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Analysis of Rayleigh-Bénard Turbulence by New Shell Models

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Abstract

Two new shell models are proposed to investigate a turbulence driven by a thermal instability (Rayleigh-Bénard turbulence). One is the nearest neighbour real component model(I) and the other is the nearest and the second neighbour complex component model(II). Using these models, quantities such as a mean values of total fluctuation energy, its standard deviation, time averaged wave spectrum, frequency spectrum, the maximum instantaneous Lyapunov exponent are evaluated and the characteristics of turbulence are analyzed. Intermittency nature of the models are also investigated and are compared with results obtained by the model which describes Navier-Stokes turbulence.

Key words : *Shell model, Rayleigh-Bénard turbulence, Chaos, Intermittency*

1. Introduction

The analysis of nonlinear dynamics such as chaos or complexity is a hot topic in physics. Problem related with the convective nonlinearity which are seen in many fields are generic ones. For example, an appearance of chaos or a transition to turbulence from laminar flow via a convective nonlinearity are investigated considerably in the field of fluids turbulence. To challenge such a problem, we should clarify the points; physically, what kind of quantities are relevant to characterize the nonlinear instability, turbulence and transport. To what extent, the quantity can be evaluated in the turbulent state as a meaningful parameter¹⁾.

To study the turbulent characteristics and the chaotic nature, various kind of models are used. Here, we categorize hierarchical model equations, i.e., (I) a model with a few degrees of freedom, e.g., the Lorenz model^{2, 3)}, (II) a model with an intermediate degrees of freedom like a shell model^{4, 5, 6)} and (III) a model with many degrees of freedom such as a Navier-Stokes equation⁷⁾. The model (I) is a very simple one and reveals chaotic nature for some Rayleigh number and Prandtl number. However, the validity of this model is limited in a low Rayleigh number. The model (II) still has a number of properties in common with the

Navier-Stokes equation and has an advantage in numerical simulations, although it does not take a real spatial structure in turbulence into account. Ohkitani and Yamada have numerically shown that Gledzer-Ohkitani-Yamada(GOY) model (a complex version of a shell model) has a chaotic characteristic⁸⁾. The GOY model is reduced from Navier-Stokes equation, retaining only one complex mode as a representative of all the Fourier modes in the shell of wave numbers between an octave, and only nonlinear couplings to the next nearest shell are kept. In this model, the turbulence is sustained by the external pumping. Jensen et al. have introduced a shell model for the small-scale statistics of a scalar field passively carried by a three dimensional turbulence fluid; this model consists of GOY model and passive scalar field⁹⁾. They calculated the structure functions and the probability distribution functions of increments of the passive scalar which show an intermittent nature. It is very time-consuming to solve model (III) numerically and is limited in a low Reynolds number due to the computer capability. Therefore the model (III) is unfavorable for parameter survey.

In this thesis, we adopt a shell model and extend it to the one which is relevant to studying a turbulence driven by the thermal instability (Rayleigh-Bénard turbulence)¹⁰⁾. We call it Yagi model in the following. Here our interest is a turbulence driven by the instability, not by the external force, which is quite poorly understood at the moment. Based on this shell model,

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we present an example of numerical analysis for a fluid in high Rayleigh number regime, which covers not only energy containing regime but also inertial range. This Yagi model enables us to treat a fully developed turbulence which contains enough positive Lyapunov exponents and to elucidate the characteristics of turbulence such as spectrum cascade, intermittency and so on. Not only coarse grained quantities but also micro quantities, (i) a mean value of total energy, (ii) it's standard deviation, (iii) time averaged wave spectrum, (iv) a probability distribution function, (v) frequency spectrum, (vi) an instantaneous maximum Lyapunov exponent, (vii) distribution of instantaneous Lyapunov exponents are evaluated. Based on these quantities, the characteristics of the turbulence driven by the thermal instability is analyzed.

The organization of this thesis is as follows. The Section 2. is devoted to a review about the experimental results, theory and models of turbulence. In Section 2.1, the most basic experimental laws of fully developed turbulence in neutral fluids, i.e., the $2/3$ law and the energy dissipation law are introduced and the Kolmogorov 1941 theory which is based on the Navier-Stokes equation is explained related with the experimental laws. In Section 2.2, Landau's objection to one aspect of the Kolmogorov 1941 theory is explained. In Section 2.3, we introduce the basic concepts and show experimental results about the intermittency in the inertial range, based on velocity measurements. In Section 2.4, two shell models are revisited. The deviation of Desnyansky and Novikov shell model, which is the most simplest model, is derived in Section 2.4.1. The GOY shell model, belonging to a class of deterministic nonlinear model which can describe an intermittency, and the numerical analysis of it are reviewed in Section 2.4.2. In Section 2.5, Rayleigh-Bénard convection problem is considered. As an example, numerical results of Rayleigh-Bénard turbulence based on extended Lorenz model is shown as well as those based on Lorenz model. The limitation of these models are discussed and the application of a shell model for this problem is addressed.

In Section 3., main results of this thesis are given. We show a new shell model (Yagi model) extended to the one which is relevant to studying the turbulence driven by the thermal instability (Rayleigh-Bénard convection) in Section 3.1. In Section 3.2, we examine the stationary solutions and the linear stability analysis of the model. In Section 3.3, we analyze our numerical results, such as a mean value of total energy, it's standard deviation, the time averaged wave spectrum, a probability distribution function, the frequency spectrum, an instantaneous maximum Lyapunov exponent and the scaling exponents of the higher order structure functions. In

Section 3.4, the scale exponents of higher order structure functions are compared between a new shell model and GOY shell model to investigate the intermittency nature of turbulence.

Summary and discussions are given in Section 4..

2. Reviews

In this chapter, we review the most basic experimental laws of fully developed turbulence in neutral fluids, which provide a direct motivation for Kolmogorov's 1941 theory¹¹⁾. This theory (here often abbreviated to K41) is briefly explained. We also explain Landau's objection to one aspect of the 1941 theory. Intermittency, a particular case of the break-down of the 1941 theory is discussed and a shell model is introduced as an intermittency model. Most parts of this review is cited from the book¹²⁾.

Then the Rayleigh-Bénard convection is reviewed related with a shear flow problem in the SOL (Scrape Off Layer) transport in a plasma. Numerical results are shown based on an extended Lorenz model and the limitation of this model is discussed. Finally, the application of a shell model to this problem is addressed.

2.1 The Kolmogorov 1941 theory

Two basic empirical laws of fully developed turbulence were found by experimental data. One is the two-thirds law and another is the law of finite energy dissipation. The former is that the mean square longitudinal velocity increment $\langle (\delta v_{\parallel}(l))^2 \rangle$ between two points separated by a distance l (second order structure function) behaves approximately as the two-thirds power of the distance in a turbulent flow at very high Reynolds number. Here, $\delta v_{\parallel}(\mathbf{r}, l)$ is defined as $\delta v_{\parallel}(\mathbf{r}, l) = [\mathbf{v}(\mathbf{r} + l) - \mathbf{v}(\mathbf{r})] \cdot \frac{l}{l}$. The latter is that the energy dissipation per unit mass dE/dt has a finite positive limit for $\nu \rightarrow 0$, where ν represents the viscosity.

There had been no fully deductive theory which started from the Navier-Stokes equation and led to the experimental laws. In his 1941 turbulence paper Kolmogorov found that an exact relation can be derived for the third order longitudinal structure function¹¹⁾. He assumed homogeneity, isotropy and the finiteness of the energy dissipation. Without any further assumptions he derived following result from the Navier-Stokes equation: in the limit of infinite Reynolds number, the third order longitudinal structure function of homogeneous isotropic turbulence, evaluated for increments l small compared to the integral scale, is given in terms of the mean energy dissipation per unit mass ϵ (assumed to remain finite and nonvanishing) by

$$\langle (\delta v_{\parallel}(l))^3 \rangle = -\frac{4}{5}\epsilon l. \quad (1)$$

This is the one of the most important results, called four-fifth law, in fully developed turbulence. Under the assumption of homogeneity, the turbulent flow is self-similar at small scales, i.e. it possesses a unique scaling exponent h . Thus, there exists a scaling exponent h such that

$$\delta v_{||}(\lambda l) = \delta \lambda^h v_{||}(l). \quad (2)$$

With this equation, under rescaling of the increment l by a factor λ , the left hand side of (1) changes by a factor λ^{3h} , while the right hand side changes by a factor λ . This immediately implies $h = 1/3$. From the self-similarity and $h = 1/3$ it is predicted that the (longitudinal) structure functions of order p , defined by $S_p(l) = \langle (\delta v_{||}(l))^p \rangle$, are written as $S_p(l) \propto l^{p/3}$. Since $(\epsilon l)^{p/3}$ have exactly the same dimension as S_p , we have

$$S_p(l) = C_p \epsilon^{p/3} l^{p/3}, \quad (3)$$

where the C_p are dimensionless. However there are many experimental and numerical evidences that the exponents are different from the value $p/3$. These are regarded as the problems of intermittency.

2.2 Landau's objection

An important issue in fully developed turbulence is Landau's objection to universality. In this section, we shall introduce the modern reinterpretation of Landau's objection. Landau tried to show that the C_p in (3), for $p \neq 3$, cannot be universal, i.e., they must depend on the detailed geometry of production of the turbulence.

