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AN INTERACTION OF MEAN FLOW AND INTERNAL GRAVITY WAVE WITHOUT BOUSSINESQ APPROXIMATION

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1. Introduction

Boussinesq approximation have been usually used in studies with respect to internal gravity waves (e. g. Lighthill[1]). Boussinesq approximation is satisfied when a density variation is small (Gill[2]). In actual atmospheric and oceanic phenomena, however, the density variation is not very small occasionally. In this paper, we introduce some properties of an internal gravity wave without Boussinesq approximation. As a result the Boussinesq approximation is a short wave approximation. We also apply the properties to a wave-mean flow interaction which was studied by Lindzen[3] with the Boussinesq Approximation. An instability occurs with a longer wave than a wave in a Boussinesq fluid.

2. Introductory Remarks

Our basic state consists in a no-Boussinesq fluid with a constant buoyancy frequency N, given by

$$N^2 = -\frac{g}{\rho_0(z)}\frac{d\rho_0(z)}{dz}\ , \tag{2.1}$$

where ρ_0 is the basic density distribution, g the acceleration due to gravity, and

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z the height; we consider a two-dimensional flow (see Fig. 1 for a coordinate system). We have the basic density distribution which satisfies (2.1) as

$$\rho_0(z) \propto \exp\left(-\frac{N^2}{g}z\right) . \tag{2.2}$$

Basic equations are

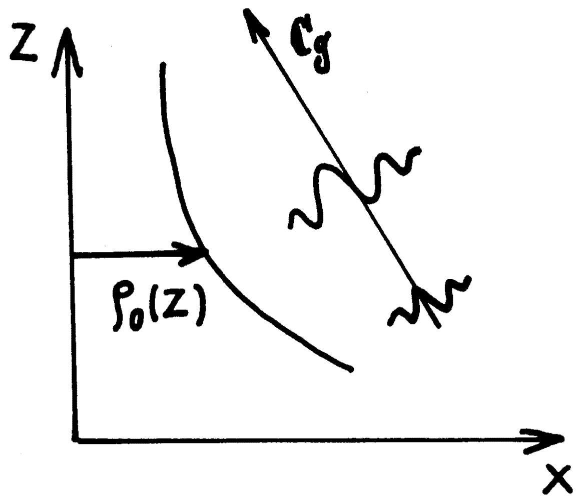

Fig. 1 Schematic diagram.

$$\rho_0\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} , \tag{2.3}$$

$$\rho_0\frac{\partial w}{\partial t} = -\rho g-\frac{\partial p}{\partial z} , \tag{2.4}$$

$$\frac{\partial \rho}{\partial t}+w\frac{d\rho_0}{dz} = 0 , \tag{2.5}$$

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z} = 0 , \tag{2.6}$$

where we assumed the hydrostatic approximation, ρ and p are perturbations from their basic density and pressure fields, u the horizontal velocity, and w the vertical velocity. After a manipulation, we get an equation for w:

$$\left\{\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2}{\partial x^2}+\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0\frac{\partial}{\partial z}\right)\right]+N^2\frac{\partial^2}{\partial x^2}\right\}w = 0 . \tag{2.7}$$

Boussinesq approximation is based on the observation that if w varies with height much more rapidly than ρ_0, then

$$\frac{\partial^2 w}{\partial z^2} \gg \frac{N^2}{g} \frac{\partial w}{\partial z} . \tag{2.8}$$

This inequality means that a typical vertical scale H is larger than the scale height $Hs(= g/N^2)$. In the Boussinesq approximation, the buoyancy frequency is

$$N_*^2 = -\frac{g}{\rho_*} \frac{d\rho_0}{dz} , \tag{2.9}$$

where ρ_* is a constant density. If the buoyancy frequency is constant, the density variation is a linear function with respect to height. Comparing this result with (2.2), the Boussinesq approximation is satisfied in small scale phenomena.

