# AN APPROXIMATE CALCULATION OF LAMINAR BOUNDARY LAYER FLOW OF AN INCOMPRESSIBLE FLUID: Determination of the Flowfield Compatible with a Prescribed Condition

Yamada, Hikoji
Research Institute for Applied Mechanics, Kyushu University : Formerly professor

Okabe, Jun-ichi
Research Institute for Applied Mechanics, Kyushu University : Professor

Yamamoto, Masaaki
Computing Center, Central Research Laboratory, Osaka Institute of Technology : Staff

https://doi.org/10.5109/6779673

出版情報：Reports of Research Institute for Applied Mechanics. 30 (95), pp.21-39, 1982-12. 九州大学応用力学研究所
バージョン：
権利関係：

Reports of Research Institute for Applied Mechanics
Vol. XXX, No. 95, 1982

# AN APPROXIMATE CALCULATION OF LAMINAR BOUNDARY LAYER FLOW OF AN INCOMPRESSIBLE FLUID

## Determination of the Flowfield Compatible with a Prescribed Condition

By Hikoji YAMADA,* Jun-ichi OKABE† and Masaaki YAMAMOTO‡

The moment equations are derived from boundary layer equations as an amelioration of Kármán-Pohlhausen approximate theory. Defining quantities prescribed for the boundary layer are introduced explicitly into the moment equations, and the flowfield is determined by solving the system of simultaneous ordinary differential equations of the first order thus obtained in terms of the field parameters. In this paper, approximation making use of two parameters is adopted, which is expected to be satisfactory for practical purposes.

***Key words***: Laminar boundary layer flow, Incompressible fluid, Moment equations, Kármán-Pohlhausen approximate theory

### 1. Introduction

Recently, the so-called *inverse problem* for the boundary layer, in which the mainstream compatible with a prescribed distribution of, for example, skin friction on the surface of a body is to be specified accordingly, comes up for discussion. It was shown meanwhile that we are enabled to work out the points of flow separation and reattachment, not as singular points, including the flow taking place in the region of backflow.[1~4] Based upon this fact, a comprehensive computation procedure will be disclosed in this paper, which makes possible for us with practically satisfactory accuracy not only to solve the inverse problem in a most general manner, but also to define the boundary layer flow under arbitrarily given conditions, sufficient for determining the flowfield. This is, in fact, an extension of the method proposed by the first author about 30 years ago as an amelioration of Kármán-Pohlhausen theory, with a view to solving the *standard problem* (these nomenclatures are due to Keller and Cebeci[1]), namely the computation of the boundary layer out of a given distribution of the mainstream.[5] The technique then employed was the so-called method of moments,

* Formerly, professor at Research Institute for Applied Mechanics, Kyushu University
† Professor, Research Institute for Applied Mechanics, Kyushu University
‡ Staff, Computing Center, Central Research Laboratory, Osaka Institute of Technology

a kind of the method of weighted residuals. It is the purpose of the present paper to show that we are now in a position to treat directly the inverse and the related problems as well, by transforming appropriately the moment equations yielded from the equations of boundary layers.

After explaining anew the derivation of the system of the governing equations as far as necessary for the standard problem, since the original paper is now inaccessible, we shall turn to treatment of the inverse and other relevant problems.

## 2. Computation formulas

In the first place, the solution of the standard problem will be explained. Taking as usual the $x$-axis along the surface of a body, $x=0$ at the point of attachment of the streamline, and the $y$-axis perpendicularly into the fluid, we are led to the boundary layer equations in terms of $u(x, y)$, the velocity component in the direction of $x$ increasing:

$$D(u) \equiv u\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\int_0^y \frac{\partial u}{\partial x}\mathrm{d}y - u_1 u_1' - \nu\frac{\partial^2 u}{\partial y^2} = 0\,, \tag{1}$$

where $u_1(x)$ denotes the velocity of the mainstream prevailing outside the boundary layer, $\nu$ the kinematic viscosity of the fluid, and a prime implies differentiation with respect to $x$. It is assumed incidentally that the fluid is incompressible and the flow is stationary. The boundary conditions are written as

$$\left.\begin{aligned} &u=0 && \text{at} \quad y=0\,, \\ \text{and} \quad &u=u_1, \quad \partial u/\partial y=0 && \text{at} \quad y=\delta\,, \end{aligned}\right\} \tag{2}$$

where $\delta(x)$ stands for the boundary layer thickness at the station $x$. Further, if it is required that equation (1) holds exactly on the surface of the body $(y=0)$ and on the outer edge of the boundary layer $(y=\delta)$ as well, then supplementary conditions

$$\left.\begin{aligned} &\frac{\partial^2 u}{\partial y^2} = -\frac{u_1 u_1'}{\nu} && \text{at} \quad y=0\,, \\ \text{and} \quad &\frac{\partial^2 u}{\partial y^2} = 0 && \text{at} \quad y=\delta, \end{aligned}\right\} \tag{2'}$$

are added to conditions (2).

Now, introducing another variable $\eta$ defined as

$$\eta = y/\delta \tag{3}$$

in place of $y$, we write

$$\left.\begin{aligned} &u(x,\ y) = u_1(x) f(x,\ \eta)\,, \\ \text{where} \quad &f(x,\ \eta) = a_1(x)\,\eta + a_2(x)\,\eta^2 + a_3(x)\,\eta^3 + \cdots. \end{aligned}\right\} \tag{4}$$

Assuming that conditions (2) and (2′) should be both satisfied, we are led to the conclusion that the function $f(x, \eta)$ should be expressed in the form

$$f(x, \eta)=F(\eta)+\omega G(\eta)+\vartheta H(\eta)+\varphi K(\eta)+\cdots; \tag{5}$$

the parameters $\omega$, $\vartheta$, $\varphi$, $\cdots$, where $\omega$ is defined as

$$\omega=u_1{}'\delta^2/6\nu, \tag{6}$$

are associated with the velocity distribution and are all functions of $x$, whose precise forms are to be specified later. $F(\eta)$, $G(\eta)$, $H(\eta)$, $K(\eta)$, $\cdots$ are all known polynomials of $\eta$. If we are successful in evaluating the parameters $\omega$, $\vartheta$, $\varphi$, $\cdots$, then all quantities pertinent to the boundary layer in question can be determined.