The ϵ in (3) is a mean dissipation rate, the mean being taken over the attractor of the flow, i.e., the mean is a time-average. Let us now construct a superensemble, made of $N > 1$ experiments with different positive values of mean dissipation rate, denoted by ϵ_i ($i = 1, 2, \dots, N$). The differences could be caused, for instance, by the flows having different integral scales. Let us tentatively assume that the C_p are universal. We denote by $S_p^i(l)$ the structure functions for the i th flow. We have, by (3):

$$S_p^i(l) = C_p (\epsilon_i)^{p/3} l^{p/3} \quad (4)$$

Now, let us assume that it is legitimate to apply (3) to the superensemble. We define

$$S_p^{super}(l) = \frac{1}{N} \sum_i S_p^i(l) \quad (5)$$

and

$$\epsilon^{super} = \frac{1}{N} \sum_i \epsilon_i, \quad (6)$$

the superaveraged structure functions and dissipation rate, respectively. From (3) and (5), we obtain:

$$\left(\frac{1}{N} \sum_i \epsilon_i \right)^{p/3} = \frac{1}{N} \sum_i (\epsilon_i)^{p/3}. \quad (7)$$

This relation is contradictory, except when $p = 3$.

2.3 Intermittency

A central assumption of Kolmogorov 1941 theory is self-similarity of the random velocity field at inertial-range scales. As we shall see this symmetry may well be broken. The meaning of concept of self-similarity as applied to a random function is illustrated in figure 1¹²⁾ in which a sample of a self-similar random function $v(t)$, here the Brownian motion function, is shown with two successive enlargements. It must be stressed that the general aspect (actually, the statistical properties) within the magnification window is independent of where the window is positioned. In contrast, figure 2¹²⁾ shows a function, called the Devil's stair case which is not self-similar in the above sense : magnification of windows 1 and 2 produce completely different results. When dealing with Devil's staircase, the smaller the window, the more carefully it must be positioned to produce a nontrivial function. The function shown in figure 2 is said to be intermittent : it displays activity during only a function of the time, which decreases with the scale under consideration.

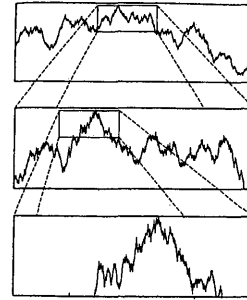


Fig. 1 A portion of the graph of the Brownian motion curve, enlarged twice, illustrating its self-similarity.

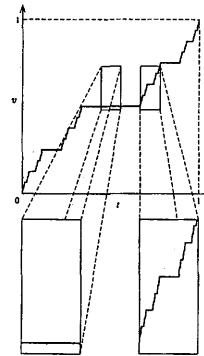


Fig. 2 The Devil's staircase: an intermittent function.

The notion of intermittency may be quantified when the random function $v(t)$ is stationary. It is then convenient to work with the high-pass filtered signal $v_n(t)$,

defined (in the temporal domain) by

$$v(t) = \int_{\mathbf{R}} d\omega e^{i\omega t} \hat{v}_{\omega}, \quad (8)$$

$$v_{\omega}^{\gamma}(t) = \int_{|\omega| > \Omega} d\omega e^{i\omega t} \hat{v}_{\omega}. \quad (9)$$

Here, we shall say that the random function $v(t)$ is intermittent at small scales if the flatness

$$F(\Omega) = \frac{\langle (v_{\omega}^{\gamma}(t))^4 \rangle}{\langle (v_{\omega}^{\gamma}(t))^2 \rangle^2} \quad (10)$$

grows without bound with the filter frequency Ω . Note that, by stationarity, $F(\Omega)$ does not depend on the time argument. To justify this definition, we observe that the inverse of the flatness is a measure of the fraction of the time (high-pass) signal is on. Consider the signals in figure 3¹²⁾. Part (a) shows a stationary signal $v(t)$ and part (b) shows the same signal chopped-off a fraction $1 - \gamma$ of the total time. More precisely, $v_{\gamma}(t)$ is obtained from $v(t)$ by setting it equal to zero except during on intervals which are randomly selected in such a way that the time the signal is on represents a fraction γ of the total time. Assuming that all relevant moments exist, it follows that

$$\langle v_{\gamma}^2 \rangle = \gamma \langle v^2 \rangle, \quad (11)$$

$$\langle v_{\gamma}^4 \rangle = \gamma \langle v^4 \rangle. \quad (12)$$

Hence,

$$F_{\gamma} = \frac{\langle v_{\gamma}^4 \rangle}{\langle v_{\gamma}^2 \rangle^2} = \frac{1}{\gamma} \frac{\langle v^4 \rangle}{\langle v^2 \rangle^2}. \quad (13)$$

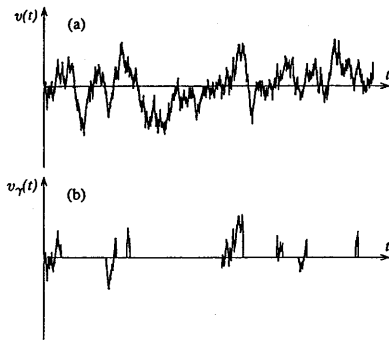


Fig. 3 A stationary signal (a); the same chopped-off a fraction $1 - \gamma$ of the time (b).

Observe that with our definition, neither Gaussian nor self-similar signals are intermittent, because their flatness $F(\Omega)$ is independent of Ω . In the Gaussian case, this is because the Gaussian property is conserved by any linear operation such as filtering and because a Gaussian random variable has a flatness of 3. Let us now assume that $v(t)$ has self-similar increments with

scaling exponent h . It is then easily shown that, for any $\lambda > 0$,

$$v_{\lambda\Omega}^{\gamma} = \lambda^{-h} v_{\Omega}^{\gamma}. \quad (14)$$

As a consequence, when Ω is changed into $\lambda\Omega$ in (10), both the numerator and the denominator are multiplied by λ^{-4h} , leaving the flatness unchanged.

Is turbulence self-similar or is it intermittent? Visual inspection of figure 4¹³⁾, a sample of turbulent signal from a jet, suggests that it is more like the self-similar example of figure 1. If, however, the same signal is subject to high-pass filtering and the filter frequency Ω is chosen high enough, intermittent features appear as shown in figure 5¹³⁾. This sort of intermittency becomes very conspicuous only when the scale associated with Ω is comparable to or smaller than the Kolmogorov dissipation scale. It is thus a characteristic of the dissipation range and does not imply violation of self-similar K41 theory of the inertial range.

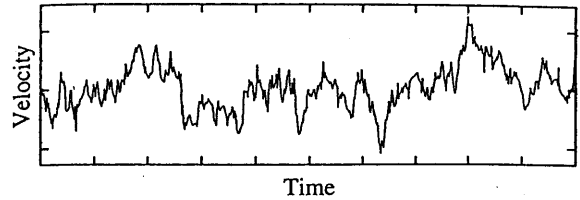


Fig. 4 Velocity signal from a jet with $R_{\lambda} \simeq 700$.

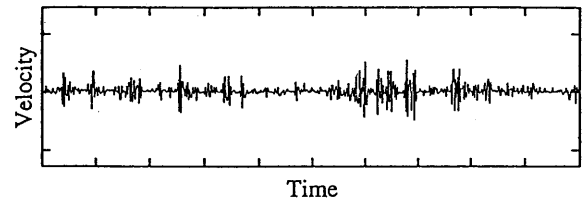


Fig. 5 Same signal as in figure 4, subject to high-pass filtering, showing intermittent bursts.

There is little about the existence of intermittency in the dissipation range. A subtler question concerns the existence of intermittency in the inertial range, since the latter would invalidate the K41 theory. Historically, the first attempts to detect such intermittency involved measurements of fluctuations of the (local) energy dissipation. Such quantities involve simultaneously dissipation-range scales and inertial-range scales. Their interpretation is thus very delicate.

The K41 theory predicts that the structure function of order p scales with an exponent $p/3$ over inertial-range separations, something which can be tested experimentally. However, in his 1941 papers Kolmogorov made explicit prediction only for second and third order structure functions. This may be one of the reasons that measurements of structure functions of order $p \geq 4$ were not attempted until the early 1970s¹⁴⁾. Measurements of high-order structure functions are actually rather difficult. Indeed, by definition, they require accurate measurements of high-order moments of velocity increments. These involves the tail of the corresponding probability distribution functions, i.e. very rare events. Hence, it is necessary to process very long records of the turbulent signal. Early measurements of structure functions were limited by recording capabilities. As shown by Anselmet, Gagne, Hopfinger and Antonia¹⁵⁾ this can lead to vastly overestimating the discrepancy between the measured exponents and their K41 prediction. They give, nevertheless reasonably convincing evidence that significant discrepancies remain. Their experiments were performed in the laboratory at Taylor-scale based Reynolds number R_λ of up to 852. The experiment was repeated by Gagne and Hopfinger in the S1 wind tunnel at $R_\lambda = 2700$ ¹⁶⁾. Although this gave a much wider inertial range, it mostly confirmed the previous results by Anselmet, Gagne, Hopfinger and Antonia¹⁵⁾.

Figure 6¹³⁾ shows the results for the (longitudinal) structure functions of order 2, 3 and 6, obtained by Gagne and Hopfinger. The second order structure function for the same data was presented in figure 7¹²⁾. The presentation of data in figure 6 uses compensating power-law factors to identify exponents of structure functions. The results suggest that structure functions follow power-laws in the inertial range:

$$S_p(l) = \langle (\delta v_{||}(l))^p \rangle \propto l^{\zeta_p}, \quad (15)$$

where the ζ_p are called the exponents of structure functions.

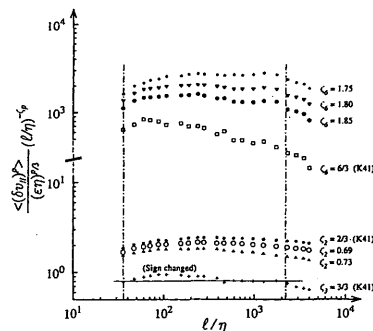


Fig. 6 Structure functions of order 2,3 and 6.

