant under the gauge transformation Eq.(4.16).

In the conformal coordinate, the Maxwell action is denoted as

$$S_A = \int d^4x \sqrt{-g} \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right], \tag{4.20}$$

and the variation of the gauge field is

$$\begin{aligned} \delta_A S_A &= \int d^4x \sqrt{-g} \left[-\frac{1}{2} F^{\mu\nu} \, \delta F_{\mu\nu} \right] \\ &= \int d^4x \sqrt{-g} \left[-\frac{1}{2} F^{\mu\nu} \left(\partial_{\mu} \delta A_{\nu} - \partial_{\nu} \delta A_{\mu} \right) \right] \\ &= \int d^4x \sqrt{-g} \left[-F^{\mu\nu} \, \partial_{\mu} \delta A_{\nu} \right] \\ &= \int d^4x \delta A_{\nu} \left[\partial_{\mu} (\sqrt{-g} F^{\mu\nu}) \right] = 0. \end{aligned} \tag{4.21}$$

Thus, we have the following forms of the equation of motion,

$$\partial_{\mu} (\sqrt{-g} F^{\mu\nu}) = 0, \quad \text{i.e. } \partial_{\mu} F^{\mu\nu} + \frac{1}{\sqrt{-g}} (\sqrt{-g})_{,\mu} F^{\mu\nu} = 0, \quad \text{i.e. } \nabla_{\mu} F^{\mu\nu} = 0. \tag{4.22}$$

The Christoffel symbol that appears here is simplified as follows,

$$
\begin{aligned}
\Gamma^{\mu}_{\mu\alpha} &= \frac{1}{2} g^{\mu\beta}(g_{\beta\mu,\alpha} + g_{\beta\alpha,\mu} - g_{\mu\alpha,\beta}) \\
&= \frac{1}{\sqrt{-g}}(\sqrt{-g})_{,\alpha}
\end{aligned}
\tag{4.23}
$$

One can also use the following pedagogical relations,

$$
g^{\mu\nu} = \frac{1}{g}\frac{\delta g}{\delta g_{\mu\nu}} \ , \quad \delta g = g g^{\mu\nu}\delta g_{\mu\nu} \ , \quad \frac{\delta g}{g} = g^{\mu\nu}\delta g_{\mu\nu} \ , \quad \therefore \ \frac{\partial_\alpha g}{g} = g^{\mu\nu}\partial_\alpha g_{\mu\nu} \tag{4.24}
$$

In the case of the gauge field is coupled to the scalar field, the case is corresponding to the Model-2 discussed in the later part of this thesis, we have the action like

$$
S_A = \int d^4x \sqrt{-g}\left[-\frac{1}{4} f(\phi)^2 F_{\mu\nu}F^{\mu\nu}\right]. \tag{4.25}
$$

Since the valuable of the coupling function is ϕ or its time evolution, the variation of the gauge field is performed in the same way,

$$
\begin{aligned}
\delta_A S_A &= \int d^4x \, \sqrt{-g}\, f^2(\phi) \left[-\frac{1}{2} F^{\mu\nu}\, \delta F_{\mu\nu}\right] \\
&= \int d^4x \, \sqrt{-g}\, f^2(\phi) \left[-\frac{1}{2} F^{\mu\nu}\, (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu)\right] \\
&= \int d^4x \, \sqrt{-g}\, f^2(\phi) \left[-F^{\mu\nu}\, \partial_\mu \delta A_\nu\right] \\
&= \int d^4x \, \delta A_\nu \left[\partial_\mu(\sqrt{-g}\, f^2(\phi)\, F^{\mu\nu})\right] = 0,
\end{aligned}
\tag{4.26}
$$

resulting the covariant form of the Maxwell equation

$$
\nabla_\mu (f^2(\phi) F^{\mu\nu}) = 0. \tag{4.27}
$$

4.2 Equation of motion of the gravitational wave

The equation of motion of the gravitational wave can be derived by the metric perturbation. The Einstein-Hilbert action is

$$S_g = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}R. \tag{4.28}$$

The metric perturbation gives

$$\delta_g S_g = \frac{1}{2\kappa^2}\int d^4x\Big((\delta\sqrt{-g})\,R + \sqrt{-g}\,(\delta g^{\mu\nu})\,R_{\mu\nu}\Big) \tag{4.29}$$

$$= \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\,\delta g^{\mu\nu}\Big(-\frac{1}{2}\,g_{\mu\nu}\,R + R_{\mu\nu}\Big) \tag{4.30}$$

Here, we have

$$\delta\sqrt{-g} = \frac{1}{2}\cdot\frac{-\delta g}{\sqrt{-g}} = \frac{1}{2}\cdot\frac{-g}{\sqrt{-g}}\,g^{\mu\nu}\,\delta g_{\mu\nu} = \frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\,\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta\,g^{\mu\nu}, \tag{4.31}$$

where we use relations

$$g^{\mu\nu} = \frac{1}{g}\frac{\delta g}{\delta g_{\mu\nu}}\ ,\quad \delta g = g g^{\mu\nu}\delta g_{\mu\nu}, \tag{4.32}$$

$$g^{\mu\alpha}g_{\alpha\nu} = \delta^{\mu}_{\nu}\ ,\quad \delta g^{\mu\alpha}g_{\alpha\nu} = -g^{\mu\alpha}\delta g_{\alpha_\nu}. \tag{4.33}$$

The metric perturbation of the action of U(1) gauge field is

$$\begin{aligned}\delta_g S_A = \int d^4x\Bigg((\delta\sqrt{-g})\Big(-\frac{1}{4}g^{\mu\rho}\,g^{\nu\rho}F_{\mu\nu}F_{\rho\sigma}\Big) + \sqrt{-g}\Big(-\frac{1}{4}g^{\nu\sigma}\,F_{\mu\nu}F_{\rho\sigma}\Big)\delta g^{\mu\rho}\\ + \sqrt{-g}\Big(-\frac{1}{4}g^{\mu\rho}\,F_{\mu\nu}F_{\rho\sigma}\Big)\delta g^{\nu\sigma}\Bigg)\\ = \int d^4x\,\sqrt{-g}\,\delta g^{\mu\nu}\Bigg(-\frac{1}{2}g_{\mu\nu}\Big(-\frac{1}{4}g^{\alpha\rho}g^{\beta\sigma}F_{\alpha\beta}F_{\rho\sigma}\Big)-\frac{1}{4}g^{\alpha\sigma}F_{\mu\alpha}F_{\nu\sigma} - \frac{1}{4}g^{\alpha\rho}F_{\alpha\mu}F_{\rho\nu}\Bigg)\\ = \int d^4x\,\sqrt{-g}\,\delta g^{\mu\nu}\Bigg(-\frac{1}{2}\,g^{\alpha\beta}\,F_{\mu\alpha}\,F_{\nu\beta} + \frac{1}{8}g_{\mu\nu}\,F_{\alpha\beta}\,F^{\alpha\beta}\Bigg).\end{aligned} \tag{4.34}$$

Therefore, the variation with respect to the metric perturbation derives:

$$\delta_g(S_A + S_g) = \frac{1}{2\kappa^2} \int d^4x\, \sqrt{-g}\, \delta g^{\mu\nu} \left(G_{\mu\nu} - 2\kappa^2 \left(\frac{1}{2}\, g^{\alpha\beta}\, F_{\mu\alpha}\, F_{\nu\beta} - \frac{1}{8} g_{\mu\nu}\, F_{\alpha\beta}\, F^{\alpha\beta} \right) \right) = 0, \tag{4.35}$$

it gives the Einstein equation

$$G_{\mu\nu} = \kappa^2 \left(g^{\alpha\beta}\, F_{\mu\alpha}\, F_{\nu\beta} - \frac{1}{4}\, g_{\mu\nu}\, F_{\alpha\beta}\, F^{\alpha\beta} \right). \tag{4.36}$$

When we take into account the perturbative approach to solve the field equation, Green's function is necessary. In the next subsection, the way to derive Green's function is mentioned.

4.3 Green's function

Before deriving the Green's function in the de Sitter space, let us briefly review the foundation of the Green's function clearly written in the textbook [8]. Mos of the field equation including that of gravitational wave take the form of second order differential equation such as

$$\frac{d^2y}{dx^2} + f_1(x)\frac{dy}{dx} + f_2(x)y = f_3(x). \tag{4.37}$$

The above equation reduces to the form of

$$\frac{d}{dx}\left\{ p(x)\frac{dy}{dx} \right\} + q(x)y = 0, \tag{4.38}$$

when we assume

$$p(x) = \exp\left[\int f_1(x)dx \right], \tag{4.39}$$

$$q(x) = f_2(x) \exp\left[\int f_1(x)dx \right], \tag{4.40}$$

$$f_3(x) = 0. \tag{4.41}$$

When we introduce functions of x, $u = u(x)$ and $v = v(x)$, one can easily have

$$\frac{d}{dx}\left(pu\frac{dv}{dx}\right) + quv = p\frac{du}{dx}\frac{dv}{dx} + uL[v], \tag{4.42}$$

here we introduced the notation

$$L[v] \equiv \frac{d}{dx}\left\{p(x)\frac{dy}{dx}\right\} + q(x)v. \tag{4.43}$$

After integrating Eq. (4.42),

$$\int_a^b \left(p\frac{du}{dx}\frac{dv}{dx} - quv\right)dx = \left[pu\frac{dv}{dx}\right]_b^a - \int_a^b uL(v)dx. \tag{4.44}$$

When we replace u and v, we have

$$\int_a^b \left(p\frac{dv}{dx}\frac{du}{dx} - qvu\right)dx = \left[pv\frac{du}{dx}\right]_b^a - \int_a^b vL(u)dx. \tag{4.45}$$

Eqs.(4.44) and (4.45) result

$$\int_a^b \Big[vL(u) - uL(v)\Big]dx = \left[p\left\{v\frac{du}{dx} - u\frac{dv}{dx}\right\}\right]_b^a, \tag{4.46}$$

which corresponds to Green's theorem. Green's function is often introduced to solve the boundary value problem like

$$\begin{aligned} &L(y) = 0, \\ &y(a) = 0\ ,\quad y(b) = 0, \end{aligned} \tag{4.47}$$

in the case of

$$\begin{vmatrix} y_1(a) & y_2(a) \\ y_1(b) & y_2(b) \end{vmatrix} \neq 0, \tag{4.48}$$

where $y_1(x)$ and $y_2(x)$ are fundamental solutions of Eq.(4.47). Green's function $G(x, \xi)$ of the boundary value problem Eq.(4.47) is supposed to satisfy

$$\left(\frac{dG}{dx}\right)_{x=\xi+0} - \left(\frac{dG}{dx}\right)_{x=\xi-0} = -\frac{1}{p(\xi)}, \tag{4.49}$$

where $p(\xi) \neq 0$, and its derivative and itself are continues at the area except $x = \xi$. Let us suppose that the two independent solutions are chosen to satisfy $y_1(a) = 0$ and $y_2(b) = 0$. Then the Green function at $a \leqq x < \xi$ can be taken as $C_1\, y_1(x)$, while Green function at $\xi < x \leqq b$ can be taken as $C_2\, y_2(x)$, where C_1 and C_2 are coefficients. At the boundary of the two areas, we have $C_1\, y_1(\xi) = C_2\, y_2(\xi)$. The first-order derivative satisfy

$$C_1 y_1'(\xi) - C_2 y_2'(\xi) = \frac{1}{p(\xi)}. \tag{4.50}$$

The condition $L(y) = 0$ gives $L(y_1) = L(y_2) = 0$ and it results we can take the integral constant to make Eq. (4.46) takes the form

$$y_2(x) y_1'(x) - y_1(x) y_2'(x) = \frac{1}{p(x)}. \tag{4.51}$$

The comparison between above relation Eq. (4.51) and Eq. (4.50) allows us to chose $C_1 = y_2(\xi)$ and $C_2 = y_1(\xi)$. Thus, Green's function is given as

$$G(x, \xi) = \begin{cases} y_1(x)\, y_2(\xi) & (x \leqq \xi) \\ y_1(\xi)\, y_2(x) & (x \geqq \xi) \end{cases}, \tag{4.52}$$

and one can easily find $G(\xi_1, \xi_2) = G(\xi_2, \xi_1)$. Here, we derive Green's function of the gravitational wave in de-Sitter spacetime. We consider the degree of the freedom of the gravitational wave $h(\eta)$ which follow the equation of motion

$$h'' + 2\frac{a'}{a}h' + k^2 = 0. \tag{4.53}$$

The derivation is described in the later part of this thesis. a is the scale factor which is related to the conformal time and Hubble constant with the relation $a = -1/H\eta$. Using the conformal time, the equation of motion is written as

$$h'' - \frac{2}{\eta}h' + k^2 h = 0, \tag{4.54}$$

and the solution is written as $h = (1 + ik\eta)e^{-ik\eta}$. When we have some source term on the right-hand side of the above equation at $\eta = \eta'$ as is shown in FIG 2, the Green's function satisfy

$$\left(\frac{d^2}{d\eta^2} - \frac{2}{\eta}\frac{d}{d\eta} + k^2\right)G(\eta, \eta') = -\delta(\eta - \eta'). \tag{4.55}$$

The boundary condition is given as $G^{\mathrm{I}}(\eta) = 0$ while junction conditions at $\eta = \eta'$ is given as

$$G^{\mathrm{II}}(\eta') = 0, \tag{4.56}$$

$$G'^{\mathrm{II}}(\eta') = -\frac{1}{2}, \tag{4.57}$$

where we take the ansatz of the Green's function in each region as

$$G^{\mathrm{I}}(\eta) = A\,h(\eta) + B\,h^*(\eta), \tag{4.58}$$

$$G^{\mathrm{II}}(\eta) = C\,h(\eta) + D\,h^*(\eta). \tag{4.59}$$

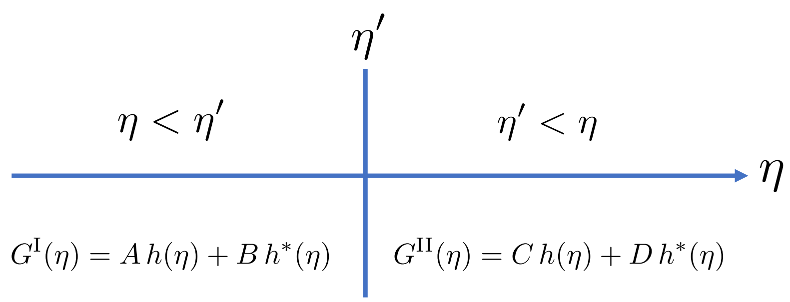

FIG 2. Configuration of the Green's function. We assume that the source term appears at $\eta = \eta'$. Note that the prime on η denotes some specific value of the time, not a derivative.

The condition Eq. (4.56) gives

$$C(1 + ik\eta')e^{-ik\eta'} + D(1 - ik\eta')e^{ik\eta'} = 0. \tag{4.60}$$

The condition Eq. (4.57) results

$$Ce^{-ik\eta'} + De^{ik\eta'} = -\frac{1}{2k^2\eta'}. \tag{4.61}$$

Eq.(4.60) and Eq.(4.61) gives

$$C = -\frac{1}{4k^2\eta}e^{ik\eta'}\left(1 - \frac{1}{ik\eta'}\right), \tag{4.62}$$

$$D = -\frac{1}{4k^2\eta}e^{-ik\eta'}\left(1 + \frac{1}{ik\eta'}\right). \tag{4.63}$$

Thus the Green's function is written as

$$G(\eta - \eta') = \theta(\eta - \eta')\left\{\frac{1}{4ik^3\eta'^2}(1 - ik\eta')(1 + ik\eta)e^{-ik(\eta-\eta')} - \frac{1}{4ik^3\eta'^2}(1 + ik\eta')(1 - ik\eta)e^{ik(\eta-\eta')}\right\}. \tag{4.64}$$

When we use the variable $y = a(\eta)h(\eta)$ in the same procedure, we have

$$G_{\rm dS}(\eta, \eta') = \frac{1}{2ik}\left(1 + \frac{i}{k\eta'}\right)\left(1 - \frac{i}{k\eta}\right)e^{-ik(\eta-\eta')} - \frac{1}{2ik}\left(1 - \frac{i}{k\eta'}\right)\left(1 + \frac{i}{k\eta}\right)e^{ik(\eta-\eta')}. \tag{4.65}$$

5 quantumness of the gravitational wave

The focus of this thesis is the quantumness of the pGWs, especially the two mode squeezed state of the graviton. As a brief introduction to the squeezed state, let us mention the squeezed state. The squeezed state is known as a quantum state that is given by operating the squeezing operator to some vacuum states. The squeezing operator is defined as

$$S(\zeta) = e^{A} \ , \quad A = (\zeta^* \hat{a}^2 - \zeta(\hat{a}^\dagger)^2)/2, \tag{5.1}$$

where ζ is defined by the phase factor θ and the squeezing parameter r $\zeta = re^{i\theta}$. The operator A satisfies the commutation relation $[A, \hat{a}] = \zeta\hat{a}^\dagger$, $[A, \hat{a}^\dagger] = \zeta^*\hat{a}$. Using the Campbell-Baker-

Hausdorf formula:

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + ... \tag{5.2}$$

the following Bogoliubov transformation is derived,

$$\hat{b} = S(\zeta)\hat{a}S^\dagger(\zeta) = \hat{a}\cosh r + \hat{a}^\dagger e^{i\theta}\sinh r \tag{5.3}$$

$$\hat{b}^\dagger = S(\zeta)\hat{a}^\dagger S^\dagger(\zeta) = \hat{a}^\dagger \cosh r + \hat{a}e^{-i\theta}\sinh r \tag{5.4}$$

When we have to discuss the classical case in a quantum context, we often use the coherent state. The coherent state is known as a eigenstate respect to the annihilation operator,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \tag{5.5}$$

where α is the complex amplitude of the coherent state defined as $\alpha = |\alpha|e^{i\phi}$. The distribution of the number operator is set to be the Poisson distribution,

$$P_n = |\langle n|\alpha\rangle|^2 = \frac{(|\alpha|^2)^n}{n!}\exp\left[-|\alpha|^2\right], \tag{5.6}$$

and this is why we use it to mimic the classical state in the context of quantum theory. In this sense, the coherent state is also written as

$$\begin{aligned} |\alpha\rangle &= \sum_n \sqrt{P(n)}\,(e^{i\phi})^n|n\rangle \\ &= \exp\left[-\frac{1}{2}|\alpha|^2\right]\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle \end{aligned} \tag{5.7}$$

The coherent state is also given after operating the displacement operator $D(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ onto the vacuum state,

$$|\alpha\rangle = D(\alpha)|0\rangle = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})|0\rangle. \tag{5.8}$$

The distribution of the number operator is often expected as some index to check the quantumness of the state. For example, the sub-Poissonian distribution represented by the Fano factor $F < 1$ corresponds to some non-classical fields, as is mentioned in [9]. Fano factor is defined as $F = (\Delta n)^2/\langle n \rangle$, where Δn denotes the variance of the particle number while $\langle n \rangle$ denotes the mean value. The Poissonian distribution corresponds to $F = 1$, the super-Poissonian distribution corresponds to $F > 1$, and the sub-Poissonian distribution corresponds to $F < 1$.

The Bogoliubov transformation of two mode squeezed state is often used to describe particle production due to the difference of the normal basis of the quantum state like the Unruh effect Hawking effect and so on. To show an example, we review a part of the discussion of a Fourier mode in the reference [10]. The Fourier mode in the expanding universe is described by two different bases in the Heisenberg picture,

$$\begin{aligned} \hat{\psi}_{\boldsymbol{k}}(\eta) &= u_{\boldsymbol{k}}(\eta)\, \hat{a}_{\boldsymbol{k}} + u^*_{\boldsymbol{k}}(\eta)\, \hat{a}^\dagger_{-\boldsymbol{k}} \\ &= v_{\boldsymbol{k}}(\eta)\, \hat{b}_{\boldsymbol{k}} + v^*_{\boldsymbol{k}}(\eta)\, \hat{b}^\dagger_{-\boldsymbol{k}}. \end{aligned}$$

The first term in each line denotes the positive frequency mode, while the second term in each line denotes the negative frequency mode. The commutation relations of each basis, $[\hat{a}_{\boldsymbol{k}_1}, \hat{a}^\dagger_{\boldsymbol{k}_2}] = \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}$, $[\hat{b}_{\boldsymbol{k}_1}, \hat{b}^\dagger_{\boldsymbol{k}_2}] = \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}$ are satisfied when the field operator is a scalar field. This corresponds to the orthonormality of the Klein-Gordon inner product of the mode functions [11]. When the field is the Dirac field, the commutation relations are replaced with the anti-commutation relations. The two different vacua are defined by annihilation operators, $\hat{a}_{\boldsymbol{k}}|\bar{0}_{\boldsymbol{k}}\rangle = 0$, $\hat{b}_{\boldsymbol{k}}|0_{\boldsymbol{k}}\rangle = 0$. In the context of cosmology, especially the squeezing process of the initial fluctuation, $|\bar{0}_{\boldsymbol{k}}\rangle$ and $|0_{\boldsymbol{k}}\rangle$ correspond to the initial time, known as the Bunch-Davies vacuum, and the vacuum in the late time. The orthonormality of the mode functions or the operators guarantees the Bogoliubov transformation,

$$\hat{a}_{\boldsymbol{k}} = e^{-i\theta_{\boldsymbol{k}}} \cosh r_{\boldsymbol{k}} \hat{b}_{\boldsymbol{k}} + e^{i\theta_{\boldsymbol{k}}+2i\varphi_{\boldsymbol{k}}} \sinh r_{\boldsymbol{k}} \hat{b}^\dagger_{-\boldsymbol{k}}, \tag{5.9}$$

where r_{k} is the squeezing parameter and $\theta_{\boldsymbol{k}}$ and $\varphi_{\boldsymbol{k}}$ is the phase factor taking real number.

The derivation of the two mode squeezed state is referred in [12]. The definition of the vacuum $\hat{a}_{\boldsymbol{k}}|\overline{0}_{\boldsymbol{k}}\rangle = 0_{\boldsymbol{k}}$ results

$$\left(\cosh r_{\boldsymbol{k}}\hat{b}_{\boldsymbol{k}} + e^{i(\theta_{\boldsymbol{k}}+\varphi_{\boldsymbol{k}})}\sinh r_{\boldsymbol{k}}\hat{b}^{\dagger}_{-\boldsymbol{k}}\right)|\overline{0}_{\boldsymbol{k}}\rangle = 0. \tag{5.10}$$

Taking into account the commutation relation $[\hat{b}_{\boldsymbol{k}_1}, \hat{b}^{\dagger}_{\boldsymbol{k}_2}] = \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}$, the above equation can be regarded as a differential equation that solution is written as

$$|\overline{0}_{\boldsymbol{k}}\rangle \propto \exp\left[e^{i(\theta_{\boldsymbol{k}}+\varphi_{\boldsymbol{k}})}\tanh r_{\boldsymbol{k}}\,\hat{b}^{\dagger}_{\boldsymbol{k}}\,\hat{b}^{\dagger}_{-\boldsymbol{k}}\right]|0_{\boldsymbol{k}}\rangle \otimes |0_{-\boldsymbol{k}}\rangle. \tag{5.11}$$

Taylor series of the above equation gives a two-mode squeezed state that consists of an infinite number of entangled particles,

$$|\overline{0}_{\boldsymbol{k}}\rangle \propto \sum_{n} e^{in(\theta_{\boldsymbol{k}}+\varphi_{\boldsymbol{k}})}(\tanh r_{\boldsymbol{k}})^{n}|n_{\boldsymbol{k}}\rangle \otimes |n_{-\boldsymbol{k}}\rangle. \tag{5.12}$$

The state reduces to the maximal entangled state when the squeezing parameter goes to infinity, $r_{\boldsymbol{k}} \to \infty$, and the non-classicality is remarkable at that time.

The inflationary cosmology predicts the existence of primordial gravitational waves (PGWs) that stems from quantum fluctuations. Hence, the detection of PGWs gives strong evidence of inflationary cosmology. In particular, if we could observe the quantum nature of PGWs, it would imply a discovery of gravitons. This point motivates several experimental projects for detecting PGWs [13, 14, 1, 2]. Remarkably, the quantum state of gravitons gets squeezed during inflation [15, 12, 16, 17, 10]. Hence, one way to prove the quantum nature of PGWs would be to find evidence of the squeezed state of gravitons. However, it has still been a challenge to detect PGWs by laser interferometers through the statistical property of the squeezed state [18, 19]. Some other ideas for detecting non-classical PGWs using their squeezed state are proposed. One is to use the Hanbury Brown-Twiss interferometry, which can distinguish non-classical particles from classical ones by measuring intensity-intensity correlations [20, 3]. Another idea is to detect primordial gravitons indirectly by measuring their noise in the interferometers [21, 22, 23, 24] or by measuring the decoherence time of a quantum object caused by the surrounding primordial gravitons [25].

If we capture some evidence of the quantumness of the gravitational wave, it means the existence of the graviton, at the same time, it supports the quantumness of the primordial fluctuation because the origin of pGWs and initial fluctuation is the same, the tensor component and the scalar component of the quantum fluctuation, respectively.

