# THE RESONANT INTERACTION BETWEEN A LONG INTERNAL GRAVITY WAVE AND A SURFACE GRAVITY WAVE PACKET

Oikawa, Masayuki
Research Institute for Applied Mechanics, Kyushu University : Associate Professor

Funakoshi, Mitsuaki
Research Institute for Applied Mechanics, Kyushu University : Research Associate

https://doi.org/10.5109/6779643

出版情報：Reports of Research Institute for Applied Mechanics. 29 (93), pp.207-219, 1982-02. 九州大学応用力学研究所

Reports of Research Institute for Applied Mechanics
Vol. XXIX, No. 93, 1982

# THE RESONANT INTERACTION BETWEEN A LONG INTERNAL GRAVITY WAVE AND A SURFACE GRAVITY WAVE PACKET

By Masayuki OIKAWA* and Mitsuaki FUNAKOSHI**

The long-short wave interactions between an internal gravity wave and a surface gravity wave packet in a two-layer fluid are investigated. When the phase speed of the internal wave coincides with the group velocity of the surface wave packet, a strong interaction occurs. The equations describing this interaction are derived both for a shallow (in comparison with a length scale of the long wave) fluid layer and for a deep one. Numerical solutions to these equations are also presented.

***Key words***: Long-short wave interaction, Internal gravity wave, Surface gravity wave, Soliton

## 1. Introduction

It is well-known that a strong interaction between long and short waves occurs when the phase speed of the long wave and the group velocity of the short wave are matched, and can be interpreted as a limiting case of the three-wave resonant interaction.[1)2)] This interaction between a long gravity wave and a capillary-gravity wave was investigated by Benney[2)] for water of infinite depth and by Djordjevic and Redekopp[3)] for that of finite depth. Djordjevic and Redekopp[3)] obtained the coupled equations

$$iA_\tau + \lambda A_{\xi\xi} = BA, \tag{1a}$$

$$B_\tau = -\alpha(|A|^2)_\xi, \tag{1b}$$

where $A$ and $B$ are proportional to the envelope of a capillary-gravity wave and to the amplitude of a long gravity wave, respectively and $\lambda$ and $\alpha$ are positive constants. This resonance is related to the singularity of the coefficient of the nonlinear term in the nonlinear Schrödinger equation describing the long-time evolution of a capillary-gravity wave packet. The stretched coordinates $\xi, \tau$ are chosen so as

* Associate Professor, Research Institute for Applied Mechanics, Kyushu University.
** Research Associate, Research Institute for Applied Mechanics, Kyushu University.

to fit this fact.

The same equations as eqs. (1) were derived by Grimshaw[4] for long and short internal waves. The model equations for the interaction of a Langmuir wave with an ion-sound wave in a plasma

$$iE_t + E_{xx}/2 = nE, \tag{2a}$$

$$n_t + n_x = -(|E|^2)_x, \tag{2b}$$

which are reduced to eqs. (1) through the transformation $\phi = Ee^{i(\frac{t}{2}-x)}$, $\xi = x-t$ and $\tau = t$, were solved exactly by Yajima and Oikawa[5] by using the inverse scattering method and shown to have the $N$-soliton solutions. The equations (1) were also solved by Ma[6] in the similar way. Ma and Redekopp[7] gave some another solutions to eqs. (1).

The long-short wave interactions in certain general model equation were discussed by Benney[8] for some different combinations of the relative magnitudes of three relevant dimensionless parameters. Newell[9] presented and analysed a model equation for long-short wave interaction which is solvable by the inverse scattering method.

In this paper, we investigate the interaction between a long internal wave and a short surface wave in a two layer fluid. When the fluid depth is sufficiently small in comparison with the wavelength of the internal wave, the equations describing the interaction become the same form as eqs. (1). On the other hand, the equations are modified by the term representing the dispersion of the internal wave when the fluid depth is sufficiently large in comparison with the wavelength of the internal wave. The uniform amplitude periodic solutions to eqs. (1) are unstable for modulational perturbations with small wavenumbers[3)7)]. For the case of a deep fluid a similar modulational instability is found. Though we cannot find simple analytical solutions to the equations in a deep fluid, numerical solutions suggest that there exist solitary wave solutions to the equations and they appear in asymptotic states for a wide class of initial conditions.

## 2. Formulation and Reduced Interaction Equations

We consider the wave propagation in a two layer fluid which consists of the upper layer with the density $\rho_1 = \rho_2(1-\Delta)$ and the thickness $h_1$ and the lower layer with the density $\rho_2$ ($> \rho_1$) and the thickness $h_2$. The fluids are assumed to be inviscid and incompressible. If the disturbed free surface is specified by $z = h_1 + \zeta_1(x, t)$, the disturbed interface by $z = \zeta_2(x, t)$ and the velocity potentials in the upper and lower layers by $\phi_1(x, z, t)$ and $\phi_2(x, z, t)$, and there is the flat rigid boundary at $z = -h_2$, the basic equations and the boundary conditions are written as

$$\frac{\partial^2\phi_1}{\partial x^2}+\frac{\partial^2\phi_1}{\partial z^2}=0, \qquad \zeta_2<z<h_1+\zeta_1, \tag{3a}$$

$$\frac{\partial\zeta_1}{\partial t}+\frac{\partial\phi_1}{\partial x}\frac{\partial\zeta_1}{\partial x}-\frac{\partial\phi_1}{\partial z}=0 \quad \text{at} \quad z=h_1+\zeta_1, \tag{3b}$$