We mention here that it is not a priori evident that

structure functions of high order exist. This depends on the shape of tails of the probability distribution function of velocity increments: if it is algebraic, moments beyond a certain order are infinite. Actually the experimental evidence is that the tails decrease in a roughly exponential manner^{14), 17), 18)}. Of course, there could be an algebraic tail lost in the noise beyond the exponential tail. Intermittency models assuming such hyperbolic tails can be constructed and studied^{19), 20), 21)}.

Let us now examine the results about structure functions (for more detail, see ¹⁵⁾ and ¹⁶⁾). The third order structure function (represented with the sign reversed to make it positive) is in good agreement with the four-fifths law: for l/η from 500 to 2000 the data fall almost exactly on the value $4/5$.

Anselmet, Gagne, Hopfinger and Antonia proposed using the range of scales over which the third order structure function is reasonably chosen to be $4/5$ as an operational definition of the inertial range¹⁵⁾. This is shown by the two vertical dash-dotted lines in figure 6, which delineate the inertial range.

With the compensated representation of figure 6, we see that it is difficult to make a choice between the three values of ζ_2 tried (including the K41 value $2/3$). The Fourier representation in terms of the energy spectrum gives a wider inertial range and thereby a more accurate definition of the exponent ζ_2 . The data for the sixth order structure function shown in figure 6 suggest that the ζ_6 is closer to 1.8 than its K41 value of $6/3$. The value $\zeta_6 = 1.8$ is also proposed in ¹⁵⁾. A consequence of $\zeta_2 \geq 2/3$ and $\zeta_6 < 6/3$ is that the (hyper-) flatness

$$F_6(l) = \frac{S_6(l)}{(S_2(l))^3} \quad (16)$$

grows as a power-law when $l \rightarrow 0$ (while staying within the inertial range). Thus intermittency, measured by $F_6(l)$, becomes arbitrariness strong at very high Reynolds numbers and small inertial-range scales. It would be harder to reach such a conclusion using the flatness $F_4(l)$ since the results of ¹⁵⁾ do not show any measurable discrepancy of ζ_4 from its K41 value of $4/3$.

To summarize this subsection, it is plausible but not certain that there are intermittency corrections to the K41 theory of the inertial range.

2.4 Shell model

Recently various cascade models of turbulence have been proposed and they gave rise to nontrivial scaling laws which almost unavoidably display intermittency. However, such models have only the remotest contact with the original Navier-Stokes equation. In contrast, shell models are deterministic, may or may not display chaos and intermittency and have a bit more of a Navier-Stokes flavor. The simplest shell model was proposed

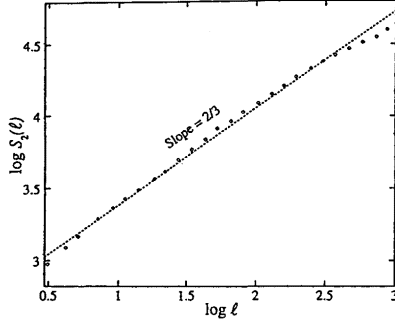


Fig. 7 Log-log plot of the second order structure function.

by Desnyansky and Novikov^{22, 23}. In this section we introduce the shell model of Desnyansky and Novikov in Section 2.4.1 and of the Gledzer-Ohkitani-Yamada shell model in Section 2.4.2.

2.4.1 Desnyansky and Novikov shell model

Let us consider a space-periodic stream of incompressible viscous fluid. The Navier-Stokes equations for the Fourier components of the velocity field are of the form

$$\frac{\partial v_j(\mathbf{k})}{\partial t} = -i(\delta_{jl} - k_j k_l k^{-2}) k_m \sum_{\mathbf{p}} v_l(\mathbf{p}) v_m(\mathbf{k} - \mathbf{p}) - \nu k^2 v_j(\mathbf{k}) + F_j(\mathbf{k}) \quad (17)$$

$$k_j v_j(\mathbf{k}) = 0 \quad (18)$$

$$v_j^*(\mathbf{k}) = v_j(-\mathbf{k}) \quad (19)$$

where k_j are wave vector components whose values are $2\pi L^{-1} n_j$ (L is the space period and n_j are integers), δ_{jl} is the Kronecker delta, $F_j(\mathbf{k})$ are Fourier components of the external force field, and (18),(19) define the conditions of solenoidality and reality of the velocity field, which are also satisfied by the field of forces $F_j(\mathbf{k})$. The twice repeated subscripts denote summation from unity to the number of measurements $s = 3$. The term containing pressure is expressed in terms of velocity by using (18),(19). The nonlinear term in the right-hand part of (17) defines the exchange energy between motions of different scales. It presents considerable difficulties in numerical simulation. The quadratic character of nonlinearity has the effect of doubling the wave number, which is the essence of cascade process. If at the instant of time the wave packet has $k = k_1$, then harmonics with $k = k_1 2^{n-1}$, $n = 1, 2, \dots$ will subsequently appear.

Let us introduce Fourier amplitudes of velocity field over the octave of wave numbers $\sqrt{2}k/2 \leq |\mathbf{k}'| \leq \sqrt{2}k$ (subsequently this is denoted by a 0 supersubscript at the summation symbol)

$$u^2(k) = \langle \sum_{\mathbf{k}'}^0 v_j(\mathbf{k}') v_j(-\mathbf{k}') \rangle \quad (20)$$

where angle brackets denotes the probable averaging over the set of realization.

From (17),(18) and (19) we have

$$u(k) \frac{\partial u(k)}{\partial t} = \langle \sum_{\mathbf{p}}^0 [-ik'_m \sum_{\mathbf{p}} v_l(-\mathbf{k}') v_l(\mathbf{p}) v_l(\mathbf{k} - \mathbf{p}') - \nu(k')^2 v_l(\mathbf{k}') v_l(\mathbf{k}') + F_l(\mathbf{k}') v_l(-\mathbf{k}')] \rangle \quad (21)$$

Since terms containing $p \propto k/2$ and $p \propto 2k$ play the main part in the formation of cascade process, it is possible to approximate the first term in the right-hand part of (21) by the expression

$$\alpha k u(k) u^2(k/2) - \bar{\alpha} k u^2(k) u(2k)$$

where α and $\bar{\alpha}$ are dimensionless constants. The condition of conservation (for $\nu = 0$ and $F_j = 0$) of the quantity

$$I = \frac{1}{2} \sum_k u^2(k) \quad (22)$$

where the summation is for $k = k_1 2^{n-1}$, $n = 1, 2, \dots$, yields

$$\bar{\alpha} = 2\alpha. \quad (23)$$

Since in a three-dimensional flow energy is transferred from large scale to small scale, hence $\alpha > 0$.

Using notation

$$\langle \sum_{\mathbf{k}'}^0 (k')^2 v_l(\mathbf{k}') v_l(-\mathbf{k}') \rangle = k^2 u^2(k) \quad (24)$$

and

$$\langle \sum_{\mathbf{k}'}^0 F_l(\mathbf{k}') v_l(-\mathbf{k}') \rangle = u(k) F(k) \quad (25)$$

we obtain the following equation:

$$\frac{\partial u(k)}{\partial t} = \alpha k [u^2(\frac{k}{2}) - 2u(k)u(2k)] - \nu k^2 u(k) + F(k) \quad (26)$$

In a finite-dimensional representation, equation (26) takes the form:

$$\frac{\partial u_n(t)}{\partial t} = \alpha k_n [u_{n-1}^2(t) - 2u_n(t)u_{n+1}(t)] - \nu k_n^2 u_n(t) + F_n(t) \quad (27)$$

where $u_0 = u_{N+1} = 0$, $u_n(t) = u(t, k)$, $k_n = k_1 2^{n-1}$, $n = 1, 2, \dots, N$ and n is the shell index.

The stationary solution of (17) was investigated in ²³. It was shown that there exists at large n a rather broad inertial range of wave numbers in which the influence of viscosity is not felt and stationary solution goes over similarity:

$$u_n = A k_n^{-\frac{1}{3}}, A = \epsilon^{1/3} \alpha^{-1/3} 2^{-\frac{2}{9}}, \quad (28)$$

where ϵ is the energy flux. From the relation between $u(k)$ and the spectral energy density $E(k)$, we obtain

$$u^2(k) = 2 \int_{k/\sqrt{2}}^{k\sqrt{2}} E(k') dk', \quad (29)$$

which conforms to the four-fifth law for the spectral energy density. With this model, numerical tests were carried out and it was shown that the similarity was obtained, however this model does not show an intermittency. This is because valubles are assumed to be real. Incorporation of the intermittency into the model equation is discussed in the next subsection.

2.4.2 Gletzer-Ohkitani-Yamada (GOY) shell model

Desnyansky-Novikov shell model was derived on the basis of the following requirements: (1) quadratic properties of nonlinear terms; (2) scale invariance of dimensionless coefficients in the equation; (3) direct interaction only between closest neighbors in the spectrum; (4) presence of the quadratic integral. Replacing the limiting (3) with interaction only between nearest and next nearest neighbours in spectrum, we can derive GOY shell model. The model equation is given as follows:

$$\begin{aligned} \frac{\partial u_n(t)}{\partial t} = & i[a_n u_{n+1}(t) u_{n+2}(t) \\ & + b_n u_{n-1}(t) u_{n+1}(t) \\ & + c_n u_{n-1}(t) u_{n-2}(t)]^* \\ & - \nu k_n^2 u_n(t) + F_n(t), \end{aligned} \quad (30)$$

where the star denotes complex conjugation,

$$a_n = k_n, b_n = -k_{n-2}, c_n = -k_{n-3}, k_n = k_1 2^{n-1} \quad (31)$$

and the force is applied to the fourth shell:

$$F_n = F \delta_{4,n}. \quad (32)$$

Ohkitani and Yamada gave qualitative and quantitative numerical evidence that the GOY model shows a chaotic nature⁸⁾; they also measured various statistical quantities indicating that the model is contains an intermittent characteristic. Jensen, Paladin and Vulpiani calculated the structure functions, defined for shell models as

$$S_p(n) = \langle |u_n|^p \rangle \quad (33)$$

and found that: (i) at inertial-range scales,

$$S_p(n) \propto k_n^{-\zeta_p}; \quad (34)$$

(ii) the exponents ζ_p have a nontrivial dependence on the order p , suggesting multifractality⁹⁾. This multifractal behavior is indeed supported by further simulations (Pisarenko, Biferale, Courvoisier, Frish and

Vergasola²⁴⁾), involving up to 250×10^6 time steps. Figure 8 and figure 9 taken from this reference show plots of the structure functions of order 2-10 (from up to bottom) and of their exponents, respectively. The calculations used 22 shells. The parameters were

$$F = (1 + i) \times 5 \times 10^{-3}, k_1 = 2^{-4}, \nu = 10^{-7}.$$

The averaging was over a sufficiently long time span (about 5000 eddy turnover times at forcing scale) to ensure that the data were essentially free noise. The error bars shown in figure 9 were generated from a least squares fit of the logarithms of the structure functions to power-laws in the inertial-range (from shell $n = 6$ to shell $n = 18$).