With this end in view, the moment method is employed in the $y$-direction, namely it is supposed that a certain number of conditions

$$\int_0^1 D(u)\,\eta^n \mathrm{d}\eta=0 \qquad (n=0,\ 1,\ 2,\cdots) \tag{7}$$

are satisfied. Then a system of simultaneous ordinary differential equations of the first order concerning with $\omega$, $\vartheta$, $\varphi$, $\cdots$ may be derived in the form

$$\begin{aligned}&\left(\frac{A_n}{\omega}+a_n\right)\omega'+b_n\vartheta'+c_n\varphi'+\cdots\\&\qquad+\left(\frac{d_n}{6\,\omega}+B_n\right)\frac{u_1{}'}{u_1}-A_n\frac{u_1{}''}{u_1{}'}=0 \qquad (n=0,\ 1,\ 2,\cdots),\end{aligned} \tag{8}$$

where $A_n$, $B_n$ and $a_n$, $b_n$, $c_n$, $d_n$, $\cdots$ are both known integral forms of the second and the first degree, respectively, of the variables $\omega$, $\vartheta$, $\varphi$, $\cdots$. The boundary layer flow can be defined completely, provided, by solving equations (8), $\omega(x)$, $\vartheta(x)$, $\varphi(x)$, $\cdots$ are determined in reference to a given $u_1(x)$, the velocity distribution of the mainstream.[5] In pariticular, $\tau_0(x)$, the friction on the surface of the body, and $\delta_1(x)$, the displacement thickness of the boundary layer, may be computed in the forms

$$\left.\begin{aligned}&\tau_0=\mu u_1(2+\omega-\vartheta-\varphi+\cdots)/\delta,\\ \text{and}\quad&\delta_1=\delta\left(\frac{3}{10}-\frac{\omega}{20}+\frac{\vartheta}{10}+\frac{3\varphi}{28}+\cdots\right),\end{aligned}\right\} \tag{9}$$

respectively, where $\mu$ denotes the coefficient of viscosity of the fluid and $\delta$ is given by

$$\delta=(6\nu\omega/u_1{}')^{1/2}. \tag{9′}$$

Incidentally, the momentum thickness and the energy thickness can be calculated in a similar manner.

Since one of the most significant quantities concerning the boundary layer is $\tau_0$, let us change our parameters in such a way that $\tau_0$ may appear quite conspicuous. Namely, when we rewrite the first equation of (9) as

$$\tau_0=\mu\frac{u_1}{\delta}\lambda, \qquad \text{i. e.} \qquad \lambda=\frac{\tau_0}{\rho}\frac{\delta}{\nu u_1}, \tag{10}$$

$\lambda$ becomes a parameter standing for the surface friction, $\rho$ being the density of the fluid. We shall introduce it in place of $\vartheta$ through the relation

$$\lambda=2+\omega-\vartheta-\varphi+\cdots. \tag{11}$$

Namely substituting for $\vartheta$ and $\vartheta'$ in equations (5) and (8) the relations

$$\left.\begin{aligned} &\vartheta=2+\omega-\lambda-\varphi+\cdots, \\ \text{and} \qquad &\vartheta'=\quad\omega'-\lambda'-\varphi'+\cdots, \end{aligned}\right\} \tag{11'}$$

we shall arrive at the following result: the velocity distribution is expressed in the form

$$u/u_1=f(x,\eta)=\mathfrak{F}(\eta)+\omega(x)\,\mathfrak{G}(\eta)+\lambda(x)\,\mathfrak{H}(\eta)+\varphi(x)\,\mathfrak{K}(\eta)+\cdots, \tag{12}$$

where $\mathfrak{F}(\eta)$, $\mathfrak{G}(\eta)$, $\mathfrak{H}(\eta)$, $\mathfrak{K}(\eta)$, $\cdots$ are known polynomials, which are found to be

$$\left.\begin{aligned} &\mathfrak{F}(\eta)=10\eta^3-15\eta^4+6\eta^5, \\ &\mathfrak{G}(\eta)=-3\eta^2(1-\eta)^3, \\ &\mathfrak{H}(\eta)=\eta-6\eta^3+8\eta^4-3\eta^5, \\ &\mathfrak{K}(\eta)=-\eta^3(1-\eta)^3, \\ &\cdots \qquad\qquad \cdots, \end{aligned}\right\} \tag{13}$$

and the moment equations (8) are transformed into

$$\begin{aligned} &\left(\frac{A_n}{\omega}+a_n+b_n\right)\omega'-b_n\lambda'-(b_n-c_n)\,\varphi'+\cdots \\ &\qquad=-\left(\frac{d_n}{6\,\omega}+B_n\right)\frac{u_1'}{u_1}+A_n\frac{u_1''}{u_1'} \qquad (n=0,\ 1,\ 2,\cdots), \end{aligned} \tag{14}$$

while the second equation of (9) reduces to

$$\delta_1=\delta\left(\frac{1}{2}+\frac{\omega}{20}-\frac{\lambda}{10}+\frac{\varphi}{140}+\cdots\right). \tag{15}$$

In view of equations (9′) and (10), it is readily understood that $\omega$ and $u_1'$ have to vanish concurrently and that the same connection is also in existence between $\lambda$ and $\tau_0$. It is more important to notice, as is evident by eliminating $\delta$ out of (9′) and (10), that a general relationship holds in the form

$$\frac{\tau_0}{\rho}=\left(\frac{\nu u_1^2 u_1'}{6\omega}\right)^{1/2}\lambda. \tag{16}$$

In order to bring this equation into a simpler form, let us introduce $T(x)$ and $v(x)$ defined by

$$\left.\begin{aligned} T&=\frac{1}{\nu}\left(\frac{\tau_0}{\rho}\right)^2, \\ \text{and}\qquad v&=u_1^3 \end{aligned}\right\} \tag{17}$$

in places of $\tau_0$ and $u_1$, respectively. Then equation (16) may be written

$$\frac{T}{\lambda^2}=\frac{v'}{18\omega}\,. \tag{18}$$

It is quite easy to transform $u_1$ into $v$ in equations (14).

### 3. Computation formulas *(continued)*

After a number of tentative computations performed by the first author, it was concluded that quite a high accuracy could be achieved not only in an accelerated flow but also in a retarded flow through the approximation in which only two equations corresponding to $n=0$ and 1 were employed and accordingly all parameters other than $\omega$ and $\lambda$ were assumed zero. In the followings, therefore, mathematical treatment will be confined to this approximation. If, however, more parameters should be adopted, essential of our computations would remain unchanged, only the accuracy of approximation would be improved. The form of our fundamental equation (18) does not depend upon the number of parameters taken account of; the numerical values of $\omega$ and $\lambda$ will be modified slightly when parameters more than two are introduced.