$$\frac{\partial\phi_1}{\partial t}+g\zeta_1+\frac{1}{2}\left(\frac{\partial\phi_1}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial\phi_1}{\partial z}\right)^2=0 \quad \text{at} \quad z=h_1+\zeta_1, \tag{3c}$$

$$\frac{\partial^2\phi_2}{\partial x^2}+\frac{\partial^2\phi_2}{\partial z^2}=0, \qquad -h_2<z<\zeta_2 \tag{3d}$$

$$\frac{\partial\zeta_2}{\partial t}+\frac{\partial\phi_2}{\partial x}\frac{\partial\zeta_2}{\partial x}-\frac{\partial\phi_2}{\partial z}=0 \quad \text{at} \quad z=\zeta_2, \tag{3e}$$

$$\frac{\partial\zeta_2}{\partial t}+\frac{\partial\phi_1}{\partial x}\frac{\partial\zeta_2}{\partial x}-\frac{\partial\phi_1}{\partial z}=0 \quad \text{at} \quad z=\zeta_2, \tag{3f}$$

$$\rho_1\left\{\frac{\partial\phi_1}{\partial t}+g\zeta_2+\frac{1}{2}\left(\frac{\partial\phi_1}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial\phi_1}{\partial z}\right)^2\right\}-\rho_2\left\{\frac{\partial\phi_2}{\partial t}+g\zeta_2+\frac{1}{2}\left(\frac{\partial\phi_2}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial\phi_2}{\partial z}\right)^2\right\}=0 \quad \text{at} \quad z=\zeta_2, \tag{3g}$$

$$\frac{\partial\phi_2}{\partial z}=0 \quad \text{at} \quad z=-h_2, \tag{3h}$$

where $g$ is the gravitational acceleration. Since we consider weak nonlinear effects, it is convenient to expand the boundary conditions at $z=h_1+\zeta_1$ and $z=\zeta_2$ about $z=h_1$ and $z=0$, respectively. The result is given in the appendix A. Linear approximation of eqs. (A. 1) yields the dispersion relation

$$D(\omega, k)\equiv\omega^4[1+(1-\Delta)\sigma_1\sigma_2]-gk\,\omega^2(\sigma_1+\sigma_2)+\Delta g^2k^2\sigma_1\sigma_2=0, \tag{4}$$

where $\sigma_1=\mathrm{th}\,(kh_1)$, $\sigma_2=\mathrm{th}\,(kh_2)$. The equation (4) gives the surface mode and the internal mode. For $h_2\to\infty$, $h_1=h$, eq. (4) reduces to

$$(\omega^2-gk)\left(\omega^2-\frac{\Delta gk\,\sigma}{1+(1-\Delta)\sigma}\right)=0, \quad \sigma=\mathrm{th}(kh). \tag{5}$$

### 2.1. Shallow fluid

We consider the interaction of a surface wave and an internal wave in the shallow fluid. We assume that $\varepsilon\equiv a/\lambda\ll1$, $\lambda/h_1\sim1$, $h_2/h_1\sim1$ and $\lambda/l\ll1$. Here $a$ and $\lambda$ are a characteristic amplitude and a characteristic length scale of the surface wave, respectively, and $l$ is a characteristic length scale of the internal wave. We expand the solution in the following form:

$$\phi_1=\varepsilon^{2/3}\phi_1^{(0)}+\varepsilon\phi_1^{(1)}+\varepsilon^{4/3}\phi_1^{(2)}+\varepsilon^{5/3}\phi_1^{(3)}+\varepsilon^2\phi_1^{(4)}+\varepsilon^{7/3}\phi_1^{(5)}+\varepsilon^{8/3}\phi_1^{(6)}+\cdots, \tag{6a}$$

$$\zeta_1=\varepsilon^{2/3}\zeta_1^{(0)}+\varepsilon\zeta_1^{(1)}+\cdots, \tag{6b}$$

$$\phi_2=\varepsilon^{2/3}\phi_2^{(0)}+\varepsilon\phi_2^{(1)}+\cdots, \tag{6c}$$

$$\zeta_2=\varepsilon^{2/3}\zeta_2^{(0)}+\varepsilon\zeta_2^{(1)}+\cdots. \tag{6d}$$

The variables $\zeta_1^{(n)}, \zeta_2^{(n)}$ (and $\phi_1^{(n)}, \phi_2^{(n)}$), $(n=0, 1, 2,\cdots)$ are regarded as functions of the stretched coordinates

$$\xi=\varepsilon^{2/3}(x-c_g t), \qquad \tau=\varepsilon^{4/3}t, \tag{7}$$

as well as $x, t$ (and $z$). Substituting eqs. (6a, c) into eqs. (A. 1) and equating the coefficients of like powers of $\varepsilon$, we obtain a linear boundary value problem for $\phi_1^{(n)}, \phi_2^{(n)}$ from the terms of $O(\varepsilon^{(n+2)/3})$.