Intermittency in shell models is the subject of much current work and quite poorly understood at the moment. So far, we review hot topics in neutral fluids turbulence. In the next subsection, we introduce another kind of turbulence such as Rayleigh-Bénard convection and discuss the application of a shell model for it.

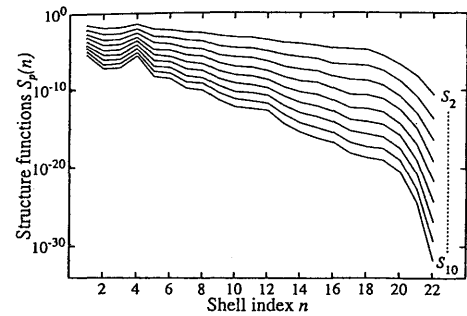


Fig. 8 Structure functions of order 2-10 (top to bottom) for the GOY model. The shell index n is $\log_2$ wavenumber. The data are averaged over 250×10^6 time steps.

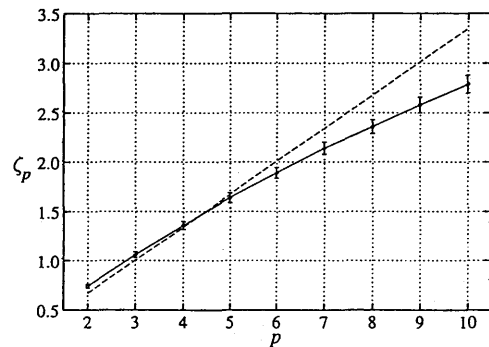


Fig. 9 Exponent ζ_p for the structure function versus order p . The dashed line corresponds to K41 ($\zeta_p = p/3$).

2.5 Rayleigh-Bénard turbulence based on extended Lorenz model

A chaos in the Lorenz model for Rayleigh-Bénard convection has been studied extensively. The Lorenz

model has an important concept in dynamics of dissipative system such as attractors in the phase space. A defining attribute of an attractor on which the dynamics is chaotic is that it displays exponentially sensitive dependence on initial conditions. A convenient indicator of the sensitivity to small orbit perturbations characteristic of chaotic attractors is the Lyapunov exponent. It gives a methods of characterization of the chaotic attractors, which are indispensable in the research of turbulence with small dimensions.

Recently, the extended Lorenz model is proposed as the model describing the dynamics of Scrape Off Layer (SOL) in tokamak plasmas. Such a model has also described Edge Localized Mode (ELM) observed during High confinement mode (H-mode). Comparison between the model with five Fourier components and Reduced Magneto Hydro Dynamics (RMHD) model in which the shear flow effect is taken into account were performed²⁵⁾. The five components model itself was also analyzed numerically in detail²⁶⁾. The five components model is derived from RMHD model, i.e., the vorticity equation and the equation of the pressure, which is relevant to analyze the behavior of edge plasma with shear flow in the SOL. In the SOL the condition $\partial/\partial z = 0$ can be assumed, where z is the direction of the magnetic field line. The perpendicular surface to the magnetic field line is labeled by the coordinates (x, y) , where x corresponds to the radial direction of torus and y corresponds to the poloidal direction of torus. The situation is considered that the effective gravity due to the magnetic field inhomogeneity acts on the x direction and the plasma shear flow is in the y direction. This situation corresponds to Rayleigh-Bénard convection. Two dimensional (2D) slab model is used here. For the boundary condition, a periodic boundary condition is used in the y direction and the boundary in the x direction is $\pm d/2$, where $x = -d/2$ corresponds to the separatrix and d is the width of SOL (figure 10²⁶⁾). The reduction of variables of RMHD model gives both the Lorenz model and the extended Lorenz model with five variables (the five components model), which takes the autonomous shear flow effect into account. If the shear flow is absent, the five components model reduces to the three components model.

The stream function ϕ is assumed to be described by three Fourier components model as

$$\begin{aligned} \phi(x, y, t) = & \phi_1(t) \cos(\pi x) \sin(qy) + \phi_2(t) \cos(\pi x) \\ & + \phi_3(t) \sin(2\pi x) \cos(qy), \end{aligned} \quad (35)$$

where q indicates (normalized) wave number of cell(roll)-arrangement, while it is described by only one Fourier component for the Lorenz model ($\phi_2 = \phi_3 = 0$). For both models the pressure perturbation has the form

of

$$p(x, y, t) = p_1(t) \cos(\pi x) \cos(qy) + p_2(t) \sin(2\pi x) \quad (36)$$

where the term $\sin(2\pi x)$ is added in the pressure field which corresponds to the higher harmonic component of Fourier mode generated by the convective nonlinearity.

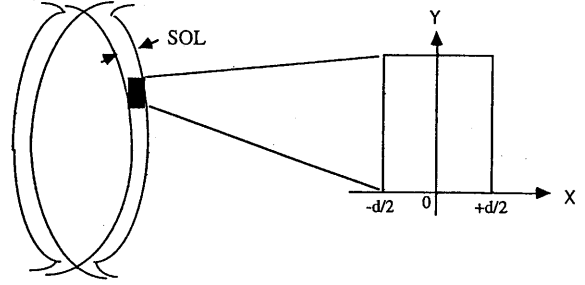


Fig. 10 The schematic view of SOL in poloidal section. Here 2D slab is assumed. Radial direction corresponds to the y direction in this model.

Introducing the normalized variables X, Y, Z, V and W which represent $\phi_1, \phi_2, \phi_3, p_1$ and p_2 respectively, as

$$\tau = (\pi^2 + q^2)t, \quad (37)$$

$$X = -\frac{\pi q}{\sqrt{2}(\pi^2 + q^2)} \phi_1, \quad (38)$$

$$Y = -\frac{\pi q^2}{\sqrt{2}(\pi^2 + q^2)^3} p_1, \quad (39)$$

$$Z = \frac{\pi q^2}{(\pi^2 + q^2)^3} p_2, \quad (40)$$

$$V = \frac{\pi q}{\sqrt{3}(\pi^2 + q^2)} \sqrt{\frac{3\pi^2 + q^2}{\pi^2 + q^2}} \phi_2, \quad (41)$$

and

$$W = \frac{\sqrt{3}\pi q}{2\sqrt{2}(\pi^2 + q^2)} \sqrt{\frac{3\pi^2 + q^2}{\pi^2 + q^2}} \phi_3, \quad (42)$$

the five components model is written as follows:

$$\frac{dX}{d\tau} = VW + P_r(Y - X), \quad (43)$$

$$\frac{dY}{d\tau} = -XZ + rX - Y, \quad (44)$$

$$\frac{dZ}{d\tau} = XY - bZ, \quad (45)$$

$$\frac{dV}{d\tau} = -XW - \gamma_v V, \quad (46)$$

and

$$\frac{dW}{d\tau} = -cXV - \gamma_w W, \quad (47)$$

with

$$r = \frac{q^2}{(\pi^2 + q^2)^3} R_a, \quad (48)$$

$$b = \frac{4\pi^2}{\pi^2 + q^2}, \quad (49)$$

$$\gamma_V = \frac{\pi^2}{\pi^2 + q^2} P_r, \quad (50)$$

$$\gamma_W = \frac{4\pi^2 + q^2}{\pi^2 + q^2} P_r, \quad (51)$$

and

$$c = \frac{3q^2}{4(\pi^2 + q^2)}. \quad (52)$$

The Lorenz model is reproduced by setting $V = W = 0$. Here Reyleigh number R_a and Prandtl number P_r are introduced by

$$R_a = \frac{g\kappa d^4}{\nu\chi} \quad (53)$$

and

$$P_r = \frac{\nu}{\chi} \quad (54)$$

Rayleigh number R_a is the ratio of the magnitude of driving force of interchange instability due to the gravity and the pressure gradient to the magnitude of dissipation due to the heat conductivity and the viscosity. The larger Reyleigh number is, the more the system unstable and turbulent becomes. On the other hand, the ratio of the viscosity to the heat conductivity, P_r is related to the similarity of the solutions. In the problem of thermal convection, it is known that the convective vortex called 'Bénard cell' is created when the Reyleigh number exceeds a certain critical value and that the chaos is generated by this convection in the larger Reyleigh number regime.

Numerical simulations of the Lorenz model and the five components model (43)-(47) are carried out and their results are shown below.