When we put all parameters other than $\omega$ and $\lambda$ to be zero and write $u_1$ in terms of $v$, equations (14) are found to be simplified into

$$\left.\begin{aligned} \left(\frac{A_0}{\omega}+a_0+b_0\right)\omega'-b_0\lambda'&=A_0\frac{v''}{v'}-D_0\frac{v'}{3v}, \\ \left(\frac{A_1}{\omega}+a_1+b_1\right)\omega'-b_1\lambda'&=A_1\frac{v''}{v'}-D_1\frac{v'}{3v}, \end{aligned}\right\} \tag{19}$$

where $\qquad D_n=d_n/6\omega+2A_n+B_n \qquad (n=0 \text{ and } 1),$ (19′)

and $A_n$, $B_n$ and $a_n$, $b_n$, $d_n$ are all polynomials of $\omega$ and $\lambda$; the numerical coefficients are shown in Table 1. Solving equations (19) with respect to $\omega'$ and $\lambda'$, we obtain

$$\left.\begin{aligned} \omega' &= \frac{1}{\Delta^{(+)}}\left\{-\begin{vmatrix} A_0 & b_0 \\ A_1 & b_1 \end{vmatrix}\frac{v''}{v'}+\begin{vmatrix} D_0 & b_0 \\ D_1 & b_1 \end{vmatrix}\frac{v'}{3v}\right\}, \\ \lambda' &= \frac{1}{\Delta^{(+)}}\left\{-\begin{vmatrix} A_0 & a_0+b_0 \\ A_1 & a_1+b_1 \end{vmatrix}\frac{v''}{v'}+\begin{vmatrix} D_0 & A_0\omega^{-1}+a_0+b_0 \\ D_1 & A_1\omega^{-1}+a_1+b_1 \end{vmatrix}\frac{v'}{3v}\right\}, \end{aligned}\right\} \tag{20}$$

where $$\Delta^{(+)} = -\begin{vmatrix} A_0\omega^{-1}+a_0 & b_0 \\ A_1\omega^{-1}+a_1 & b_1 \end{vmatrix} = -\frac{1}{\omega}\begin{vmatrix} A_0 & b_0 \\ A_1 & b_1 \end{vmatrix} - \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix}. \tag{20'}$$

The sign $(+)$ attached to $\Delta$ implies the determinant of coefficients taking place in a problem of determining the boundary layer for a given $v(x)$, namely the standard problem.

Table 1. Coefficients of fundamental polynomials $A_n$, $B_n$, $a_n$, $b_n$ and $d_n$ ($n=0$ and 1) (The coefficients of the term $\omega$, for example, of $A_0$ through $d_1$ are given under the heading $\omega$. All the numerical values in this table are multiplied by 100.)

| | constant term | $\omega$ | $\lambda$ | $\omega^2$ | $\omega\lambda$ | $\lambda^2$ |
|---|---|---|---|---|---|---|
| $A_0$ | 5.41126 | −0.54113 | 1.73160 | −0.19481 | 0.74675 | −0.75036 |
| $A_1$ | 3.59849 | −0.17857 | 0.77561 | −0.14367 | 0.54221 | −0.52056 |
| $B_0$ | 71.64502 | 2.83550 | −3.07359 | −0.77922 | 2.98701 | −3.00144 |
| $B_1$ | 26.89394 | 1.78571 | −1.89755 | −0.44968 | 1.66883 | −1.58225 |
| $a_0$ | 2.38095 | 0.71429 | −1.50794 | | | |
| $a_1$ | 1.40873 | 0.42262 | −0.78571 | | | |
| $b_0$ | −3.46320 | −1.49351 | 3.00144 | | | |
| $b_1$ | −2.12843 | −0.87229 | 1.58225 | | | |
| $d_0$ | 0 | 0 | −100.00000 | | | |
| $d_1$ | −100.00000 | 0 | 0 | | | |

On the right-hand side of equations (20), $v$ is assumed non-vanishing as is usually the case and $v'$ appearing in the denominator acquires the form $v'/\omega$ when connected with $\omega$ in $\Delta^{(+)}$ and gives rise to no singularity, as shown by equation (18). Further, since $\Delta^{(+)}$ is contained in the denominator, we have to scrutinize its zeros. It is proved, however, if equations (19) are taken into account, that the terms contained in the braces on the right-hand sides of equations (20) also cancel necessarily at these points. We may conclude, therefore, that equations (20) have definite solutions in the domain relevant to our problem, and raise no essential difficulty from an analytical point of view. However, we have to deal with the system of these equations numerically, since they are rather complicated.

With this end in view, we should expect that cases sometimes happen where we are necessitated to draw on the $(\omega, \lambda)$-plane a path of numerical integration across the $\lambda$-

axis (i. e. $\omega=0$ ; on this axis, it is readily known that $v'=0$), or across the curves $\Delta^{(+)}(\omega, \lambda)=0$ (let us call them the $\Delta^{(+)}$-curves in what follows). The $\Delta^{(+)}$-curves are shown in Fig. 1 with full lines. We can realize that they hold such positions as might become an obstacle if the scope of the problem we are interested in is expanded. Removal of these difficulties will be discussed later.*

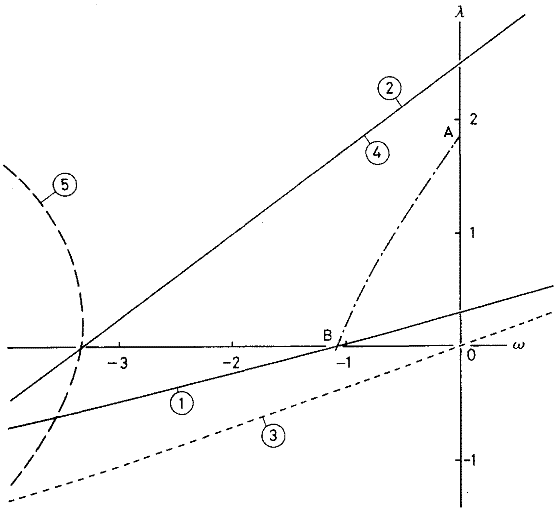


Fig. 1. Curves $\Delta^{(+)}=0$, $\Delta^{(-)}=0$ and $\Delta^{(0)}=0$

Note — ① : the first branch of the curve $\Delta^{(+)}=0$, ② : the second branch of $\Delta^{(+)}=0$, ③ : the first branch of $\Delta^{(-)}=0$, ④ : the second branch of $\Delta^{(-)}=0$ (almost coincident with ②) and ⑤ : the first branch of $\Delta^{(0)}=0$. The second branch of $\Delta^{(0)}=0$ and the third branches of $\Delta^{(+)}=0$ and $\Delta^{(-)}=0$ are located at the right and the bottom, respectively (all outside the picture). The curve AB represents the locus of the velocity profile taking place in linearly retarded flows. The points A and B situated on the $\lambda$-, and the $\omega$-axes correspond, respectively, to the leading edge and the separation point.

* In a previous paper, the first author treated numerically the standard problem for Howarth's flow, and arrived at a good agreement with the original solution due to Howarth.[5,6] In this case, the result of the author's computation showed consistency with Goldstein's well-known postulation that the separation point is a singular point. The reason of this fact is simply that continuation of the computation beyond the separation point did not appear feasible at that time owing to the very existence of the $\Delta^{(+)}$-curve in the immediate neighborhood of the separation point. In reality, however, in approximate integration due to the moment method, the separation point is defined by $\lambda=0$ and is generally an ordinary point ; see (II) Separation of a decelerated flow of **6** of this paper.