Both for $\phi_1^{(0)}, \phi_2^{(0)}$ and for $\phi_1^{(1)}, \phi_2^{(1)}$, the equations and the boundary conditions are given by the linearized forms of eqs. (A. 1). We assume that $\phi_1^{(0)}, \phi_2^{(0)}$ consist of a long wave alone and $\phi_1^{(1)}$, $\phi_2^{(1)}$ a short wave alone. Therefore, we take

$$\begin{cases} \phi_1^{(0)}=\Phi_1(\xi,\tau), \quad \phi_2^{(0)}=\Phi_2(\xi,\tau), \\ \zeta_1^{(0)}=\zeta_2^{(0)}=0, \end{cases} \tag{8}$$

and

$$\begin{cases} \phi_1^{(1)}=\dfrac{i\omega}{k}A(\xi,\tau)e^{i\theta}\{\mathrm{sh}k(h_1-z)-\dfrac{gk}{\omega^2}\mathrm{ch}\,k(h_1-z)\}+\mathrm{c.c.}, \\ \zeta_1^{(1)}=A(\xi,\tau)e^{i\theta}+\mathrm{c.c.}, \\ \phi_2^{(1)}=-\dfrac{i\omega}{k\mathrm{sh}kh_2}\Big(\mathrm{ch}\,kh_1-\dfrac{gk}{\omega^2}\mathrm{sh}\,kh_1\Big)A(\xi,\tau)e^{i\theta}\mathrm{ch}\,k(z+h_2)+\mathrm{c.c.}, \\ \zeta_2^{(1)}=\Big(\mathrm{ch}\,kh_1-\dfrac{gk}{\omega^2}\mathrm{sh}\,kh_1\Big)A(\xi,\tau)e^{i\theta}+\mathrm{c.c.}, \end{cases} \tag{9}$$

where $\theta=kx-\omega t$, $k, \omega$ satisfying the dispersion relation of the surface mode and c.c. denotes the complex conjugate of the preceding term. It is noted that the long wave given by eqs. (8) gives rise to a horizontal velocity of $O(\varepsilon^{4/3})$. In the next order, we obtain

$$\begin{cases} \phi_1^{(2)}=\Phi_1^{(2)}(\xi,\tau), \quad \phi_2^{(2)}=\Phi_2^{(2)}(\xi,\tau), \\ \zeta_1^{(2)}=\dfrac{c_g}{g}\dfrac{\partial\Phi_1}{\partial\xi}, \quad \zeta_2^{(2)}=\dfrac{c_g}{\Delta g}\Big[\dfrac{\partial\Phi_2}{\partial\xi}-(1-\Delta)\dfrac{\partial\Phi_1}{\partial\xi}\Big]. \end{cases} \tag{10}$$

The equations and the boundary conditions for $\phi_1^{(3)}, \phi_2^{(3)}$ are inhomogeneous and the inhomogeneous terms are proportional to $e^{i\theta}$. The condition for no-secularity yields

$$c_g = -\frac{\partial D}{\partial k}\bigg/\frac{\partial D}{\partial \omega} = \frac{d\omega}{dk}. \tag{11}$$

That is, $c_g$ is the group velocity of the short wave. Then, we obtain

$$\phi_1^{(3)} = \left[\frac{\omega}{k}\frac{\partial A}{\partial \xi}(h_1-z)\left\{\mathrm{ch}k(h_1-z) - \frac{gk}{\omega^2}\mathrm{sh}k(h_1-z)\right\}\right.$$
$$+\frac{i\omega}{k}B(\xi,\tau)\left\{\mathrm{sh}k(h_1-z) - \frac{gk}{\omega^2}\mathrm{ch}k(h_1-z)\right\}$$
$$\left.+\left(\frac{2gc_g}{\omega^2} - \frac{g}{k\omega}\right)\frac{\partial A}{\partial \xi}\mathrm{ch}k(h_1-z)\right]e^{i\theta} + \text{c.c.}, \quad \text{etc.}$$

Though the inhomogeneous terms in the equations and the boundary conditions for $\phi_1^{(4)}$, $\phi_2^{(4)}$ include the second harmonic terms as well as the long wave terms, it is sufficient here to consider the long wave terms alone. In order that the boundary value problem for the long wave component of $\phi_1^{(4)}, \phi_2^{(4)}$ have a solution, $\Phi_1$ and $\Phi_2$ must satisfy the relations

$$(gh_1 - c_g^2)\frac{\partial^2 \Phi_1}{\partial \xi^2} + gh_2\frac{\partial^2 \Phi_2}{\partial \xi^2} = 0, \tag{12a}$$

$$(\Delta gh_1 - c_g^2)\frac{\partial^2 \Phi_1}{\partial \xi^2} + c_g^2\frac{\partial^2 \Phi_2}{\partial \xi^2} = 0. \tag{12b}$$

The compatibility condition for these becomes

$$c_g^2 = c_p^2, \tag{13}$$

where $c_p$ is the phase speed of the internal wave in the long wave limit and given by

$$c_p = \left[\frac{g}{2}\left(h_1 + h_2 - \sqrt{(h_1+h_2)^2 - 4\Delta h_1 h_2}\right)\right]^{\frac{1}{2}}. \tag{14}$$

We assume $k$ to satisfy the resonance condition $c_g = c_p$. Then,

$$\begin{cases} \phi_1^{(4)} = -\dfrac{1}{2}\dfrac{\partial^2 \Phi_1}{\partial \xi^2}z^2 + h_1\left(1 - \dfrac{c_g^2}{gh_1}\right)\dfrac{\partial^2 \Phi_1}{\partial \xi^2}z + \Phi_1^{(4)}(\xi,\tau), \\ \phi_2^{(4)} = -\dfrac{1}{2}\dfrac{\partial^2 \Phi_2}{\partial \xi^2}(z+h_2)^2 + \Phi_2^{(4)}(\xi,\tau), \end{cases} \tag{15}$$

except for the second harmonic terms.