Here, let us review one of the ideas to detect the quantumness of the pGWs. The system is illustrated in FIG.3. The procedure of the experiment supposed in Ref.[25] is the following. At first, a photon is shot into the beam splitter from the left side of the picture (See FIG.3). Then, the photon goes x-direction or y-direction with an equal probability. The photon hits mirror 1 when the photon goes to the x-direction, while the photon hits mirror 2 when the photon goes to the y-direction. Mirrors are connected to springs, and thus when a photon hits mirrors, oscillation state ξ_1 and ξ_2 are generated. We define the time when the photon hits the mirror as t_i. We assume that pGWs go through mirrors continuously. We describe the quantum state of the system at the initial time t_i as

$$|\psi(t_i)\rangle = \left[\frac{1}{\sqrt{2}}|\xi_1\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|0\rangle \otimes |\xi_2\rangle\right]_{\text{mirror}} \otimes |0\rangle_{\text{graviton}}, \tag{5.13}$$

where the state in the bracket on the right-hand side is the superposition state, while the state $|0\rangle_{\text{graviton}}$ represents the vacuum state of gravitons. The reduced density operator of the mirror is obtained when we trace out the states of gravitons as

$$\begin{aligned}\rho(t_i) &= \mathrm{Tr}_g\Big[|\psi(t_i)\rangle\langle\psi(t_i)|\Big] \\ &= \frac{1}{2}|\xi_1\rangle\langle\xi_1| \otimes |0\rangle\langle 0| + \frac{1}{2}|0\rangle\langle\xi_1| \otimes |\xi_2\rangle\langle 0| + \frac{1}{2}|\xi_1\rangle\langle 0| \otimes |0\rangle\langle\xi_2| + \frac{1}{2}|0\rangle\langle 0| \otimes |\xi_2\rangle\langle\xi_2| \\ &= \rho_{11}(t_i) + \rho_{21}(t_i) + \rho_{12}(t_i) + \rho_{22}(t_i),\end{aligned} \tag{5.14}$$

where we defined $\rho_{11}(t_i) = \frac{1}{2}|\xi_1\rangle\langle\xi_1| \otimes |0\rangle\langle 0|$, $\rho_{21}(t_i) = \frac{1}{2}|0\rangle\langle\xi_1| \otimes |\xi_2\rangle\langle 0|$, $\rho_{12}(t_i) = \frac{1}{2}|\xi_1\rangle\langle 0| \otimes |0\rangle\langle\xi_2|$, and $\rho_{22}(t_i) = \frac{1}{2}|0\rangle\langle 0| \otimes |\xi_2\rangle\langle\xi_2|$. ρ_{12} and ρ_{21} is the interference term representing the initial entangled state between mirror 1 and mirror 2.

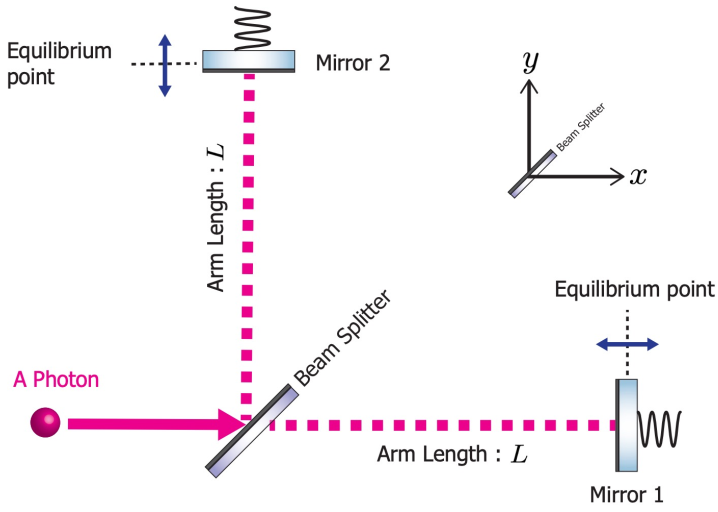

FIG 3. The schematic picture of the interferometer. The picture is taken from Ref [25].

Because mirror 1 and mirror 2 interact with the surrounding gravitons, the vacuum state of the gravitons are changed as $|h;\xi_1\rangle$ and $|h;\xi_2\rangle$, respectively. The state of the total system after the interaction can be expressed as

$$|\psi(t_f)\rangle = \frac{1}{\sqrt{2}}|\xi_1\rangle|0\rangle \otimes |h;\xi_1\rangle + \frac{1}{\sqrt{2}}|0\rangle|\xi_2\rangle \otimes |h;\xi_2\rangle. \tag{5.15}$$

The mirrors get entangled with gravitons due to the interaction, and the superposition state of mirrors is decohered due to the interaction. Then, the reduced density operator of the total system takes the form

$$\begin{aligned}\rho(t_f) &= \mathrm{Tr}_g\Big[|\psi(t_f)\rangle\langle\psi(t_f)|\Big] \\ &= \rho_{11}(t_i) + e^{\mathrm{Im}\Phi^*}\rho_{21}(t_i) + e^{-\mathrm{Im}\Phi}\rho_{12}(t_i) + \rho_{22}(t_i).\end{aligned} \tag{5.16}$$

The influence of the gravitons appears as Φ, which is called decoherence functional. The decoherence time of the entangled state of mirror 1 and mirror 2 is derived from the decoherence function, and it depends on the state of the environmental gravitons. In the case that the graviton is squeezed during conventional inflation, the decoherence time is estimated as about 20 seconds. Other parameters are set as follows, the angular frequency of mirrors $\sim$ 1kHz, the arm length $\sim$ 40 km, the mass of the mirror $\sim$ 40 kg. In the case that the graviton is a coherent state, the decoherence time gets greater. When we regard the coherent state as a classical state, we can say that the decoherence time when the graviton is quantum is greater than the decoherence time when the graviton is classical. In other words, the quantumness of the background graviton would be detected by measuring the decoherence time of the mirror system.

We can expect that the gravitons that went through inflation keep their squeezed states until today unless the environmental effects on them are considered. It is well known that the generation of relic gravitons can be interpreted as the squeezing process of a quantum state during inflation [15, 12, 16, 17]. Since the degree of squeezing is extremely high, the quantum state is highly entangled between two modes with opposite wave number vectors due to the conservation of momentum. Therefore, whether the background graviton is squeezed state or not is very important to test the quantumness of the pGWs in the way proposed in Ref [25]. In fact, the squeezed gravitons can significantly enhance the quantum noise in interferometers [21, 22, 23, 24, 25]. Hence, we need to show the degree of the squeezing generated during inflation survives under the decoherence processes in the evolution of the universe. So far, the decoherence process due to short wavelength modes of a field has been investigated [26, 27, 28, 29]. However, it is argued that the decoherence obtained by tracing out the short wavelength modes is false decoherence [30]. Thus, it is worth studying different decoherence processes.

If the gravitons were surrounded by matter fields during inflation, they may not be able to keep their squeezed states anymore. If the squeezed state is broken due to some environmental effect, the detection of the quantumness with the approaches mentioned above would be difficult. We can think of a scalar field (inflaton field) and a vector field as the matter field during inflation. Since the inflaton field couples with PGWs in the form of the gradient, the coupling with the vector field would be more effective.

In the next section, we see some observational background of the cosmic magnetic field.

6 Observational background of the cosmic magnetic field

From the point of view of observations, primordial magnetic fields may have existed during inflation. In fact, there are observations that cannot be explained without the presence of primordial magnetic fields [31, 32, 33, 34, 35, 36]. Several observation gives us the range of current value of the cosmic magnetic field. The expected area of cosmic magnetic field is about 10^{-16} Gauss $< B_0 < 10^{-9}$ Gauss.

The upper bound $B_0 < 10^{-9}$ Gauss is given by several kinds of observations. For instance, there is a very tiny anisotropy of the CMB temperature fluctuation, and it gives the value of the upper bound since the existence of the background magnetic field gives rise to the anisotropy of the CMB temperature fluctuation [37, 38]. Another example is the Faraday rotation. If there is a background magnetic field, the angle of the polarization plane of the photon rotates, it is known as Faraday rotation. Since the Faraday rotation of the photon traveled a far distance due to the background magnetic field is not observed, we can get the upper bound of the magnetic field [39]. These ways to get the upper bound are quite intuitive and clear. Then, the next concern is the lower bound, and some consideration is required. The next figure is adapted from Ref. [40].

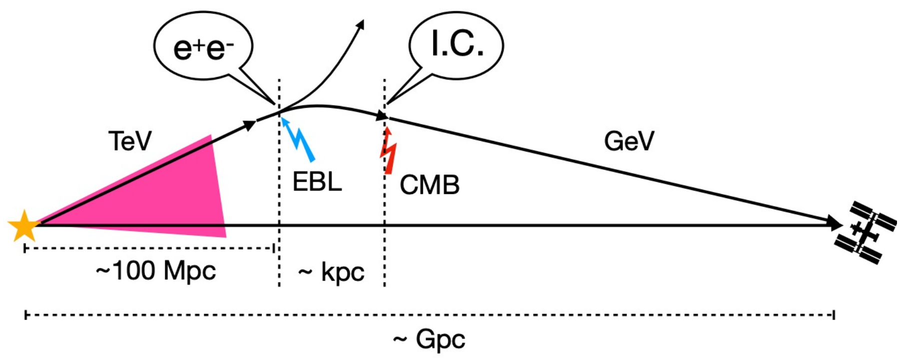

FIG 4. Schematic picture of the mechanism of the emission of TeV gamma-ray and GeV gamma-ray from blazar in a distant galaxy. The picture is taken from Ref [40].

Several paths of the TeV photon in the picture FIG.Tanmayn. At first, a blazar in a distant galaxy emits the TeV gamma-ray. The created lepton pair proceeds the distance in kpc order and is terminated by an inverse Compton IC scattering. The mean free path of the electron is the scale of $\sim$ kpc. The GeV photon that is created by the Inverse Compton

(IC) scattering propagates to the detector, perhaps on board a satellite. The path of the lepton pair is much smaller than the pass of the GeV photon ~ 100 Mpc and the pass of TeV photon ~ Gpc. Here, only the lepton pair probes the intergalactic magnetic field. When there are inter-galactic magnetic fields with enough strength, the Lorentz force bent the pass of the lepton pair. In that case, the GeV gamma rays are no longer directed toward the observer. Then, the GeV gamma-ray will not be observed. In, Ref [33] the observational data of the lower bound of the strength of intergalactic magnetic fields is elaborated. The following figure is adapted from Ref.[33].

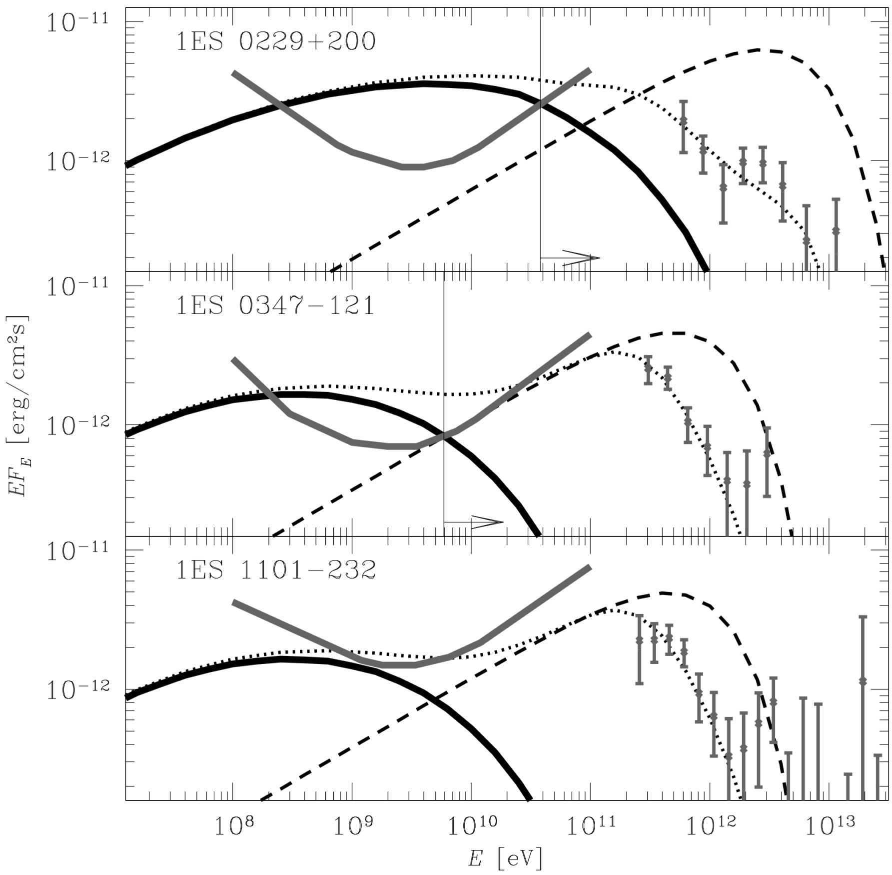

FIG 5. Thick solid black curves show the expected cascade emission from TeV blazars, grey curves show the Fermi upper limits, and grey data points show the data of the High Energy Stereoscopic System (HESS). This figure is taken from Ref [33].

The dashed curves in each panel show the source spectra. Dotted curves show the electromagnetic cascade spectra originated by pair creation on extra-galactic background light. If there are no intergalactic magnetic fields, some data points will be observed between the

Fermi upper limits and the solid line of the expected cascade emission from TeV blazars. Since the GeV gamma-ray emission from the electromagnetic cascade initiated by the TeV gamma-ray in the intergalactic medium is not observed, one can obtain the lower bound of the intergalactic magnetic field. Furthermore, if a coherence length of magnetic fields in mega-parsec scales were found, we need to consider magnetic fields generated during inflation [41, 42].

The presence of background magnetic fields causes the conversion of gravitons into photons and vice versa [4, 43]. These photons could be the dark photon [44]. The graviton photon conversion would affect the squeezed state of gravitons. Therefore, we need to investigate the effect of the conversion process on the squeezed state of gravitons.

Thus, as a source of the decoherence, we assume the presence of a sizable magnetic field at the beginning of inflation. The squeezed state of gravitons may turn into the squeezed state of photons due to the graviton-photon conversion. Hence, it is important to clarify to what extent the squeezing of the relic gravitons survives at present.

In this thesis, we investigate whether gravitons surrounded by primordial magnetic fields can keep their squeezed states. The purpose of this paper is to compute the degree of squeezing parameters of graviton and photon and cross squeezing parameter between gravitons and photons during inflation. We then consider the conversion process of the squeezed gravitons into photons during inflation in the case of minimal coupling between gravitons and photons [4, 43, 45, 46].

Part II

Background magnetic field Model-1

This part is organized as follows: In section 1, we derive basic equations for analyzing the conversion process of gravitons into photons during inflation. In section 2, we explain the perturbative formalism for solving a coupled system between gravitons and photons in order to obtain the time evolution of mode functions. In section 3, we derive Bogoliubov transformations due to the squeezing process in the presence of primordial magnetic fields. In section 4, we deduce formulae for the squeezing parameters and reveal the time evolution of the squeezing parameters numerically and analytically. We also discuss the implications of our results. The final section is devoted to the conclusion.

1 Graviton-photon conversion during inflation

We represent the graviton in a spatially flat expanding background by the tensor mode perturbation in the three-dimensional metric,

$$ds^2 = a^2(\eta)\left[-d\eta^2 + (\delta_{ij} + h_{ij})\, dx^i dx^j\right] , \tag{1.1}$$

where η is the conformal time and the metric perturbation h_{ij} satisfies the transverse traceless conditions $h_{ij}{}^{,j} = h^i{}_i = 0$. The spatial indices $i, j, k, \cdots$ are raised and lowered by δ^{ij} and $\delta_{k\ell}$.

The Einstein-Hilbert action and the action for the electromagnetic field is given by

$$S = S_g + S_A = \frac{M_{\rm pl}^2}{2} \int d^4x \sqrt{-g}\, R - \frac{1}{4} \int d^4x \sqrt{-g}\, F^{\mu\nu} F_{\mu\nu} , \tag{1.2}$$

where $M_{\rm pl} = 1/\sqrt{8\pi G}$ is the Planck mass. The gauge field A_μ represents the photon and the field strength is defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Expanding the Einstein-Hilbert action up to the second order in perturbations h_{ij}, we find

$$\delta S_g = \frac{M_{\rm pl}^2}{8} \int d^4x\, a^2 \left[h^{ij\prime}\, h'_{ij} - h^{ij,k} h_{ij,k} \right] . \tag{1.3}$$

Here, a prime denotes the derivative with respect to the conformal time. The action for the photon up to second order in perturbations A_i reads

$$\delta S_A = \frac{1}{2} \int d^4x \left[A_i'^2 - A_{k,i}^2 \right] , \tag{1.4}$$

where the photon field satisfies the Coulomb gauge $A_0 = 0$ and $A^i{}_{,i} = 0$. The action for the interaction between the graviton and the photon up to second order in perturbations h_{ij}, A^i is found to be

$$\delta S_{\rm I} = \int d^4x \left[\varepsilon_{i\ell m} B_m h^{ij} \left(\partial_j A_\ell - \partial_\ell A_j \right) \right] . \tag{1.5}$$

Note that we assumed $B_m = \varepsilon_{mj\ell}\, \partial_j A_\ell$ is a constant background magnetic field that existed at the beginning of inflation.

At quadratic order, it is convenient to expand $h_{ij}(\eta, x^i)$ and $A_i(\eta, x^i)$ in the Fourier modes,

$$h_{ij}(\eta, x^i) = \frac{2}{M_{\rm pl}} \sum_P \frac{1}{(2\pi)^{3/2}} \int d^3k\, h^P_{\boldsymbol{k}}(\eta)\, e^P_{ij}(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} , \tag{1.6}$$

$$A_i(\eta, x^i) = \sum_P \frac{\pm i}{(2\pi)^{3/2}} \int d^3k\, A^P_{\boldsymbol{k}}(\eta)\, e^P_i(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{1.7}$$

where three-vectors are denoted by bold math type and $e^P_{ij}(\boldsymbol{k})$ and $e^P_i(\boldsymbol{k})$ are the polarization tensors and vectors for the $\boldsymbol{k}$ mode respectively normalized as $e^{ijP}(\boldsymbol{k}) e^Q_{ij}(\boldsymbol{k}) = \delta^{PQ}$ and $e^{iP}(\boldsymbol{k}) e^Q_i(\boldsymbol{k}) = \delta^{PQ}$ with $P, Q = +, \times$. Using the canonical variable $y^P_{\boldsymbol{k}}(\eta) = a(\eta) h^P_{\boldsymbol{k}}(\eta)$, we can rewrite the quadratic actions (2.4), (2.7) and (2.10) as

$$\delta S_g = \frac{1}{2}\sum_P \int d^3k\, d\eta \left[|y_{\boldsymbol{k}}^{P}{}'|^2 - k^2|y_{\boldsymbol{k}}^P|^2 - \frac{a'}{a} y_{\boldsymbol{k}}^P y_{-\boldsymbol{k}}^{P}{}' - \frac{a'}{a} y_{-\boldsymbol{k}}^P y_{\boldsymbol{k}}^{P}{}' + \left(\frac{a'}{a}\right)^2 |y_{\boldsymbol{k}}^P|^2 \right], \quad (1.8)$$

$$\delta S_A = \frac{1}{2}\sum_P \int d^3k\, d\eta \left[|A_{\boldsymbol{k}}^{P}{}'|^2 - k^2|A_{\boldsymbol{k}}^P|^2 \right], \quad (1.9)$$

$$\delta S_I = \frac{2}{M_{\rm pl}}\sum_{P,Q} \int d^3k\, d\eta\, \frac{1}{a} \left[\varepsilon_{i\ell m}\, B_m\, y_{\boldsymbol{k}}^P A_{-\boldsymbol{k}}^Q\, e_{ij}^P(\boldsymbol{k}) \left\{ ik_\ell\, e_j^Q(-\boldsymbol{k}) - ik_j\, e_\ell^Q(-\boldsymbol{k}) \right\} \right], \quad (1.10)$$

where $k = |\boldsymbol{k}|$. Polarization vectors $e^{i+}, e^{i\times}$ and a vector k^i/k constitute an orthnormal basis. Without loss of generality, we assume the constant background magnetic field is in the $(k^i, e^{i\times})$-plane.

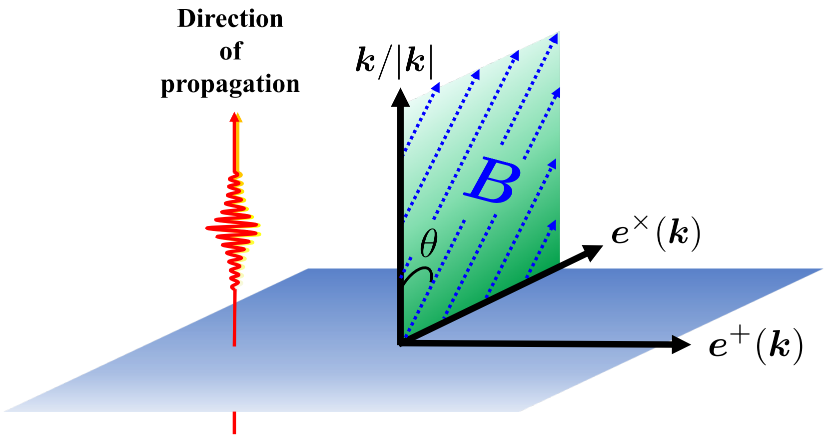

FIG 6. Configuration of the polarization vector $\boldsymbol{e}^P(\boldsymbol{k})$, wave number $\boldsymbol{k}$, and background magnetic field $\boldsymbol{B}$.

The polarization tensors can be written in terms of polarization vectors e^{i+} and $e^{i\times}$ as

$$e_{ij}^+(\boldsymbol{k}) = \frac{1}{\sqrt{2}}\left\{ e_i^+(\boldsymbol{k})e_j^+(\boldsymbol{k}) - e_i^\times(\boldsymbol{k})e_j^\times(\boldsymbol{k}) \right\}, \quad (1.11)$$

$$e_{ij}^\times(\boldsymbol{k}) = \frac{1}{\sqrt{2}}\left\{ e_i^+(\boldsymbol{k})e_j^\times(\boldsymbol{k}) + e_i^\times(\boldsymbol{k})e_j^+(\boldsymbol{k}) \right\}. \quad (1.12)$$

In the following, we assume

$$e_i^\times(-\boldsymbol{k}) = -e_i^\times(\boldsymbol{k}). \quad (1.13)$$

The action (2.11) is then reduced into

$$\delta S_I \;=\; \frac{\sqrt{2}}{M_{\rm pl}} \int d^3k\, d\eta\, \frac{1}{a} \left[\, \lambda(\boldsymbol{k})\, y^{+}_{\boldsymbol{k}}(\eta)\, A^{+}_{-\boldsymbol{k}}(\eta) + \lambda(\boldsymbol{k})\, y^{\times}_{\boldsymbol{k}}(\eta)\, A^{\times}_{-\boldsymbol{k}}(\eta) \,\right] , \tag{1.14}$$

where we defined the coupling between graviton and photon as

$$\lambda(\boldsymbol{k}) \equiv \frac{\sqrt{2}}{M_{\rm pl}} \varepsilon^{i\ell m}\, e^{+}_{i}\, k_\ell\, B_m\,. \tag{1.15}$$

Here, the conditions for the graviton and photon to be real read, $h^{+,\times}_{-\boldsymbol{k}}(\eta) = h^{*+,\times}_{\boldsymbol{k}}(\eta)$ and $A^{+,\times}_{-\boldsymbol{k}}(\eta) = -A^{*+,\times}_{\boldsymbol{k}}(\eta)$. Below, we focus on the plus polarization and omit the index P unless there may be any confusion.

In the case of de Sitter space, the scale factor is given by $a(\eta) = -1/(H\eta)$ where $-\infty < \eta < 0$. The variation of the actions (1.8), (1.9) and (2.14) with respect to the graviton and the photon fields gives

$$y''_{\boldsymbol{k}} + \left(k^2 - \frac{2}{\eta^2}\right) y_{\boldsymbol{k}} = \lambda H \eta A_{\boldsymbol{k}}\,, \tag{1.16}$$

$$A''_{\boldsymbol{k}} + k^2 A_{\boldsymbol{k}} = \lambda H \eta\, y_{\boldsymbol{k}}\,. \tag{1.17}$$

If we define the Lagrangian in the actions (1.8) and (1.9) by $\delta S_g = \int d\eta\, L_g$ and $\delta S_A = \int d\eta\, L_A$, the conjugate momenta of graviton $p_{\boldsymbol{k}}$ and photon $\pi_{\boldsymbol{k}}$ are respectively given by

$$p_{\boldsymbol{k}}(\eta) = \frac{\partial L_g}{\partial y'_{-\boldsymbol{k}}} = y'_{\boldsymbol{k}}(\eta) + \frac{1}{\eta} y_{\boldsymbol{k}}(\eta)\ , \tag{1.18}$$

$$\pi_{\boldsymbol{k}}(\eta) = \frac{\partial L_A}{\partial A'_{-\boldsymbol{k}}} = A'_{\boldsymbol{k}}(\eta)\ . \tag{1.19}$$

Now we promote variables $y_{\boldsymbol{k}}(\eta), A_{\boldsymbol{k}}(\eta)$ and their momenta $p_{\boldsymbol{k}}(\eta), \pi_{\boldsymbol{k}}(\eta)$ into operators. The annihilation operator for the graviton is expressed by canonical variables as

$$\hat{a}_y(\eta, \boldsymbol{k}) = \sqrt{\frac{k}{2}} \hat{y}_{\boldsymbol{k}}(\eta) + \frac{i}{\sqrt{2k}} \hat{p}_{\boldsymbol{k}}(\eta)\,. \tag{1.20}$$

In the same way, the annihilation operator for the photon is given by

$$\hat{a}_A(\eta, \boldsymbol{k}) = \sqrt{\frac{k}{2}}\hat{A}_{\boldsymbol{k}}(\eta) + \frac{i}{\sqrt{2k}}\hat{\pi}_{\boldsymbol{k}}(\eta)\,. \tag{1.21}$$

The commutation relations $[\hat{a}_y(\eta, \boldsymbol{k}), \hat{a}_y^\dagger(\eta, -\boldsymbol{k}')] = \delta(\boldsymbol{k} + \boldsymbol{k}')$ and $[\hat{a}_A(\eta, \boldsymbol{k}), \hat{a}_A^\dagger(\eta, -\boldsymbol{k}')] = \delta(\boldsymbol{k} + \boldsymbol{k}')$ guarantee the canonical commutation relations $[y_{\boldsymbol{k}}(\eta), p_{\boldsymbol{k}'}(\eta)] = i\delta(\boldsymbol{k} - \boldsymbol{k}')$ and $[A_{\boldsymbol{k}}(\eta), \pi_{\boldsymbol{k}'}(\eta)] = i\delta(\boldsymbol{k} - \boldsymbol{k}')$. Notice that the annihilation operator becomes time-dependent through the time dependence of canonical variables. Thus, the vacuum defined by $\hat{a}(\eta, \boldsymbol{k})|0\rangle = 0$ is time dependent as well and the vacuum in this formalism turns out to be defined at every moment.