In contrast with the standard problem, in which details of the boundary layer are to be worked out corresponding to the mainstream $v(x)$, recently the inverse problem becomes frequently a subject of discussions. Let us rewrite the calculation scheme adove-mentioned so as to be appropriate to the inverse problem. To this end, $\tau_0(x)$ should be introduced as the given quantity instead of $u_1(x)$, which is carried out through equation (18). Namely, differentiating this equation with respect to $x$, we obtain

$$\frac{T'}{T}=\frac{v''}{v'}-\frac{\omega'}{\omega}+2\frac{\lambda'}{\lambda}. \tag{21}$$

So that substitution of this relationship for $v''/v'$ in equations (19) gives after some rearrangements

$$(a_n+b_n)\,\omega'+\left(\frac{2}{\lambda}A_n-b_n\right)\lambda' = A_n\frac{T'}{T}-D_n\frac{v'}{3v} \qquad (n=0 \text{ and } 1). \tag{22}$$

These two equations when combined with equation (18), or

$$v'=18\omega T/\lambda^2, \tag{22$'$}$$

constitute a system of simultaneous ordinary differential equations of the first order with three unknowns: $\omega$, $\lambda$ and $v$. They suffice precisely to form a closed system. If we succeed in evaluating the unknowns in terms of $x$ by integrating these equations, all the details of the boundary layer can be clarified.

Solving (22) and (22$'$) algebraically with respect to $\omega'$, $\lambda'$ and $v'$, we obtain

$$\left.\begin{aligned} \omega'&=\frac{1}{\Delta^{(-)}}\left\{-\begin{vmatrix} A_0 & b_0 \\ A_1 & b_1 \end{vmatrix}\frac{T'}{T}-\begin{vmatrix} D_0 & 2A_0\lambda^{-1}-b_0 \\ D_1 & 2A_1\lambda^{-1}-b_1 \end{vmatrix}\frac{v'}{3v}\right\}, \\ \lambda'&=\frac{1}{\Delta^{(-)}}\left\{-\begin{vmatrix} A_0 & a_0+b_0 \\ A_1 & a_1+b_1 \end{vmatrix}\frac{T'}{T}+\begin{vmatrix} D_0 & a_0+b_0 \\ D_1 & a_1+b_1 \end{vmatrix}\frac{v'}{3v}\right\}, \end{aligned}\right\} \tag{23}$$

and $\quad v'=18\omega T\lambda^{-2}$,

where

$$\Delta^{(-)}=-\begin{vmatrix} 2A_0\lambda^{-1}+a_0 & a_0+b_0 \\ 2A_1\lambda^{-1}+a_1 & a_1+b_1 \end{vmatrix} =-\begin{vmatrix} A_0 & a_0+b_0 \\ A_1 & a_1+b_1 \end{vmatrix}\frac{2}{\lambda}-\begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix}. \tag{23$'$}$$

The sign $(-)$ attached to $\Delta$ involves the determinant of coefficients related to the inverse problem. Similarly to the case of equations (20), the zeros of $T$ contained in the denominators constitute no singularity when combined with $\lambda$ in $\Delta^{(-)}$, and the zeros of $\Delta^{(-)}$ are cancelled by those of the numerators of equations (23). The curves $\Delta^{(-)}=0$, or let us call them the $\Delta^{(-)}$-curves in the followings, are given in Fig. 1 by dotted lines, and it is indicated that if we attempt to treat miscellaneous kinds of problems through this procedure, some cases can happen where the numerical integration of equations

(23) might not be carried out in the original form.

Since $\omega'$ and $\lambda'$ contained in equation (21) are identical with those given by equations (20) and (23), we can eliminate them out of these equations and are led finally to a relationship between $T'/T$ and $v''/v'$:

$$\Delta^{(+)}\frac{T'}{T}-\Delta^{(-)}\frac{v''}{v'}=\begin{vmatrix} D_0 & \frac{2}{\lambda}\left(\frac{A_0}{\omega}+a_0+b_0\right)-\frac{b_0}{\omega} \\ D_1 & \frac{2}{\lambda}\left(\frac{A_1}{\omega}+a_1+b_1\right)-\frac{b_1}{\omega} \end{vmatrix}\frac{v'}{3v}. \tag{24}$$

At an arbitrary point, therefore, intermediate during integration, $T'/T$ and $v''/v'$ are equivalent to each other in the sense that one may be computed from the other. So that it is left to our choice which should be regarded as the unknown. Only in the neighborhood of the $\Delta^{(+)}$-, or the $\Delta^{(-)}$-curve, numerical accuracy of this conversion should be deteriorated.

## 4. Numerical calculation

The calculation formulas derived in the preceding sections are quite general. The initial values of $\omega$, $\lambda$ and $v$ (or $T$) are determined from the conditions imposed upon the given quantity $v$ (or $T$) in the neighborhood of the front end ($x=0$), and accordingly computations may be performed downstream. In this paper, out of a number of boundary layer flows, cases where the incident uniform flow meets the surface of an immersed body tangentially will be taken up with a view to illustrating a practical scheme of computations. In any case otherwise, the front end is not likely to give rise to any additional difficulty.

In both of the standard and the inverse problems, the representative dimensional quantities in the equations are $x$, $v\ (=u_1{}^3)$ and $T(=\tau_0{}^2/\rho^2\nu)$ standing respectively for the length, the velocity of the mainstream and the friction on the surface. Reducing these quantities non-dimensional by means of their constant values $l$, $v_0$ and $T_0$, say, we shall denote them by $\xi$, $\bar{v}$ and $\bar{T}$, respectively. Then equations (19) and (22) (for $n=0$ and 1) are now looked upon as non-dimensional preserving the original forms, provided $v$ and $T$ therein are understood as $\bar{v}$ and $\bar{T}$ respectively and a prime is regarded in the following sections as implying differentiation with respect to $\xi$. Only equation (22′) which gives the relationship between $v'$ and $T$ becomes

$$\left.\begin{aligned} &\bar{v}'=18C\omega\bar{T}/\lambda^2, \\ \text{where}\quad &C=lT_0/v_0\,; \end{aligned}\right\} \tag{25}$$

the numerical value of $C$ depends upon the choice of constant quantities $l$, $T_0$ and $v_0$. Since we are now interested only in the case of tangential incidence of flow, taking into consideration the situation around the leading edge, we shall make use of the expansions

$$\left.\begin{aligned} v/v_0 &\equiv \overline{v}(\xi) = \quad 1+v_1\xi+v_2\xi^2+\cdots, \\ \text{and} \quad T/T_0 &\equiv \overline{T}(\xi) = \xi^{-1}+t_0+t_1\xi+t_2\xi^2+\cdots; \end{aligned}\right\} \tag{26}$$

a representative length $l$ will be taken appropriately to define a non-dimensional distance

$$\xi = x/l. \tag{27}$$

Similary to equations (26), also for $\omega$ and $\lambda$ the expansions

$$\left.\begin{aligned} \omega &= \quad \omega_1\xi+\omega_2\xi^2+\cdots, \\ \text{and} \quad \lambda &= \lambda_0+\lambda_1\xi+\lambda_2\xi^2+\cdots \end{aligned}\right\} \tag{28}$$

will be assumed. Substituting equations (26) and (28) for $\overline{v}$, $\overline{T}$, $\omega$ and $\lambda$ in equations (22) and (25), we can obtain the first several terms numerically. Namely, together with

$$C = lT_0/v_0 = 0.112449,^* \tag{29}$$

we have also for the coefficients of the expansions

$$\left.\begin{aligned} & v_1 = 0.24708t_0, \quad v_2 = 0.16132t_1 - 0.01238t_0^2; \\ & \omega_1 = 0.42431t_0, \quad \omega_2 = 0.55406t_1 - 0.19217t_0^2; \\ & \lambda_0 = 1.86438, \quad \lambda_1 = 0.60344t_0 \\ \text{and} \quad & \lambda_2 = 0.69157t_1 - 0.27254t_0^2. \end{aligned}\right\} \tag{30}$$

Equations (30) are given as a computation scheme for the case where $\overline{T}(\xi)$ is given. When $\overline{v}(\xi)$ is given, on the contrary, we have only to rewrite the formula assuming that $v_1$, $v_2$,$\cdots$ are given constants.