The inhomogeneous terms in the equations and the boundary conditions for $\phi_1^{(5)}, \phi_2^{(5)}$ are also proportional to $e^{i\theta}$. The condition for no-secularity leads to

$$i\frac{\partial A}{\partial \tau}+\frac{1}{2}\omega''\frac{\partial^2 A}{\partial \xi^2}=-\alpha\zeta_2^{(2)}A, \quad \omega''=\frac{d^2\omega}{dk^2}, \tag{16}$$

where $\alpha$ is given by eq. (B. 1) in the appendix B. In deriving eq. (16), we have used the relations

$$\zeta_1^{(2)}=-\frac{c_g^2}{gh_1-c_g^2}\zeta_2^{(2)}, \quad \frac{\partial \Phi_1}{\partial \xi}=-\frac{gc_g}{gh_1-c_g^2}\zeta_2^{(2)}, \quad \frac{\partial \Phi_2}{\partial \xi}=\frac{c_g}{h_2}\zeta_2^{(2)}, \tag{17}$$

which can be obtained from eqs. (10) and (12) under usual boundary conditions.

From the equations and the boundary conditions for the long wave component of $\phi_1^{(6)}$, $\phi_2^{(6)}$, we obtain as the compatibility condition

$$\frac{\partial \zeta_2^{(2)}}{\partial \tau}=\beta\frac{\partial |A|^2}{\partial \xi}, \tag{18}$$

where eqs. (17) have been used and $\beta$ is given by eq. (B. 2) in the appendix B.

The equations (16) and (18) give the desired coupled set of equations. Numerical computations show that $\alpha>0$ and $\beta>0$. As shown easily, $\omega''<0$. Since $gh_1-c_g^2>0$, the signs of $\zeta_1^{(2)}$ and $\zeta_2^{(2)}$ are opposite each other. The equations (16) and (18) are equivalent to eqs. (1) and also to eqs. (2) which are exactly solvable by using the inverse scattering method[5)6)]. The soliton solutions to eqs. (16) and (18) are

$$A=\left(\frac{-\omega''p}{\alpha\beta}\right)^{1/2} q \operatorname{sech} q(\xi-p\tau-\xi_0)\exp\left[i\left\{\frac{p}{\omega''}\xi+\frac{1}{2}\left(\omega''q^2-\frac{p^2}{\omega''}\right)\tau+\theta_0\right\}\right], \tag{19a}$$

$$\zeta_2^{(2)}=\frac{\omega''}{\alpha}q^2\operatorname{sech}^2 q(\xi-p\tau-\xi_0), \tag{19b}$$

where $q$, $\xi_0$ and $\theta_0$ are arbitrary real constants and $p$ is an arbitrary positive constant. The equations (19) show that the soliton travels with the velocity larger than $c_g$ and it is accompanied by the depression of the interface. The $N$-soliton solutions were obtained by Yajima and Oikawa[5)] and by Ma[6)]. The equations (16) and (18) also have the envelope-hole solitary wave solutions with negative velocities and the breather solutions[7)].

### 2.2. Deep fluid

In the preceding subsection, we supposed that the length scale of the long internal wave is sufficiently larger than the fluid depth. Therefore, the preceding analysis is not applicable to the case that the fluid depth is sufficiently larger than the length scale of the long internal wave. We consider here this case and assume that $\varepsilon\equiv a/\lambda\ll 1$, $\lambda/h_1\sim 1$, $\lambda/l\ll 1$ and $h_2=\infty$. From now on, we omit the subscript from $h_1$. The vertical length scale of long wave component in the lower layer is the same as the horizontal one. This suggests that it is necessary to introduce the stretched coordinate $Z=\varepsilon^{2/3}z$ in addition to $\xi, \tau$. In the linear approximation, the boundary condition (A. 1d) means continuity of the vertical velocity, which suggests that the leading term of the velocity potential of the long wave component in the lower layer is $O(\varepsilon^{4/3})$ because the vertical velocity of the long wave component in the upper layer is $O(\varepsilon^2)$. Therefore, we assume that the solution is of the form

$$\phi_1=\varepsilon^{2/3}\phi_1^{(0)}+\varepsilon\phi_1^{(1)}+\varepsilon^{4/3}\phi_1^{(2)}+\cdots, \tag{20a}$$

$$\phi_2= \qquad \varepsilon\phi_2^{(1)}+\varepsilon^{4/3}\phi_2^{(2)}+\cdots. \tag{20b}$$

Introducing $Z=\varepsilon^{2/3}z$ as well as the variables $\xi, \tau$ given by eqs. (7), we regard the long wave component of $\phi_2^{(n)}$ as a function of $\xi, Z, \tau$.

We assume that $\phi_1^{(0)}$ consists of a long internal mode alone and $\phi_1^{(1)}$, $\phi_2^{(1)}$ consist of a short surface mode alone. Therefore, we take

$$\phi_1^{(0)}=\Phi_1(\xi,\tau), \qquad \zeta_1^{(0)}=0, \tag{21}$$

$$\left\{\begin{array}{l} \phi_m^{(1)}=-\dfrac{i\omega}{k}A(\xi,\tau)\mathrm{e}^{i\theta}\mathrm{e}^{k(z-h)}+\text{c.c.}, \quad (m=1,2), \\ \zeta_1^{(1)}=A(\xi,\tau)\mathrm{e}^{i\theta}+\text{c.c.}, \quad \zeta_2^{(1)}=\zeta_1^{(1)}\mathrm{e}^{-kh}, \\ \theta=kx-\omega t, \end{array}\right. \tag{22}$$

where $k, \omega$ satisfy the dispersion relation of the surface mode $\omega=(gk)^{1/2}$. In $O(\varepsilon^{4/3})$, we obtain

$$\left\{\begin{array}{l} \phi_1^{(2)}=\Phi_1^{(2)}(\xi,\tau), \quad \phi_2^{(2)}=\Phi_2^{(2)}(\xi,Z,\tau), \\ \zeta_1^{(2)}=\dfrac{c_g}{g}\dfrac{\partial\Phi_1}{\partial\xi}, \quad \zeta_2^{(2)}=-\dfrac{c_g}{\Delta g}(1-\Delta)\dfrac{\partial\Phi_1}{\partial\xi}. \end{array}\right. \tag{23}$$