In these simulations, the normalized Reyleigh number

$$r = \frac{R_a}{8\pi^2} \quad (55)$$

is chosen to be an order parameter to characterize the system behavior and the other parameters are fixed as

$$q = \pi, b = 2.0, \gamma_V = 2.0, \gamma_W = 10.0, c = 0.15, \quad (56)$$

and

$$P_r = 4.0. \quad (57)$$

Initial values are given in the Lorenz model as

$$(X, Y, Z) = (1.0, 1.0, 1.0)$$

and those in the five components model are

$$(X, Y, Z, V, W) = (1.0, 1.0, 1.0, 1.0, 1.0).$$

Both models are compared based on the Lyapunov analysis. The Lyapunov exponents for the Lorenz model and for the five components model are calculated. Parameter r is surveyed from zero to 4000. If the maximum Lyapunov exponent exists to be positive, then the system becomes chaotic. The periodic solution is found if the maximum exponent is zero. If the negative value of the exponent is found, the solution converges to the fixed points. The Lyapunov exponents versus r (normalized Reyleigh number) in the case with $P_r = 4.0$ in figure 11²⁶⁾. The exponents for the Lorenz model and for the five components model are shown in (a) and (b), respectively. In the Lorenz model, 1) the regime $0 < r < 29$ has the fixed point, 2) the regime $29 < r < 71$ has the chaotic solution, 3) the regime $71 < r < 91$ has either the chaotic or periodic solution, and 4) the regime of $91 < r$ has only the periodic solution which falls into the limit cycle.

Concerning the five components model, the similar behavior of the Lyapunov exponent is observed to that of the Lorenz model for the parameter regime of $r < 140$. As seen in figure 11(b), the positive value of the maximum Lyapunov exponent appears and disappears in the regime of $306 < r < 2200$. The chaotic behavior and the limit cycle solution as well as the period doubling process of the solution is observed. Finally, for the regime of $r > 2200$ the five components model also falls into the limit cycle with zero value of the exponent.

The three components may not be enough to describe a chaos in the regime of higher Rayleigh number or of well developed turbulence. And five components model can apply only to a certain extent of the Rayleigh number. The six components model or the seven components model have been already proposed for the extended Lorenz model with the shear flow effect^{27), 28), 29)}. A model with a small number of degrees of freedom, such as three, five, six^{27), 28)}, and seven²⁹⁾, has the limit to describe a chaos. Therefore model with intermediate or large number of degrees of freedom should be needed to investigate the turbulence in high Rayleigh number region. In the next chapter, a new shell model which describes Rayleigh-Bénard turbulence is discussed.

3. Comparison study of shell models for Rayleigh-Bénard turbulence and of GOY shell model

3.1 Derivation of model equation

Here we develop a simplified model which describes Rayleigh-Bénard turbulence, heuristically. More quantitative discussion of model equation is given in Appendix

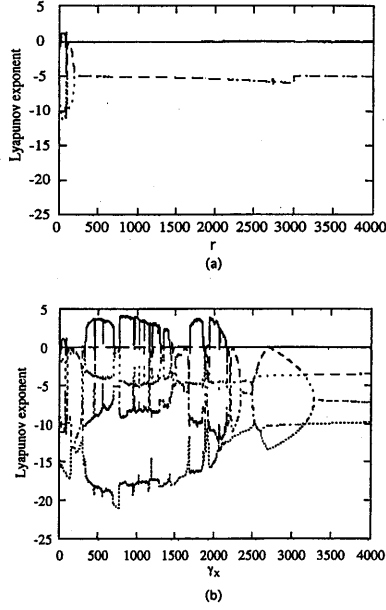


Fig. 11 The relation between the Lyapunov exponent and r for $P_r = 4.0$. One of the exponent is the zero line in (a) the Lorenz model and (b) the five components model.

A1..

We assume that the fluid layer of height h under the gravity acceleration g is submitted to a temperature gradient $\beta = \Delta T/h > 0$ where $\Delta T = T_b - T_t (> 0)$ is the temperature difference between the bottom (subscript 'b') and top (subscript 't') plates. The temperature field is written as

$$T = T_0(z) + \theta, \quad (58)$$

where $T_0(z) = T_b - \beta z$ is a purely conducting temperature, z , the distance from bottom plate and θ , a fluctuating temperature. The density distribution is governed by

$$\rho_0 = \rho(T_0(z)) = \rho_b(1 + \alpha\beta z), \quad (59)$$

where α is the thermal expansion coefficient and ρ_b is the reference density at the bottom. Then the differential buoyancy is given as $-g(\rho - \rho_0) = -g(\rho(T) - \rho(T_0)) = \rho_b\alpha g\theta$. Neglecting not the nonlinear effect but the vertical dependence of the fluctuations, the vertical momentum equation is obtained by

$$\frac{\partial v_z}{\partial t} + v_z \frac{\partial v_z}{\partial z} = \nu \frac{\partial^2 v_z}{\partial x^2} + \alpha g \theta, \quad (60)$$

where v_z is the vertical velocity, ν the kinematics viscosity. The temperature evolution equation is given by

$$\frac{\partial \theta}{\partial t} + v_z \frac{\partial \theta}{\partial z} = \kappa \frac{\partial^2 \theta}{\partial x^2} + \beta v_z, \quad (61)$$

where κ is the heat diffusivity and the first term in the right hand side comes from the term $\mathbf{v} \cdot \nabla T_0$. Using

the normalizations: $x/h \rightarrow x, z/h \rightarrow z, t\kappa/h^2, v_z h/\kappa \rightarrow v_z, \theta\alpha g h^3/(\kappa\nu) \rightarrow \theta$, we obtain

$$\frac{\partial v_z}{\partial t} + v_z \frac{\partial v_z}{\partial z} = P_r \frac{\partial^2 v_z}{\partial x^2} + P_r \theta \quad (62)$$

and

$$\frac{\partial \theta}{\partial t} + v_z \frac{\partial \theta}{\partial z} = \frac{\partial^2 \theta}{\partial x^2} + R_a v_z \quad (63)$$

where $P_r = \nu/\kappa$ is the Prandtl number and $R_a = \alpha g \beta h^4/(\nu\kappa)$, the Rayleigh number. Equations (62) and (63) describe Rayleigh-Bénard turbulence.

We now use the Fourier transform of the v_z, θ give by

$$v_z = \sum_k u(k) e^{ik(x+z)} \quad (64)$$

and

$$\theta = \sum_k p(k) e^{ik(x+z)}, \quad (65)$$

where isotropic turbulence is assumed, i.e., $k_x = k_z = k$. The nonlinear terms in (62) and (63) are written as

$$v_z \frac{\partial v_z}{\partial z} = i \sum_k \sum_{k'} k' u(k-k') u(k') e^{ik'(x+z)}, \quad (66)$$

$$v_z \frac{\partial \theta}{\partial z} = i \sum_k \sum_{k'} k' u(k-k') p(k') e^{ik'(x+z)}. \quad (67)$$

Then, we obtain the equation for the Fourier components of the velocity and temperature fluctuations as

$$\begin{aligned} \frac{du(k)}{dt} &= -i \sum_{k'} k' u(k-k') u(k') \\ &\quad + P_r p(k) - P_r k^2 u(k), \end{aligned} \quad (68)$$

$$\begin{aligned} \frac{dp(k)}{dt} &= -i \sum_{k'} k' u(k-k') p(k') \\ &\quad + R_a u(k) - k^2 p(k). \end{aligned} \quad (69)$$

From (68) and (69), we now derive two new shell models in the following. One is the nearest neighbour real component model which is the extension of the Desnyansky and Novikov model. The other is the nearest and the second nearest neighbour complex component model, which is the extension of the original GOY model supplemented by the passive scalar component. The model equations are shown.

According to the same treatment discussed in section 2.4, the nearest neighbour and real component model is obtained

$$\begin{aligned} \frac{du_n}{dt} &= k_n(u_{n-1}^2 - 2u_n u_{n+1}) \\ &\quad - P_r k_n^2 u_n + P_r p_n, \end{aligned} \quad (70)$$

$$\begin{aligned} \frac{dp_n}{dt} &= k_n(u_{n-1} p_{n-1} - 2u_n p_{n+1}) \\ &\quad - k_n^2 p_n + R_a u_n. \end{aligned} \quad (71)$$

If we neglect the third term in the right hand side of (70) and add the forcing term to it, the Desnyansky and Novikov model (27) is recovered.

The nearest and the second nearest neighbour complex component model is given by

$$\begin{aligned} \frac{du_n}{dt} &= i(k_n u_{n+1} u_{n+2} - k_{n-2} u_{n-1} u_{n+1} \\ &\quad - k_{n-3} u_{n-1} u_{n-2})^* - P_r k_n^2 u_n + P_r p_n, \quad (72) \\ \frac{dp_n}{dt} &= i[k_{n-1}(u_{n-1} p_{n+1} - u_{n+1} p_{n-1}) \\ &\quad - k_{n-2}(u_{n-2} p_{n-1} + u_{n-1} p_{n-2}) \\ &\quad + k_n(u_{n+1} p_{n+2} + u_{n+2} p_{n+1})]^* \\ &\quad - k_n^2 p_n + R_a u_n \end{aligned} \quad (73)$$

where $k_n = C \cdot 2^{n-1}$ ($n = 1, 2, \dots, N$), N is the number of shells, C is a numerical constant and

$$u_{-1} = u_0 = u_{N+1} = u_{N+2} = 0, \quad (74)$$

$$p_{-1} = p_0 = p_{N+1} = p_{N+2} = 0. \quad (75)$$

If we neglect the third terms in RHS of (72) and (73) and add the force terms to them, respectively, then (72) corresponds to the GOY model (30) and the (73) corresponds to passive scalar field which involves with GOY model⁹⁾.

In this thesis, we introduce these new models above which are appropriate to describe Rayleigh-Bénard turbulence and are the extended versions of Desnyansky and Novikov model and of GOY model. In the following sections, we analyze these model equations.

3.2 Theoretical analysis of model equation

In this section we firstly discuss the linear stability analysis. In the shell models which describe the fully developed turbulence of neutral fluid, the variable, $u_n(k)$ obeys the scaling law, $u_n(k) \sim k_n^{-1/3}$ in the stationary state. For comparison, we investigate the stationary similarity mode of our model equation.