A calculation procedure in which polynomial approximation is applied for the region next to the leading edge, and is continued further to the second region by employing new polynomials, and so on, the method of *piecewise polynomials*, so to speak, is very appropriate to our present problem, from a point of view that the procedure allows us without difficulty not only pertinent considerations for the points

---

* The value of $C$ determined making use of the exact solution due to Blasius for the flat plate boundary layer is 0.11026, yielding discrepancy of about 2 % out of the value mentioned here. This error may be regarded as a measure for the approximation in which only two parameters $\omega$ and $\lambda$ are taken into account.

$v'=0$ and $T=0$ but also traverse of the $\Delta^{(\pm)}$-curves. However, developments of the method will be made in a report to be published in future, because some preparations for numerical analysis become indispensable. Here, we take an alternative: numerical integrations are started at a point where numerical values of the variables yielded out of the series expansions around the leading edge still remain accurate satisfactorily. Among a number of numerical methods conceivable, integration due to Runge, Kutta and Gill was used for the most part; there is no reason for confining ourselves to this specific method. In fact, cases may happen, of course, where other techniques are necessitated.

As mentioned above, our problems are now treatments of the zeros of $v'$ and $T$ and traverse of the $\Delta^{(\pm)}$-curves as well. For the zeros of $v'$ and $T$, we have only to make use of the fact effectively that $\omega$ and $\lambda$, respectively, vanish here concurrently. With regard to the traverse of the $\Delta^{(\pm)}$-curves, on the other hand, it is necessary to convert the specifying quantity for the boundary layer in such a way that we have $\Delta^{(-)}$ in our computation formula when we have to traverse the $\Delta^{(+)}$-curve, and vice versa. For example, in order to remove the obstacle arising from the $\Delta^{(-)}$-curve, we have only to change the specifying quantity from $T$ to $v$. In doing so, we have to define beforehand $v''/v'$ as a function of $\xi$, according to the relation (18) or (21). This conversion, as shown already in equation (24), is likely to fail in providing necessary accuracy as the $\Delta^{(\pm)}$-curves are approached. It is, therefore, advisable to take another way: to assume tentatively through extraporation a formula for $v''/v'$ on the basis of knowledges gathered up to the present step of integration we have arrived at, to perform the integration making use of it over another step forward or till the traverse of the $\Delta^{(-)}$-curve is completed, and finally to compare the resulting value of $T'/T$ from the given value with a view to correcting the formula itself.

A method which enables us to deal with the zeros of $v'$ and $T$ and also to traverse the $\Delta^{(\pm)}$-curves at the same time is the introduction of a positive function, $\Phi$, say, which will be discussed in the next section.

## 5. Positive function $\Phi$

As explained before, the zeros of $v'(x)$ and $T(x)$, if let intact, constitute seemingly singular points for the system of equations in the standard and the inverse problems for the boundary layer, and it is to be noted that these zeros take place quite frequently in the domain relevant to practical applications. These singularities can be removed, however, through introduction of $\Phi(x)$, a function connecting the given quantity with the unknown in the form

$$\Phi(x)=v'/18\omega \tag{31}$$

for the standard problem, and by equation (18) we may write alternatively

$$\Phi(x)=T/\lambda^2 \tag{31'}$$

for the inverse problem as well. Thus $\Phi$ may be understood as resulting from $v'$ or $T$ by removing its zeros. According to equations (9) or (10), we have also the relation

$$\Phi(x)=\nu v^{2/3}\delta^{-2}, \tag{31''}$$

and through $\delta$ contained in this expression, we may adopt the displacement thickness, the momentum thickness, or the energy thickness of the boundary layer as a constituent factor for $\Phi$. Besides, the distinctive feature of $\Phi$ is that it is always positive and non-vanishing. A bar is placed upon $\Phi$ in order to indicate its non-dimensional form defined by $\overline{v}/18C\omega$ or $\overline{T}/\lambda^2$.

Making use of $\overline{\Phi}$, we may rewrite equations (22) and (25) in the following forms, where the independent variable has been changed from $x$ to $\xi$.

$$\left.\begin{aligned}&(a_n+b_n)\,\omega'-b_n\lambda'=A_n\frac{\overline{\Phi}'}{\overline{\Phi}}-D_n\omega\frac{\overline{v}'}{3\overline{v}\omega}\quad(n=0 \text{ and } 1),\\ \text{and}\qquad &\overline{v}'=18C\omega\overline{\Phi}.\end{aligned}\right\} \tag{32}$$

We have succeeded in this way not only in removal of singular points of the equations but also in unified treatment of just the same equations irrespective of the formulation of the problem. At the same time, $\Delta^{(0)}$, the determinant consisting from the coefficients of equations (32) (for $n=0$ and 1), namely

$$\Delta^{(0)}=-\begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix}, \tag{33}$$

becomes $\Delta^{(0)}=0.18896+0.03056\omega-0.02743\lambda$

$$-0.00785\omega^2-0.00417\omega\lambda+0.02763\lambda^2, \tag{33'}$$

if the values of the coefficients shown in Table 1 are employed. The curves defined by $\Delta^{(0)}=0$, or abbreviated as the $\Delta^{(0)}$-curves in the followings, are shown in Fig. 1 with broken lines; on the $(\omega, \lambda)$-plane they lie far away from the domain we are now interested in and as a matter of fact are not likely to give rise to any trouble.

Another feature associated with $\overline{\Phi}$ is that it is very easy to interchange the given quantities with each other at an arbitrary point of $\xi$ and to proceed further. For example, the boundary layer computed up to a point $\xi=\xi_0$, say, with assigned $\overline{v}(\xi)$, the velocity of the mainstream prevailing outside, may be continued as the boundary layer, for which $\overline{T}(\xi)$, the skin friction, is assigned over the region of $\xi$ exceeding $\xi_0$, simply through interchanging mutually the given quantities contained in $\overline{\Phi}$ at the point $\xi=\xi_0$.