We assume a plane wave solution

$$w = W \exp\left[i(kx+mz-\omega t)\right] , \tag{2.10}$$

substitute it into (2.7), and we have

$$(k^2+m^2) - \frac{N^2}{\omega^2} k^2 + i\frac{N^2}{g} m = 0 . \tag{2.11}$$

Because the eigenvalue equation has an imaginary coefficient, we decompose m into real and imaginary parts ($m = m_r + im_i$), which is associated with exponential growth or decay with height. Using (2.11) we have two relations

$$m_i = -1/2H_s , \tag{2.12}$$

$$\omega^2 = \frac{N^2 k^2}{\kappa^2 + (1/2H_s)^2} , \tag{2.13}$$

where κ^2 is k^2+m^2. If we use the Boussinesq approximation, we have $m_i = 0$ and $\omega^2 = N^2k^2/\kappa^2$. The Boussinesq approximation, therefore, is a short wave approximation: wave length λ is smaller than the scale height H_s. On the other hand we assume a long wave approximation (i. e. $\lambda \gg H_s$), we get $\omega^2 = N^2k^2(2H_s)^2$. This wave is a nondispersive wave which propagates vertically. The reader should notice that the wave is not a boundary wave. Using the dispersion relationship with the Boussinesq approximation we can derive that the group velocity $\boldsymbol{c}_g$ is normal to the wavenumber vector $\boldsymbol{k}$. Using (2.13), we have

$$\boldsymbol{c}_g \cdot \boldsymbol{k} = \frac{N^2}{\omega} \frac{k^2/4H_s^2}{\kappa^2 + 1/4H_s^2} . \tag{2.14}$$

The two vectors are tilted with an angle θ:

$$\cos\theta = \frac{k}{m_r \kappa^2} \frac{1}{4H_s^2} + 0(H_s^{-4}) . \tag{2.15}$$

therefore we have $\theta < \pi/2$ (see Fig. 2). The angle is zero as m_r tends to zero (vertical wave length tends to infinity) in eq. (2.14) This wave is an interfacial wave which propagates horizontally and which vertical structure is an exponential type. With the complex vertical wavenumber the wave propagates vertically as the amplitude changes. The energy density remains constant with (2.2) and (2.12). It is consistent with the absence of energy sources. The solution

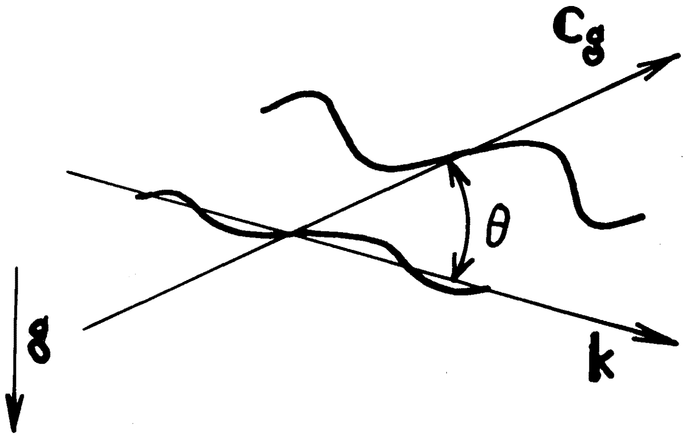

Fig. 2 Propagation angle.

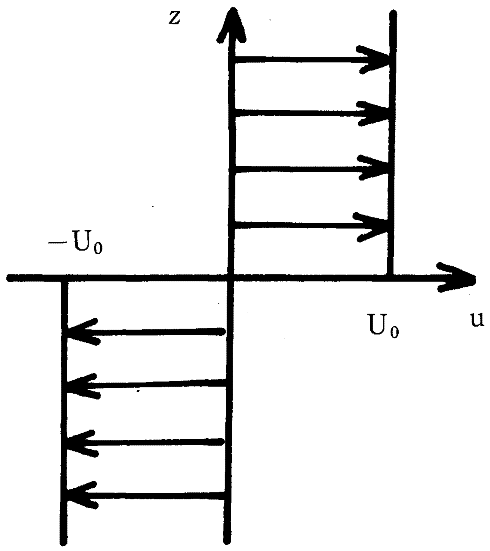

Fig. 3 Definition sketch of the mean velocity field.