3.2.1 Linear stability analysis

By linearizing (70) and (71), or (72) and (73) we obtain

$$\frac{du_n}{dt} = -P_r k_n^2 u_n + P_r p_n, \quad (76)$$

$$\frac{dp_n}{dt} = R_a u_n - k_n^2 p_n, \quad (77)$$

for both two models. These linearized differential equations are easily solved by assuming an exponential time dependence involving an unknown growth rate s : $\{u_n, p_n\} \rightarrow \{\hat{u}_n, \hat{p}_n\} \exp(s_n t)$, where $\{\hat{u}_n, \hat{p}_n\}$ are now pure numbers. This converts the time differential problem into an algebraic 2×2 homogeneous linear system

$$(s_n + P_r k_n^2) \hat{u}_n - P_r \hat{p}_n = 0, \quad (78)$$

$$-R_a \hat{u}_n + (s_n + k_n^2) \hat{p}_n = 0. \quad (79)$$

A perturbation will exist if this system has a nontrivial solution, which can be the case only if its determinant vanishes:

$$\begin{aligned} 0 &= (s_n + P_r k_n^2)(s_n + k_n^2) - P_r R_a \\ &= s_n^2 + k_n^2(1 + P_r)s_n + P_r(k_n^4 - R_a) \end{aligned} \quad (80)$$

This equation is quadratic in s_n . It can have two roots, $s_n^\pm$, with

$$s_n^+ + s_n^- = \Sigma = -k_n^2(1 + P_r), \quad (81)$$

$$s_n^+ s_n^- = \Pi = P_r(k_n^4 - R_a). \quad (82)$$

s_n^+ and s_n^- can be real or complex according to the sign of the determinant

$$\Delta = k_n^4(1 + P_r)^2 - 4P_r(k_n^4 - R_a) \quad (83)$$

$$= k_n^4(P_r - 1)^2 + 4P_r R_a. \quad (84)$$

Δ can be negative only when R_a is negative, i.e., when the layer is heated from above. When R_a is negative and sufficiently large, the two roots are complex conjugate but their real part $\Sigma/2$ remains negative so that the corresponding perturbation is damped; there cannot be unstable oscillatory modes.

When R_a is positive, Δ remains positive and the roots are real but their Σ stays negative. In order to detect the instability, we need only to discuss the sign of the product Π . When it is positive both roots are negative and we have two stable modes. This occurs when R_a is small, especially when $R_a \rightarrow 0$, i.e., close to thermal equilibrium. When Π is negative, root s_n^- remains negative but s_n^+ becomes positive; one of the modes grows exponentially and the layer is unstable. This happens when R_a is large so that the release of gravitational energy by buoyancy can overcome dissipation losses.

Marginal stability corresponds to $s_n^+ = 0$, that is to say $\Pi = 0$:

$$R_a = k_n^4. \quad (85)$$

The condition in which the system is unstable is given by

$$R_a > k_n^4, \quad (86)$$

and that in which the system is stable is given by

$$R_a < k_n^4. \quad (87)$$

3.2.2 Stationary similarity mode

(a) nearest neighbor real component model

In the stationary state, since $du_n/dt = 0$ and $dp_n/dt = 0$, the model equations (70) and (71) are in the forms of

$$k_n(u_{n-1}^2 - 2u_n u_{n+1}) - P_r u_n + P_r p_n = 0 \quad (88)$$

$$k_n(u_{n-1} p_{n-1} - 2u_n p_{n+1}) - k_n^2 p_n + R_a u_n = 0. \quad (89)$$

Here, we assume the wavenumber dependences of u_n and p_n , such that, $u_n = Ak_n^a$ and $p_n = Bk_n^b$, respectively, and find the stationary similarity mode. Substituting these equations into (88) and (89), we obtain the following equations

$$0 = (k_n^{2a+1}2^{-2a} - k_n^{2a+1}2^{a+1})A^2 - P_r Ak_n^{a+2} + P_r Bk_n^b, \quad (90)$$

$$0 = (k_n^{a+b+1}2^{-a-b} - k_n^{a+b+1}2^{b+1})AB - Bk_n^{b+2} + R_a Ak_n^a. \quad (91)$$

When $a = -1/3$ and $b = -1/3$, the nonlinear terms vanish in (90) and (91). Then, we obtain

$$Ak_n^{\frac{5}{3}} - Bk_n^{-\frac{1}{3}} = 0, \quad (92)$$

$$-R_a Ak_n^{-\frac{1}{3}} + Bk_n^{\frac{5}{3}} = 0. \quad (93)$$

Substituting (93) into (92) and eliminating k_n , we have the relation equation between A and B as

$$-R_a A^2 + B^2 = 0, \quad (94)$$

i.e.,

$$B = R_a^{1/2} A. \quad (95)$$

(b) nearest and second nearest neighbor complex component model

Before analyzing the stationary similarity mode of the model equations (72) and (73), we rewrite the model equations in terms of the real and imaginary parts of u_n and p_n . Then, the following equations are obtained:

$$\begin{aligned} \frac{du_n^r}{dt} &= k_n(u_{n+1}^r u_{n+2}^i + u_{n+1}^i u_{n+2}^r) \\ &\quad - k_{n-2}(u_{n-1}^r u_{n+1}^i + u_{n-1}^i u_{n+1}^r) \\ &\quad - k_{n-3}(u_{n-1}^r u_{n-2}^i + u_{n-1}^i u_{n-2}^r) \\ &\quad - P_r k_n^2 u_n^r + P_r p_n^r, \end{aligned} \quad (96)$$

$$\begin{aligned} \frac{du_n^i}{dt} &= k_n(u_{n+1}^r u_{n+2}^r - u_{n+1}^i u_{n+2}^i) \\ &\quad - k_{n-2}(u_{n-1}^i u_{n+1}^i - u_{n-1}^r u_{n+1}^r) \\ &\quad - k_{n-3}(u_{n-1}^i u_{n-2}^i - u_{n-1}^r u_{n-2}^r) \\ &\quad - P_r k_n^2 u_n^i + P_r p_n^i, \end{aligned} \quad (97)$$

$$\begin{aligned} \frac{dp_n^r}{dt} &= k_{n-1}(u_{n-1}^r p_{n+1}^i + u_{n-1}^i p_{n+1}^r) \\ &\quad - k_{n-1}(u_{n+1}^r p_{n-1}^i + u_{n+1}^i p_{n-1}^r) \\ &\quad - k_{n-2}(u_{n-2}^r p_{n-1}^i + u_{n-2}^i p_{n-1}^r) \\ &\quad - k_{n-2}(u_{n-1}^r p_{n-2}^i + u_{n-1}^i p_{n-2}^r) \\ &\quad + k_n(u_{n+1}^r p_{n+2}^i + u_{n+1}^i p_{n+2}^r) \\ &\quad + k_n(u_{n+2}^r p_{n+1}^i + u_{n+2}^i p_{n+1}^r) \\ &\quad - k_n^2 p_n^r + R_a u_n^r, \end{aligned} \quad (98)$$

and

$$\begin{aligned} \frac{dp_n^i}{dt} &= k_{n-1}(u_{n-1}^r p_{n+1}^r - u_{n-1}^i p_{n+1}^i) \\ &\quad - k_{n-1}(u_{n+1}^r p_{n-1}^r - u_{n+1}^i p_{n-1}^i) \\ &\quad - k_{n-2}(u_{n-2}^r p_{n-1}^r - u_{n-2}^i p_{n-1}^i) \\ &\quad - k_{n-2}(u_{n-1}^r p_{n-2}^r - u_{n-1}^i p_{n-2}^i) \\ &\quad + k_n(u_{n+1}^r p_{n+2}^r - u_{n+1}^i p_{n+2}^i) \\ &\quad + k_n(u_{n+2}^r p_{n+1}^r - u_{n+2}^i p_{n+1}^i) \\ &\quad - k_n^2 p_n^i + R_a u_n^i, \end{aligned} \quad (99)$$

where superscripts 'r' and 'i' represent the real part and the imaginary part of the variables, respectively.

Here, we assume the wavenumber dependences of the variables given by $u_n^r = Ak_n^a$, $u_n^i = Bk_n^b$, $p_n^r = Ck_n^c$ and $p_n^i = Dk_n^d$, respectively, to obtain the stationary similarity mode. Substituting these relation equations into (96)-(99) and taking a similar process as in the case of (a), we obtain $a = b = c = d = -1/3$. The relations between the coefficients of u_n^r , u_n^i , p_n^r and p_n^i are given by

$$C = R_a^{1/2} A, \quad (100)$$

$$D = R_a^{1/2} B. \quad (101)$$

In both cases (a) and (b), the energy spectrum $E(k_n)$, which is defined as $E(k_n) = \sum_n (|u_n|^2 + |p_n|^2)/(2k_n)$, scales like $k^{-5/3}$. These special solutions do not involve the nonlinear terms and the Kolmogorov's type scaling laws is obtained.

3.3 Numerical analysis

Numerical simulations of the new two shell models are carried out and their results are shown. Two models are given by (I) (70) with (71) and (II) (72) with (73), respectively.

3.3.1 Linear analysis

Figures 12 and 13 show the wavenumber dependence of the linear growth rate of the model equations. The linear growth rate does not show the difference between the nearest neighbour real component model(I) and the nearest and second nearest neighbour complex component model(II).