In this paper, we suppose $B_m/M_{\rm pl} \ll 1$ so that the coupling between graviton and photon (2.15) is weak. Then we solve the Eqs. (2.17) and (2.18) iteratively up to the second order in $y_{\boldsymbol{k}}$ and $A_{\boldsymbol{k}}$ in the next section.

2 Time evolution of mode functions

Using the basic equations presented in the previous section, we perturbatively derive mode functions in this section.

2.1 Zeroth order

By letting $\lambda = 0$ in Eqs. (2.17) and (2.18), the equations of the zeroth order approximation become

$$\hat{y}_{\boldsymbol{k}}^{(0)\prime\prime} + \left(k^2 - \frac{2}{\eta^2}\right)\hat{y}_{\boldsymbol{k}}^{(0)} = 0\,, \tag{2.1}$$

$$\hat{A}_{\boldsymbol{k}}^{(0)\prime\prime} + k^2 \hat{A}_{\boldsymbol{k}}^{(0)} = 0\,, \tag{2.2}$$

where the superscript (0) denotes the zeroth order. The solutions to the above equations are

$$\hat{y}_{\boldsymbol{k}}^{(0)}(\eta) = u_{\boldsymbol{k}}^{(0)}(\eta)\ \hat{c} + u_{\boldsymbol{k}}^{(0)*}(\eta)\ \hat{c}^\dagger\,, \tag{2.3}$$

$$\hat{A}_{\boldsymbol{k}}^{(0)}(\eta) = v_{\boldsymbol{k}}^{(0)}(\eta)\ \hat{d} + v_{\boldsymbol{k}}^{(0)*}(\eta)\ \hat{d}^\dagger, \tag{2.4}$$

where $\hat{c}\,(\hat{d})$ and its conjugate $\hat{c}^{\dagger}\,(\hat{d}^{\dagger})$ are constant operators of integration. We choose the properly normalized positive frequency mode in the remote past as a basis, which is expressed as

$$u^{(0)}_{\boldsymbol{k}}(\eta) = \frac{1}{\sqrt{2k}}\left(1-\frac{i}{k\eta}\right)e^{-ik\eta}\,, \qquad v^{(0)}_{\boldsymbol{k}}(\eta) = \frac{1}{\sqrt{2k}}e^{-ik\eta}. \tag{2.5}$$

2.2 First order

Inserting the solutions of zeroth order approximation (3.7) and (3.8) into the r.h.s of Eqs. (2.17) and (2.18) as the source terms, the equations of the first order approximation are written as

$$\hat{y}^{(1)\prime\prime}_{\boldsymbol{k}} + \left(k^2 - \frac{2}{\eta^2}\right)\hat{y}^{(1)}_{\boldsymbol{k}} = \lambda H\eta \hat{A}^{(0)}_{\boldsymbol{k}}\,, \tag{2.6}$$

$$\hat{A}^{(1)\prime\prime}_{\boldsymbol{k}} + k^2 \hat{A}^{(1)}_{\boldsymbol{k}} = \lambda H\eta \hat{y}^{(0)}_{\boldsymbol{k}}\,. \tag{2.7}$$

The effect of photon comes in Eq. (2.6). Using the Green function

$$G_{\rm dS}(\eta,\eta') = \frac{1}{2ik}\left(1+\frac{i}{k\eta'}\right)\left(1-\frac{i}{k\eta}\right)e^{-ik(\eta-\eta')} - \frac{1}{2ik}\left(1-\frac{i}{k\eta'}\right)\left(1+\frac{i}{k\eta}\right)e^{ik(\eta-\eta')}\,, \tag{2.8}$$

we obtain the solution as

$$\begin{aligned}
\hat{y}^{(1)}_{\boldsymbol{k}}(\eta) &= -\int_{\eta_i}^{\eta} d\eta' G_{\rm dS}(\eta,\eta')\lambda H\eta' \hat{A}^{(0)}_{\boldsymbol{k}}(\eta') \\
&= -\int_{\eta_i}^{\eta} d\eta' G_{\rm dS}(\eta,\eta')\lambda H\eta' v^{(0)}_{\boldsymbol{k}}(\eta')\ \hat{d} - \int_{\eta_i}^{\eta} d\eta' G_{\rm dS}(\eta,\eta')\lambda H\eta' v^{(0)*}_{\boldsymbol{k}}(\eta')\ \hat{d}^{\dagger} \\
&\equiv u^{(1)}_{\boldsymbol{k}}(\eta)\ \hat{d} + u^{(1)*}_{\boldsymbol{k}}(\eta)\ \hat{d}^{\dagger}\,,
\end{aligned} \tag{2.9}$$

where η_i is an initial time. From the first line to the second line, we used Eq. (3.8). In the last line, we defined the first-order correction due to the source of the photon to the positive frequency mode of graviton by

$$u^{(1)}_{\boldsymbol{k}}(\eta) \equiv -\int_{\eta_i}^{\eta} d\eta' G_{\rm dS}(\eta,\eta')\lambda H\eta' v^{(0)}_{\boldsymbol{k}}(\eta')\ . \tag{2.10}$$

After integration, we have

$$
\begin{aligned}
u^{(1)}_{\boldsymbol{k}}(\eta) &= \frac{\lambda H}{8\sqrt{2}\eta k^{9/2}}\Big[e^{-ik\eta}\Big\{2i\eta^3k^3+\eta k\Big(2\eta_i k(2-i\eta_i k)+3i\Big)-2\eta_i k(\eta_i k+2i)+3\Big\} \\
&\quad -e^{ik(\eta-2\eta_i)}(\eta k+i)(2\eta_i k-3i)\Big] \ . \qquad (2.11)
\end{aligned}
$$

Similarly, the effect of graviton comes in Eq. (2.7). By using the Green function

$$
G_{\mathrm{M}}(\eta,\eta') = -\frac{1}{k}\sin k(\eta-\eta')\,, \qquad (2.12)
$$

we have

$$
\begin{aligned}
\hat{A}^{(1)}_{\boldsymbol{k}}(\eta) &= -\int_{\eta_i}^{\eta} d\eta' G_{\mathrm{M}}(\eta,\eta')\lambda H\eta'\hat{y}^{(0)}_{\boldsymbol{k}}(\eta') \\
&= -\int_{\eta_i}^{\eta} d\eta' G_{\mathrm{M}}(\eta,\eta')\lambda H\eta' u^{(0)}_{\boldsymbol{k}}(\eta)\ \hat{c} - \int_{\eta_i}^{\eta} d\eta' G_{\mathrm{M}}(\eta,\eta')\lambda H\eta' u^{(0)*}_{\boldsymbol{k}}(\eta)\ \hat{c}^{\dagger} \\
&\equiv v^{(1)}_{\boldsymbol{k}}(\eta)\ \hat{c} + v^{(1)*}_{\boldsymbol{k}}(\eta)\ \hat{c}^{\dagger}\ , \qquad (2.13)
\end{aligned}
$$

where we used Eq. (3.7) from the first line to the second line. We also defined the first-order correction due to the source of graviton to the positive frequency mode of the photon in the third line by

$$
v^{(1)}_{\boldsymbol{k}}(\eta) \equiv -\int_{\eta_i}^{\eta} d\eta' G_{\mathrm{M}}(\eta,\eta')\lambda H\eta' u^{(0)}_{\boldsymbol{k}}(\eta')\ . \qquad (2.14)
$$

More explicitly, the above is written as

$$
v^{(1)}_{\boldsymbol{k}}(\eta) = \frac{\lambda H}{8\sqrt{2}k^{7/2}}\Big[e^{-ik\eta}\Big\{2ik^2(\eta^2-\eta_i^2)+k(6\eta-4\eta_i)-3i\Big\}+e^{ik(\eta-2\eta_i)}(-2\eta_i k+3i)\Big]\ . \qquad (2.15)
$$

2.3 Second order

By plugging the solution of the first order approximation (2.9) and (2.13) into the r.h.s of Eqs. (2.17) and (2.18) as the source terms, the equations of the second order approximation are

$$y_{\boldsymbol{k}}^{(2)\prime\prime} + \left(k^2 - \frac{2}{\eta^2}\right) y_{\boldsymbol{k}}^{(2)} = \lambda H \eta A_{\boldsymbol{k}}^{(1)} \,, \tag{2.16}$$

$$A_{\boldsymbol{k}}^{(2)\prime\prime} + k^2 A_{\boldsymbol{k}}^{(2)} = \lambda H \eta \, y_{\boldsymbol{k}}^{(1)} \,. \tag{2.17}$$

At this order, the effect of graviton itself comes in Eq. (2.16). The solution is written by the Green function $G_{\rm dS}$ such as

$$\begin{aligned}
\hat{y}_{\boldsymbol{k}}^{(2)}(\eta) &= -\int_{\eta_i}^{\eta} d\eta' G_{\rm dS}(\eta,\eta')\lambda H\eta' \hat{A}_{\boldsymbol{k}}^{(1)}(\eta') \\
&= -\int_{\eta_i}^{\eta} d\eta' G_{\rm dS}(\eta,\eta')\lambda H\eta' v_{\boldsymbol{k}}^{(1)}(\eta') \ \hat{c} - \int_{\eta_i}^{\eta} d\eta' G_{\rm dS}(\eta,\eta')\lambda H\eta' v_{\boldsymbol{k}}^{(1)*}(\eta') \ \hat{c}^\dagger \\
&= -\int_{\eta_i}^{\eta} d\eta' G_{\rm dS}(\eta,\eta')\lambda H\eta' \left(-\int_{\eta_i}^{\eta'} d\eta'' G_{\rm M}(\eta',\eta'')\lambda H\eta'' u_{\boldsymbol{k}}^{(0)}(\eta'') \right) \hat{c} \\
&\quad - \int_{\eta_i}^{\eta} d\eta' G_{\rm dS}(\eta,\eta')\lambda H\eta' \left(-\int_{\eta_i}^{\eta'} d\eta'' G_{\rm M}(\eta',\eta'')\lambda H\eta'' u_{\boldsymbol{k}}^{(0)*}(\eta'') \right) \hat{c}^\dagger \\
&\equiv u_{\boldsymbol{k}}^{(2)}(\eta) \ \hat{c} + u_{\boldsymbol{k}}^{(2)*}(\eta) \ \hat{c}^\dagger ,
\end{aligned} \tag{2.18}$$

where we used Eqs. (2.13) and (2.14) in the second and the third lines respectively. In the last line, we defined

$$\begin{aligned}
u_{\boldsymbol{k}}^{(2)}(\eta) &\equiv -\int_{\eta_i}^{\eta} d\eta' G_{\rm dS}(\eta,\eta')\lambda H\eta' v_{\boldsymbol{k}}^{(1)}(\eta') \\
&= \int_{\eta_i}^{\eta} d\eta' G_{\rm dS}(\eta,\eta')\lambda H\eta' \int_{\eta_i}^{\eta'} d\eta'' G_{\rm M}(\eta',\eta'')\lambda H\eta'' u_{\boldsymbol{k}}^{(0)}(\eta'').
\end{aligned} \tag{2.19}$$

By performing the integration, the explicit form of the $u_{\boldsymbol{k}}^{(2)}(\eta)$ is found to be

$$\begin{aligned} u_{\boldsymbol{k}}^{(2)}(\eta) &= -\frac{\lambda^2 H^2}{192\sqrt{2}\eta k^{15/2}} \\ &\times \Bigg[3e^{ik(\eta-2\eta_i)} \Bigg\{ 2\eta^3 k^3(-3-2i\eta_i k) + \eta k \Big(-35 + 2\eta_i k(\eta_i k(11+2i\eta_i k) - 23i) \Big) \\ &\quad +2\eta_i k(23+\eta_i k(-2\eta_i k + 11i)) - 35i \Bigg\} \\ &+e^{-ik\eta} \Bigg\{ k\Big(-105\eta + 72\eta_i + 6\eta k^4 \left(\eta^2-\eta_i^2\right)^2 - 2ik^3 \left(17\eta^4 - 12\eta^3\eta_i - 8\eta\eta_i^3 + 3\eta_i^4\right) \\ &\quad +k^2\left(16\eta_i^3 - 52\eta^3\right) + 72i\eta\eta_i k \Big) + 105i \Bigg\} \Bigg] \,. \end{aligned} \tag{2.20}$$

Similarly, the effect of photon itself comes in Eq. (2.17) and the solution is given by

$$\begin{aligned} \hat{A}_{\boldsymbol{k}}^{(2)}(\eta) &= -\int_{\eta_i}^{\eta} d\eta' G_{\mathrm{M}}(\eta,\eta')\lambda H\eta' \ \hat{y}_{\boldsymbol{k}}^{(1)}(\eta') \\ &= -\int_{\eta_i}^{\eta} d\eta' G_{\mathrm{M}}(\eta,\eta')\lambda H\eta' u_{\boldsymbol{k}}^{(1)}(\eta') \ \hat{d} - \int_{\eta_i}^{\eta} d\eta' G_{\mathrm{M}}(\eta,\eta')\lambda H\eta' \ u_{\boldsymbol{k}}^{(1)*}(\eta') \ \hat{d}^\dagger \\ &= -\int_{\eta_i}^{\eta} d\eta' G_{\mathrm{M}}(\eta,\eta')\lambda H\eta' \ \left(-\int_{\eta_i}^{\eta'} d\eta'' G_{\mathrm{dS}}(\eta',\eta'')\lambda H\eta'' v_{\boldsymbol{k}}^{(0)}(\eta'') \right)\hat{d} \\ &\quad - \int_{\eta_i}^{\eta} d\eta' G_{\mathrm{M}}(\eta,\eta')\lambda H\eta' \ \left(-\int_{\eta_i}^{\eta'} d\eta'' G_{\mathrm{dS}}(\eta',\eta'')\lambda H\eta'' v_{\boldsymbol{k}}^{(0)*}(\eta'') \right)\hat{d}^\dagger \\ &= v_{\boldsymbol{k}}^{(2)}(\eta) \ \hat{d} + v_{\boldsymbol{k}}^{(2)*}(\eta) \ \hat{d}^\dagger, \end{aligned} \tag{2.21}$$

where we used Eqs. (2.9) in the second line and (2.10) in the third line and in the last line. We defined

$$\begin{aligned} v_{\boldsymbol{k}}^{(2)}(\eta) &\equiv -\int_{\eta_i}^{\eta} d\eta' G_{\mathrm{M}}(\eta,\eta')\lambda H\eta' u_{\boldsymbol{k}}^{(1)}(\eta') \\ &= \int_{\eta_i}^{\eta} d\eta' G_{\mathrm{M}}(\eta,\eta')\lambda H\eta' \int_{\eta_i}^{\eta'} d\eta'' G_{\mathrm{dS}}(\eta',\eta'')\lambda H\eta'' v_{\boldsymbol{k}}^{(0)}(\eta'') \,. \end{aligned} \tag{2.22}$$

The integral of the above reduces to

$$
\begin{aligned}
v_{\boldsymbol{k}}^{(2)}(\eta) &= -\frac{\lambda^2 H^2}{64\sqrt{2}k^{13/2}} \\
&\quad\times\Bigg[e^{-ik\eta}\Bigg(2k^4\left(\eta^2-\eta_i^2\right)^2-4i\eta k^3(\eta-\eta_i)(\eta+3\eta_i) \\
&\qquad +12\eta_i k^2(\eta_i-2\eta)-12ik(\eta-2\eta_i)-15\Bigg) \\
&\qquad +e^{ik(\eta-2\eta_i)}\left(2k(3+2i\eta_i k)\left(\eta_i^2 k-\eta(\eta k+3i)\right)+6i\eta_i k+15\right)\Bigg]\,. \qquad (2.23)
\end{aligned}
$$

3 Bogoliubov transformations

By solving Eqs.(2.17) and (2.18) iteratively up to the second order, we can take into account the backreaction of graviton and photon respectively. For the graviton, the field and its conjugate momentum are now given by

$$
\begin{aligned}
\hat{y}_{\boldsymbol{k}}(\eta) &= \left(u_{\boldsymbol{k}}^{(0)}+u_{\boldsymbol{k}}^{(2)}\right)\hat{c}+u_{\boldsymbol{k}}^{(1)}\,\hat{d}+\text{h.c.}\,, &(3.1)\\
\hat{p}_{\boldsymbol{k}}(\eta) &= \left(u_{\boldsymbol{k}}^{(0)\prime}+u_{\boldsymbol{k}}^{(2)\prime}\right)\hat{c}+u_{\boldsymbol{k}}^{(1)\prime}\,\hat{d}+\frac{1}{\eta}\left\{\left(u_{\boldsymbol{k}}^{(0)}+u_{\boldsymbol{k}}^{(2)}\right)\hat{c}+u_{\boldsymbol{k}}^{(1)}\,\hat{d}\right\}+\text{h.c.}\,, &(3.2)
\end{aligned}
$$

where we used Eq. (3.1) and h.c. represents Hermitian conjugate. For the photon, the field and its conjugate momentum become

$$
\begin{aligned}
\hat{A}_{\boldsymbol{k}}(\eta) &= \left(v_{\boldsymbol{k}}^{(0)}+v_{\boldsymbol{k}}^{(2)}\right)\hat{d}+v_{\boldsymbol{k}}^{(1)}\,\hat{c}+\text{h.c.}\,, &(3.3)\\
\hat{\pi}_{\boldsymbol{k}}(\eta) &= \left(v_{\boldsymbol{k}}^{(0)\prime}+v_{\boldsymbol{k}}^{(2)\prime}\right)\hat{d}+v_{\boldsymbol{k}}^{(1)\prime}\,\hat{c}+\text{h.c.}\,, &(3.4)
\end{aligned}
$$

where we used Eq. (3.2). Then the annihilation operators for the graviton and photon are obtained by using Eqs. (3.3) and (3.4) such as

$$
\hat{a}_y(\eta,\boldsymbol{k})=\left(\psi_p^{(0)}+\psi_p^{(2)}\right)\hat{c}+\left(\psi_m^{(0)*}+\psi_m^{(2)*}\right)\hat{c}^\dagger+\psi_p^{(1)}\hat{d}+\psi_m^{(1)*}\hat{d}^\dagger\,, \qquad (3.5)
$$

$$
\hat{a}_A(\eta,\boldsymbol{k})=\left(\phi_p^{(0)}+\phi_p^{(2)}\right)\hat{d}+\left(\phi_m^{(0)*}+\phi_m^{(2)*}\right)\hat{d}^\dagger+\phi_p^{(1)}\hat{c}+\phi_m^{(1)*}\hat{c}^\dagger\,. \qquad (3.6)
$$

Here, we defined new variables

$$\psi_p^{(j)} = \sqrt{\frac{k}{2}} u_{\boldsymbol{k}}^{(j)}(\eta) + \frac{i}{\sqrt{2k}} \Big(u_{\boldsymbol{k}}^{(j)\prime}(\eta) + \frac{1}{\eta} u_{\boldsymbol{k}}^{(j)}(\eta) \Big), \tag{3.7}$$

$$\psi_m^{(j)} = \sqrt{\frac{k}{2}} u_{\boldsymbol{k}}^{(j)}(\eta) - \frac{i}{\sqrt{2k}} \Big(u_{\boldsymbol{k}}^{(j)\prime}(\eta) + \frac{1}{\eta} u_{\boldsymbol{k}}^{(j)}(\eta) \Big), \tag{3.8}$$

$$\phi_p^{(j)} = \sqrt{\frac{k}{2}} v_{\boldsymbol{k}}^{(j)}(\eta) + \frac{i}{\sqrt{2k}} v_{\boldsymbol{k}}^{(j)\prime}(\eta), \tag{3.9}$$

$$\phi_m^{(j)} = \sqrt{\frac{k}{2}} v_{\boldsymbol{k}}^{(j)}(\eta) - \frac{i}{\sqrt{2k}} v_{\boldsymbol{k}}^{(j)\prime}(\eta), \tag{3.10}$$

where $j = 0, 1, 2$ denotes the order of perturbations.

We see that all mode functions other than the zeroth order given in Eqs. (2.10), (2.14) (2.19) and (2.22) vanish at the initial time η_i. Thus only the zeroth order of the above Eqs. (3.30) $\sim$ (3.10) remains at the initial time. This means that annihilation operators in Eqs.(3.28) and (3.29) at the initial time are expressed by the zeroth order variables

$$\hat{a}_y(\eta_i, \boldsymbol{k}) = \left(1 - \frac{i}{2k\eta_i}\right) e^{-ik\eta_i}\, \hat{c} + \frac{i}{2k\eta_i} e^{ik\eta_i}\, \hat{c}^\dagger, \tag{3.11}$$

$$\hat{a}_A(\eta_i, \boldsymbol{k}) = e^{-ik\eta_i} \hat{d}. \tag{3.12}$$

Combining Eqs. (3.10) and (3.11) with their complex conjugate, we can express the $\hat{c}$ and $\hat{d}$ by the initial creation and annihilation operators as

$$\hat{c} = \left(1 + \frac{i}{2k\eta_i}\right) e^{ik\eta_i}\, \hat{a}_y(\eta_i, \boldsymbol{k}) - \frac{i}{2k\eta_i} e^{ik\eta_i}\, \hat{a}_y^\dagger(\eta_i, -\boldsymbol{k}) \ , \tag{3.13}$$

$$\hat{d} = e^{ik\eta_i}\, \hat{a}_A(\eta_i, \boldsymbol{k}) \ . \tag{3.14}$$

Plugging the above back into Eqs.(3.28) and (3.29), the time evolution of the annihilation operator of the graviton is described by the Bogoliubov transformation in the form

$$
\begin{aligned}
\hat{a}_y(\eta,\boldsymbol{k}) =& \left[\left(\psi_p^{(0)}+\psi_p^{(2)}\right)\left(1+\frac{i}{2k\eta_i}\right)e^{ik\eta_i}+\left(\psi_m^{(0)*}+\psi_m^{(2)*}\right)\frac{i}{2k\eta_i}e^{-ik\eta_i}\right]\hat{a}_y(\eta_i,\boldsymbol{k}) \\
&+\left[\left(\psi_p^{(0)}+\psi_p^{(2)}\right)\left(-\frac{i}{2k\eta_i}\right)e^{ik\eta_i}+\left(\psi_m^{(0)*}+\psi_m^{(2)*}\right)\left(1-\frac{i}{2k\eta_i}\right)e^{-ik\eta_i}\right]\hat{a}_y^\dagger(\eta_i,-\boldsymbol{k}) \\
&+\psi_p^{(1)}e^{ik\eta_i}\hat{a}_A(\eta_i,\boldsymbol{k})+\psi_m^{(1)*}e^{-ik\eta_i}\hat{a}_A^\dagger(\eta_i,-\boldsymbol{k}),
\end{aligned}
\tag{3.15}
$$

and the time evolution of the annihilation operator of the photon is expressed by the Bogoliubov transformation such as

$$
\begin{aligned}
\hat{a}_A(\eta,\boldsymbol{k}) =& \left(\phi_p^{(1)}\left(1+\frac{i}{2k\eta_i}\right)e^{ik\eta_i}+\phi_m^{(1)*}\frac{i}{2k\eta_i}e^{-ik\eta_i}\right)\hat{a}_y(\eta_i,\boldsymbol{k}) \\
&+\left(-\phi_p^{(1)}\frac{i}{2k\eta_i}e^{ik\eta_i}+\phi_m^{(1)*}\left(1-\frac{i}{2k\eta_i}\right)e^{-ik\eta_i}\right)\hat{a}_y^\dagger(\eta_i,-\boldsymbol{k}) \\
&+\left(\phi_p^{(0)}+\phi_p^{(2)}\right)e^{ik\eta_i}\hat{a}_A(\eta_i,\boldsymbol{k})+\left(\phi_m^{(0)*}+\phi_m^{(2)*}\right)e^{-ik\eta_i}\hat{a}_A^\dagger(\eta_i,-\boldsymbol{k}).
\end{aligned}
\tag{3.16}
$$

These Bogoliubov transformations show the particle production during inflation and the mixing between graviton and photon.