(2.10) with (2.12) indicates that an upward propagating wave has larger amplitude than do downward one. This vertical amplitude change should be notice in the atmosphere and ocean observations.

3. Wave-Mean Flow Interaction

We apply the above propeties to an over-reflection of an internal gravity wave. We decompose the horizontal velocity u into a mean (U_0) and perturbation (u) velocities (see Fig. 3 for a definition sketch). The basic equations are

$$\rho_0 \frac{Du}{Dt} = -\frac{\partial p}{\partial x} , \tag{3.1}$$

$$\rho_0 \frac{Dw}{Dt} = -\rho g - \frac{\partial p}{\partial z} , \tag{3.2}$$

$$\frac{D\rho}{Dt} + w \frac{d\rho_0}{dz} = 0 , \tag{3.3}$$

and the continuity equation (2.6), where the material derivative D/Dt is $\partial/\partial t + U_0 \partial/\partial x$. The equation for w is

$$\frac{D^2}{Dt^2}\left(\varDelta - \frac{N^2}{g}\frac{\partial}{\partial z}\right) w + N^2 \frac{\partial^2 w}{\partial x^2} = 0 , \tag{3.4}$$

where $\varDelta$ is the Laplacian. If we assume a plane wave solution $w = W(z) \exp [ik(x-ct)]$ with a complex phase speed $c = c_r + ic_i$, we get

$$\left\{\frac{d^2}{dz^2} - \frac{N^2}{g}\frac{d}{dz} + \left[\frac{N^2}{(U_0-c)^2} - k^2\right]\right\} W = 0 . \tag{3.5}$$

The second term means the scale height effect. Using a transformation

$$W(z) = \hat{W}(z) \exp\left(\int^z N^2(z') dz'/2g\right) \tag{3.6}$$

yields

$$\frac{d^2 \hat{W}}{dz^2} + \left[\frac{N^2}{(U_0-c)^2} - k_*^2\right] \hat{W} = 0 , \tag{3.7}$$

where

$$k_*^2 = k^2 + \frac{N^4}{4g^2} - \frac{N}{g}\frac{dN}{dz} . \tag{3.8}$$

If we use the Boussinesq approximation, the right hand side is k^2. This corresponds to a longer wave than a wave in Boussinesq fluid.

A process of obtaining the solution is straightforward with boundary conditions: continuities of pressure and vertical displacement at $z = 0$, and radiation conditions (cf. Lindzen[3]). We show results alone.

(1) Neutral solution: $c_i = 0$

(i) $k_*^2 < N^2/U_0^2$

We have $\hat{W}(z) \propto \exp(im_r z)$ and $m_i > 0$, which solution satisfies a radiation condition. This wave is radiated from the interface. This is a resonant over-reflection.

(ii) $k_*^2 > N^2/U_0^2$

In this case we have no neutral solutions which satisfy a boundary (radiation) condition.

(2) Time variable solution: $c_i \neq 0$

We have a dispersion relationship:

$$m_1^2 = k_*^2 - \frac{N^2}{(U_0 - ic_i)^2}\ , \tag{3.9}$$

$$m_2^2 = c.c.(m_1^2)\ , \tag{3.10}$$

where $c.c.(\ )$ means a complex conjugate. The solution $\hat{W}(z)$ is proportional to $\exp(\mp m_i z)$, which means the vertical amplitude change.

(i) $k_*^2 < N^2/U_0^2$

We have $c_i = 0$, $m_{1r}^2 = 0$, and $m_{1i}^2 = (N^2/U_0^2) - k_*^2$.