Figure 12 is the case for various values of Rayleigh number with the fixed Prandtl number, $P_r = 1.0$. The solid line corresponds to the case of $R_a = 2 \times 10^4$. The dotted-dashed line shows the case of $R_a = 10^6$ and the dotted line shows the case of $R_a = 10^7$. The growth rate increases according to the increase of the Rayleigh number. The system is linearly stable in the region of $n > 8(k_n > 2^8)$ for $R_a = 2 \times 10^4$, in the region of $n > 9(k_n > 2^9)$ for $R_a = 10^6$ and in the region of $n > 10(k_n > 2^{10})$ for $R_a = 10^7$, respectively.

Figure 13 is the case for various values of Prandtl number with the fixed Rayleigh number, $R_a = 2 \times 10^4$. The solid line is the case of $P_r = 2.0$. The dotted-dashed line shows the case $P_r = 5.0$ and the dotted line corresponds to the case of $P_r = 8.0$. The growth rate increases according to the increase of the Prandtl number. The maximum linear growth rate is determined by the longest wavelength mode. It is found that the normalized profile of the growth rate versus the wave number shows the similar profile for different Rayleigh numbers.

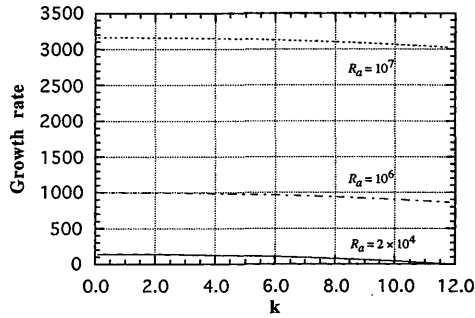


Fig. 12 The wavenumber dependence of the linear growth rate of the model equation. The solid line: $R_a = 2 \times 10^4$, the dotted-dashed line: $R_a = 10^6$ and dotted line: $R_a = 10^7$.

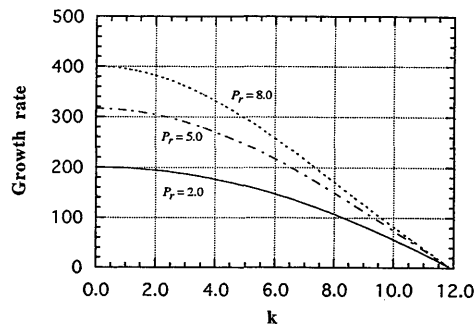


Fig. 13 The wavenumber dependence of the linear growth rate of the model equation. The solid line: $P_r = 2.0$, the dotted-dashed line: $P_r = 5.0$ and the dotted line: $P_r = 8.0$.

3.3.2 Nonlinear analysis

In this section, the following parameters are used for numerical simulations; $C = 1.0$ and the numbers of shells $N = 13$ for $R_a = 2 \times 10^4$, $N = 14$ for $R_a = 10^6$ and $N = 14$ for $R_a = 10^7$.

(a) Time averaged wavenumber spectrum

Figure 14 shows the log-log plot of the time averaged wave number spectra of the variables, $u(k)$ and $p(k)$ by the nearest neighbour real component model(I) for the case of $R_a = 2 \times 10^4$ and $P_r = 1.0$. The circles correspond to the wavenumber spectrum of $u(k)$, the triangles to the wavenumber spectrum of $p(k)$, respec-

tively. Time averaging is done from $t = 0.2$ to $t = 0.5$. In the wavenumber region, $2 \leq k \leq 64$ (mode number $2 \leq n \leq 7$), both variables obey the scaling law, $u(k) \sim k^{-1/3}$ and $p(k) \sim k^{-1/3}$, these are plotted by dotted lines. The Kolmogorov's type scaling law is obtained.

Figure 15 shows the log-log plot of the time averaged wavenumber spectra of $u(k)$ and $p(k)$ by the nearest and the second nearest neighbour complex component model(II) for the case of $R_a = 2 \times 10^4$ and $P_r = 1.0$. The circles corresponds to the spectrum of $u(k)$, the triangles correspond to the spectrum of $p(k)$, respectively. Time averaging is performed from $t = 0.2$ to $t = 10.0$. In the wave number region, $2 \leq k \leq 16$ (mode number $2 \leq n \leq 5$), both variables seem to obey the scaling law, $u(k) \sim k^{-1/3}$ and $p(k) \sim k^{-1/3}$ as are shown by the dotted lines, although they fluctuate around the line proportional to $k^{-1/3}$.

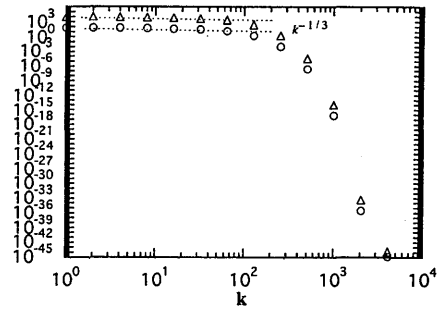


Fig. 14 The log-log plot of the time averaged wavenumber spectra of the variables by the nearest neighbour real component model(I) for $R_a = 2.0 \times 10^4$ and $P_r = 1.0$.

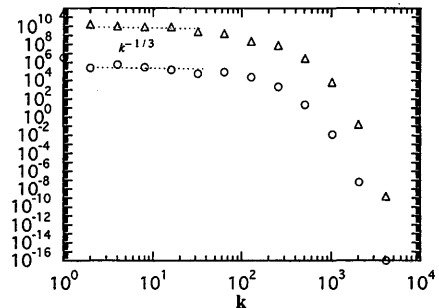


Fig. 15 The log-log plot of the time averaged wavenumber spectra of $u(k)$ and $p(k)$ by the nearest and the second nearest neighbour complex component model(II) for $R_a = 2 \times 10^4$ and $P_r = 1.0$.

(b) Total energy

Firstly, we investigate the Rayleigh number dependence of the saturation level of total energy. Figure 16 shows a plot of total energy $E(t)$ versus t calculated by the nearest neighbour real component model(I) in the case

of $P_r = 1.0$. The solid line corresponds to the case of $R_a = 2 \times 10^4$. The dotted-dashed line shows the case of $R_a = 10^6$ and the dotted line shows the case of $R_a = 10^7$. Here, the total energy is defined by

$$E(t) = \frac{1}{2} \sum_n (|u_n|^2 + |p_n|^2). \quad (102)$$

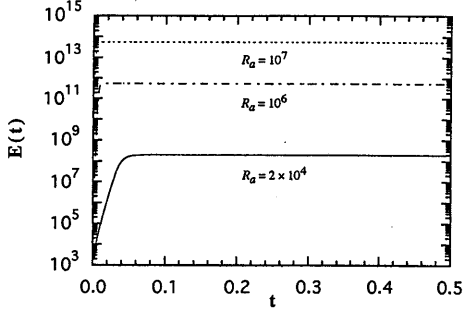


Fig. 16 The time dependence of the total energy $E(t)$ of the nearest neighbor real component model(I) for the fixed Prandtl number, $P_r = 1.0$.

For each case, the total energy converges at a constant value in the nonlinear phase. In other words, the attractor of this model converges to the fixed point. The larger Rayleigh number is, the higher the saturation level of total energy becomes.

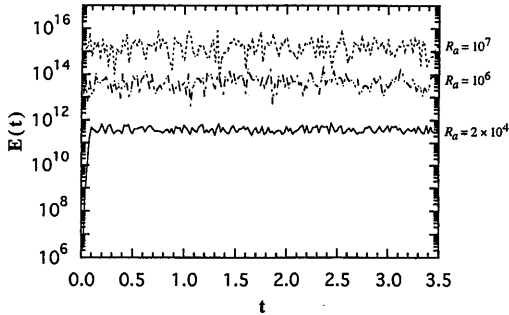


Fig. 17 The time dependence of the total energy of the nearest and the second nearest neighbor complex component model(II) with $P_r = 1.0$.

Figure 17 shows a plot of total energy $E(t)$ versus t calculated by the nearest and second nearest neighbour complex component model(II) for the same parameters in figure 16. The solid line corresponds to $R_a = 2 \times 10^4$, the dotted-dashed line to $R_a = 10^6$ and the dotted line to $R_a = 10^7$. The total energy tends to saturate in the nonlinear phase, however the fluctuation around the mean value of the total energy is observed, which shows the chaotic nature of the system. Both the mean value and the fluctuation amplitude increase with the Rayleigh number.

The saturation level of the total energy is roughly determined by the maximum linear growth rate and the

nonlinear dumping rate and the higher saturation level is obtained for the model of the nearest and the second nearest neighbour complex component model(II). This is because the two different models do not show the same results even if the parameters of R_a and P_r are given. Model (I) consists of real variables and model (II) of complex variables. The scheme of shell model may cause the differences in the nonlinear interaction rate (the nonlinear damping rate) and/or in the nonlinear interaction regime in k space permitting for future analysis. The same argument with respect to the absolute of the saturation level applies to the following comparison analysis.

Next, we investigate the Prandtl number dependence of the saturation level of total energy. Figure 18 shows the time dependence of the total energy by the nearest neighbor real component model(I) for the fixed Rayleigh number, $R_a = 2 \times 10^4$. Cases with various Prandtl numbers are plotted. The solid line represents the case of $P_r = 2.0$, the dotted-dashed line $P_r = 5.0$ and the dotted line $P_r = 8.0$ are also plotted. The total energy converges to a constant value in nonlinear phase. The total energy becomes higher with the increase of Prandtl number as is expected from the linear analysis.

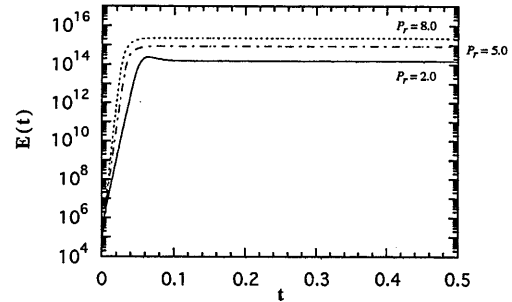


Fig. 18 The time dependence of the total energy by the nearest neighbour real component model(I) for fixed Rayleigh number, $R_a = 2 \times 10^4$.