It is useful to use a matrix form for later calculations. In fact, the Bogoliubov transformation (3.32) and (3.33) and their conjugate can be accommodated into the simple 4×4 matrix form M

$$
\begin{pmatrix} a_y(\eta) \\ a_y^\dagger(\eta) \\ a_A(\eta) \\ a_A^\dagger(\eta) \end{pmatrix} = M \begin{pmatrix} a_y(\eta_i) \\ a_y^\dagger(\eta_i) \\ a_A(\eta_i) \\ a_A^\dagger(\eta_i) \end{pmatrix} = \left\{ \begin{pmatrix} A_0 & 0 \\ 0 & D_0 \end{pmatrix} + \begin{pmatrix} 0 & B_1 \\ C_1 & 0 \end{pmatrix} + \begin{pmatrix} A_2 & 0 \\ 0 & D_2 \end{pmatrix} \right\} \begin{pmatrix} a_y(\eta_i) \\ a_y^\dagger(\eta_i) \\ a_A(\eta_i) \\ a_A^\dagger(\eta_i) \end{pmatrix} .
\tag{3.17}
$$

Here, the zeroth order Bogoliubov transformation consists of 2×2 matrices A_0 and D_0 given by

$$A_0 = \begin{pmatrix} K^* & -L^* \\ -L & K \end{pmatrix} \ , \qquad D_0 = \begin{pmatrix} e^{ik\,(\eta-\eta_i)} & 0 \\ 0 & e^{-ik\,(\eta-\eta_i)} \end{pmatrix} \ , \tag{3.18}$$

where we defined

$$K = \left(1+\frac{i}{2k\eta}\right)\left(1-\frac{i}{2k\eta_i}\right)e^{ik(\eta-\eta_i)} - \frac{1}{4k^2\eta\eta_i}e^{-ik(\eta-\eta_i)} \ , \tag{3.19}$$

$$L = -\frac{i}{2k\eta_i}\left(1+\frac{i}{2k\eta}\right)e^{ik(\eta-\eta_i)} + \frac{i}{2k\eta}\left(1+\frac{i}{2k\eta_i}\right)e^{-ik(\eta-\eta_i)} \ . \tag{3.20}$$

The first order Bogoliubov transformation is written by 2×2 matrices B_1 and C_1 such as

$$B_1 = \begin{pmatrix} e^{ik\eta_i}\psi_p^{(1)} & e^{-ik\eta_i}\psi_m^{(1)*} \\ e^{ik\eta_i}\psi_m^{(1)} & e^{-ik\eta_i}\psi_p^{(1)*} \end{pmatrix} \tag{3.21}$$

and

$$C_1 = \begin{pmatrix} \left(1+\frac{i}{2k\eta_i}\right)e^{ik\eta_i}\phi_p^{(1)} + \frac{i}{2k\eta_i}e^{-ik\eta_i}\phi_m^{(1)*} & \left(1-\frac{i}{2k\eta_i}\right)e^{-ik\eta_i}\phi_m^{(1)*} - \frac{i}{2k\eta_i}e^{ik\eta_i}\phi_p^{(1)} \\ \left(1+\frac{i}{2k\eta_i}\right)e^{ik\eta_i}\phi_m^{(1)} + \frac{i}{2k\eta_i}e^{-ik\eta_i}\phi_p^{(1)*} & \left(1-\frac{i}{2k\eta_i}\right)e^{-ik\eta_i}\phi_p^{(1)*} - \frac{i}{2k\eta_i}e^{ik\eta_i}\phi_m^{(1)} \end{pmatrix} . \tag{3.22}$$

Finally, the second-order Bogoliubov transformation A_2 and D_2 are

$$A_2 = \begin{pmatrix} \left(1+\frac{i}{2k\eta_i}\right)e^{ik\eta_i}\psi_p^{(2)} + \frac{i}{2k\eta_i}e^{-ik\eta_i}\psi_m^{(2)*} & \left(1-\frac{i}{2k\eta_i}\right)e^{-ik\eta_i}\psi_m^{(2)*} - \frac{i}{2k\eta_i}e^{ik\eta_i}\psi_p^{(2)} \\ \left(1+\frac{i}{2k\eta_i}\right)e^{ik\eta_i}\psi_m^{(2)} + \frac{i}{2k\eta_i}e^{-ik\eta_i}\psi_p^{(2)*} & \left(1-\frac{i}{2k\eta_i}\right)e^{-ik\eta_i}\psi_p^{(2)*} - \frac{i}{2k\eta_i}e^{ik\eta_i}\psi_m^{(2)} \end{pmatrix} \tag{3.23}$$

and

$$D_2 = \begin{pmatrix} e^{ik\eta_i}\phi_p^{(2)} & e^{-ik\eta_i}\phi_m^{(2)*} \\ e^{ik\eta_i}\phi_m^{(2)} & e^{-ik\eta_i}\phi_p^{(2)*} \end{pmatrix} . \tag{3.24}$$

4 Time evolution of squeezing parameters

In the previous section, we obtained the Bogoliubov transformation that mixes the operators $\hat{a}_y(\eta)$, $\hat{a}_A(\eta)$ and their Hermitian conjugates $\hat{a}_y^\dagger(\eta)$, $\hat{a}_A^\dagger(\eta)$. Note that the initial Bunch-Davies state is defined by

$$\hat{a}_y(\eta_i, \boldsymbol{k})|\mathrm{BD}\rangle = \hat{a}_A(\eta_i, \boldsymbol{k})|\mathrm{BD}\rangle = 0\,. \tag{4.1}$$

Note that the initial quantum state could be taken as the different state from the Bunchi-Davies vacuum. For instance, we can take alpha vacua as the initial state, which can be interpreted as excited states. In this case, we would obtain a different result depending on the parameters of the alpha vacua. In order to impose these conditions, we need to invert the Bogoliubov transformations (3.32) and (3.33) into the form

$$\hat{a}_y(\eta_i, \boldsymbol{k}) = \alpha_y\, \hat{a}_y(\eta, \boldsymbol{k}) + \beta_y\, \hat{a}_y^\dagger(\eta, -\boldsymbol{k}) + \gamma_A\, \hat{a}_A(\eta, \boldsymbol{k}) + \delta_A\, \hat{a}_A^\dagger(\eta, -\boldsymbol{k})\,, \tag{4.2}$$

$$\hat{a}_A(\eta_i, \boldsymbol{k}) = \gamma_y\, \hat{a}_y(\eta, \boldsymbol{k}) + \delta_y\, \hat{a}_y^\dagger(\eta, -\boldsymbol{k}) + \alpha_A\, \hat{a}_A(\eta, \boldsymbol{k}) + \beta_A\, \hat{a}_A^\dagger(\eta, -\boldsymbol{k})\,, \tag{4.3}$$

where α_y, β_y, γ_A, δ_A, γ_y, δ_y, α_A and β_A are the Bogoliubov coefficients and we will find these coefficients in the next subsection.

4.1 Inversion of the Bogoliubov transformation

The matrix M in Eq. (3.34) can be expanded perturbatively as

$$M = M^{(0)} + M^{(1)} + M^{(2)} = M^{(0)}\left[1 + M^{(0)-1}M^{(1)} + M^{(0)-1}M^{(2)}\right]\,, \tag{4.4}$$

where

$$M^{(0)} = \begin{pmatrix} A_0 & 0 \\ 0 & D_0 \end{pmatrix}, \qquad M^{(1)} = \begin{pmatrix} 0 & B_1 \\ C_1 & 0 \end{pmatrix}, \qquad M^{(2)} = \begin{pmatrix} A_2 & 0 \\ 0 & D_2 \end{pmatrix}. \tag{4.5}$$

Then the inverse of the M is given by

$$M^{-1} = \left[1 - M^{(0)-1}M^{(1)} - M^{(0)-1}M^{(2)} + M^{(0)-1}M^{(1)}M^{(0)-1}M^{(1)}\right] M^{(0)-1}\,. \tag{4.6}$$

Using the above general formula, the inverse of the M is obtained in the form

$$M^{-1} = \begin{pmatrix} A_0^{-1} - A_0^{-1}A_2A_0^{-1} + A_0^{-1}B_1D_0^{-1}C_1A_0^{-1} & -A_0^{-1}B_1D_0^{-1} \\ -D_0^{-1}C_1A_0^{-1} & D_0^{-1} - D_0^{-1}D_2D_0^{-1} + D_0^{-1}C_1A_0^{-1}B_1D_0^{-1} \end{pmatrix} . \tag{4.7}$$

We see that A_0^{-1} and D_0^{-1} are necessary to calculate the elements of the M^{-1}. They are given by

$$A_0^{-1} = \begin{pmatrix} K & L^* \\ L & K^* \end{pmatrix} , \qquad D_0^{-1} = \begin{pmatrix} e^{-ik\,(\eta-\eta_i)} & 0 \\ 0 & e^{ik\,(\eta-\eta_i)} \end{pmatrix} . \tag{4.8}$$

From Eqs. (3.42) and (3.43), the M^{-1} is also written as

$$M^{-1} = \begin{pmatrix} \alpha_y & \beta_y & \gamma_A & \delta_A \\ \beta_y^* & \alpha_y^* & \delta_A^* & \gamma_A^* \\ \gamma_y & \delta_y & \alpha_A & \beta_A \\ \delta_y^* & \gamma_y^* & \beta_A^* & \alpha_A^* \end{pmatrix} , \tag{4.9}$$

where

$$\alpha_y = \alpha_y^{(0)} + \alpha_y^{(2)} , \qquad \beta_y = \beta_y^{(0)} + \beta_y^{(2)} , \qquad \gamma_A = \gamma_A^{(1)} , \qquad \delta_A = \delta_A^{(1)} , \tag{4.10}$$

$$\alpha_A = \alpha_A^{(0)} + \alpha_A^{(2)} , \qquad \beta_A = \beta_A^{(2)} , \qquad \gamma_y = \gamma_y^{(1)} , \qquad \delta_y = \delta_y^{(1)} . \tag{4.11}$$

The zeroth order elements are given by

$$\alpha_y^{(0)} = \left(1 + \frac{i}{2k\eta}\right)\left(1 - \frac{i}{2k\eta_i}\right) e^{ik(\eta-\eta_i)} - \frac{1}{4k^2\eta\eta_i} e^{-ik(\eta-\eta_i)} , \tag{4.12}$$

$$\beta_y^{(0)} = \frac{i}{2k\eta_i}\left(1 - \frac{i}{2k\eta}\right) e^{-ik(\eta-\eta_i)} - \frac{i}{2k\eta}\left(1 - \frac{i}{2k\eta_i}\right) e^{ik(\eta-\eta_i)} , \tag{4.13}$$

$$\alpha_A^{(0)} = e^{ik(\eta-\eta_i)} , \qquad \beta_A^{(0)} = 0 . \tag{4.14}$$

The first-order elements are written as

$$\gamma_A^{(1)} = -\left(K\psi_p^{(1)} + L^*\psi_m^{(1)}\right)e^{ik\eta}\ , \tag{4.15}$$

$$\delta_A^{(1)} = -\left(K\psi_m^{(1)*} + L^*\psi_p^{(1)*}\right)e^{-ik\eta}\ , \tag{4.16}$$

$$\begin{aligned} \gamma_y^{(1)} &= -K\left[\left(1+\frac{i}{2k\eta_i}\right)e^{ik\eta}\phi_p^{(1)} + \frac{i}{2k\eta_i}e^{ik(\eta-2\eta_i)}\phi_m^{(1)*}\right] \\ &\quad -L\left[\left(1-\frac{i}{2k\eta_i}\right)e^{ik(\eta-2\eta_i)}\phi_m^{(1)*} - \frac{i}{2k\eta_i}e^{ik\eta}\phi_p^{(1)}\right]\ , \end{aligned} \tag{4.17}$$

$$\begin{aligned} \delta_y^{(1)} &= -L^*\left[\left(1+\frac{i}{2k\eta_i}\right)e^{ik\eta}\phi_p^{(1)} + \frac{i}{2k\eta_i}e^{ik(\eta-2\eta_1)}\phi_m^{(1)*}\right] \\ &\quad -K^*\left[\left(1-\frac{i}{2k\eta_i}\right)e^{ik(\eta-2\eta_i)}\phi_m^{(1)*} - \frac{i}{2k\eta_i}e^{ik\eta}\phi_p^{(1)}\right]\ . \end{aligned} \tag{4.18}$$

The second order is

$$\begin{aligned} \alpha_y^{(2)} &= -K\left(KA_{11} + L^*A_{21}\right) - L\left(KA_{12} + L^*A_{22}\right) \\ &\quad +(C_{11}K + C_{12}L)\left(K\psi_p^{(1)} + L^*\psi_m^{(1)}\right)e^{ik\eta} \\ &\quad +(C_{21}K + C_{22}L)\left(K\psi_m^{(1)*} + L^*\psi_p^{(1)*}\right)e^{-ik\eta}\ , \end{aligned} \tag{4.19}$$

$$\begin{aligned} \beta_y^{(2)} &= -L^*\left(KA_{11} + L^*A_{21}\right) - K^*\left(KA_{12} + L^*A_{22}\right) \\ &\quad +(C_{11}L^* + C_{12}K^*)\left(K\psi_p^{(1)} + L^*\psi_m^{(1)}\right)e^{ik\eta} \\ &\quad +(C_{21}L^* + C_{22}K^*)\left(K\psi_m^{(1)*} + L^*\psi_p^{(1)*}\right)e^{-ik\eta}\ , \end{aligned} \tag{4.20}$$

$$\begin{aligned} \alpha_A^{(2)} &= -e^{ik(2\eta-\eta_i)}\phi_p^{(2)} \\ &\quad +(C_{11}K + C_{12}L)e^{ik(2\eta-\eta_i)}\psi_p^{(1)} + (C_{11}L^* + C_{12}K^*)e^{ik(2\eta-\eta_i)}\psi_m^{(1)}\ , \end{aligned} \tag{4.21}$$

$$\begin{aligned} \beta_A^{(2)} &= -e^{-ik\eta_i}\phi_m^{(2)*} \\ &\quad +(C_{11}K + C_{12}L)e^{-ik\eta_i}\psi_m^{(1)*} + (C_{11}L^* + C_{12}K^*)e^{-ik\eta_i}\psi_p^{(1)*}\ , \end{aligned} \tag{4.22}$$

where we have defined

$$
\begin{aligned}
A_{11} &= \psi_p^{(2)}\left(1+\frac{i}{2k\eta_i}\right)e^{ik\eta_i}+\psi_m^{(2)*}\frac{i}{2k\eta_i}e^{-ik\eta_i}\,, && (4.23)\\
A_{12} &= -\psi_p^{(2)}\frac{i}{2k\eta_i}e^{ik\eta_i}+\psi_m^{(2)*}\left(1-\frac{i}{2k\eta_i}\right)e^{-ik\eta_i}\,, && (4.24)\\
A_{21} &= \psi_p^{(2)*}\frac{i}{2k\eta_i}e^{-ik\eta_i}+\psi_m^{(2)}\left(1+\frac{i}{2k\eta_i}\right)e^{ik\eta_i}\,, && (4.25)\\
A_{22} &= \psi_p^{(2)*}\left(1-\frac{i}{2k\eta_i}\right)e^{-ik\eta_i}-\psi_m^{(2)}\frac{i}{2k\eta_i}e^{ik\eta_i}\,, && (4.26)
\end{aligned}
$$

and

$$
\begin{aligned}
C_{11} &= \phi_p^{(1)}\left(1+\frac{i}{2k\eta_i}\right)e^{ik\eta_i}+\phi_m^{(1)*}\frac{i}{2k\eta_i}e^{-ik\eta_i}\,, && (4.27)\\
C_{12} &= \phi_m^{(1)*}\left(1-\frac{i}{2k\eta_i}\right)e^{-ik\eta_i}-\phi_p^{(1)}\frac{i}{2k\eta_i}e^{ik\eta_i}\,, && (4.28)\\
C_{21} &= \phi_p^{(1)*}\frac{i}{2k\eta_i}e^{-ik\eta_i}+\phi_m^{(1)}\left(1+\frac{i}{2k\eta_i}\right)e^{ik\eta_i}\,, && (4.29)\\
C_{22} &= \phi_p^{(1)*}\left(1-\frac{i}{2k\eta_i}\right)e^{-ik\eta_i}-\phi_m^{(1)}\frac{i}{2k\eta_i}e^{ik\eta_i}\,. && (4.30)
\end{aligned}
$$

4.2 Squeezing operator

In the previous subsection, we obtained the Bogoliubov coefficients of Eqs. (3.42) and (3.43) up to the second order. If we apply the Eqs. (3.42) and (3.43) to the definition of the Bunch-Davies vacuum (3.41) and use the relations $[\hat{a}_y(\eta,\boldsymbol{k}),\hat{a}_y^\dagger(\eta,-\boldsymbol{k'})]=\delta(\boldsymbol{k}+\boldsymbol{k'})\,,$ $[\hat{a}_A(\eta,\boldsymbol{k}),\hat{a}_A^\dagger(\eta,-\boldsymbol{k'})]=\delta(\boldsymbol{k}+\boldsymbol{k'})$ and $[\hat{a}_y(\eta,\boldsymbol{k}),\hat{a}_A(\eta,-\boldsymbol{k'})]=0$, the Bunch-Davies vacuum can be written by using squeezing parameters Λ,Ξ and Ω such as

$$
|\mathrm{BD}\rangle=\prod_{k=-\infty}^{\infty}\exp\left[\frac{\Lambda}{2}\,\hat{a}_y^\dagger(\eta,\boldsymbol{k})\hat{a}_y^\dagger(\eta,-\boldsymbol{k})+\Xi\,\hat{a}_y^\dagger(\eta,\boldsymbol{k})\hat{a}_A^\dagger(\eta,-\boldsymbol{k})+\frac{\Omega}{2}\,\hat{a}_A^\dagger(\eta,\boldsymbol{k})\hat{a}_A^\dagger(\eta,-\boldsymbol{k})\right]|0\rangle, \tag{4.31}
$$

where $|0\rangle$ is the instantaneous vacuum defined by

$$
\hat{a}_y(\eta,\boldsymbol{k})|0\rangle=\hat{a}_A(\eta,\boldsymbol{k})|0\rangle=0\,. \tag{4.32}
$$

This describes a four-mode squeezed state of pairs of graviton y and photon A. In a different context, a four-mode squeezed state of two free massive scalar fields is discussed in [47, 48]. If we expand the exponential function in the Taylor series, we find

$$|\mathrm{BD}\rangle = \prod_{\boldsymbol{k}} \sum_{p\,,q\,,r=0}^{\infty} \frac{\Lambda^p\,\Xi^q\,\Omega^r}{2^{p+r}p!\,q!\,r!}|p+q\rangle_{y,\boldsymbol{k}} \otimes |p\rangle_{y,-\boldsymbol{k}} \otimes |r\rangle_{A,\boldsymbol{k}} \otimes |q+r\rangle_{A,-\boldsymbol{k}}\,. \tag{4.33}$$

This is a four-mode squeezed state which consists of an infinite number of entangled particles in the $\mathcal{H}_{y,\boldsymbol{k}} \otimes \mathcal{H}_{y,-\boldsymbol{k}} \otimes \mathcal{H}_{A,\boldsymbol{k}} \otimes \mathcal{H}_{A,-\boldsymbol{k}}$ space. In particular, in the highly squeezing limit $\Lambda\,,\Xi\,,\Omega \to 1$, the Bunch-Davies vacuum becomes the maximally entangled state from the point of view of the instantaneous vacuum.

Now we find the squeezing parameters. The condition $\hat{a}_y(\eta_i,\boldsymbol{k})|\mathrm{BD}\rangle = 0$ of Eq. (3.41) yields

$$\alpha_y\Lambda + \beta_y + \gamma_A\Xi = 0\ , \qquad \alpha_y\Xi + \gamma_A\Omega + \delta_A = 0\,, \tag{4.34}$$

and another condition $\hat{a}_A(\eta_i,\boldsymbol{k})|\mathrm{BD}\rangle = 0$ of Eq. (3.41) gives

$$\alpha_A\Xi + \gamma_y\Lambda + \delta_y = 0\ , \qquad \alpha_A\Omega + \beta_A + \gamma_y\Xi = 0\,. \tag{4.35}$$

Then, we obtain the three squeezing parameters Λ, Ξ and Ω of the form

$$\Lambda = \frac{\gamma_A\delta_y - \beta_y\alpha_A}{\alpha_y\alpha_A - \gamma_y\gamma_A}\ , \qquad \Xi = \frac{\beta_y\gamma_y - \alpha_y\delta_y}{\alpha_y\alpha_A - \gamma_y\gamma_A}\ , \qquad \Omega = \frac{\gamma_y\delta_A - \beta_A\alpha_y}{\alpha_y\alpha_A - \gamma_y\gamma_A}\,. \tag{4.36}$$

We have four relations for three parameters Λ, Ξ, and Ω. The remaining relation is turned out to be guaranteed by the commutation relation:

$$[\hat{a}_y(\eta,\boldsymbol{k})\ ,\hat{a}_A(\eta,\boldsymbol{k})] = -\alpha_A\delta_A + \beta_A\gamma_A - \gamma_y\beta_y + \alpha_y\delta_y = 0\,. \tag{4.37}$$

Thus, we find that Eq. (3.50) is the unique solution. Since the Bogoliubov coefficients are given up to the second order as in Eqs. (4.10) and (4.11), the squeezing parameters can be expanded up to the second order such as

$$\Lambda = -\frac{\beta_y^{(0)}}{\alpha_y^{(0)}}\left[1-\frac{\alpha_y^{(2)}}{\alpha_y^{(0)}}+\frac{\beta_y^{(2)}}{\beta_y^{(0)}}+\frac{\gamma_y^{(1)}\gamma_A^{(1)}}{\alpha_y^{(0)}\alpha_A^{(0)}}-\frac{\gamma_A^{(1)}\delta_y^{(1)}}{\beta_y^{(0)}\alpha_A^{(0)}}\right] , \tag{4.38}$$

$$\Xi = \frac{\beta_y^{(0)}\gamma_y^{(1)}}{\alpha_y^{(0)}\alpha_A^{(0)}}-\frac{\delta_y^{(1)}}{\alpha_A^{(0)}} , \tag{4.39}$$

$$\Omega = \frac{\delta_A^{(1)}\gamma_y^{(1)}}{\alpha_y^{(0)}\alpha_A^{(0)}}-\frac{\beta_A^{(2)}}{\alpha_A^{(0)}} . \tag{4.40}$$

In this way, we obtained the squeezing parameters perturbatively up to the second order. We will discuss the behavior of the squeezing of graviton Λ, the squeezing of mixing between graviton and photon Ξ, and the squeezing of photon Ω in the next section.

4.3 Numerical and analytical results

The results of numerical calculations for the amplitude and the phase of the squeezing parameters Λ, Ξ, and Ω are plotted in FIGs. 7, 8, 9, 10, 11, and 12, respectively, where we normalized the scale factor at the end of inflation as $a(\eta_f) = 1$. The evolution of the amplitude of Λ in FIG. 7 shows graviton is squeezed, that is, graviton pair production occurs during inflation ($\eta < 0$). We see that sub-horizon modes oscillate rapidly and no graviton pair production seems to occur before horizon exit. In the presence of coupling with magnetic fields ($\lambda \neq 0$), the amplitude of oscillation is relatively small as represented by the blue line. After horizon exit, the oscillation ceases and graviton pair production starts to occur and eventually Λ becomes one. This means that almost the maximum entangled pair of the graviton is produced. This behavior does not change even for $\lambda \neq 0$. The evolution of phase of Λ is plotted in FIG. 8, in which we see the phase converge to zero. This is consistent with the result in [17]. The time evolution of the amplitude of Ξ in FIG. 9 shows that one of the pair of gravitons is converted to a photon and graviton-photon pair production occurs. We see that some amount of pair production occurs when the mode leaves the horizon but the graviton-photon pair production decreases rapidly by the end of inflation. The evolution of the phase of Ξ is plotted in FIG. 10 is found to oscillate rapidly but eventually becomes constant after horizon exit. A similar behavior appears in the evolution of phase of Λ in

FIG. 8. However, the final phase is found to depend on the initial condition in this case. The time evolution of the amplitude of Ω in FIG. 11 tells us that the photon is squeezed, that is, graviton pair production is converged to photon pair production. Interestingly, photon pair production occurs rapidly only at the initial time and no more production occurs after that. The behavior of the phase evolution of Ω in FIG.12 is similar to that of Ξ.

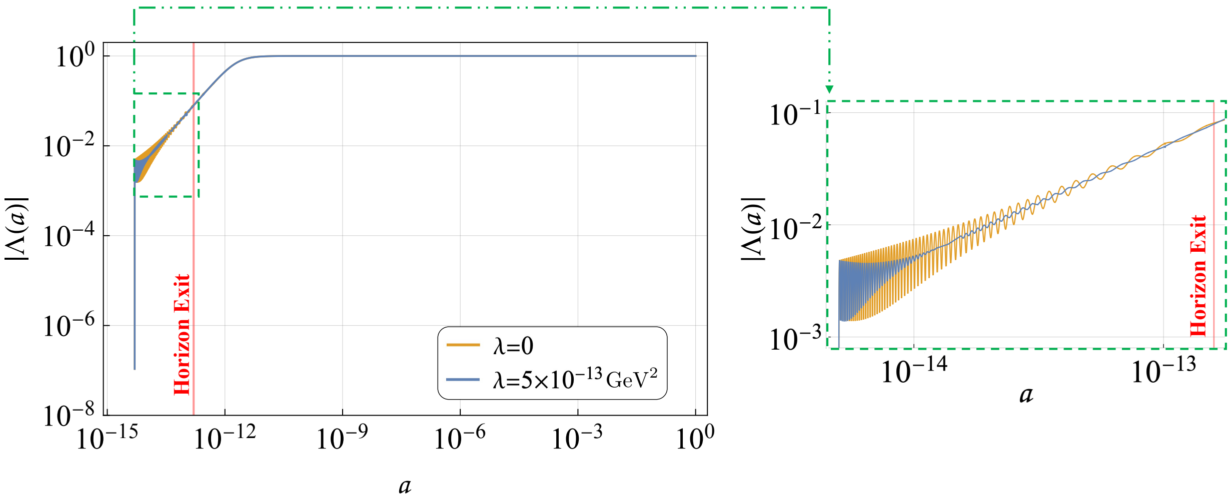

FIG 7. Squeezing parameter of graviton pair Λ during inflation as a function of the scale factor $a(\eta)$. We set $\lambda = 5 \times 10^{-13}\text{GeV}^2$(blue line) and $\lambda = 0$ GeV^2(yellow line). Other parameters are set as $k = 10^2$GeV, $H = 10^{14}$GeV, $\eta_i = -2\text{GeV}^{-1}$, $\eta_f = -10^{-14}\text{GeV}^{-1}$, $a(\eta_i) = (2 \times 10^{14})^{-1}$, and $a(\eta_f) = 1$. The red grid line shows the scale factor $a = 1.59... \times 10^{-13}$ at the time of horizon exit $\eta = -2\pi/k$.