(ii) $k_*^2 > N^2/U_0^2$

For this case we have a growth (or decay) rate

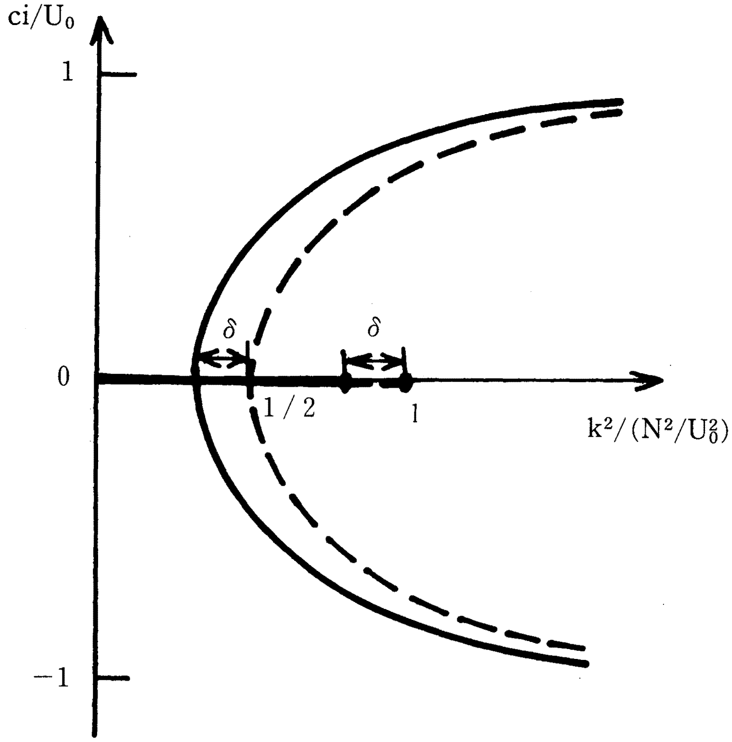

Fig. 4 Stability diagram. Broken lines are obtained by Lindzen[3] with the Boussinesq approximation. δ is an order of H_s^{-2}.

$$c_i^2 = U_0^2 - \frac{N^2}{2k_*^2} , \tag{3.11}$$

and

$$m_1^2 = -k^2 \frac{(U_0 + ic_i)^2}{(U_0 - ic_i)^2} . \tag{3.12}$$

These results lead to a stability diagram Fig. 4. All solutions are respect to longer waves than waves studied under the Boussinesq approximation.

We consider an optimal wave which has a maximum wave energy flux. Using the wave solutions we have

$$\lambda_{opt} = \frac{2\sqrt{2}\pi U_0}{N}(1+\delta) , \tag{3.13}$$

where $\delta = U_0^2/(4H_s^2N^2)$. Using data in the atmosphere (Reed & Hardy[4]) $U_0 = 28\,\mathrm{ms}^{-1}$, $N = 1.3\times10^{-2}\,\mathrm{s}^{-1}$, $H_s = 8\times10^5\,\mathrm{cm}$, we have $\lambda_{opt} = 19.4\,\mathrm{km}$. This value is consistent with an observed wavelength (15-20 km). Using values $U_0 = 4\,\mathrm{cms}^{-1}$, $N = 3.1\times10^{-3}\,\mathrm{s}^{-1}$, and $H_s = 8\times10^4\,\mathrm{cm}$, we have $\lambda_{opt} = 120\,\mathrm{m}$. Internal waves with this length scale have been observed in the ocean. Discussions about a kinetic energy and typical time scales of the neutral and unstable waves are parallel to those by Lindzen[3].

Internal waves, which have near inertial frequency, have been observed below the mixed layer in the ocean. Over-reflection as one generation mechanism is submitted (Stern[5], and Kamachi and Grimshaw[6]). They used the Bousssinesq approximation. The wave propagates vertically in the main thermocline below the mixed layer. The vertical amplitude change effect may have to be taken into consideration, because the density variation is large in the main thermocline.

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