Only one problem is left untouched so far: in $\overline{\Phi}$, which should be a given quantity, is contained as a constituent factor an unknown quantity to be worked out as a result of the computation itself and that accordingly its derivative exists implicitly in $\overline{\Phi}'/\overline{\Phi}$. With a view to scrutinizing a treatment of this problem, equations (32) are solved with regard to $\omega'$ and $\lambda'$ yielding

$$\left.\begin{aligned}
\omega' &= \frac{1}{\Delta^{(0)}}\left\{-\begin{vmatrix} A_0 & b_0 \\ A_1 & b_1 \end{vmatrix}\frac{\overline{\Phi}'}{\overline{\Phi}} + \begin{vmatrix} D_0 & b_0 \\ D_1 & b_1 \end{vmatrix}\frac{\overline{v}'}{3\overline{v}}\right\}, \\
\lambda' &= \frac{1}{\Delta^{(0)}}\left\{-\begin{vmatrix} A_0 & a_0+b_0 \\ A_1 & a_1+b_1 \end{vmatrix}\frac{\overline{\Phi}'}{\overline{\Phi}} + \begin{vmatrix} D_0 & a_0+b_0 \\ D_1 & a_1+b_1 \end{vmatrix}\frac{\overline{v}'}{3\overline{v}}\right\}, \\
\text{and}\qquad \overline{v}' &= 18C\omega\overline{\Phi}.
\end{aligned}\right\} \tag{34}$$

Since $\overline{\Phi}'/\overline{\Phi}$ appearing on the right-hand side may be written, with the aid of equation (31) or (31′) in the form

$$\frac{\overline{\Phi}'}{\overline{\Phi}} = \frac{\overline{v}''}{\overline{v}'} - \frac{\omega'}{\omega}, \qquad \text{or} \qquad \frac{\overline{\Phi}'}{\overline{\Phi}} = \frac{\overline{T}'}{\overline{T}} - 2\frac{\lambda'}{\lambda}, \tag{35}$$

substituting the value of $\omega'$ (or $\lambda'$), derived out of the first (or the second) equation, for the left-hand side of the first (or the second) equation of (34), we can obtain

$$\left.\begin{aligned}
\frac{\overline{\Phi}'}{\overline{\Phi}} &= \frac{1}{\omega\Delta^{(+)}}\left\{\Delta^{(0)}\frac{\overline{v}''}{\overline{v}'}\omega - \begin{vmatrix} D_0 & b_0 \\ D_1 & b_1 \end{vmatrix}\frac{\overline{v}'}{3\overline{v}}\right\}, \\
\text{or}\qquad \frac{\overline{\Phi}'}{\overline{\Phi}} &= \frac{1}{\lambda\Delta^{(-)}}\left\{\Delta^{(0)}\frac{\overline{T}'}{\overline{T}}\lambda - 2\begin{vmatrix} D_0 & a_0+b_0 \\ D_1 & a_1+b_1 \end{vmatrix}\frac{\overline{v}'}{3\overline{v}}\right\}.
\end{aligned}\right\} \tag{36}$$

Namely, at each mesh-point in the numerical computation, where the values of $\omega$, $\lambda$, $\overline{v}$ and $\overline{\Phi}$ are assigned, $\overline{\Phi}'/\overline{\Phi}$ is evaluated in the first place; $\omega'$ and $\lambda'$ are worked out accordingly from equations (34).

With the aid of the expression

$$\overline{v}''\omega/\overline{v}' = \overline{v}''/18C\overline{\Phi}, \tag{37}$$

valid in the neighborhood of $\overline{v}'=0$, or likewise through the relationship

$$\overline{T}'\lambda/\overline{T} = \overline{T}'/(\overline{T}\ \overline{\Phi})^{1/2}, \tag{38}$$

where $\overline{T}(\xi)$ is assumed as usual in the form

$$\overline{T} = (\xi_s - \xi)^2\Theta(\xi), \tag{38'}$$

($\xi_s$ denoting the zero of $\overline{T}$, the separation point in other words, and $\Theta(\xi_s)>0$) valid in the neighborhood of $\overline{T}=0$, it is readily observed that the value of $\overline{\Phi}'/\overline{\Phi}$ is determined definitely at all points including $\overline{v}'=0$ or $\overline{T}=0$ and that therefore the existence of the extreme value in the mainstream velocity, or the existence of the point of flow separation or reattachment on the surface of the body does not give rise to any obstacle for the computation. Thus we may expect that our scheme has a considerable merit: the system of equations (34) has no singular points in the relevant domain and can be employed irrespectively of the given quantity, the only difference being that $\overline{\Phi}'/\overline{\Phi}$ be calculated in terms of either given $\overline{v}$ or given $\overline{T}$. But equations (36) yield a rough or even an erroneous value if the curve $\Delta^{(+)}=0$ or $\Delta^{(-)}=0$ is approached or traversed when the first or the second equation, respectively, of (36) is dealt with. Equating, on the other hand, both of equations (36) in order to find a relationship between $\bar{v}''/\bar{v}'$ and $\bar{T}'/\bar{T}$ reduces to nothing but equation (24) before-mentioned and is fruitless in resolving our difficulty.

We take an alternative, therefore. Instead of fixing the value of $\bar{\Phi}'$ contained in equations (34) prior to determining those of $\omega'$ and $\lambda'$, let us regard from now on $(\omega, \lambda, \bar{\Phi})$ as a set of the parameters to be determined simultaneously for the boundary layer, which means that provided a quantity, $\bar{v}(\xi)$ for example, is given which suffices to define the boundary layer uniquely, then $(\omega, \lambda, \bar{\Phi})$ is to be determined accordingly. Namely, the third equation of (34) becomes the supplementary equation for determining $\bar{\Phi}$, and the system of equations is closed. In the numerical integrations, by virtue of the requirement that the three equations of (34) should hold valid in each step, $(\omega, \lambda, \bar{\Phi})$ can be determined. If $\bar{T}(\xi)$ is assigned, on the contrary, equation (31′) or

$$\bar{\Phi}=\bar{T}/\lambda^2 \tag{39}$$

should be added to equations (34) as the fourth equation, and further if the non-dimensional thickness of the boundary layer, $\bar{\delta}(\xi)$, is assigned, we have from equation (31″)

$$\left.\begin{aligned} &\bar{\Phi}=\bar{v}^{-2/3}/CH^2, \\ \text{where}\quad &H=R^{1/2}\bar{\delta}, \text{ and } \quad R=lv_0^{1/3}/\nu, \end{aligned}\right\} \tag{40}$$

which is to be added to equations (34). The displacement thickness $\bar{\delta}_1$, the momentum thickness $\bar{\delta}_2$, or the energy thickness $\bar{\delta}_3$ is correlated to $\bar{\delta}$ with the known relationship

$$\left.\begin{aligned} &H=H_n/E_n(\omega,\lambda) \\ \text{where}\quad &H_n=R^{1/2}\bar{\delta}_n \quad (n=1,\ 2 \text{ and } 3). \end{aligned}\right\} \tag{41}$$

For instance, according to equation (15), we have

$$E_1=\frac{1}{2}+\frac{\omega}{20}-\frac{\lambda}{10}. \tag{41′}$$

By substituting equation (41) for $H$ in equation (40), $\bar{\Phi}$ can be evaluated for the case where one or the other thickness of the boundary layer is assigned, which might be regarded as an extension of the inverse problem.