In $O(\varepsilon^{5/3})$, the condition for no-secularity yields $c_g=d\omega/dk=\omega/2k$, and then we have

$$\phi_m^{(3)}=\left\{-\frac{\omega}{k}\frac{\partial A}{\partial\xi}(z-h)+B(\xi,\tau)\right\}\mathrm{e}^{i\theta}\mathrm{e}^{k(z-h)}+\text{c.c.}, \quad (m=1,2). \tag{24}$$

In $O(\varepsilon^2)$, from the compatibility condition we obtain

$$c_g^2=\Delta\, gh=c_p^2 \tag{25}$$

where $c_p=(\Delta gh)^{1/2}$ is the phase speed of the internal wave in the long wave limit, $l/h\to\infty$. The relation (25) is satisfied by taking $k, \omega$ so as to satisfy the resonance condition $c_g=c_p$. Then,

$$\left\{\begin{aligned}&\phi_1^{(4)}=-\frac{1}{2}\frac{\partial^2\Phi_1}{\partial\xi^2}z^2+f(\xi,\tau)z+\Phi_1^{(4)}(\xi,\tau),\\&\phi_2^{(4)}=\Phi_2^{(4)}(\xi,Z,\tau),\end{aligned}\right.\tag{26}$$

and

$$f(\xi,\tau)=\left[\frac{\partial\Phi_2^{(2)}}{\partial Z}\right]_{Z=0}=(1-\Delta)h\frac{\partial^2\Phi_1}{\partial\xi^2},\quad \frac{\partial\Phi_2^{(2)}}{\partial Z}\to 0\quad(Z\to-\infty).\tag{27}$$

In $O(\varepsilon^{7/3})$, the condition for no-secularity leads to

$$i\frac{\partial A}{\partial\tau}+\frac{1}{2}\omega''\frac{\partial^2A}{\partial\xi^2}=\frac{(1-\Delta)k(1-e^{-2kh})}{1-\Delta(1-e^{-2kh})}A\frac{\partial\Phi_1}{\partial\xi},\tag{28}$$

$$\omega''=\frac{d^2\omega}{dk^2}=-\frac{\omega}{4k^2}.$$

In $O(\varepsilon^{8/3})$, we obtain from the compatibility condition

$$2\frac{\partial^2\Phi_1}{\partial\xi\partial\tau}+c_g\left[\frac{\partial^2\Phi_2^{(2)}}{\partial\xi^2}\right]_{Z=0}+4\Delta\omega^2(1-e^{-2kh})\frac{\partial|A|^2}{\partial\xi}=0,\tag{29}$$

and also

$$\frac{\partial^2\Phi_2^{(2)}}{\partial\xi^2}+\frac{\partial^2\Phi_2^{(2)}}{\partial Z^2}=0,\quad Z<0.\tag{30}$$

From eqs. (30) and (27), we have

$$\left[\frac{\partial^2\Phi_2^{(2)}}{\partial\xi^2}\right]_{Z=0}=\frac{(1-\Delta)h}{\pi}\frac{\partial}{\partial\xi}\mathrm{P}\int_{-\infty}^{\infty}\frac{1}{\xi'-\xi}\frac{\partial^2}{\partial\xi'^2}\Phi_1(\xi',\tau)d\xi',$$

where P denotes the Cauchy principal value. Substituting this into eq. (29) we obtain

$$\frac{\partial^2\Phi_1}{\partial\xi\partial\tau}+\frac{(1-\Delta)hc_g}{2\pi}\frac{\partial^2}{\partial\xi^2}\mathrm{P}\int_{-\infty}^{\infty}\frac{1}{\xi'-\xi}\frac{\partial}{\partial\xi'}\Phi_1(\xi',\tau)d\xi'+2\Delta\omega^2(1-e^{-2kh})\frac{\partial|A|^2}{\partial\xi}=0.\tag{31}$$

By using $\zeta_2^{(2)}$ in place of $\Phi_1$, eqs. (28) and (31) are written as

$$i\frac{\partial A}{\partial\tau}+\frac{1}{2}\omega''\frac{\partial^2A}{\partial\xi^2}=-\alpha\zeta_2^{(2)}A,\tag{32a}$$

$$\frac{\partial\zeta_2^{(2)}}{\partial\tau}+\gamma\frac{\partial^2}{\partial\xi^2}\mathrm{H}(\zeta_2^{(2)})=\beta\frac{\partial|A|^2}{\partial\xi},\tag{32b}$$

where H represents the Hilbert transform

$$\mathrm{H}(f)=\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{f(\xi')}{\xi'-\xi}d\xi'$$

and

$$\alpha = \frac{2\Delta k\omega(1-e^{-2kh})}{1-\Delta(1-e^{-2kh})} > 0, \quad \beta = (1-\Delta)\omega(1-e^{-2kh}) > 0,$$
$$\gamma = \frac{1}{2}(1-\Delta)h\sqrt{\Delta gh} > 0, \quad \omega'' = -\frac{\omega}{4k^2} < 0. \tag{33}$$

From eq. (23),

$$\zeta_1^{(2)} = -\frac{\Delta}{1-\Delta}\zeta_2^{(2)}, \quad \frac{\partial \Phi_1}{\partial \xi} = -\frac{2\Delta\omega}{1-\Delta}\zeta_2^{(2)}. \tag{34}$$

The equations (32) are to be compared with the eqs. (16) and (18). The extra term in eq. (32b) represents the dispersion of the internal wave in a deep fluid.