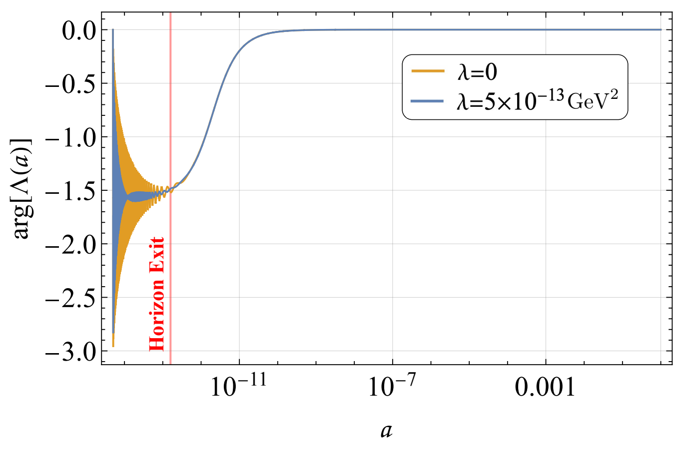

FIG 8. The phase of the squeezing parameter of graviton pair $\Lambda(a)$ during inflation as function of the scale factor $a(\eta)$. We set $\lambda = 5 \times 10^{-13}\text{GeV}^2$(blue line) and $\lambda = 0$ GeV^2(yellow line). Other parameters are set as $k = 10^2$GeV, $H = 10^{14}$GeV, $\eta_i = -2\text{GeV}^{-1}$, $\eta_f = -10^{-14}\text{GeV}^{-1}$, $a(\eta_i) = (2 \times 10^{14})^{-1}$, and $a(\eta_f) = 1$.

Now, we investigate the behavior of those squeezing parameters for $k\eta \ll 1$ and $k\eta_i \gg 1$

analytically. The leading and sub-leading terms of Λ and Ξ can be calculated as

$$\Lambda = 1 + \mathcal{O}\left(\frac{\lambda^2 H^2 \eta_i^2}{k^4}\right) , \qquad \Xi = 0 + \mathcal{O}\left(\frac{\lambda H \eta}{k^2}\right) . \tag{4.41}$$

We find that sub-leading terms of Λ and Ξ are negligibly small near the end of inflation and which is consistent with the numerical results in FIGs. 7 and 9. This result tells us that the conversion from graviton pair production to graviton-photon pair production is hard to occur. For the squeezing parameter Ω, we find

$$\Omega = i e^{2ik\eta_i} \frac{5\lambda^2 H^2 \eta_i^3}{32k^3} . \tag{4.42}$$

If we use the numerical values $\lambda = 5 \times 10^{-13}\,\mathrm{GeV}^2$, $k = 10^2\,\mathrm{GeV}$, $H = 10^{14}\,\mathrm{GeV}$, and $\eta_i = -2\,\mathrm{GeV}^{-1}$, we find $|\Omega| \sim 0.003$ and which agrees with the numerical result in FIG. 11. Thus only a small amount of conversion from graviton pair production to photon pair production occurs at the end of inflation. These results support the validity of our iterative method to derive squeezing parameters.

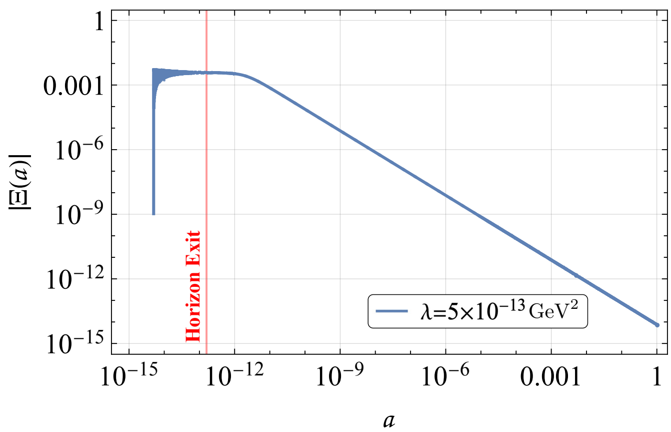

FIG 9. Squeezing parameter of graviton-photon pair Ξ during inflation as a function of the scale factor a. We set $\lambda = 5\times10^{-13}\mathrm{GeV}^2$ (blue line). Other parameters are set as $k = 10^2\mathrm{GeV}$, $H = 10^{14}\mathrm{GeV}$, $\eta_i = -2\mathrm{GeV}^{-1}$, $\eta_f = -10^{-14}\mathrm{GeV}^{-1}$, $a(\eta_i) = (2\times10^{14})^{-1}$, and $a(\eta_f) = 1$.

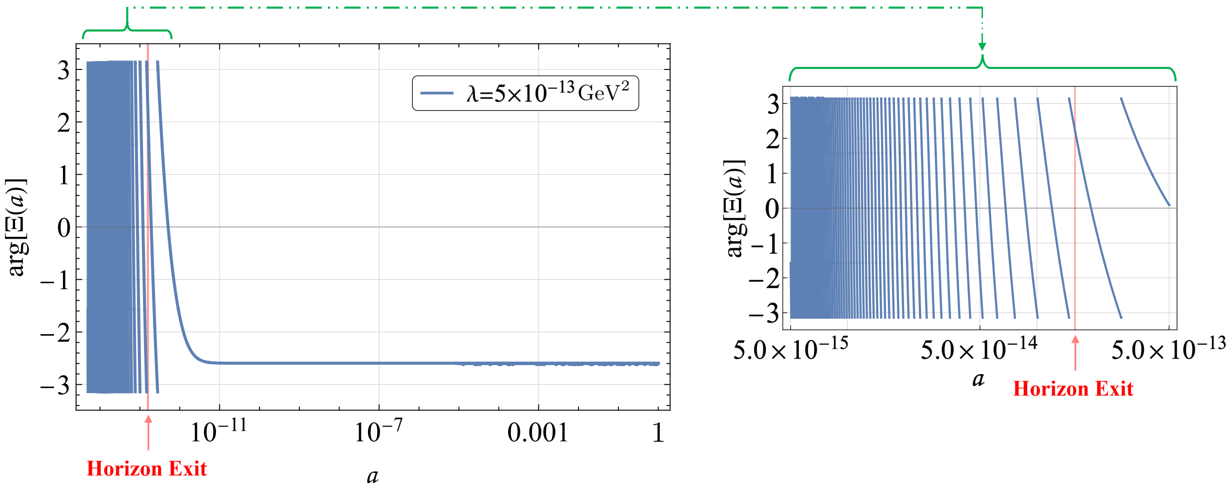

FIG 10. The phase of the squeezing parameter of photon pair $\Xi(a)$ during inflation as a function of the scale factor $a(\eta)$. We set $\lambda = 5\times10^{-13}\text{GeV}^2$, $k = 10^2\text{GeV}$, $H = 10^{14}\text{GeV}$, $\eta_i = -2\text{GeV}^{-1}$, $\eta_f = -10^{-14}\text{GeV}^{-1}$, $a(\eta_i) = (2\times10^{14})^{-1}$, and $a(\eta_f) = 1$.

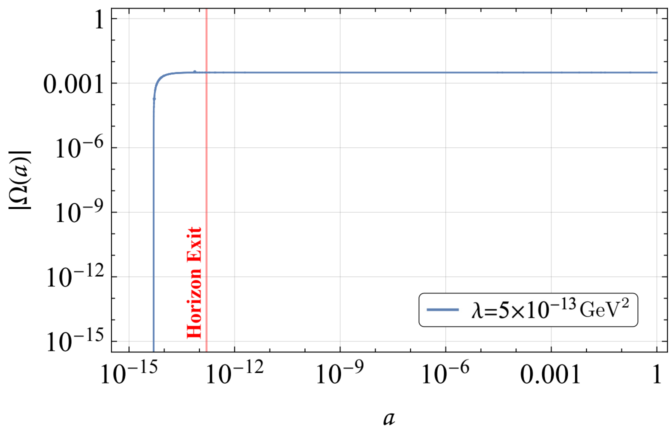

FIG 11. Squeezing parameter of photon pair Ω during inflation as a function of the scale factor $a(\eta)$. We set $\lambda = 5\times10^{-13}\text{GeV}^2$, $k = 10^2\text{GeV}$, $H = 10^{14}\text{GeV}$, $\eta_i = -2\text{GeV}^{-1}$, $\eta_f = -10^{-14}\text{GeV}^{-1}$, $a(\eta_i) = (2\times10^{14})^{-1}$, and $a(\eta_f) = 1$.

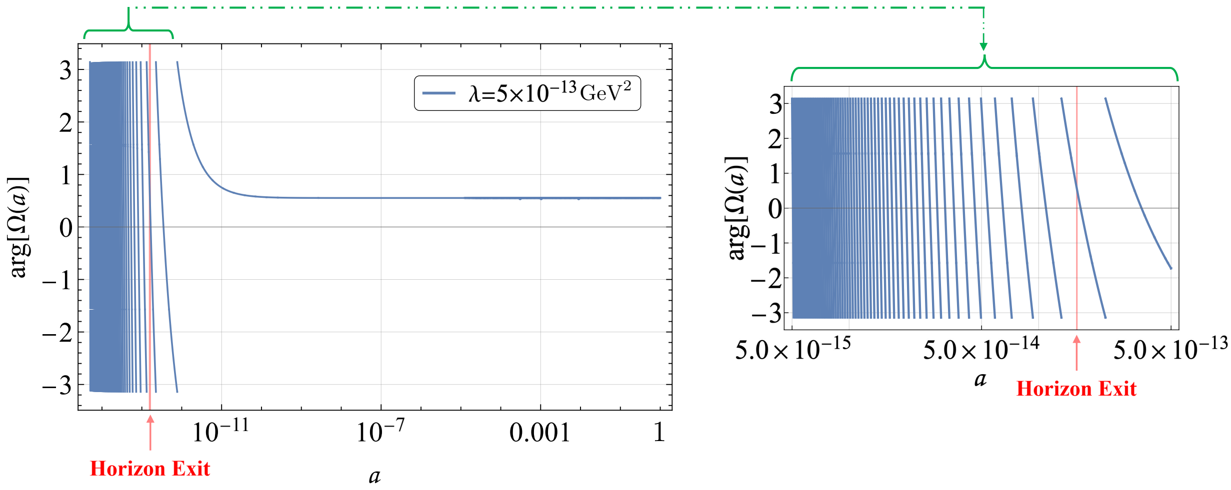

FIG 12. The phase of the squeezing parameter of photon pair Ω during inflation as a function of the scale factor $a(\eta)$. We set $\lambda = 5 \times 10^{-13}\text{GeV}^2$, $k = 10^2\text{GeV}$, $H = 10^{14}\text{GeV}$, $\eta_i = -2\text{GeV}^{-1}$, $\eta_f = -10^{-14}\text{GeV}^{-1}$, $a(\eta_i) = (2 \times 10^{14})^{-1}$, and $a(\eta_f) = 1$.

Let us discuss the implications of our numerical and analytical results. If the squeezing of graviton decreases as time evolves, it implies that the decoherence of graviton occurs. However, we found that the squeezing parameter of the graviton pair increases and becomes $\Lambda \to 1$, so it seems that the decoherence is hard to occur. This behavior can be understood as follows. Since the effective coupling $\lambda H \eta$ between graviton and photon in Eqs. (2.17) and (2.18) decreases and eventually becomes negligible as $\eta \to 0$ during inflation, practically graviton-photon conversion stops. Even after the graviton-photon conversion stops, the squeezing process of the graviton pair continues as time evolves during inflation, so the squeezing of graviton pair Λ continues to grow irrespective of the presence of the magnetic field as shown in FIG. 7. Next, from FIG. 9, we see the squeezing parameter of the graviton-photon pair vanishes $\Xi \to 0$ as time evolves. This is consistent with Eq. (4.41). This is because graviton-photon pair production is possible only in the presence of magnetic fields due to spin conservation. In our setup, however, the energy density of the background magnetic field decreases proportional to $a(\eta)^{-4}$ as the universe expands. Hence, the rapid decay of magnetic fields leads to the rapid decay of Ξ. Finally, we consider the evolution of the squeezing parameter of photon pair Ω. By using the coupling constant $\lambda \simeq Bk/M_{\text{pl}}$ in

Eq. (2.15) and the scale factor at the initial time $a_i \equiv -1/(H\eta_i)$, the Ω reads

$$\Omega \simeq \frac{B^2}{a_i^4 M_{\rm pl}^2 H^2} \frac{1}{k\eta_i} \ . \tag{4.43}$$

The first factor is the ratio of the energy density of the background magnetic field at the time η_i to that of the inflaton field. The second factor is the ratio of the mode of graviton to the Hubble radius. In order to have inflation, the energy density of the magnetic field has to be smaller than that of inflaton fields, that is, $B^2/a_i^4 \ll M_{\rm pl}^2 H^2$. And all modes of the graviton are inside the horizon initially, that is, $1/k \ll \eta_i$. Hence, the Ω never exceeds unity after time evolution, which is consistent with FIG. 11. Moreover, since graviton-photon conversion stops, the squeezing of photon pair Ω converges to a constant value as shown in FIG. 11.

4.4 Schmidt decomposition

In the previous section, we found that the squeezing of the graviton-photon pair is produced but eventually disappears during inflation. In this subsection, we reveal the entanglement between the graviton-photon pairs. We compute the entanglement entropy between gravitons and photons by tracing over the photons using the method developed in [49, 50].

The initial state is expressed in Eq. (3.46) and it is difficult to trace over the photon degree of freedom. Thus, we perform the following Bogoliubov transformation

$$\hat{C}_{y,\boldsymbol{k}} = \Phi\, \hat{a}_{y,\boldsymbol{k}} + \Psi\, \hat{a}^\dagger_{y,-\boldsymbol{k}} \ , \qquad \hat{C}_{A,\boldsymbol{k}} = \Upsilon\, \hat{a}_{A,\boldsymbol{k}} + \Theta\, \hat{a}^\dagger_{A,-\boldsymbol{k}} \ , \tag{4.44}$$

where $|\Phi|^2 - |\Psi|^2 = 1$, $|\Upsilon|^2 - |\Theta|^2 = 1$ so that the state $|\overline{\rm BD}\rangle$ becomes in the Schmidt form

$$|\overline{\rm BD}\rangle = \prod_{\boldsymbol{k}=-\infty}^{\infty} \exp\big[\, \rho\, \hat{C}^\dagger_{y,\boldsymbol{k}}\, \hat{C}^\dagger_{A,-\boldsymbol{k}} \big] |0'\rangle_{y,\boldsymbol{k}} |0'\rangle_{A,-\boldsymbol{k}} \, . \tag{4.45}$$

Note that we consider different Bogoliubov coefficients between $(\Phi\,, \Psi)$ and $(\Upsilon\,, \Theta)$ because the Λ and Ξ in Eq. (3.46) are complex parameters. Here new vacuum states are defined by

$$\hat{C}_{y,\boldsymbol{k}}\, |0'\rangle_{y,\boldsymbol{k}} = 0 \ , \quad \hat{C}_{A,\boldsymbol{k}}\, |0'\rangle_{A,\boldsymbol{k}} = 0 \ . \tag{4.46}$$

Performing the new operators $\hat{C}_{y,\boldsymbol{k}}$ and $\hat{C}_{x,\boldsymbol{k}}$ on Eq. (4.2), we obtain the following relations,

$$\hat{C}_{y,\boldsymbol{k}}\, |\overline{\mathrm{BD}}\rangle = \rho\, \hat{C}^{\dagger}_{A,-\boldsymbol{k}}\, |\overline{\mathrm{BD}}\rangle, \tag{4.47}$$

$$\hat{C}_{A,\boldsymbol{k}}\, |\overline{\mathrm{BD}}\rangle = \rho\, \hat{C}^{\dagger}_{y,-\boldsymbol{k}}\, |\overline{\mathrm{BD}}\rangle\ . \tag{4.48}$$

By using Eq. (4.1), the above relations lead to the equations for the Bogoliubov coefficients as

$$\begin{pmatrix} \Lambda & 1 & 0 & -\rho\Xi \\ \Xi & 0 & -\rho & -\rho\Omega \\ -\rho^* & -\rho^*\Lambda^* & \Xi^* & 0 \\ 0 & -\rho^*\Xi^* & \Omega^* & 1 \end{pmatrix} \begin{pmatrix} \Phi \\ \Psi \\ \Upsilon^* \\ \Theta^* \end{pmatrix} = 0. \tag{4.49}$$

In order to find a nontrivial solution, the determinant of the above 4 by 4 matrix has to be zero. That is, $|\rho|^2$ satisfies

$$|\rho|^2 = Q - \sqrt{Q^2 - 1}\,, \tag{4.50}$$

where we have defined

$$Q = \frac{\left(|\Lambda|^2 - 1\right)\left(|\Omega|^2 - 1\right) + |\Xi|^4 - 2\mathrm{Re}(\Xi^2\Lambda^*\Omega^*)}{2|\Xi|^2}\ . \tag{4.51}$$

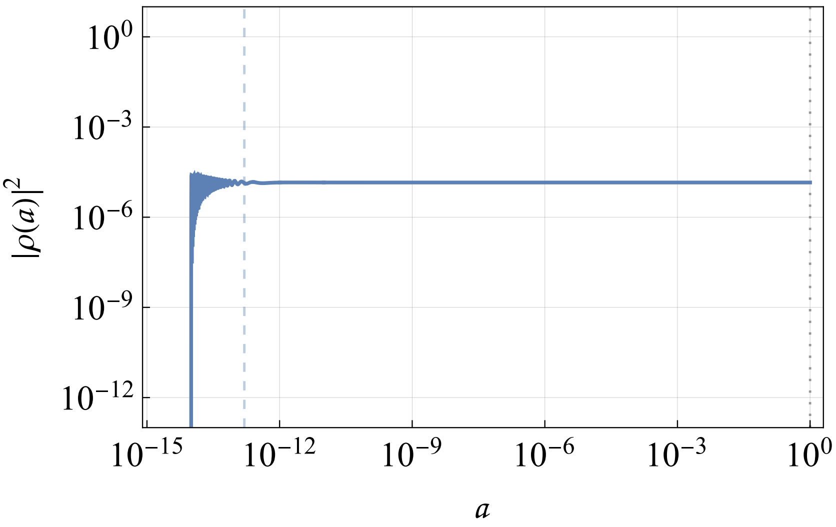

FIG 13. Plots of the parameter $|\rho(a)|^2$ as a function of $a(\eta)$.

In FIG. 13, we plotted $|\rho|^2$ versus $a(\eta)$ for the value of $k = 10^2\text{GeV}^2$ under a fixed value of $|\boldsymbol{B}|$. Here, λ is automatically determined once k is fixed because of Eq. (3.51) where we take $\theta = \pi/2$. We see that $|\rho|^2$ reduces to the small value after the horizon exit. Hence, the squeezing of graviton-photon pair in the basis $|0'\rangle_{y,\boldsymbol{k}}|0'\rangle_{A,-\boldsymbol{k}}$ turns out to be not a maximum entangled state. This can be understood as the result of the Ω reduced to a small value. Since the particle creation of the photon is stopped after the horizon exit, the number of gravitons get larger than the number of photons. Thus, the number of photons which entangle with graviton run out after the horizon exit. This is a specific characteristic result of Model-1. We have a different result with Model-2, as is shown in Part III.

4.5 Entanglement entropy

Since gravitons and photons are coupled to each other through λ as in Eqs. (2.17) and (2.18), they are expected to get entangled eventually. In the previous subsection, we find the squeezing of the graviton-photon pair don't becomes maximum in the basis of $|0'\rangle_{y,\boldsymbol{k}}|0'\rangle_{A,-\boldsymbol{k}}$. In order to clarify whether they get entangled or not, we compute the entanglement entropy as a measure of entanglement, since the entanglement entropy is basis independent.

We define the density operator of the vacuum $|\overline{\mathrm{BD}}\rangle$ in Eq. (4.2) by

$$\begin{aligned}\sigma &= |\overline{\mathrm{BD}}\rangle\langle\overline{\mathrm{BD}}| \\ &= \left(1-|\rho|^2\right)\prod_{\boldsymbol{k},-\boldsymbol{k}}\sum_{n',m'=0}^{\infty}\rho^{n'}\rho^{*m'}\,|n'\rangle_{y,\boldsymbol{k}}\,|n'\rangle_{A,-\boldsymbol{k}}\ {}_{y,\boldsymbol{k}}\langle m'|\ {}_{A,-\boldsymbol{k}}\langle m'|\ . \end{aligned} \tag{4.52}$$

The reduced density operator for the gravitons is obtained by tracing over the degree of freedom of photons such as

$$\begin{aligned}\sigma_y &= \mathrm{Tr}_A\,|\overline{\mathrm{BD}}\rangle\langle\overline{\mathrm{BD}}| = \sum_i {}_{A,\boldsymbol{k}'}\langle\, i\,|\overline{\mathrm{BD}}\rangle\langle\overline{\mathrm{BD}}|\, i\,\rangle_{A,\boldsymbol{k}'} \\ &= \left(1-|\rho|^2\right)\sum_{n'=0}^{\infty}|\rho|^{2n}\,|n'\,\rangle_{y,\boldsymbol{k}}\ {}_{y,\boldsymbol{k}}\langle n'\,|\ . \end{aligned} \tag{4.53}$$

The entanglement entropy between the graviton and photon can be characterized by

$$\begin{aligned}S &= -\mathrm{Tr}_y\,\sigma_y\log\sigma_y = -\sum_{n'=0}^{\infty}\left(1-|\rho|^2\right)|\rho|^{2n'}\left(\log\left(1-|\rho|^2\right)+n'\log|\rho|^2\right) \\ &= -\log\left(1-|\rho|^2\right)-\frac{|\rho|^2}{1-|\rho|^2}\log|\rho|^2\ . \end{aligned} \tag{4.54}$$

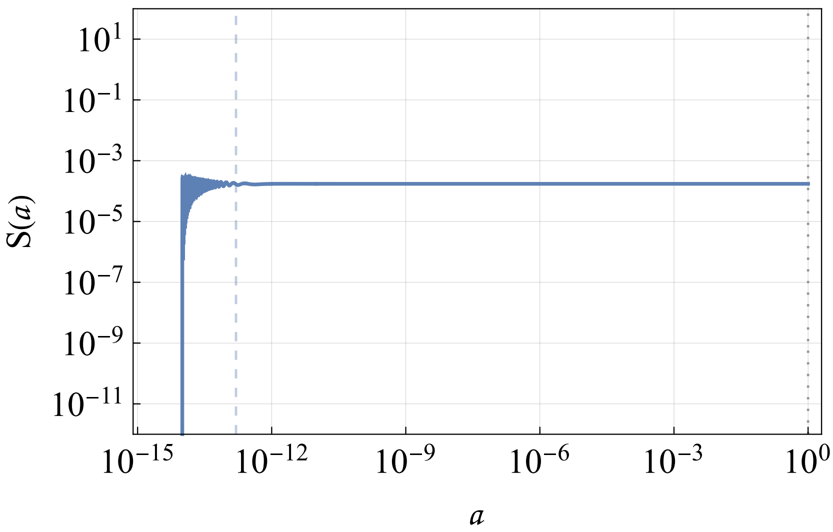

FIG 14. Entanglement entropy between graviton and photon as a function of $a(\eta)$.

In FIG. 14, we plotted the entanglement entropy for the value of $k = 10^2\mathrm{GeV}^2$ under a fixed value of $|\boldsymbol{B}|$, which clearly shows that the graviton and photon are poorly entangled during inflation.

Part III

Background magnetic field Model-2

In the previous part, we assumed the presence of primordial magnetic fields at the beginning of inflation and examined the evolution of the squeezing parameters of gravitons and photons in the process of graviton to photon conversion mediated by the background magnetic field [51]. There, it turned out that the squeezing of gravitons was robust against the conversion process. It is because the background magnetic field rapidly decays due to inflation. Then we concluded that gravitons keep their squeezed states even in the presence of the background magnetic fields. However, if magnetic fields decay slowly during inflation, gravitons may lose their squeezed states. In this case, graviton to photon conversion never ends as long as magnetic fields survive during inflation. Hence, in this paper, we study the conversion process of gravitons in the presence of magnetic fields that decays slowly during inflation and see if the gravitons can keep their squeezed states until today. Remarkably, we find that the magnetic fields generate the maximal entanglement between gravitons and photons. As a consequence, the quantum state of gravitons becomes a mixed state instead of the squeezed (pure) state. Namely, the quantum entanglement between gravitons and photons partially destroys the squeezed state of gravitons.