Although variation more or less in the technique of numerical computation may become necessary according to the kind of a given quantity, difficulties to be encountered in constructing the solution of the system of these equations have been removed by and large. Under the most unfavorable circumstances, the method of successive approximation may be necessitated over the whole domain for the numerical solution of the problem, but in cases where $v$ or $T$ is given, use of the method can be confined practically to small regions relevent to traverse of the curves $\Delta^{(\pm)}=0$. We might expect, however, on the other hand, that for problems in which the curves $\Delta^{(\pm)}=0$ have to be traversed a number of times, unified use of the successive approximation would be rather practical. On the contrary, for problems much simpler in which computation procedure for assigned $v$ or $T$ can be employed invariably, the scheme (20) or (23) respectively is, of course, more straightforward.

Use of the scheme incorporated in $\Phi$-function may be extended further so as to enable us to investigate the boundary layer flow for the case where $\Phi(x)$ is assigned for the first place over the whole domain. Thus possibility of theoretical study of boundary layers having miscellaneous specific features might be expected. In this case, the solution valid in the leading edge region are given once again by equations (26) through (30) and expansions of $\overline{\Phi}$ or $\overline{T}/\lambda^2$, calculated out of these equations are written in the form

$$\left.\begin{aligned} &\overline{\Phi}=\phi_{-1}\xi^{-1}+\phi_0+\phi_1\xi+\cdots, \\ \text{where}\quad &\phi_{-1}=0.28769,\quad \phi_0=0.10146\,t_0, \\ &\phi_1=0.07426\,t_1-0.01171\,t_0^2,\quad \cdots. \end{aligned}\right\} \tag{42}$$

These $\phi$'s are now given quantities, and equations (26) through (30) are solutions of equations (32) or (34). It is quite easy to rewrite other expansion coefficients in terms of $\phi$'s.

### 6. Two illustrative examples

Confining ourselves in this paper to critical survey of our method by means of some simple illustrative examples, we shall explain in more detail treatments of general problems of practical importance in a paper to be published in near future.

(I) Flow with assigned boundary layer thickness

According to equations (40) in the preceding section, in terms of the boundary layer thickness $H(\xi)$, we can write

$$\overline{\Phi}=\overline{v}^{2/3}/CH^2\,; \tag{i}$$

we have therefore

$$\frac{\overline{\Phi}'}{\overline{\Phi}}=2\left(\frac{\overline{v}'}{3\overline{v}}-\frac{H'}{H}\right). \tag{ii}$$

Substituting (i) and (ii) for $\overline{\Phi}$ and $\overline{\Phi}'/\overline{\Phi}$ respectively in equations (34), we have

$$\left.\begin{aligned}
\omega' &= \frac{1}{\Delta^{(0)}}\begin{vmatrix} 2A_0\dfrac{H'}{H}+\left(\dfrac{d_0}{6}+B_0\omega\right)\dfrac{\overline{v}'}{3\overline{v}\omega} & b_0 \\ 2A_1\dfrac{H'}{H}+\left(\dfrac{d_1}{6}+B_1\omega\right)\dfrac{\overline{v}'}{3\overline{v}\omega} & b_1 \end{vmatrix}, \\
\lambda' &= \frac{1}{\Delta^{(0)}}\begin{vmatrix} 2A_0\dfrac{H'}{H}+\left(\dfrac{d_0}{6}+B_0\omega\right)\dfrac{\overline{v}'}{3\overline{v}\omega} & a_0+b_0 \\ 2A_1\dfrac{H'}{H}+\left(\dfrac{d_1}{6}+B_1\omega\right)\dfrac{\overline{v}'}{3\overline{v}\omega} & a_1+b_1 \end{vmatrix}, \\
\text{and} \quad \overline{v}' &= \frac{18\overline{v}^{2/3}\omega}{H^2}, \quad \text{i.e.} \quad \frac{\overline{v}'}{3\overline{v}\omega}=\frac{6}{\overline{v}^{1/3}H^2};
\end{aligned}\right\} \tag{iii}$$

integration of this system of equations can be carried out, provided $H(\xi)$ is prescribed.

The solution in the form of series expansions valid for the leading edge region is given by equations (26) through (30). Equation (i), or

$$H=\overline{v}^{1/3}(C\overline{\Phi})^{-1/2}=\overline{v}^{1/3}\lambda\,(C\overline{T})^{-1/2}, \tag{iv}$$

mav be written as

$$\left.\begin{aligned}
&H=h_0\xi^{1/2}(1+h_1\xi+h_2\xi^2+\cdots), \\
\text{where}\quad &h_0=5.5598, \quad h_1=-0.09397t_0, \\
&h_2=0.04155t_0^2-0.07529t_1, \quad \cdots,
\end{aligned}\right\} \tag{v}$$

if these series expansions are introduced.

The first example is related to the inverse problem for a linearly retarded flow. By the use of the numerical table of $H(\xi)$ defined as

$$H=\{-6\,\omega(\xi)\}^{1/2}, \quad \text{i.e.} \quad H'=H\omega'/2\,\omega \tag{vi}$$

for the standard problem (see also the footnote on p. 27),[5] integration was performed starting at $\xi=0.0300$ from the initial values predicted by the solution due to series expansions up to a point lying in the neighborhood of the flow separation. Values of $\omega$, $\lambda$ and $\overline{v}$ thus obtained showed satisfactory agreement with the solution yielded from the standard problem.[5] This might be understood as a reaffirmation of the fact that the solution disclosed here for the inverse problem is exactly the inversed process to the computation for the standard problem. In Table 2, numerical values of $H$ and $E_1$ are given for the case where $\overline{u}_1(\xi)$ is assigned. From equations (17), $\overline{u}_1$ is nothing but $\overline{v}^{1/3}$. $E_1$, the multiplying factor deriving the displacement thickness $H_1$ from the boundary layer thickness $H$, is defined in equations (41) and (41′).

Our next problem is an accelerated flow defined by

$$H(\xi)=5.5598\xi^{1/2}(1-\xi/2). \tag{vii}$$

In the leading edge region, it is found from equations (v) that

$$\left.\begin{aligned} &h_1=-0.5, \text{ and } h_2=0, \\ \text{i.e.} \quad &t_0=5.3208, \text{ and } t_1=15.625. \end{aligned}\right\} \tag{viii}$$

Making use of these values in equations (30), we are enabled to find the first several terms of the series expansions for $\overline{v}$, $\overline{T}$, $\omega$ and $\lambda$. Connected with their values at $\xi = 0.0300$ predicted from these expansions, numerical integrations due to Runge, Kutta and Gill were carried out. Table 3 incorporates a part of results out of the computation.