## 3. Numerical Solutions and Some Remarks

The equations (32) can be reduced to the forms

$$i\frac{\partial S}{\partial T} - \frac{\partial^2 S}{\partial X^2} = -LS, \tag{35a}$$

$$\frac{\partial L}{\partial T} + \delta\frac{\partial^2}{\partial X^2}\mathrm{H}(L) = \frac{\partial |S|^2}{\partial X}, \tag{35b}$$

by an appropriate scale transformation. The coefficient $\delta$ is a positive constant. The equations (16) and (18) also can be reduced to the eqs. (35) with $\delta=0$. In the case $\delta=0$, the eqs. (35) have an infinite set of conservation laws[5)6)]. On the other hand, in the case $\delta \neq 0$, they seem to have only the following four conserved quantities,

$$I_1 = \int_{-\infty}^{\infty} L dX, \quad I_2 = \int_{-\infty}^{\infty} |S|^2 dX, \quad I_3 = \int_{-\infty}^{\infty} \left\{L^2 - i\left(S^*\frac{\partial S}{\partial X} - S\frac{\partial S^*}{\partial X}\right)\right\} dX,$$
$$I_4 = \int_{-\infty}^{\infty} \left\{\left|\frac{\partial S}{\partial X}\right|^2 + L|S|^2 - \frac{\delta}{2}L\frac{\partial}{\partial X}\mathrm{H}(L)\right\} dX. \tag{36}$$

Therefore, we cannot expect the eqs. (35) with non-zero $\delta$ to be completely integrable.

The uniform amplitude periodic solutions to eqs. (35) with $\delta=0$ are unstable for modulational perturbations with small wavenumbers[3)7)]. The similar instability occurs in the case $\delta \neq 0$. The uniform amplitude periodic solutions to eqs. (35) are given by

$$S = S_0 e^{iL_0T}, \quad L = L_0, \quad (S_0, L_0: \text{real constants}). \tag{37}$$

We consider the following perturbed solution

$$S=(S_0+\varepsilon a)\exp(iL_0T+i\varepsilon\theta),\qquad L=L_0+\varepsilon b, \tag{38}$$

where $\varepsilon$ is a small parameter. Substituting these into eqs. (35) and assuming the solution of the form

$$(a,\theta,b)=(\hat{a},\hat{\theta},\hat{b})\exp[i(\kappa X-\nu T)],\qquad(\kappa>0),$$

we obtain as the condition for instability

$$\kappa^3<27\,S_0^2\,[(\delta^2+3)(\sqrt{\delta^2+3}-\delta)+12\,\delta]^{-1}. \tag{39}$$

The numerical solutions to eqs. (35) are shown in Figs. 1 and 2. These are calculated under the periodic boundary condition with the period 80. In this case, the Hilbert transform in eq. (35b) must be replaced by

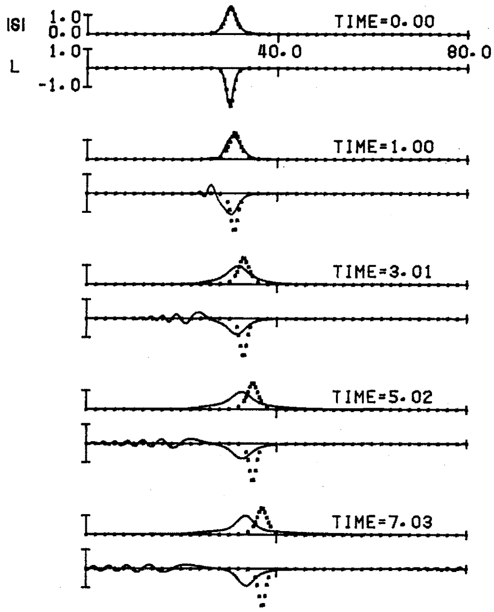


Fig. 1 Time development of solution for the initial data (41). ——: $\delta=1$, ……: $\delta=0$.

$$\mathrm{T}(L)=\frac{1}{2l}\mathrm{P}\int_{-l}^{l}L(X',T)\cot\frac{\pi}{2l}(X'-X)dX', \tag{40}$$

where $2l$ is the period. The initial data in Fig. 1 is

$$S=\sqrt{2}\,\mathrm{sech}(X-30)\exp(-iX/2), \tag{41a}$$

$$L=-2\,\mathrm{sech}^2(X-30), \tag{41b}$$

which is the value at $T=0$ of a soliton solution to eqs. (35) with $\delta=0$ and to propagate with the velocity $p=1$. The solid lines show the solution for $\delta=1$ and the dotted lines that for $\delta=0$. Figure 2 shows the generation of an internal wave by a surface wave. The initial data in Fig. 2 is given by $S$ in eq. (41a) and $L=0$. Figure 2 (a) shows the solution for $\delta=1$ and Fig. 2 (b) that for $\delta=0$. In the case $\delta=1$, an oscillatory tail of $L$ spreads behind owing to dispersion, but it seems that a coupled solitary type of wave is formed asymptotically. It is interesting to note in the case $\delta=0$ (Fig. 2 (b)) that the positive phase of $L$ remains there without propagating and the value of $|S|$ at corresponding location tends to zero, while the negative phase of $L$ forms a coupled soliton to propagate ahead.