The organization of this part is as follows. In section 1, we introduce a model describing the situation where magnetic fields are persistently generated. Then, we review the graviton-photon conversion during inflation. In section 2, we solve the dynamics and calculate Bogoliubov coefficients describing the time evolution of the quantum state. We obtain a four-mode squeezed state as a consequence of graviton to photon conversion. In section 3, we calculate entanglement entropy between gravitons and photons. We discuss the quantum state at present in the presence of the entanglement. In particular, we reveal the effects of quantum entanglement on the power spectrum of PGWs.

1 Graviton-photon conversion

We begin with the Einstein-Hilbert action and the action for a $U(1)$ gauge field coupled to a scalar field:

$$S = S_g + S_\phi + S_A = \int d^4x\sqrt{-g}\left[\frac{M_{\rm pl}^2}{2}R - \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - V(\phi) - \frac{1}{4}f^2(\phi)F^{\mu\nu}F_{\mu\nu}\right], \quad (1.1)$$

where $M_{\rm pl} = 1/\sqrt{8\pi G}$ is the Planck mass. The gauge field A_μ represents photons and the field strength is defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

The background inflationary dynamics are determined by the metric

$$ds^2 = a^2(\eta)\left[-d\eta^2 + \delta_{ij}dx^i dx^j\right], \quad (1.2)$$

and the inflaton $\phi(\eta)$. Once the background is given, the coupling function can be regarded as a function of the conformal time η; $f = f(\eta)$. We also assume the presence of constant magnetic fields $B_i = \text{constant}$. It should be emphasized that the physical magnetic fields are not B_i but fB_i. In the next section, we consider the quantum evolution of gravitons and photons in the above background.

1.1 Primordial gravitational waves

We consider gravitons in a spatially flat expanding background represented by tensor mode perturbations in the three-dimensional metric h_{ij},

$$ds^2 = a^2(\eta)\left[-d\eta^2 + (\delta_{ij} + h_{ij})\,dx^i dx^j\right], \quad (1.3)$$

where h_{ij} satisfies the transverse traceless conditions $h_{ij}{}^{,j} = h^i{}_i = 0$. The spatial indices $i, j, k, \cdots$ are raised and lowered by δ^{ij} and $\delta_{k\ell}$. In the case of de Sitter space, the scale factor is given by $a(\eta) = -1/(H\eta)$ where $-\infty < \eta < 0$.

Expanding the Einstein-Hilbert action up to the second order in perturbations h_{ij}, we have

$$\delta S_g = \frac{M_{\rm pl}^2}{8} \int d^4x\, a^2 \left[h^{ij\prime}\, h_{ij}^{\prime} - h^{ij,k} h_{ij,k} \right] \,, \tag{1.4}$$

where a prime denotes the derivative with respect to the conformal time. At this quadratic order of the action, it is convenient to expand $h_{ij}(\eta, x^i)$ in Fourier modes,

$$h_{ij}(\eta, x^i) = \frac{2}{M_{\rm pl}} \sum_P \frac{1}{(2\pi)^{3/2}} \int d^3k\, h_{\boldsymbol{k}}^P(\eta)\, e_{ij}^P(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \,, \tag{1.5}$$

where three-vectors are denoted by bold math type and $e_{ij}^P(\boldsymbol{k})$ are the polarization tensors for the $\boldsymbol{k}$ mode normalized as $e^{ijP}(\boldsymbol{k}) e_{ij}^Q(\boldsymbol{k}) = \delta^{PQ}$ with $P, Q = +, \times$. Then the action (2.4) in the Fourier modes becomes

$$\delta S_g = \frac{1}{2} \sum_P \int d^3k\, d\eta\, a^2 \left[|h_{\boldsymbol{k}}^{P\prime}|^2 - k^2 |h_{\boldsymbol{k}}^P|^2 \right] \,. \tag{1.6}$$

1.2 Primordial magnetic fields

Next, we consider the action for the photon up to the second order in perturbations A_i, which is given by

$$\delta S_A = \frac{1}{2} \int d^4x\ f^2 \left[A_i^{\prime 2} - A_{k,i}^2 \right] \,, \tag{1.7}$$

where the photon field satisfies the Coulomb gauge $A_0 = 0$ and $A^i{}_{,i} = 0$.

If we expand the $A_i(\eta, x^i)$ in the Fourier modes, we find

$$A_i(\eta, x^i) = \sum_P \frac{\pm i}{(2\pi)^{3/2}} \int d^3k\, A_{\boldsymbol{k}}^P(\eta)\, e_i^P(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \,, \tag{1.8}$$

where $e_i^P(\boldsymbol{k})$ are the polarization vectors for the $\boldsymbol{k}$ mode normalized as $e^{iP}(\boldsymbol{k}) e_i^Q(\boldsymbol{k}) = \delta^{PQ}$ with $P, Q = +, \times$. The sign of $\pm i$ corresponds to the $P, Q = +, \times$. The action (2.7) in the

Fourier modes is

$$\delta S_A = \frac{1}{2}\sum_P \int d^3k\, d\eta\, f^2 \left[\, |A_{\boldsymbol{k}}^{P\prime}|^2 - k^2 |A_{\boldsymbol{k}}^P|^2 \,\right] . \tag{1.9}$$

1.3 Graviton-photon interaction

The action for the interaction between the graviton and the photon up to second order in perturbations h_{ij}, A^i is found to be

$$\delta S_{\rm I} = \int d^4x \left[\varepsilon_{i\ell m} f^2 B_m h^{ij} \left(\partial_j A_\ell - \partial_\ell A_j\right)\right] . \tag{1.10}$$

Note that we assumed the existence of the background magnetic field $B_m = \varepsilon_{mj\ell}\, \partial_j A_\ell$ at the beginning of inflation.

In the Fourier mode defined in Eqs. (2.5) and (2.8),

$$\delta S_I = \frac{2}{M_{\rm pl}} \sum_{P,Q} \int d^3k\, d\eta\, f^2 \left[\varepsilon_{i\ell m}\, B_m\, h_{\boldsymbol{k}}^P A_{-\boldsymbol{k}}^Q\, e_{ij}^P(\boldsymbol{k}) \left\{ ik_\ell\, e_j^Q(-\boldsymbol{k}) - ik_j\, e_\ell^Q(-\boldsymbol{k}) \right\}\right] , \tag{1.11}$$

where $k = |\boldsymbol{k}|$. Polarization vectors $e^{i+}, e^{i\times}$ and a vector k^i/k constitute an orthonormal basis. Without loss of generality, we assume the constant background magnetic field is in the $(k^i, e^{i\times})$-plane as depicted in FIG. 15. The polarization tensors can be written in terms of polarization vectors e^{i+} and $e^{i\times}$ as

$$e_{ij}^+(\boldsymbol{k}) = \frac{1}{\sqrt{2}} \left\{ e_i^+(\boldsymbol{k}) e_j^+(\boldsymbol{k}) - e_i^\times(\boldsymbol{k}) e_j^\times(\boldsymbol{k}) \right\} , \tag{1.12}$$

$$e_{ij}^\times(\boldsymbol{k}) = \frac{1}{\sqrt{2}} \left\{ e_i^+(\boldsymbol{k}) e_j^\times(\boldsymbol{k}) + e_i^\times(\boldsymbol{k}) e_j^+(\boldsymbol{k}) \right\} . \tag{1.13}$$

Below, we assume $e_i^\times(-\boldsymbol{k}) = -e_i^\times(\boldsymbol{k})$. The action (2.11) is then written as

$$\delta S_I \;=\; \int d^3k\, d\eta\, f^2\, \lambda(\boldsymbol{k}) \left[\, h_{\boldsymbol{k}}^+(\eta)\, A_{-\boldsymbol{k}}^+(\eta) + \, h_{\boldsymbol{k}}^\times(\eta)\, A_{-\boldsymbol{k}}^\times(\eta) \,\right] , \tag{1.14}$$

where we defined the coupling between graviton and photon as

$$\lambda(\boldsymbol{k}) \equiv \frac{\sqrt{2}}{M_{\rm pl}} \varepsilon^{i\ell m}\, e_i^{+}\, k_\ell\, B_m\,. \tag{1.15}$$

Here, the conditions for the graviton and photon to be real read, $h_{-\boldsymbol{k}}^{+,\times}(\eta) = h_{\boldsymbol{k}}^{*+,\times}(\eta)$ and $A_{-\boldsymbol{k}}^{+,\times}(\eta) = -A_{\boldsymbol{k}}^{*+,\times}(\eta)$. In the following, we focus on the plus polarization and omit the index P unless there may be any confusion.

1.4 Total action in canonical variables

If we use the canonical variable $y_{\boldsymbol{k}}^P(\eta) = a\, h_{\boldsymbol{k}}^P(\eta)$ and $x_{\boldsymbol{k}}^P(\eta) = f\, A_{\boldsymbol{k}}^P(\eta)$, the total action of Eqs. (2.6), (2.9) and (2.14) are written as

$$\begin{aligned}
\delta S &= \delta S_y + \delta S_x + \delta S_I \\
&= \frac{1}{2}\int d^3k\, d\eta \left[|y_{\boldsymbol{k}}'|^2 - \left(k^2 - \left(\frac{a'}{a}\right)^2\right)|y_{\boldsymbol{k}}|^2 - \frac{a'}{a}\left(y_{\boldsymbol{k}}\, y_{-\boldsymbol{k}}' + y_{-\boldsymbol{k}}\, y_{\boldsymbol{k}}'\right)\right] \\
&\quad + \frac{1}{2}\int d^3k\, d\eta \left[|x_{\boldsymbol{k}}'|^2 - \left(k^2 - \left(\frac{f'}{f}\right)^2\right)|x_{\boldsymbol{k}}|^2 - \frac{f'}{f}\left(x_{\boldsymbol{k}}\, x_{-\boldsymbol{k}}' + x_{-\boldsymbol{k}}\, x_{\boldsymbol{k}}'\right)\right] \\
&\quad + \int d^3k\, d\eta \left[\frac{f}{a}\lambda(\boldsymbol{k})\, y_{\boldsymbol{k}}\, x_{-\boldsymbol{k}}\right]\,.
\end{aligned} \tag{1.16}$$

The variation of the actions (2.16) with respect to the graviton and the photon fields gives

$$y_{\boldsymbol{k}}'' + \left(k^2 - \frac{a''}{a}\right) y_{\boldsymbol{k}} = -\lambda f \frac{x_{\boldsymbol{k}}}{a(\eta)}, \tag{1.17}$$

$$x_{\boldsymbol{k}}'' + \left(k^2 - \frac{a''}{a}\right) x_{\boldsymbol{k}} = -\lambda f \frac{y_{\boldsymbol{k}}}{a(\eta)}. \tag{1.18}$$

In this paper, we suppose $B_m/M_{\rm pl} \ll 1$ so that the coupling between graviton and photon (2.15) is weak. Then we solve the Eqs. (2.17) and (2.18) iteratively up to the second order in $y_{\boldsymbol{k}}$ and $x_{\boldsymbol{k}}$ in the next section.

We assume the gauge kinetic function in the form

$$f(\eta) = a(\eta)^{-2c}, \tag{1.19}$$

where c is a constant parameter. We take $c = -1/2$ to make the analysis easier. For this parameter, the power spectrum of the electromagnetic fields A_μ is scale-invariant [52]. On the other hand, the spectrum of magnetic fields becomes $P(B) \propto k^2$. We have a choice of taking $c = -1$ in order to have a scale-invariant primordial magnetic field. However, we will see $c = -1/2$ is sufficient to get a significant modification of the quantum state of gravitons.

2 Graviton-photon conversion

We begin with the Einstein-Hilbert action and the action for a $U(1)$ gauge field coupled to a scalar field:

$$S = S_g + S_\phi + S_A = \int d^4x\sqrt{-g}\left[\frac{M_{\rm pl}^2}{2}R - \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - V(\phi) - \frac{1}{4}f^2(\phi)F^{\mu\nu}F_{\mu\nu}\right], \quad (2.1)$$

where $M_{\rm pl} = 1/\sqrt{8\pi G}$ is the Planck mass. The gauge field A_μ represents photons and the field strength is defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

The background inflationary dynamics is determined by the metric

$$ds^2 = a^2(\eta)\left[-d\eta^2 + \delta_{ij}dx^i dx^j\right], \quad (2.2)$$

and the inflaton $\phi(\eta)$. Once the background is given, the coupling function can be regarded as a function of the conformal time η; $f = f(\eta)$. We also assume the presence of constant magnetic fields $B_i =$ constant . It should be emphasized that the physical magnetic fields are not B_i but fB_i. In the next section, we consider the quantum evolution of gravitons and photons in the above background.

2.1 Primordial gravitational waves

We consider gravitons in a spatially flat expanding background represented by tensor mode perturbations in the three-dimensional metric h_{ij},

$$ds^2 = a^2(\eta)\left[-d\eta^2 + (\delta_{ij} + h_{ij})\,dx^i dx^j\right], \quad (2.3)$$

where h_{ij} satisfies the transverse traceless conditions $h_{ij}{}^{,j} = h^i{}_i = 0$. The spatial indices $i, j, k, \cdots$ are raised and lowered by δ^{ij} and $\delta_{k\ell}$. In the case of de Sitter space, the scale factor is given by $a(\eta) = -1/(H\eta)$ where $-\infty < \eta < 0$.

Expanding the Einstein-Hilbert action up to the second order in perturbations h_{ij}, we have

$$\delta S_g = \frac{M_{\rm pl}^2}{8} \int d^4x \, a^2 \left[h^{ij\prime} h^\prime_{ij} - h^{ij,k} h_{ij,k} \right] , \tag{2.4}$$

where a prime denotes the derivative with respect to the conformal time. At this quadratic order of the action, it is convenient to expand $h_{ij}(\eta, x^i)$ in Fourier modes,

$$h_{ij}(\eta, x^i) = \frac{2}{M_{\rm pl}} \sum_P \frac{1}{(2\pi)^{3/2}} \int d^3k \, h^P_{\boldsymbol{k}}(\eta) \, e^P_{ij}(\boldsymbol{k}) \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} , \tag{2.5}$$

where three-vectors are denoted by bold math type and $e^P_{ij}(\boldsymbol{k})$ are the polarization tensors for the $\boldsymbol{k}$ mode normalized as $e^{ijP}(\boldsymbol{k}) e^Q_{ij}(\boldsymbol{k}) = \delta^{PQ}$ with $P, Q = +, \times$. Then the action (2.4) in the Fourier modes becomes

$$\delta S_g = \frac{1}{2} \sum_P \int d^3k \, d\eta \, a^2 \left[\, |h^{P\prime}_{\boldsymbol{k}}|^2 - k^2 |h^P_{\boldsymbol{k}}|^2 \, \right] . \tag{2.6}$$

2.2 Primordial magnetic fields

Next, we consider the action for the photon up to the second order in perturbations A_i, which is given by

$$\delta S_A = \frac{1}{2} \int d^4x \; f^2 \left[A_i^{\prime 2} - A_{k,i}^2 \right] , \tag{2.7}$$

where the photon field satisfies the Coulomb gauge $A_0 = 0$ and $A^i{}_{,i} = 0$.

If we expand the $A_i(\eta, x^i)$ in the Fourier modes, we find

$$A_i(\eta, x^i) = \sum_P \frac{\pm i}{(2\pi)^{3/2}} \int d^3k \, A^P_{\boldsymbol{k}}(\eta) \, e^P_i(\boldsymbol{k}) \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} , \tag{2.8}$$

where $e^P_i(\boldsymbol{k})$ are the polarization vectors for the $\boldsymbol{k}$ mode normalized as $e^{iP}(\boldsymbol{k}) e^Q_i(\boldsymbol{k}) = \delta^{PQ}$

with $P, Q = +, \times$. The sign of $\pm i$ corresponds to the $P, Q = +, \times$. The action (2.7) in the Fourier modes is

$$\delta S_A = \frac{1}{2} \sum_P \int d^3k \, d\eta \, f^2 \left[|A_{\boldsymbol{k}}^{P\prime}|^2 - k^2 |A_{\boldsymbol{k}}^P|^2 \right] . \tag{2.9}$$

2.3 Graviton-photon interaction

The action for the interaction between the graviton and the photon up to second order in perturbations h_{ij}, A^i is found to be

$$\delta S_{\rm I} = \int d^4x \left[\varepsilon_{i\ell m} f^2 B_m h^{ij} \left(\partial_j A_\ell - \partial_\ell A_j \right) \right] . \tag{2.10}$$

Note that $B_m = \varepsilon_{mj\ell} \partial_j A_\ell$ is a constant background magnetic field that we assumed the presence at the beginning of inflation.

In the Fourier mode defined in Eqs. (2.5) and (2.8),

$$\delta S_I = \frac{2}{M_{\rm pl}} \sum_{P,Q} \int d^3k \, d\eta \, f^2 \left[\varepsilon_{i\ell m} \, B_m \, h_{\boldsymbol{k}}^P A_{-\boldsymbol{k}}^Q \, e_{ij}^P(\boldsymbol{k}) \left\{ ik_\ell \, e_j^Q(-\boldsymbol{k}) - ik_j \, e_\ell^Q(-\boldsymbol{k}) \right\} \right] , \tag{2.11}$$

where $k = |\boldsymbol{k}|$. Polarization vectors $e^{i+}, e^{i\times}$ and a vector k^i/k constitute an orthonormal basis. Without loss of generality, we assume the constant background magnetic field is in the $(k^i, e^{i\times})$-plane as depicted in FIG. 15.

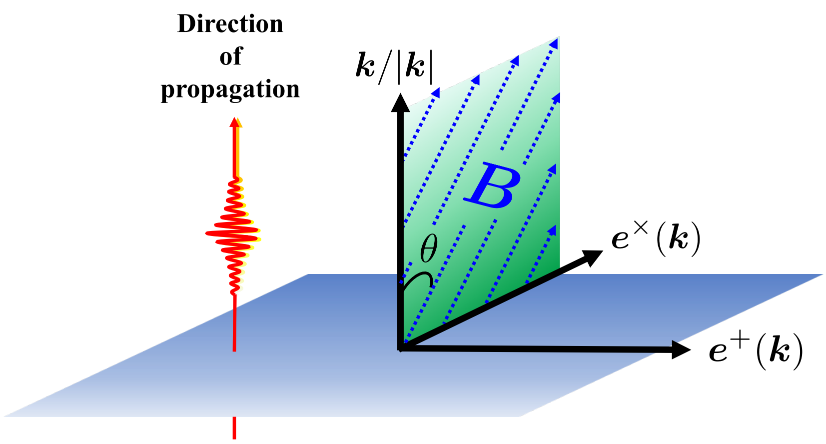

FIG 15. Configuration of the polarization vector $\boldsymbol{e}^P(\boldsymbol{k})$, wave number $\boldsymbol{k}$, and background magnetic field $\boldsymbol{B}$.

The polarization tensors can be written in terms of polarization vectors e^{i+} and $e^{i\times}$ as

$$e^{+}_{ij}(\boldsymbol{k}) = \frac{1}{\sqrt{2}}\Big\{e^{+}_{i}(\boldsymbol{k})e^{+}_{j}(\boldsymbol{k}) - e^{\times}_{i}(\boldsymbol{k})e^{\times}_{j}(\boldsymbol{k})\Big\}\,, \tag{2.12}$$

$$e^{\times}_{ij}(\boldsymbol{k}) = \frac{1}{\sqrt{2}}\Big\{e^{+}_{i}(\boldsymbol{k})e^{\times}_{j}(\boldsymbol{k}) + e^{\times}_{i}(\boldsymbol{k})e^{+}_{j}(\boldsymbol{k})\Big\}\,. \tag{2.13}$$

Below, we assume $e^{\times}_{i}(-\boldsymbol{k}) = -e^{\times}_{i}(\boldsymbol{k})$. The action (2.11) is then written as

$$\delta S_I \;=\; \int d^3k\, d\eta\, f^2\, \lambda(\boldsymbol{k})\, \big[\, h^{+}_{\boldsymbol{k}}(\eta)\, A^{+}_{-\boldsymbol{k}}(\eta) + \; h^{\times}_{\boldsymbol{k}}(\eta)\, A^{\times}_{-\boldsymbol{k}}(\eta)\big]\ , \tag{2.14}$$

where we defined the coupling between graviton and photon as

$$\lambda(\boldsymbol{k}) \equiv \frac{\sqrt{2}}{M_{\rm pl}}\varepsilon^{i\ell m}\, e^{+}_{i}\, k_\ell\, B_m\,. \tag{2.15}$$

Here, the conditions for the graviton and photon to be real read, $h^{+,\times}_{-\boldsymbol{k}}(\eta) = h^{*+,\times}_{\boldsymbol{k}}(\eta)$ and $A^{+,\times}_{-\boldsymbol{k}}(\eta) = -A^{*+,\times}_{\boldsymbol{k}}(\eta)$. In the following, we focus on the plus polarization and omit the index P unless there may be any confusion.

2.4 Total action in canonical variables

If we use the canonical variable $y^{P}_{\boldsymbol{k}}(\eta) = a\, h^{P}_{\boldsymbol{k}}(\eta)$ and $x^{P}_{\boldsymbol{k}}(\eta) = f\, A^{P}_{\boldsymbol{k}}(\eta)$, the total action of Eqs. (2.6), (2.9) and (2.14) are written as

$$\begin{aligned}\delta S \;&=\; \delta S_y + \delta S_x + \delta S_I \\ &=\; \frac{1}{2}\int d^3k\, d\eta \left[\, |y'_{\boldsymbol{k}}|^2 - \left(k^2 - \left(\frac{a'}{a}\right)^2\right)|y_{\boldsymbol{k}}|^2 - \frac{a'}{a}\left(y_{\boldsymbol{k}}\, y'_{-\boldsymbol{k}} + y_{-\boldsymbol{k}}\, y'_{\boldsymbol{k}}\right)\right] \\ &\quad + \frac{1}{2}\int d^3k\, d\eta \left[\, |x'_{\boldsymbol{k}}|^2 - \left(k^2 - \left(\frac{f'}{f}\right)^2\right)|x_{\boldsymbol{k}}|^2 - \frac{f'}{f}\left(x_{\boldsymbol{k}}\, x'_{-\boldsymbol{k}} + x_{-\boldsymbol{k}}\, x'_{\boldsymbol{k}}\right)\right] \\ &\quad + \int d^3k\, d\eta \left[\, \frac{f}{a}\, \lambda(\boldsymbol{k})\, y_{\boldsymbol{k}}\, x_{-\boldsymbol{k}}\right]\,. \end{aligned} \tag{2.16}$$

The variation of the actions (2.16) with respect to the graviton and the photon fields gives

$$y''_{\boldsymbol{k}} + \left(k^2 - \frac{a''}{a}\right) y_{\boldsymbol{k}} = -\lambda f \frac{x_{\boldsymbol{k}}}{a}, \tag{2.17}$$
$$x''_{\boldsymbol{k}} + \left(k^2 - \frac{a''}{a}\right) x_{\boldsymbol{k}} = -\lambda f \frac{y_{\boldsymbol{k}}}{a}. \tag{2.18}$$

We assume the gauge kinetic function in the form

$$f(\eta) = a(\eta)^{-2c}, \tag{2.19}$$

where c is a constant parameter. We take $c = -1/2$ to make the analysis easier. For this parameter, the power spectrum of the electromagnetic fields A_μ is scale-invariant [52]. We have a choice of taking $c = -1$ in order to have a scale-invariant primordial magnetic field. However, we will see $c = -1/2$ is sufficient to get a significant modification of the quantum state of gravitons.

3 Time evolution of quantum state

Using the basic equations presented in the previous section, we solve the time evolution of the quantum state in de Sitter space in this section. We assume the effect of the coupling λ instantaneously appears at η_i, namely, $\lambda(\eta) = \lambda\,\theta(\eta - \eta_i)$.

When we define the Lagrangian in the actions (2.16) by $\delta S = \int d\eta\, L$, the conjugate momenta of graviton $p_{\boldsymbol{k}}$ and photon $\pi_{\boldsymbol{k}}$ are respectively given by

$$p_{\boldsymbol{k}}(\eta) = \frac{\partial L}{\partial y'_{-\boldsymbol{k}}} = y'_{\boldsymbol{k}}(\eta) + \frac{1}{\eta} y_{\boldsymbol{k}}(\eta)\ , \tag{3.1}$$
$$\pi_{\boldsymbol{k}}(\eta) = \frac{\partial L}{\partial x'_{-\boldsymbol{k}}} = x'_{\boldsymbol{k}}(\eta) + \frac{1}{\eta} x_{\boldsymbol{k}}(\eta)\ . \tag{3.2}$$

Now we promote variables $y_{\boldsymbol{k}}(\eta)\,, x_{\boldsymbol{k}}(\eta)$ and their momenta $p_{\boldsymbol{k}}(\eta), \pi_{\boldsymbol{k}}(\eta)$ into operators. Annihilation operators for the graviton and photon are respectively expressed by canonical variables as

$$\hat{a}_y(\eta,\boldsymbol{k}) = \sqrt{\frac{k}{2}}\hat{y}_{\boldsymbol{k}}(\eta) + \frac{i}{\sqrt{2k}}\hat{p}_{\boldsymbol{k}}(\eta)\,, \tag{3.3}$$

$$\hat{a}_x(\eta,\boldsymbol{k}) = \sqrt{\frac{k}{2}}\hat{x}_{\boldsymbol{k}}(\eta) + \frac{i}{\sqrt{2k}}\hat{\pi}_{\boldsymbol{k}}(\eta)\,. \tag{3.4}$$

The commutation relations $[\hat{a}_y(\eta,\boldsymbol{k}),\hat{a}_y^\dagger(\eta,-\boldsymbol{k}')] = \delta(\boldsymbol{k}+\boldsymbol{k}')$ and $[\hat{a}_x(\eta,\boldsymbol{k}),\hat{a}_x^\dagger(\eta,-\boldsymbol{k}')] = \delta(\boldsymbol{k}+\boldsymbol{k}')$ guarantee the canonical commutation relations $[\hat{y}_{\boldsymbol{k}}(\eta),\hat{p}_{\boldsymbol{k}'}(\eta)] = i\delta(\boldsymbol{k}-\boldsymbol{k}')$ and $[\hat{x}_{\boldsymbol{k}}(\eta),\hat{\pi}_{\boldsymbol{k}'}(\eta)] = i\delta(\boldsymbol{k}-\boldsymbol{k}')$. Note that the annihilation operator becomes time-dependent through the time dependence of canonical variables. Thus, the vacuum defined by $\hat{a}(\eta,\boldsymbol{k})|0\rangle = 0$ is time dependent as well, and the vacuum in this formalism turns out to be defined at every moment. Our aim is to find the formula for Bogoliubov coefficients relating $\hat{a}_y(\eta,\boldsymbol{k}),\hat{a}_x(\eta,\boldsymbol{k})$ and $\hat{a}_y(\eta_i,\boldsymbol{k}),\hat{a}_x(\eta_i,\boldsymbol{k})$.