In case where the assigned thickness relates to displacement, momentum, or energy, $H$ is found to be the combined function composed from the given $H_n(\xi)$ and $E_n(\omega, \lambda)$ ($n=1$, 2 and 3), as shown in equations (41). It can be treated in the same manner as $\overline{\Phi}$, and numerical integration can be done step by step. Or alternatively, since variation of $E_n(\omega, \lambda)$ experienced when we enter the next new step is expected to remain small in general as seen in Tables 2 and 3, we might integrate throughout the whole range of $\xi$ with the use of assumed values of $E_n(\xi)$, correct the assumption out of the result of the computation, and repeat the integration. This might be a simpler way of constructing the solution.

Table 2 Values of $H$ and $E_1$ for a linearly retarded flow $\overline{u}_1=1-\xi$

| $\xi$ | $H$ | $E_1$ |
|---|---|---|
| 0.00 | 0.0000 | 0.3136 |
| 0.01 | 0.5623 | 0.3186 |
| 0.02 | 0.8053 | 0.3240 |
| 0.03 | 0.9996 | 0.3295 |
| 0.04 | 1.1707 | 0.3354 |
| 0.05 | 1.3289 | 0.3418 |
| 0.06 | 1.4804 | 0.3489 |
| 0.07 | 1.6283 | 0.3568 |
| 0.08 | 1.7763 | 0.3658 |
| 0.09 | 1.9278 | 0.3762 |
| 0.10 | 2.0890 | 0.3890 |

Table 3 Values of $\overline{u}_1$ and $E_1$ for an accelerated flow $H = 5.5598\ \xi^{1/2}(1-\xi/2)$

| $\xi$ | $\overline{u}_1$ | $E_1$ |
|---|---|---|
| 0.00 | 1.000 | 0.3136 |
| 0.01 | 1.004 | 0.3115 |
| 0.02 | 1.009 | 0.3093 |
| 0.03 | 1.013 | 0.3075 |
| 0.04 | 1.018 | 0.3051 |
| 0.05 | 1.023 | 0.3027 |
| 0.06 | 1.028 | 0.3003 |
| 0.07 | 1.033 | 0.2979 |
| 0.08 | 1.039 | 0.2955 |
| 0.09 | 1.044 | 0.2931 |
| 0.10 | 1.050 | 0.2906 |

(II) Separation of a decelerated flow

According to our tentative computation for the front part of the boundary layer of a linearly retarded flow, so-called Howarth's boundary layer,[6] $\overline{\Phi}(\xi)$ for that part of the boundary layer may be expressed approximately by the formula

$$\overline{\Phi}=0.28770\xi^{-1}-1.2100. \tag{ix}$$

Connecting it at $\xi=0.0300$ with the solution derived from series expansions in the range $\xi\leq 0.0300$ for Howarth's boundary layer, we are now in a position to compute further the boundary layer developing downstream. The results are summarized in Table 4. In fact, the velocity of the maintsream prevailing outside the boundary layer is found to obey approximately the relation $\overline{u}_1=1-\xi$, i.e. $\overline{u}_1'=-1$ for the range $\xi \lesssim 0.10$. In the region downstream, however, it is found that the rate of retardation decreases gradually and that the separation takes place midway $\xi=0.1440$ and $0.1442$ (Howarth's boundary layer separates at $\xi=0.120$). By means of direct computation of $\Delta^{(+)}$ or plot of points on the $(\omega, \lambda)$-plane (Fig. 1), it can be verified that the point of intersection of the integral curve with the $\Delta^{(+)}$-curve lies in the immediate neighborhood of $\xi=0.142$.

Incidentally, if the computation should be continued further beyond the range of Table 4, the rate of deceleration decreases remarkably as $\xi$ increasing. It is found that the minimum speed ($\overline{u}_1'=0$, $\omega=0$, $\lambda<0$) is attained at a point in the neighborhood of $\xi=0.180$ and that afterwards the speed is accelerated again gradually, yielding the point of reattachment ($\overline{u}_1'>0$, $\omega>0$, $\lambda=0$) near $\xi=0.200$. Thus, it is concluded that the

Table 4 Values of characteristic quantities of the boundary layer defined by $\overline{\Phi}=0.28770\ \xi^{-1}-1.2100$

| $\xi$ | $\omega$ | $\lambda$ | $\overline{u}_1$ | $\overline{u}_1'$ |
|---|---|---|---|---|
| 0.03 | −0.17639 | 1.61298 | 0.9689 | −1.062 |
| 0.04 | −0.22321 | 1.53513 | 0.9589 | −0.980 |
| 0.06 | −0.36405 | 1.33135 | 0.9390 | −0.999 |
| 0.08 | −0.51935 | 1.08600 | 0.9191 | −0.990 |
| 0.10 | −0.68508 | 0.79553 | 0.8996 | −0.958 |
| 0.12 | −0.83971 | 0.45551 | 0.8814 | −0.866 |
| 0.140 | −0.91912 | 0.07741 | 0.8655 | −0.699 |
| 0.142 | −0.91802 | 0.03907 | 0.8642 | |
| 0.1440 | −0.91449 | 0.00094 | 0.8628 | |
| 0.1442 | −0.91400 | −0.00286 | 0.8627 | |
| 0.15 | −0.88672 | −0.11108 | 0.8591 | |
| 0.16 | −0.76429 | −0.28134 | 0.8542 | −0.416 |

flow pattern corresponds to a simplified model of a separation bubble.

## 7. Concluding remarks

In the preceding sections, a simplified computation procedure for time-independent laminar boundary layer in two dimensions of an incompressible fluid was explained in some detail. Verification of accuracy of the method and application to practical problems were scarcely referred to.* It is hoped, therefore, to have another chance to discuss these problems and an extension of the method to three-dimensional boundary layers as well.

## Acknowledgments

The authors greatly appreciate assistances offered by Mr. S. Inoue and Miss S. Hoshino in preparing the manuscript of this paper.

## References

1 ) Keller, H. B. and Cebeci, T.: *An inverse problem in boundary-layer flows: numerical determination of pressure gradient for a given wall shear,* J. Compt. Phys. **10** (1972) 151.

2 ) Horton, H. P.: *Separating laminar boundary layers with prescribed wall shear,* AIAA J. **12** (1974) 1772.

3 ) Carter, J. E.: *Inverse solutions for laminar boundary-layer flows with separation and reattachment,* NASA TR R-447 (1975) pp. 61.

4 ) Williams, J. C., III: *Incompressible boundary-layer separation,* Ann. Rev. Fluid Mech. **9** (1977) 113.

5 ) Yamada, H.: *A method of approximate integration of the laminar boundary layer equations,* Rep. Res. Inst. Fluid Engrg. **6** (1950) 87.

6 ) Howarth, L.: *On the solution of the laminar boundary layer equations,* Proc. Roy. Soc. A **164** (1938) 547.

(Received Febrary 18, 1982)

* In this connection, see the following papers : Yamamoto, M. : *An approximate calculation of inverse problems of laminar boundary layers by the integral method* (I) and (II) (in Japanese), to appear in Journal of the Japan Society for Aeronautical and Space Sciences.