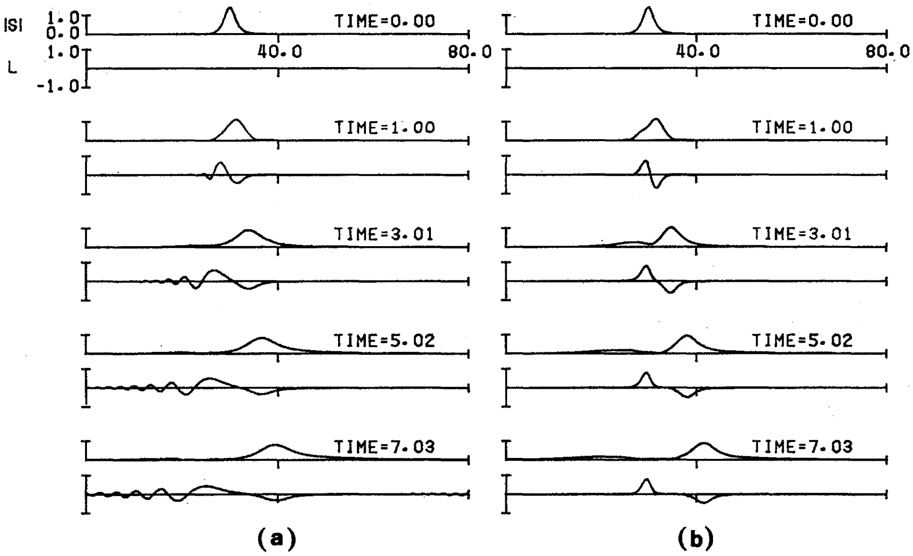


Fig. 2 Time development of solution for the initial data (41a) and $L=0$. (a) $\delta=1$, (b) $\delta=0$.

**Acknowledgements**

The numerical calculations were made by means of the FACOM M-200 of the Computer Center of Kyushu University and the MELCOM COSMO-900 of the Research Institute for Applied Mechanics in Kyushu University.

This work was supported by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture.

**References**

1) Phillips, O. M.: *The Dynamics of the Upper Ocean*, 2 nd ed. (Cambridge University Press, Cambridge, 1977) pp. 80–81.

2) Benney, D. J.: *Significant interactions between small and large scale surface waves*, Studies Appl. Math. **55** (1976) 93.
3) Djordjevic, V. D. and Redekopp, L. G.: *On two-dimensional packets of capillary-gravity waves*, J. Fluid Mech. **79** (1977) 703.
4) Grimshaw, R. H. J.: *The modulation of an internal gravity-wave packet, and the resonance with the mean motion*, Studies Appl. Math. **56** (1977) 241.
5) Yajima, N. and Oikawa, M.: *Formation and interaction of sonic-Langmuir solitons—inverse scattering method—*, Prog. Theor. Phys. **56** (1976) 1719.
6) Ma, Y. -C.: *The complete solution of the long-wave — short-wave resonance equations*, Studies Appl. Math. **59** (1978) 201.
7) Ma, Y. -C. and Redekopp, L. G.: *Some solutions pertaining to the resonance interaction of long and short waves*, Phys. Fluids **22** (1979) 1872.
8) Benney, D. J.: *A general theory for interactions between short and long waves*, Studies Appl. Math. **56** (1977) 81.
9) Newell, A. C.: *Long waves-short waves; a solvable model*, SIAM J. Appl. Math. **35** (1978) 650.

(Received November 30, 1981)

## Appendix

**A.** After expansion of the boundary conditions at $z=h_1+\zeta_1$ and $z=\zeta_2$ about $z=h_1$ and $z=0$ and elimination of $\zeta_1$ and $\zeta_2$, the equations and the boundary conditions (3) are reduced to

$$\frac{\partial^2\phi_1}{\partial x^2}+\frac{\partial^2\phi_1}{\partial z^2}=0, \quad 0<z<h_1, \tag{A. 1a}$$

$$\frac{\partial^2\phi_1}{\partial t^2}+g\frac{\partial\phi_1}{\partial z}=-2\frac{\partial\phi_1}{\partial x}\frac{\partial^2\phi_1}{\partial t\partial x}-2\frac{\partial\phi_1}{\partial z}\frac{\partial^2\phi_1}{\partial t\partial z}$$
$$+\frac{1}{g}\frac{\partial}{\partial z}\left[\frac{\partial\phi_1}{\partial t}\left(\frac{\partial^2\phi_1}{\partial t^2}+g\frac{\partial\phi_1}{\partial z}\right)\right] \quad \text{at} \quad z=h_1, \tag{A. 1b}$$

$$\frac{\partial^2\phi_2}{\partial x^2}+\frac{\partial^2\phi_2}{\partial z^2}=0, \quad -h_2<z<0, \tag{A. 1c}$$

$$\frac{\partial\phi_2}{\partial z}-\frac{\partial\phi_1}{\partial z}=\frac{1}{\Delta g}\left(\frac{\partial\phi_2}{\partial x}-\frac{\partial\phi_1}{\partial x}\right)\left((1-\Delta)\frac{\partial^2\phi_1}{\partial t\partial x}-\frac{\partial^2\phi_2}{\partial t\partial x}\right)$$
$$-\frac{1}{\Delta g}\left((1-\Delta)\frac{\partial\phi_1}{\partial t}-\frac{\partial\phi_2}{\partial t}\right)\left(\frac{\partial^2\phi_2}{\partial z^2}-\frac{\partial^2\phi_1}{\partial z^2}\right) \quad \text{at } z=0, \tag{A. 1d}$$