3.1 Boundary conditions

In this subsection, we specify boundary conditions of solutions of Eqs. (2.17) and (2.18). Notice that $\lambda = 0$ before the initial time η_i.

Let us first consider Eqs. (2.17) and (2.18) of the form

$$\hat{y}_{\boldsymbol{k}}^{(0)\prime\prime} + \left(k^2 - \frac{2}{\eta^2}\right)\hat{y}_{\boldsymbol{k}}^{(0)} = 0\,, \tag{3.5}$$

$$\hat{x}_{\boldsymbol{k}}^{(0)\prime\prime} + \left(k^2 - \frac{2}{\eta^2}\right)\hat{x}_{\boldsymbol{k}}^{(0)} = 0\,, \tag{3.6}$$

where the superscript (0) denotes $\lambda = 0$. Since Eqs. (3.5) and (3.6) are the same form, the mode function for the graviton and the photon at the zeroth order becomes identical. Then the solutions of the above equations can be written as

$$\hat{y}_{\boldsymbol{k}}^{(0)}(\eta) = u_{\boldsymbol{k}}(\eta)\;\hat{c} + u_{\boldsymbol{k}}^*(\eta)\;\hat{c}^\dagger\,, \tag{3.7}$$

$$\hat{x}_{\boldsymbol{k}}^{(0)}(\eta) = u_{\boldsymbol{k}}(\eta)\;\hat{d} + u_{\boldsymbol{k}}^*(\eta)\;\hat{d}^\dagger, \tag{3.8}$$

where $\hat{c}\,(\hat{d})$ and its conjugate $\hat{c}^\dagger(\hat{d}^\dagger)$ are constant operators of integration. We choose the properly normalized positive frequency mode in the remote past as a basis, which is expressed as

$$u_{\boldsymbol{k}}(\eta) = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta}. \tag{3.9}$$

Thus, annihilation operators at the initial time are expressed by the zeroth order variables

$$\hat{a}_y(\eta_i, \boldsymbol{k}) = \left(1 - \frac{i}{2k\eta_i}\right) e^{-ik\eta_i}\, \hat{c} + \frac{i}{2k\eta_i} e^{ik\eta_i}\, \hat{c}^\dagger, \tag{3.10}$$

$$\hat{a}_x(\eta_i, \boldsymbol{k}) = \left(1 - \frac{i}{2k\eta_i}\right) e^{-ik\eta_i}\, \hat{d} + \frac{i}{2k\eta_i} e^{ik\eta_i}\, \hat{d}^\dagger. \tag{3.11}$$

Combining Eqs. (3.10) and (3.11) with their complex conjugate, we can express the $\hat{c}$ and the $\hat{d}$ by the initial creation and annihilation operators as

$$\hat{c} = \left(1 + \frac{i}{2k\eta_i}\right) e^{ik\eta_i}\, \hat{a}_y(\eta_i, \boldsymbol{k}) - \frac{i}{2k\eta_i} e^{ik\eta_i}\, \hat{a}_y^\dagger(\eta_i, -\boldsymbol{k}) \ , \tag{3.12}$$

$$\hat{d} = \left(1 + \frac{i}{2k\eta_i}\right) e^{ik\eta_i}\, \hat{a}_x(\eta_i, \boldsymbol{k}) - \frac{i}{2k\eta_i} e^{ik\eta_i}\, \hat{a}_x^\dagger(\eta_i, -\boldsymbol{k}) \ . \tag{3.13}$$

Thus, we have obtained the boundary conditions. By solving Eqs. (2.17) and (2.18) with $f = a$ analytically, we will take into account the effect of interaction between gravitons and photons in the next subsection.

3.2 Bogoliubov coefficients

To properly take into account the boundary conditions at η_i, it is convenient to diagonalize equations of motion (2.17), (2.18) as

$$Y_{\boldsymbol{k}}'' + \left(k^2 - \frac{2}{\eta^2} + \lambda\right) Y_{\boldsymbol{k}} = 0, \tag{3.14}$$

$$X_{\boldsymbol{k}}'' + \left(k^2 - \frac{2}{\eta^2} - \lambda\right) X_{\boldsymbol{k}} = 0, \tag{3.15}$$

where we defined $Y_{\boldsymbol{k}} = (x_{\boldsymbol{k}} + y_{\boldsymbol{k}})/2$ and $X_{\boldsymbol{k}} = (x_{\boldsymbol{k}} - y_{\boldsymbol{k}})/2$. The solutions of Eqs.(3.14) and (3.15) are written as

$$\tilde{Y}_{\boldsymbol{k}}(\eta,\lambda) = c_1\, Y_{\boldsymbol{k}}(\eta,\lambda) + c_2\, Y^*_{\boldsymbol{k}}(\eta,\lambda) \ , \tag{3.16}$$

$$\tilde{X}_{\boldsymbol{k}}(\eta,\lambda) = c_1\, X_{\boldsymbol{k}}(\eta,\lambda) + c_2\, X^*_{\boldsymbol{k}}(\eta,\lambda) \ , \tag{3.17}$$

where c_1 and c_2 are constants of integration. We defined

$$Y_{\boldsymbol{k}}(\eta,\lambda) = \frac{1}{\sqrt{2\sqrt{k^2+\lambda}}}\left(1 - \frac{i}{\sqrt{k^2+\lambda}\,\eta}\right) e^{-i\sqrt{k^2+\lambda}\,\eta} \ , \tag{3.18}$$

$$X_{\boldsymbol{k}}(\eta,\lambda) = \frac{1}{\sqrt{2\sqrt{k^2-\lambda}}}\left(1 - \frac{i}{\sqrt{k^2-\lambda}\,\eta}\right) e^{-i\sqrt{k^2-\lambda}\,\eta} \ . \tag{3.19}$$

Note that we take the same constants of integration for $\tilde{Y}_{\boldsymbol{k}}$ and $\tilde{X}_{\boldsymbol{k}}$ so that $\tilde{Y}_{\boldsymbol{k}}(\eta,\lambda)$ and $\tilde{X}_{\boldsymbol{k}}(\eta,\lambda)$ are interchanged by switching λ to $-\lambda$. Then we can construct the odd and even solution with respect to λ as

$$F^{(\mathrm{odd})}_{\boldsymbol{k}}(\eta,\lambda) = \frac{1}{2}\left(\tilde{Y}_{\boldsymbol{k}}(\eta,\lambda) - \tilde{X}_{\boldsymbol{k}}(\eta,\lambda)\right) , \tag{3.20}$$

$$F^{(\mathrm{even})}_{\boldsymbol{k}}(\eta,\lambda) = \frac{1}{2}\left(\tilde{Y}_{\boldsymbol{k}}(\eta,\lambda) + \tilde{X}_{\boldsymbol{k}}(\eta,\lambda)\right) , \tag{3.21}$$

respectively. Here, we call Eq. (3.20) odd solution because a minus sign comes out by switching λ to $-\lambda$. The coefficients c_1 and c_2 are given by the junction condition at $\eta = \eta_i$.

Let us solve the dynamics with the boundary conditions

$$\hat{y}_{\boldsymbol{k}}(\eta,\lambda)|_{\eta<\eta_i} = u_{\boldsymbol{k}}(\eta) \ , \tag{3.22}$$

$$\hat{x}_{\boldsymbol{k}}(\eta,\lambda)|_{\eta<\eta_i} = u_{\boldsymbol{k}}(\eta) \ . \tag{3.23}$$

where we assumed that the positive frequency mode of de Sitter space is realized before the time η_i. After the time η_i, the gravitons and photons start to interact with each other. Taking into account the relations $Y_{\boldsymbol{k}} = (x_{\boldsymbol{k}} + y_{\boldsymbol{k}})/2$ and $X_{\boldsymbol{k}} = (x_{\boldsymbol{k}} - y_{\boldsymbol{k}})/2$, we can deduce the graviton field and its conjugate momentum as

$$\hat{y}_{\boldsymbol{k}}(\eta,\lambda) = F_{\boldsymbol{k}}^{(\text{even})}(\eta,\lambda)\,\hat{c} + F_{\boldsymbol{k}}^{(\text{odd})}(\eta,\lambda)\,\hat{d} + \text{H.c.}\,, \tag{3.24}$$

$$\begin{aligned}\hat{p}_{\boldsymbol{k}}(\eta,\lambda) &= F_{\boldsymbol{k}}^{(\text{even})\prime}(\eta,\lambda)\,\hat{c} + F_{\boldsymbol{k}}^{(\text{odd})\prime}(\eta,\lambda)\,\hat{d} \\ &\quad + \frac{1}{\eta}\Big(F_{\boldsymbol{k}}^{(\text{even})}(\eta,\lambda)\,\hat{c} + F_{\boldsymbol{k}}^{(\text{odd})}(\eta,\lambda)\,\hat{d}\Big) + \text{H.c.}\,,\end{aligned} \tag{3.25}$$

where we used Eq. (3.1) and H.c. represents Hermitian conjugate. We see that the operator of photon $\hat{d}$ comes into the graviton $\hat{y}_{\boldsymbol{k}}$ and $\hat{p}_{\boldsymbol{k}}$ together with $F_{\boldsymbol{k}}^{(\text{odd})}$. For the photon, the field and its conjugate momentum become

$$\hat{x}_{\boldsymbol{k}}(\eta) = F_{\boldsymbol{k}}^{(\text{even})}(\eta,\lambda)\,\hat{d} + F_{\boldsymbol{k}}^{(\text{odd})}(\eta,\lambda)\,\hat{c} + \text{H.c.}\,, \tag{3.26}$$

$$\begin{aligned}\hat{\pi}_{\boldsymbol{k}}(\eta) &= F_{\boldsymbol{k}}^{(\text{even})\prime}(\eta,\lambda)\,\hat{d} + F_{\boldsymbol{k}}^{(\text{odd})\prime}(\eta,\lambda)\,\hat{c} \\ &\quad + \frac{1}{\eta}\Big(F_{\boldsymbol{k}}^{(\text{even})}(\eta,\lambda)\,\hat{d} + F_{\boldsymbol{k}}^{(\text{odd})}(\eta,\lambda)\,\hat{c}\Big) + \text{H.c.}\,,\end{aligned} \tag{3.27}$$

where we used Eq. (3.2). We see that the operator of graviton $\hat{c}$ comes into the photon $\hat{x}_{\boldsymbol{k}}$ and $\hat{\pi}_{\boldsymbol{k}}$ together with $F_{\boldsymbol{k}}^{(\text{odd})}$. Then the annihilation operators for the graviton and photon are obtained by using Eqs. (3.3) and (3.4) such as

$$\hat{a}_y(\eta,\boldsymbol{k}) = \psi_p^{(\text{even})}\,\hat{c} + \psi_m^{(\text{even})*}\hat{c}^\dagger + \psi_p^{(\text{odd})}\hat{d} + \psi_m^{(\text{odd})*}\hat{d}^\dagger\,, \tag{3.28}$$

$$\hat{a}_x(\eta,\boldsymbol{k}) = \psi_p^{(\text{even})}\,\hat{d} + \psi_m^{(\text{even})*}\,\hat{d}^\dagger + \psi_p^{(\text{odd})}\hat{c} + \psi_m^{(\text{odd})*}\hat{c}^\dagger\,. \tag{3.29}$$

Here, we defined new variables

$$\psi_p^{(j)} = \sqrt{\frac{k}{2}}F_{\boldsymbol{k}}^{(j)}(\eta,\lambda) + \frac{i}{\sqrt{2k}}\Big(F_{\boldsymbol{k}}^{(j)\prime}(\eta,\lambda) + \frac{1}{\eta}F_{\boldsymbol{k}}^{(j)}(\eta,\lambda)\Big), \tag{3.30}$$

$$\psi_m^{(j)} = \sqrt{\frac{k}{2}}F_{\boldsymbol{k}}^{(j)}(\eta,\lambda) - \frac{i}{\sqrt{2k}}\Big(F_{\boldsymbol{k}}^{(j)\prime}(\eta,\lambda) + \frac{1}{\eta}F_{\boldsymbol{k}}^{(j)}(\eta,\lambda)\Big), \tag{3.31}$$

where $(j) = (\text{even}), (\text{odd})$ denotes the even mode and odd mode with respect to the coupling λ, respectively.

Inserting the above back into Eqs. (3.28) and (3.29), the time evolution of the annihilation operator of the graviton is described by the Bogoliubov transformation in the form

$$\begin{aligned}\hat{a}_y(\eta,\boldsymbol{k}) = &\left[\psi_p^{(\text{even})}\Big(1+\frac{i}{2k\eta_i}\Big)e^{ik\eta_i}+\psi_m^{(\text{even})*}\frac{i}{2k\eta_i}e^{-ik\eta_i}\right]\hat{a}_y(\eta_i,\boldsymbol{k})\\
&+\left[\psi_p^{(\text{even})}\Big(-\frac{i}{2k\eta_i}\Big)e^{ik\eta_i}+\psi_m^{(\text{even})*}\Big(1-\frac{i}{2k\eta_i}\Big)e^{-ik\eta_i}\right]\hat{a}_y^\dagger(\eta_i,-\boldsymbol{k})\\
&+\left[\psi_p^{(\text{odd})}\Big(1+\frac{i}{2k\eta_i}\Big)e^{ik\eta_i}+\psi_m^{(\text{odd})*}\frac{i}{2k\eta_i}e^{-ik\eta_i}\right]\hat{a}_x(\eta_i,\boldsymbol{k})\\
&+\left[\psi_p^{(\text{odd})}\Big(-\frac{i}{2k\eta_i}\Big)e^{ik\eta_i}+\psi_m^{(\text{odd})*}\Big(1-\frac{i}{2k\eta_i}\Big)e^{-ik\eta_i}\right]\hat{a}_x^\dagger(\eta_i,-\boldsymbol{k}),\end{aligned}\tag{3.32}$$

and the time evolution of the annihilation operator of a photon is expressed by the Bogoliubov transformation such as

$$\begin{aligned}\hat{a}_x(\eta,\boldsymbol{k}) = &\left[\psi_p^{(\text{even})}\Big(1+\frac{i}{2k\eta_i}\Big)e^{ik\eta_i}+\psi_m^{(\text{even})*}\frac{i}{2k\eta_i}e^{-ik\eta_i}\right]\hat{a}_x(\eta_i,\boldsymbol{k})\\
&+\left[\psi_p^{(\text{even})}\Big(-\frac{i}{2k\eta_i}\Big)e^{ik\eta_i}+\psi_m^{(\text{even})*}\Big(1-\frac{i}{2k\eta_i}\Big)e^{-ik\eta_i}\right]\hat{a}_x^\dagger(\eta_i,-\boldsymbol{k})\\
&+\left[\psi_p^{(\text{odd})}\Big(1+\frac{i}{2k\eta_i}\Big)e^{ik\eta_i}+\psi_m^{(\text{odd})*}\frac{i}{2k\eta_i}e^{-ik\eta_i}\right]\hat{a}_y(\eta_i,\boldsymbol{k})\\
&+\left[\psi_p^{(\text{odd})}\Big(-\frac{i}{2k\eta_i}\Big)e^{ik\eta_i}+\psi_m^{(\text{odd})*}\Big(1-\frac{i}{2k\eta_i}\Big)e^{-ik\eta_i}\right]\hat{a}_y^\dagger(\eta_i,-\boldsymbol{k}).\end{aligned}\tag{3.33}$$

We see that the Bogoliubov transformations for graviton and photon are symmetric. The Bogoliubov transformations show the particle production during inflation and the mixing between graviton and photon.

Let us introduce a matrix form of the Bogoliubov transformations for the calculations below. The Bogoliubov transformation (3.32) and (3.33) and their conjugate can be accommodated into the simple 4×4 matrix form M

$$\begin{pmatrix} a_y(\eta) \\ a_y^\dagger(\eta) \\ a_x(\eta) \\ a_x^\dagger(\eta) \end{pmatrix} = M \begin{pmatrix} a_y(\eta_i) \\ a_y^\dagger(\eta_i) \\ a_x(\eta_i) \\ a_x^\dagger(\eta_i) \end{pmatrix} = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \begin{pmatrix} a_y(\eta_i) \\ a_y^\dagger(\eta_i) \\ a_x(\eta_i) \\ a_x^\dagger(\eta_i) \end{pmatrix} . \tag{3.34}$$

Here, the M consists of 2×2 matrices A and B. The A is written by the even-order solutions,

$$A_{\text{even}} = \begin{pmatrix} K_{\text{even}}^* & -L_{\text{even}}^* \\ -L_{\text{even}} & K_{\text{even}} \end{pmatrix} , \tag{3.35}$$

where we defined

$$K_{\text{even}} = \psi_p^{(\text{even})*}\Big(1 - \frac{i}{2k\eta_i}\Big)e^{-ik\eta_i} - \psi_m^{(\text{even})}\frac{i}{2k\eta_i}e^{ik\eta_i}, \tag{3.36}$$

$$L_{\text{even}} = -\psi_p^{(\text{even})*}\frac{i}{2k\eta_i}e^{-ik\eta_i} - \psi_m^{(\text{even})}\Big(1 + \frac{i}{2k\eta_i}\Big)e^{ik\eta_i}. \tag{3.37}$$

The B comes from the first-order solution expressed as

$$B = \begin{pmatrix} K_{\text{odd}}^* & -L_{\text{odd}}^* \\ -L_{\text{odd}} & K_{\text{odd}} \end{pmatrix} , \tag{3.38}$$

where we defined

$$K_{\text{odd}} = \psi_p^{(\text{odd})*}\Big(1 - \frac{i}{2k\eta_i}\Big)e^{-ik\eta_i} - \psi_m^{(\text{odd})}\frac{i}{2k\eta_i}e^{ik\eta_i}, \tag{3.39}$$

$$L_{\text{odd}} = -\psi_p^{(\text{odd})*}\frac{i}{2k\eta_i}e^{-ik\eta_i} - \psi_m^{(\text{odd})}\Big(1 + \frac{i}{2k\eta_i}\Big)e^{ik\eta_i}. \tag{3.40}$$

3.3 Inversion of the Bogoliubov transformation

In the previous subsection, we obtained the Bogoliubov transformation that mixes the operators $\hat{a}_y(\eta, \boldsymbol{k})$, $\hat{a}_x(\eta, \boldsymbol{k})$ and their Hermitian conjugates $\hat{a}_y^\dagger(\eta, -\boldsymbol{k})$, $\hat{a}_x^\dagger(\eta, -\boldsymbol{k})$. Then the initial state is defined by

$$\hat{a}_y(\eta_i, \boldsymbol{k})|\overline{\mathrm{BD}}\rangle = \hat{a}_x(\eta_i, \boldsymbol{k})|\overline{\mathrm{BD}}\rangle = 0\,. \tag{3.41}$$

Here, $|\overline{\mathrm{BD}}\rangle$ is a vacuum deviated from the Bunch-Davies vacuum due to the presence of the constant background magnetic field. In order to impose these conditions, we need to invert the Bogoliubov transformations (3.32) and (3.33) into the form

$$\hat{a}_y(\eta_i, \boldsymbol{k}) = \alpha\,\hat{a}_y(\eta, \boldsymbol{k}) + \beta\,\hat{a}_y^\dagger(\eta, -\boldsymbol{k}) + \gamma\,\hat{a}_x(\eta, \boldsymbol{k}) + \delta\,\hat{a}_x^\dagger(\eta, -\boldsymbol{k})\,, \tag{3.42}$$

$$\hat{a}_x(\eta_i, \boldsymbol{k}) = \gamma\,\hat{a}_y(\eta, \boldsymbol{k}) + \delta\,\hat{a}_y^\dagger(\eta, -\boldsymbol{k}) + \alpha\,\hat{a}_x(\eta, \boldsymbol{k}) + \beta\,\hat{a}_x^\dagger(\eta, -\boldsymbol{k})\,, \tag{3.43}$$

where α, β, γ, δ are the Bogoliubov coefficients. In order to find these coefficients, we need the inverse of the matrix M, which is calculated as

$$M^{-1} = \begin{pmatrix} (A - BA^{-1}B)^{-1} & -(AB^{-1}A - B)^{-1} \\ -(AB^{-1}A - B)^{-1} & (A - BA^{-1}B)^{-1} \end{pmatrix}\,. \tag{3.44}$$

From Eqs. (3.42) and (3.43), the M^{-1} is also written as

$$M^{-1} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \beta^* & \alpha^* & \delta^* & \gamma^* \\ \gamma & \delta & \alpha & \beta \\ \delta^* & \gamma^* & \beta^* & \alpha^* \end{pmatrix}\,. \tag{3.45}$$

By comparing Eq.(3.44) with (3.45), we can obtain the Bogoliubov coefficients α, β, γ, δ numerically.

3.4 Squeezed state

In the previous subsection, we obtained the Bogoliubov coefficients of Eqs. (3.42) and (3.43). If we apply the Eqs. (3.42) and (3.43) to the definition of the initial state (3.41) and use the relations $[\hat{a}_y(\eta, \boldsymbol{k}), \hat{a}_y^\dagger(\eta, -\boldsymbol{k}')] = \delta(\boldsymbol{k} + \boldsymbol{k}')$, $[\hat{a}_x(\eta, \boldsymbol{k}), \hat{a}_x^\dagger(\eta, -\boldsymbol{k}')] = \delta(\boldsymbol{k} + \boldsymbol{k}')$ and

$[\hat{a}_y(\eta, \boldsymbol{k}), \hat{a}_x(\eta, -\boldsymbol{k}')] = 0$, the initial state can be written by using squeezing parameters Λ and Ξ of the form

$$|\overline{\mathrm{BD}}\rangle = \prod_{k=-\infty}^{\infty} \exp\left[\frac{\Lambda}{2}\,\hat{a}_y^\dagger(\eta, \boldsymbol{k})\hat{a}_y^\dagger(\eta, -\boldsymbol{k}) + \Xi\,\hat{a}_y^\dagger(\eta, \boldsymbol{k})\hat{a}_x^\dagger(\eta, -\boldsymbol{k}) + \frac{\Lambda}{2}\,\hat{a}_x^\dagger(\eta, \boldsymbol{k})\hat{a}_x^\dagger(\eta, -\boldsymbol{k})\right]|0\rangle, \tag{3.46}$$

where $|0\rangle$ is the instantaneous vacuum defined by

$$\hat{a}_y(\eta, \boldsymbol{k})|0\rangle = \hat{a}_x(\eta, \boldsymbol{k})|0\rangle = 0\,. \tag{3.47}$$

Note that Λ and Ξ are complex parameters. The squeezing parameter of a graviton-graviton pair and a photon-photon pair is written by the same Λ. This is because the Bogoliubov transformations (3.32) and (3.33) are symmetric. The squeezing of the graviton-photon pair is expressed by the Ξ. This describes a four-mode squeezed state of pairs of graviton y and photon x. In a different context, a four-mode squeezed state of two free massive scalar fields is discussed in [47, 48]. If we expand the exponential function in the Taylor series, we find

$$|\overline{\mathrm{BD}}\rangle = \prod_{\boldsymbol{k}} \sum_{p,q,r=0}^{\infty} \frac{\Lambda^{p+r}\,\Xi^q}{2^{p+r}p!\,q!\,r!}|p+q\rangle_{y,\boldsymbol{k}} \otimes |p\rangle_{y,-\boldsymbol{k}} \otimes |r\rangle_{x,\boldsymbol{k}} \otimes |q+r\rangle_{x,-\boldsymbol{k}}\,. \tag{3.48}$$

This is a four-mode squeezed state which consists of an infinite number of entangled particles in the $\mathcal{H}_{y,\boldsymbol{k}}\otimes\mathcal{H}_{y,-\boldsymbol{k}}\otimes\mathcal{H}_{x,\boldsymbol{k}}\otimes\mathcal{H}_{x,-\boldsymbol{k}}$ space. In particular, in the highly squeezing limit $\Lambda\,, \Xi \to 1$, the Bunch-Davies vacuum becomes the maximally entangled state from the point of view of the instantaneous vacuum.

Now we find the squeezing parameters Ξ and Λ. The condition $\hat{a}_y(\eta_i, \boldsymbol{k})|\overline{\mathrm{BD}}\rangle = 0$ of Eq. (3.41) yields

$$\alpha\Lambda + \beta + \gamma\Xi = 0\;, \qquad \alpha\Xi + \gamma\Lambda + \delta = 0\,, \tag{3.49}$$

and another condition $\hat{a}_x(\eta_i, \boldsymbol{k})|\overline{\mathrm{BD}}\rangle = 0$ gives the same equations. Then, we obtain the two squeezing parameters Λ and Ξ of the form

$$\Lambda = \frac{\gamma\delta - \beta\alpha}{\alpha^2 - \gamma^2} \,, \qquad \Xi = \frac{\beta\gamma - \alpha\delta}{\alpha^2 - \gamma^2} \,. \tag{3.50}$$

The results of numerical calculations for the squeezing parameters Λ and Ξ versus $a(\eta)$ with different values of k are plotted in FIGs. 16 and 17, respectively, where we normalized the scale factor at the end of the inflation as $a(\eta_f) = 1$. We note that the coupling Eq. 2.15 is expressed as

$$\lambda(\boldsymbol{k}) \equiv \frac{\sqrt{2}}{M_{\rm pl}} k \,|\mathbf{B}| \sin\theta \,, \tag{3.51}$$

where θ is the angle between the magnetic field and the wave number vectors depicted in FIG. 15. Apparently, the magnitude of the $\lambda(\boldsymbol{k})$ depends on the θ. Therefore, the squeezing parameters depend on the direction of the wave-number vector of gravitons for the fixed magnetic field. For simplicity, we take $\theta = \pi/2$ in the following. But we consider the effects of the angle in Section 4.3.

In FIG. 16, we plotted the squeezing parameter Λ as a function of the scale factor $a(\eta)$. We see that the amplitude of Λ goes to unity after the horizon exit and graviton and photon pair production become maximum during inflation. That is, the maximum entangled pairs of graviton and photon are produced. FIG. 17 shows that graviton-photon pair production occurs but the production keeps decreasing after the horizon exit.

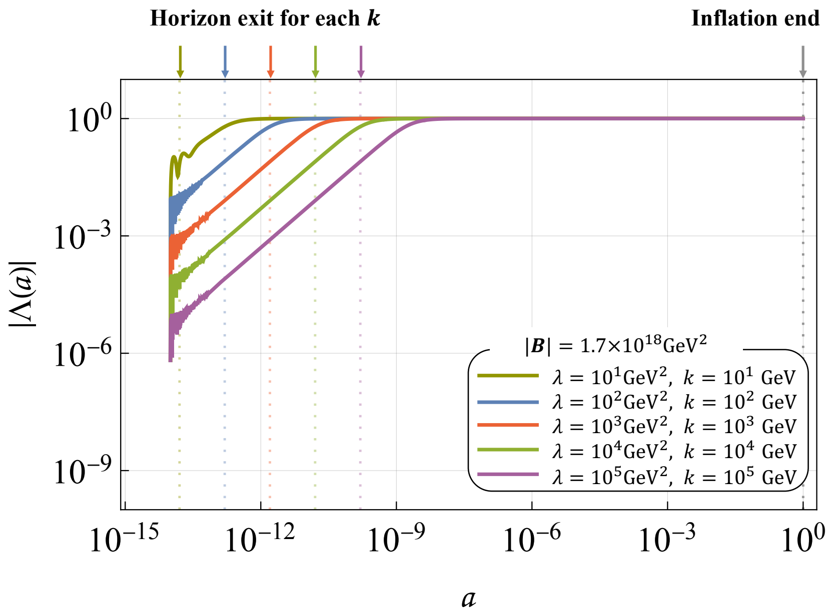

FIG 16. The squeezing parameter of gravitons or photons as a function of the scale factor of $a(\eta)$. Other parameters are set as $H = 10^{14}\,\mathrm{GeV}, \eta_i = -1\,\mathrm{GeV}^{-1}, \eta_f = -10^{-14}\,\mathrm{GeV}^{-1}, a(\eta_i) = 10^{-14}, a(\eta_f) = 1$.

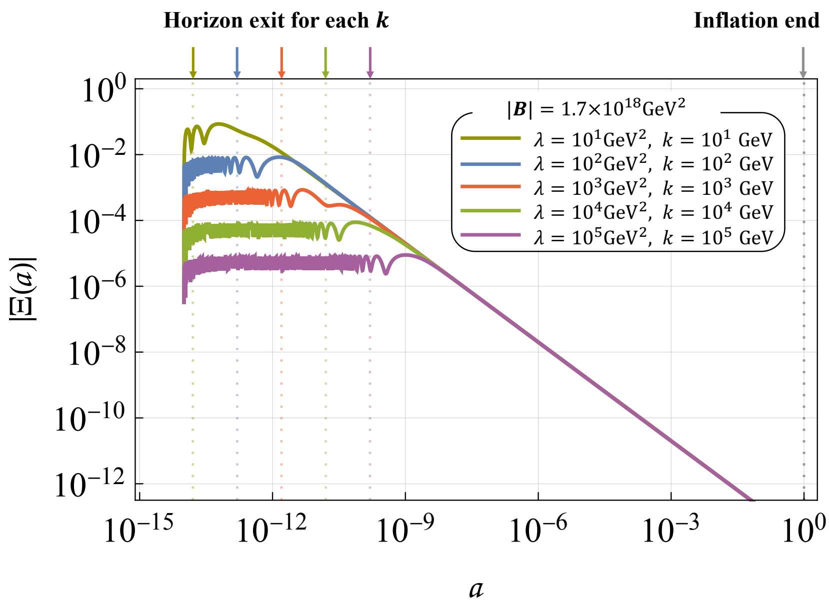

FIG 17. The squeezing parameter of graviton and photon pair as a function of $a(\eta)$. Other parameters are set as $H = 10^{14}\,\mathrm{GeV}, \eta_i = -1\,\mathrm{GeV}^{-1}, \eta_f = -10^{-14}\,\mathrm{GeV}^{-1}, a(\eta_i) = 10^{-14}, a(\eta_f) = 1$.

4 Graviton state at present

The standard lore about PGWs is that the quantum state of PGWs (gravitons) becomes squeezed during inflation due to the mechanism we discussed in the previous section. Hence, it is believed that finding the squeezed gravitons turns out to prove inflation. In this section, we see the quantum state of gravitons can be a mixed state in the presence of magnetic fields. This result tells us that we have to reconsider how to estimate the non-classicality of the primordial gravitational waves.

4.1 Schmidt decomposition

In the previous section, we found that the squeezing of the graviton-photon pair is produced but eventually disappears during inflation. In this subsection, we reveal the entanglement between the graviton-photon pairs. We compute the entanglement entropy between gravitons and photons by tracing over the photons using the method developed in [49, 50].

The initial state is expressed in Eq. (3.46) and it is difficult to trace over the photon degree of freedom. Thus, we perform the following Bogoliubov transformation

$$\hat{C}_{y,\boldsymbol{k}} = \Phi\,\hat{a}_{y,\boldsymbol{k}} + \Psi\,\hat{a}^{\dagger}_{y,-\boldsymbol{k}}\ , \qquad \hat{C}_{x,\boldsymbol{k}} = \Upsilon\,\hat{a}_{x,\boldsymbol{k}} + \Omega\,\hat{a}^{\dagger}_{x,-\boldsymbol{k}}\ , \tag{4.1}$$

where $|\Phi|^2 - |\Psi|^2 = 1$, $|\Upsilon|^2 - |\Omega|^2 = 1$ so that the state $|\overline{\rm BD}\rangle$ becomes in the Schmidt form

$$|\overline{\rm BD}\rangle = \prod_{\boldsymbol{k}=-\infty}^{\infty} \exp\left[\,\rho\,\hat{C}^{\dagger}_{y,\boldsymbol{k}}\,\hat{C}^{\dagger}_{x,-\boldsymbol{k}}\right]|0'\rangle_{y,\boldsymbol{k}}|0'\rangle_{x,-\boldsymbol{k}}\,. \tag{4.2}$$

Note that we consider different Bogoliubov coefficients between $(\Phi\,,\Psi)$ and $(\Upsilon\,,\Omega)$ because the Λ and Ξ in Eq. (3.46) are complex parameters. Here new vacuum states are defined by

$$\hat{C}_{y,\boldsymbol{k}}\,|0'\rangle_{y,\boldsymbol{k}} = 0\ , \quad \hat{C}_{x,\boldsymbol{k}}\,|0'\rangle_{x,\boldsymbol{k}} = 0\ . \tag{4.3}$$

Performing the new operators $\hat{C}_{y,\boldsymbol{k}}$ and $\hat{C}_{x,\boldsymbol{k}}$ on Eq. (4.2), we obtain the following relations,

$$\hat{C}_{y,\boldsymbol{k}}\,|\overline{\mathrm{BD}}\rangle = \rho\,\hat{C}^{\dagger}_{x,-\boldsymbol{k}}\,|\overline{\mathrm{BD}}\rangle\,, \tag{4.4}$$

$$\hat{C}_{x,\boldsymbol{k}}\,|\overline{\mathrm{BD}}\rangle = \rho\,\hat{C}^{\dagger}_{y,-\boldsymbol{k}}\,|\overline{\mathrm{BD}}\rangle\,. \tag{4.5}$$

By using Eq. (4.1), the above relations lead to the equations for the Bogoliubov coefficients as

$$\begin{pmatrix} \Lambda & 1 & 0 & -\rho\Xi \\ \Xi & 0 & -\rho & -\rho\Lambda \\ -\rho^* & -\rho^*\Lambda^* & \Xi^* & 0 \\ 0 & -\rho^*\Xi^* & \Lambda^* & 1 \end{pmatrix} \begin{pmatrix} \Phi \\ \Psi \\ \Upsilon^* \\ \Omega^* \end{pmatrix} = 0. \tag{4.6}$$

In order to find a nontrivial solution, the determinant of the above 4 by 4 matrix has to be zero. That is, $|\rho|^2$ satisfies

$$|\rho|^2 = Q - \sqrt{Q^2 - 1}\,, \tag{4.7}$$

where we have defined

$$Q = \frac{\left(|\Lambda|^2 - 1\right)^2 + |\Xi|^4 - 2\mathrm{Re}(\Xi^2\Lambda^{*2})}{2|\Xi|^2}\,. \tag{4.8}$$

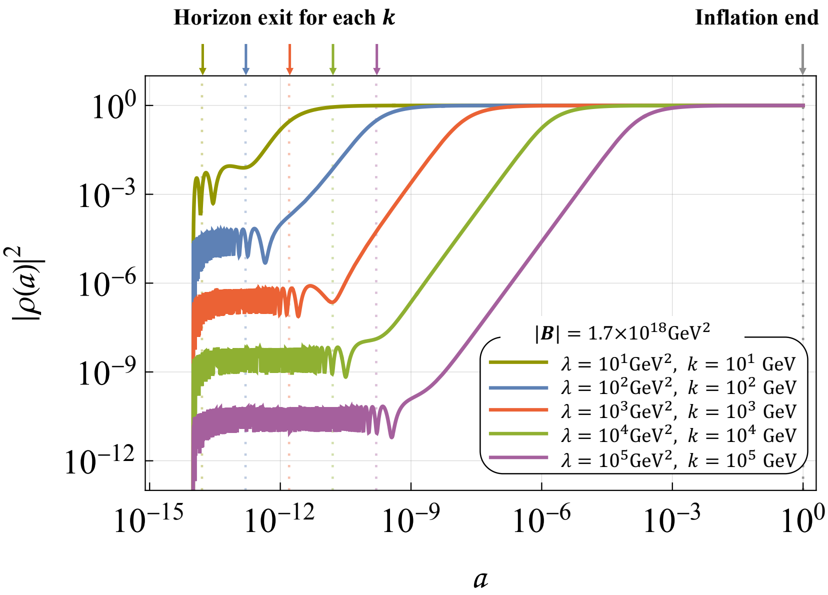

FIG 18. Plots of the parameter $|\rho(a)|^2$ as a function of $a(\eta)$. Other parameters are set as $H = 10^{14}\,\text{GeV}, \eta_i = -1\,\text{GeV}^{-1}, \eta_f = -10^{-14}\,\text{GeV}^{-1}, \lambda = 10\,\text{GeV}^2, a(\eta_i) = 10^{-14}, a(\eta_f) = 1$.

In FIG. 18, we plotted $|\rho|^2$ versus $a(\eta)$ for various values of k under a fixed value of $|\boldsymbol{B}|$. Here, λ is automatically determined once k is fixed because of Eq. (3.51) where we take $\theta = \pi/2$. We see that $|\rho|^2$ goes to unity irrespective of the value of k after the horizon exit if the value of $|\boldsymbol{B}|$ is fixed. Hence, the squeezing of graviton-photon pair in the basis $|0'\rangle_{y,\boldsymbol{k}}|0'\rangle_{x,-\boldsymbol{k}}$ turns out to be almost maximum, while Ξ in the basis of $|0\rangle$ eventually vanishes as shown in FIG. 17.

4.2 Entanglement entropy

Since gravitons and photons are coupled to each other through λ as in Eqs. (2.17) and (2.18), they are expected to get entangled eventually. In the previous subsection, we find the squeezing of the graviton-photon pair becomes almost maximum in the basis of $|0'\rangle_{y,\boldsymbol{k}}|0'\rangle_{x,-\boldsymbol{k}}$ but eventually vanish in the basis of $|0\rangle$. In order to clarify whether they get entangled or not, we compute the entanglement entropy as a measure of entanglement. The entanglement entropy is basis independent.

We define the density operator of the vacuum $|\overline{\mathrm{BD}}\rangle$ in Eq. (4.2) by

$$\begin{aligned}\sigma &= |\overline{\mathrm{BD}}\rangle\langle\overline{\mathrm{BD}}| \\ &= \left(1-|\rho|^2\right) \prod_{\boldsymbol{k},-\boldsymbol{k}} \sum_{n',m'=0}^{\infty} \rho^{n'}\,\rho^{*m'}\,|n'\rangle_{y,\boldsymbol{k}}\,|n'\rangle_{x,-\boldsymbol{k}}\;{}_{y,\boldsymbol{k}}\langle m'|\,{}_{x,-\boldsymbol{k}}\langle m'| \ . \end{aligned} \tag{4.9}$$

The reduced density operator for the gravitons is obtained by tracing over the degree of freedom of photons such as

$$\begin{aligned}\sigma_y &= \mathrm{Tr}_x\,|\overline{\mathrm{BD}}\rangle\langle\overline{\mathrm{BD}}| = \sum_i {}_{x,\boldsymbol{k}'}\langle\, i\,|\overline{\mathrm{BD}}\rangle\langle\overline{\mathrm{BD}}|\, i\,\rangle_{x,\boldsymbol{k}'} \\ &= \left(1-|\rho|^2\right) \sum_{n'=0}^{\infty} |\rho|^{2n}\,|n'\,\rangle_{y,\boldsymbol{k}}\;{}_{y,\boldsymbol{k}}\langle n'\,| \ . \end{aligned} \tag{4.10}$$

The entanglement entropy between the graviton and photon can be characterized by

$$\begin{aligned}S &= -\mathrm{Tr}_y\,\sigma_y \log\sigma_y = -\sum_{n'=0}^{\infty}\left(1-|\rho|^2\right)|\rho|^{2n'}\Big(\log\left(1-|\rho|^2\right)+n'\log|\rho|^2\Big) \\ &= -\log\left(1-|\rho|^2\right) - \frac{|\rho|^2}{1-|\rho|^2}\log|\rho|^2 \ . \end{aligned} \tag{4.11}$$

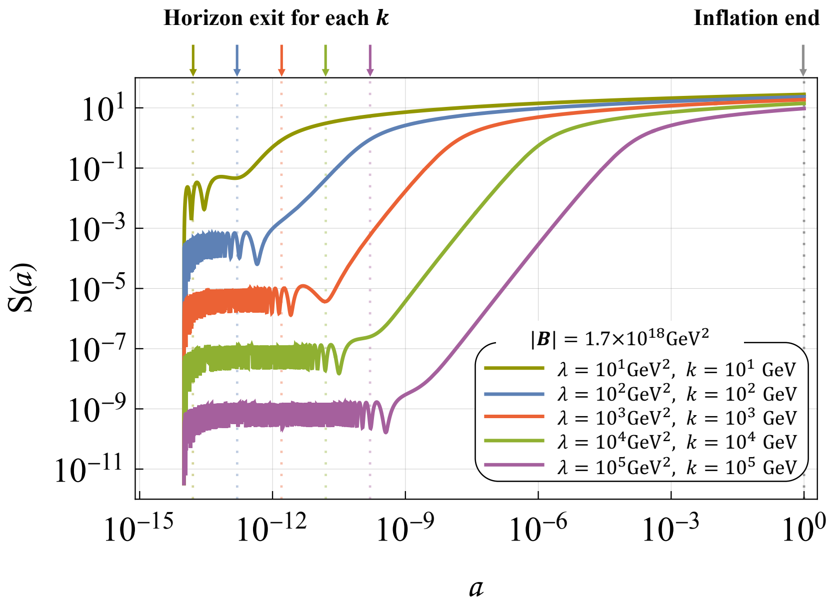

FIG 19. Entanglement entropy between graviton and photon as a function of $a(\eta)$. Other parameters are set as $H = 10^{14}\text{GeV}$, $\eta_i = -1\text{GeV}^{-1}$, $\eta_f = -10^{-14}\text{GeV}^{-1}$, $a(\eta_i) = 10^{-14}$, and $a(\eta_f) = 1$.

In FIG. 19, we plotted the entanglement entropy for various values of k under a fixed value of $|\boldsymbol{B}|$, which clearly shows that the graviton and photon are highly entangled during inflation. As well as the result of FIG. 18, the asymptotic value of $S(a)$ becomes the same irrespective of the value of k.

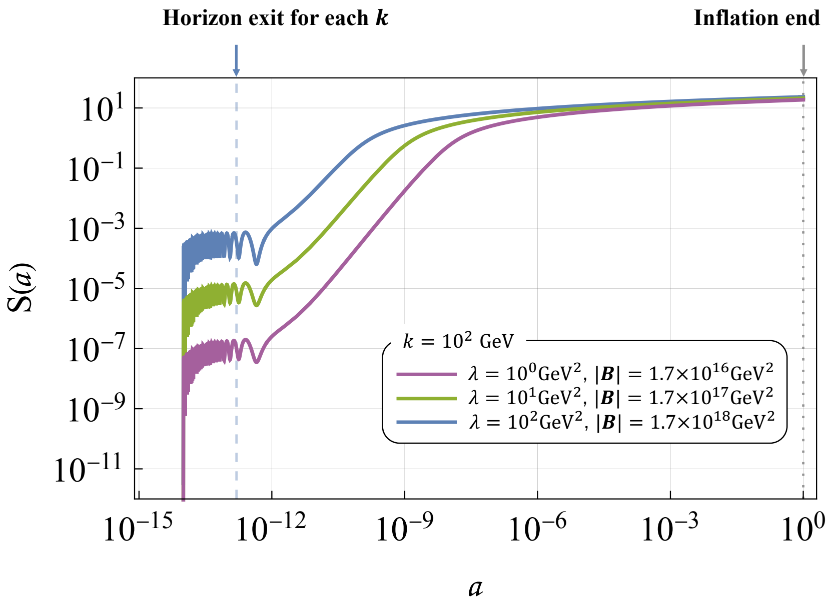

FIG 20. The entanglement entropy of graviton and photon field induced by the coupling of background magnetic field with a different magnetic field. Each line with a different color represents a different magnetic field. Other parameters are set as $H = 10^{14}\,\mathrm{GeV}, \eta_i = -1\,\mathrm{GeV}^{-1}$, and $\eta_f = -10^{-14}\,\mathrm{GeV}^{-1}$, $a(\eta_i) = 10^{-14}$, and $a(\eta_f) = 1$.

In FIG. 20, the entanglement entropy for various values of λ under a fixed value of k is plotted. In this case, the different λ corresponds to different $|\boldsymbol{B}|$ because of Eq. (3.51) where $\theta = \pi/2$.

Part IV

Conclusion

As mentioned in the introduction part, we need to properly take into account the decoherence of the relic gravitons during cosmic history. As a first modest step in this direction, we assumed the presence of a sizable magnetic field at the beginning of inflation [51, 53]. If the squeezing of graviton decreases as time evolves, it implies that the decoherence of graviton occurs. In this thesis, we investigated the squeezed state of graviton during inflation under the effects of graviton-photon conversion induced by primordial magnetic fields. In particular, we study how the squeezed state of gravitons changes when the magnetic field decays rapidly $fB \propto 1/a^2$ (Model-1) and when the magnetic field decays slowly due to the interaction of gauge field and scalar field $fB \propto 1/a$ (Model-2). We solved the dynamical evolution of a coupled system of graviton and photon with the perturbative approach for Model-1 as is mentioned in part II, and with the analytical approach for Model-2 as is mentioned in part III. In terms of the coupling function of the gauge field and scalar field, Model-1 corresponds to the coupling parameter $c = 0$ while Model-2 corresponds to the scale-invariant parameter $c = -1/2$. When the primordial magnetic field decays rapidly due to inflation, only gravitons are strongly squeezed, whereas when the magnetic field decays slowly, photons are similarly strongly squeezed. Quantum states in such cases were non-trivial before this study. The difference in the squeezing parameter of a photon in these models brings out the difference in the entanglement between photons and gravitons.

Through the analysis in part II, we found that the effects of graviton-photon conversion on the squeezed state of gravitons are limited in the case of the magnetic field that decays rapidly during inflation. This is because the physical value of the primordial magnetic field damps rapidly due to the expansion of the background space. We derived an analytic formula for the squeezing parameter of photons and found that the degree of squeezing is at a few percent at most.

Through the analysis in part III, we studied primordial gravitational waves (PGWs) in the presence of magnetic fields that survive during inflation. In contrast to conventional inflation, where only PGWs are highly squeezed, electromagnetic fields in such system are highly squeezed as well. We showed that graviton to photon conversion and vice versa never end as long as inflation lasts, and then gravitons and photons get highly entangled. We derived a reduced density matrix of the gravitons and calculated their entanglement entropy by using the reduced density matrix. We revealed that the quantum states of the primordial gravitons observed today are almost squeezed states and mixed states. The relic gravitons are expected to be squeezed during inflation. In that case, quantum noise induced by them can be significantly enhanced in current interferometers.

In both cases, we numerically plotted the squeezing parameters for the system of graviton and photon. It is shown that magnetic fields do not affect the graviton squeezing parameter. Also, we numerically checked the parameter of squeezed graviton-photon pair Ξ and found that the Ξ rapidly decays at the end of inflation. This fact was confirmed also analytically, in Eq. (4.41) for instance. We also depicted the squeezing parameter of the photon. It turned out that the amount of squeezed photon produced by the conversion was tiny in the case of Model-1, while it converges to 1 which is the completely same behavior of the graviton's squeezing parameter in the case of Model-2.

Since we found that gravitons are robust against the decoherence caused by the cosmological magnetic field, we could expect to find squeezed relic gravitons through quantum noise induced by them in interferometers [21, 22, 23, 24, 25]. We should note that the analysis in our paper can also be applicable to the dark magnetic field models [44] based on the dark photon scenario [54].

There could be classical gravitational waves initially as in Ref. [52]. Quantum mechanically, the initial condition can be represented by coherent states $|\Upsilon\rangle$, such as

$$\hat{a}_y(\eta_i, \boldsymbol{k})|\Upsilon\rangle = \Upsilon|\Upsilon\rangle \,, \qquad \hat{a}_A(\eta_i, \boldsymbol{k})|\Upsilon\rangle = 0 \,,$$

where Υ denotes a complex number. In such cases, we just replace (5.1) in Part II with the above conditions and follow the same calculation. As is mentioned in Eq. (A5) of the appendix in Ref. [55], the squeezed state is unchanged even when we replace the state with the coherent state. Thus, we conclude that the squeezing parameter is unchanged when we take some classical initial conditions for GWs.

There are several directions to be pursued. It would be intriguing to follow up on the evolution of the squeezed relic gravitons up to the radiation-dominated and matter-dominated eras. If we could show the absence of decoherence of the squeezed relic gravitons, their robustness would be proven. It would also be interesting to study the case that the primordial magnetic fields persist against the cosmic no-hair theorem during inflation [52]. On top of gravitons, the squeezing occurs for the light axion dark matter fields [55, 10]. The decoherence of axion fields due to magnetic fields can be discussed in a similar way.

In addition, even when the photon has a mass, there could be a conversion from gravitons into massive photons. In this case, graviton-photon conversion can give rise to a new mechanism for massive photon production. In particular, the converted massive vector of the extra U(1) sector could be vector dark matter. The possibility is worth studying further. We leave these issues for future work. Also, in the case of $c = -1$, the physical magnetic fields fB do not decay during inflation. Hence, we would be able to expect more drastic effects on the quantum state of gravitons. We leave the analysis of this case for future work.

Our findings have important implications for the quantum state of primordial gravitons. So far, states of primordial gravitons are regarded as squeezed pure states. However, if magnetic fields had coupled with gravitons during inflation, the primordial gravitons observed today would be mixed states. Then the estimation of observables has to be changed.

Refining the model of the magnetic field would be another direction to be pursued since the mechanism of the production of the primordial magnetic field is still controversial, and both of the models we discussed in this thesis still include some bold assumptions. For instance, we assume the uniform background magnetic field though the metric is homogeneous and isotropic. It would be better to consider some anisotropic background, such as the background metric derived from anisotropic inflation[56]. If the relations of the physical quantity such as the power spectrum and the primordial magnetic field is clarified, it would be possible to get some constrain the model of the primordial magnetic field and inflationary scenarios. Various inflation scenarios which include the mechanism to generate the primordial magnetic field is proposed [57, 60, 61, 62, 63, 64, 65, 66, 67, 58, 59]. If we can describe the squeezed graviton under the influence of the more precise description of the background magnetic field such as described in [68], the effect of the conversion process on the graviton would become more concrete.

Our results also open up the possibility of probing primordial magnetic fields through the observations of the non-classicality of primordial gravitational waves.

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