$$(1-\Delta)\frac{\partial^2\phi_1}{\partial t^2}-\Delta g\frac{\partial\phi_1}{\partial z}-\frac{\partial^2\phi_2}{\partial t^2}=-(1-\Delta)\frac{\partial}{\partial t}\left[\left(\frac{\partial\phi_1}{\partial x}\right)^2+\left(\frac{\partial\phi_1}{\partial z}\right)^2\right]$$
$$+\frac{1}{2}\frac{\partial}{\partial t}\left[\left(\frac{\partial\phi_2}{\partial x}\right)^2+\left(\frac{\partial\phi_2}{\partial z}\right)^2\right]+\frac{\partial\phi_1}{\partial x}\frac{\partial^2\phi_2}{\partial t\partial x}+\frac{\partial\phi_1}{\partial z}\frac{\partial^2\phi_2}{\partial t\partial z}$$

$$-\frac{1}{\Delta g}\frac{\partial}{\partial z}\left[\left((1-\Delta)\frac{\partial\phi_1}{\partial t}-\frac{\partial\phi_2}{\partial t}\right)\left((1-\Delta)\frac{\partial^2\phi_1}{\partial t^2}-\Delta g\frac{\partial\phi_1}{\partial z}-\frac{\partial^2\phi_2}{\partial t^2}\right)\right]$$

$$\text{at } z=0, \qquad (\text{A. 1e})$$

$$\frac{\partial\phi_2}{\partial z}=0 \qquad \text{at } z=-h_2. \qquad (\text{A. 1f})$$

The free surface and interface displacements $\zeta_1$ and $\zeta_2$ are given by

$$\zeta_1=\frac{1}{g}\left[-\frac{\partial\phi_1}{\partial t}-\frac{1}{2}\left(\frac{\partial\phi_1}{\partial x}\right)^2-\frac{1}{2}\left(\frac{\partial\phi_1}{\partial z}\right)^2+\frac{1}{g}\frac{\partial\phi_1}{\partial t}\frac{\partial^2\phi_1}{\partial z\partial t}\right]_{z=h_1}, \qquad (\text{A. 2a})$$

$$\zeta_2=\frac{1}{\Delta g}\left[(1-\Delta)\frac{\partial\phi_1}{\partial t}-\frac{\partial\phi_2}{\partial t}+\frac{1}{\Delta g}\left((1-\Delta)\frac{\partial\phi_1}{\partial t}-\frac{\partial\phi_2}{\partial t}\right)\left((1-\Delta)\frac{\partial^2\phi_1}{\partial z\partial t}-\frac{\partial^2\phi_2}{\partial z\partial t}\right)\right.$$

$$\left.+\frac{1}{2}(1-\Delta)\left\{\left(\frac{\partial\phi_1}{\partial x}\right)^2+\left(\frac{\partial\phi_1}{\partial z}\right)^2\right\}-\frac{1}{2}\left\{\left(\frac{\partial\phi_2}{\partial x}\right)^2+\left(\frac{\partial\phi_2}{\partial z}\right)^2\right\}\right]_{z=0}. \qquad (\text{A. 2b})$$

Here we have retained up to the quadratic terms, these alone being required in the present problem.

**B.** The coupling constants $\alpha$ and $\beta$ in eqs. (16) and (18) are given by

$$\frac{\alpha}{\sqrt{g/h_1^3}}=K\Omega\left[\Delta\left(1-\frac{K^2}{\Omega^4}\right)(\sigma_2-\sigma_1)+\frac{2\Delta}{\Omega V_g}\left(\frac{K}{\Omega^2}\sigma_1-1\right)\right.$$

$$+\frac{V_g^2}{V_g^2-1}\left\{\left(1-\frac{K^2}{\Omega^4}\right)\left(\sigma_1+\sigma_2-\Delta\sigma_2\left(1+\frac{K}{\Omega^2}\sigma_1\right)\right)-\frac{2}{\Omega V_g}\left((1-\Delta)(1+\sigma_1\sigma_2)\right.\right.$$

$$\left.\left.\left.+\frac{\Delta K}{\Omega^2}\sigma_1\left(1-\frac{K}{\Omega^2}\sigma_2\right)\right)\right\}\right]\Big/\left[4(1+(1-\Delta)\sigma_1\sigma_2)-\frac{2K}{\Omega^2}(\sigma_1+\sigma_2)\right], \qquad (\text{B. 1})$$

$$\frac{\beta}{\sqrt{g/h_1}}=-\frac{\delta}{2(\Delta\delta-V_g^4)}\left[(1-\Delta)V_g\left(\Omega^2-\frac{K^2}{\Omega^2}\right)-2(1-\Delta)\frac{KV_g^2}{\Omega}\right.$$

$$-2\Omega V_g^2(1-\Delta)\left(\text{ch}K-\frac{K}{\Omega^2}\text{sh}K\right)\left(\text{sh}K-\frac{K}{\Omega^2}\text{ch}K\right)$$

$$\left.-\left\{\Omega^2 V_g(V_g^2-1)\left(\frac{1}{\sigma_2^2}-1\right)+\frac{2\Omega}{\sigma_2}(V_g^2-\Delta)\right\}\left(\text{ch}K-\frac{K}{\Omega^2}\text{sh}K\right)^2\right], \qquad (\text{B. 2})$$

where

$$K=kh_1, \quad \Omega=\omega\sqrt{h_1/g}, \quad V_g=c_g/\sqrt{gh_1}, \quad \delta=h_2/h_1, \quad \sigma_1=\text{th}\,K, \quad \sigma_2=\text{th}\,\delta K.$$