Figure 19 shows the time dependence of the total energy by the nearest and the second nearest neighbour complex component model(II) with $R_a = 2 \times 10^4$. The solid line corresponds to the case of $P_r = 2.0$, the dotted-dashed line is for $P_r = 5.0$ and the dotted line shows for $P_r = 8.0$. The total energy also fluctuates as is observed in figure 17. In contrast with the Rayleigh number dependence shown in figure 3.6, however, the fluctuation amplitude relative to the mean value of the total energy is found to decrease as the Prandtl number increases. The increase of mean value of saturation level with the increase of Prandtl number is correlated with the maximum linear growth rate, as is the case of the Rayleigh number dependence.

It is found that the nearest neighbour real component model(I) does not have a chaotic attractor. Therefore the following analysis, (b), (c), (d) and (e), are done by employing the nearest and the second nearest neighbour complex component model(II). In order to investigate the fluctuation amplitude of the total energy by the nearest and the second nearest neighbour complex component model(II), we study the mean value of the total energy and its variance in the next section.

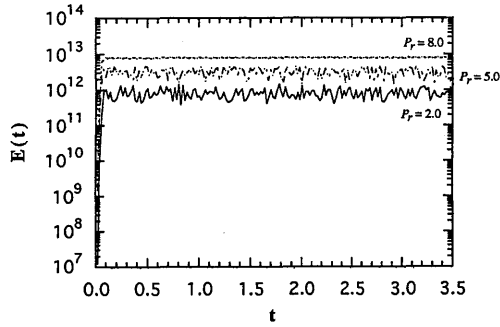


Fig. 19 The time dependence of the total energy of the nearest and the second nearest neighbour complex model(II) for fixed Rayleigh number, $R_a = 2 \times 10^4$.

(c) Mean value and its variance of total energy

Figure 20 shows the log-log plot of the mean value of the total energy and its variance versus Rayleigh number by the nearest and the second nearest neighbour complex component model(II).

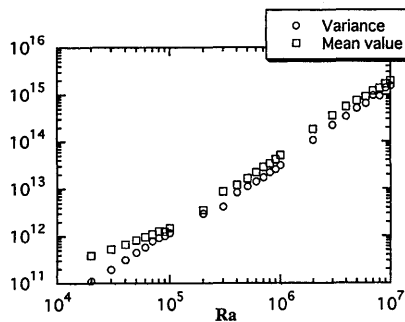


Fig. 20 The log-log plot of the mean value of total energy and its variance versus Rayleigh number by the nearest and second nearest neighbor complex component model(II) for fixed Prandtl number, $P_r = 1.0$.

The mean value of the total energy is defined by

$$\langle E \rangle = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} E(t) dt, \quad (103)$$

where $t_1 = 0.2$ and $t_2 = 10.0$ are chosen. Its variance is defined by

$$\sigma^2 = \langle E^2 \rangle - \langle E \rangle^2. \quad (104)$$

The square symbols correspond to the mean value of the total energy and the circle symbols to the variance

in figure 20. The mean value increases with the increase of the Rayleigh number. This corresponds to the rise of the saturation level of the total energy in figure 17. The variance of the total energy also increases with the increase of the Rayleigh number. The dependence of the variance on Rayleigh number is $\sim R_a^{1.53}$. Figure 21 shows the log-log plot of the mean value of the total energy and its variance versus Prandtl number. In this figure Rayleigh number is fixed to $R_a = 2 \times 10^4$. The square symbols correspond to the mean value of the total energy and the circle symbols to the variance. The mean value of the total energy increases with the increase of the Prandtl number. It is found that the variance increases in the region $0.1 < P_r < 7.5$, on the other hand, it decreases in the region $P_r > 7.5$.

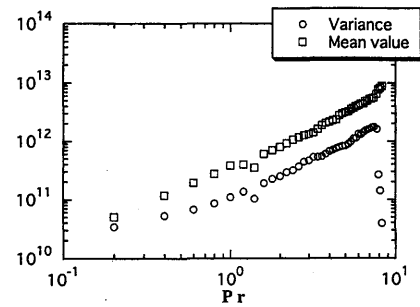


Fig. 21 The log-log plot of the mean value of total energy versus Prandtl number by the nearest and second nearest neighbor complex component model(II) for fixed Rayleigh number, $R_a = 2 \times 10^4$.

In order to study the ratio between the variance and the mean value, we show the log-log plot of the ratio of the variance to the mean value of the total energy versus Rayleigh number in figure 22. Prandtl number is fixed to the value of $P_r = 1.0$ in this case. The circle symbols show the ratio. It is found that the ratio is

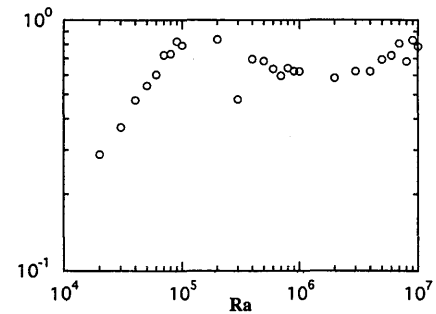


Fig. 22 The log-log plot of the ratio of the variance to the mean value of the total energy versus Rayleigh number for fixed Prandtl number, $P_r = 1.0$.

not a monotonic function of the Rayleigh number. The ratio increases with the increase of the Rayleigh number

in the region $10^4 < R_a < 10^5$, then it decreases in the region $10^5 < R_a < 2 \times 10^6$. For $R_a > 2 \times 10^6$, it again increases with the Rayleigh number.

Figure 23 shows the log-log plot of the ratio of the variance to the mean value of the total energy versus Prandtl number. Rayleigh number is fixed to the value of $R_a = 2 \times 10^4$ in this case. The circles show the ratio. The value of the ratio decreases with the increase of the Prandtl number in the region of $P_r < 3.0$. The value of the ratio stays constant in the region of $3.0 < P_r < 7.5$. The value of the ratio decreases rapidly in the region $P_r > 7.5$. These results indicate that nonlinear instability is stabilized for $P_r > 7.5$ and the attractor of this system falls into the limit cycle from chaotic attractor.

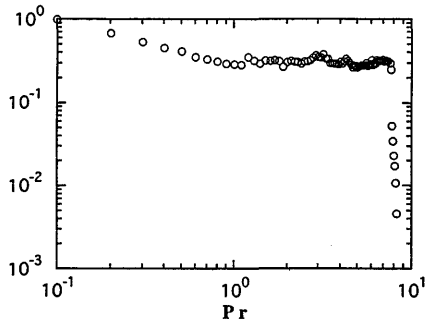


Fig. 23 The log-log plot of the ratio of the variance to the mean value of the total energy versus Rayleigh number for fixed Rayleigh number, $R_a = 2.0 \times 10^4$.

Figure 24 shows the log-log plot of the maximum Lyapunov exponent versus the Rayleigh number for the fixed Prandtl number, $P_r = 1.0$. The procedure to obtain Lyapunov exponent is explained in Appendix A2.. For these Rayleigh numbers the maximum Lyapunov

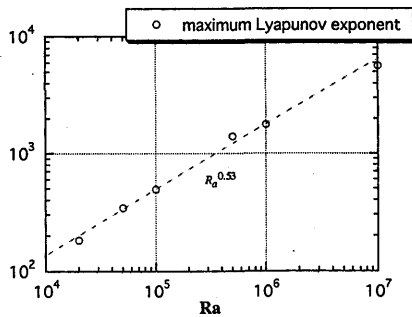


Fig. 24 The log-log plot of the maximum Lyapunov exponent versus Rayleigh number for fixed Prandtl number, $P_r = 1.0$. The dotted line corresponds to $R_a^{0.53}$.

exponent is found to be positive. This means that the system shows a chaotic nature, which is also related with an intermittency. The dependence of the maximum Lyapunov exponent of the Rayleigh number is $\sim R_a^{0.53}$, which shows a little stronger dependence than that of

the linear growth rate in the inviscid limit ($\sim R_a^{1/2}$).

Figure 25 shows the plot of the maximum Lyapunov exponent versus the Prandtl number for the fixed Rayleigh number, $R_a = 2 \times 10^4$. The values of maximum Lyapunov exponent are positive in the region, $0.5 \leq P_r \leq 7.5$. It is found that the values of the maximum Lyapunov exponent are almost zero in the region, $8.0 \leq P_r \leq 10.0$. We conjecture that both parameters, R_a and P_r , have the destabilizing effect on the linear mode, which is reflected to the mean value of saturation amplitude. On the other hand, the Rayleigh number has an effect of destabilizing on the nonlinear mode, while, the Prandtl number has an effects of both stabilizing and destabilizing on the nonlinear mode.

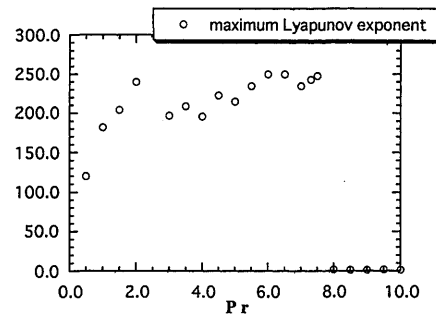


Fig. 25 The log-log plot of the maximum Lyapunov exponent versus Prandtl number for fixed Rayleigh number, $R_a = 2.0 \times 10^4$

(d) Probability density function

Figure 26 shows the probability density function of $P_{n=3}(s)$ versus the normalized temperature component for the case of $R_a = 2.0 \times 10^4$ and $P_r = 1.0$. The normalized temperature component, s is defined by

$$s = \frac{|p_n|}{\langle |p_n|^2 \rangle^{1/2}}. \quad (105)$$

We took the mode, $n = 3$ which is located in the iner-

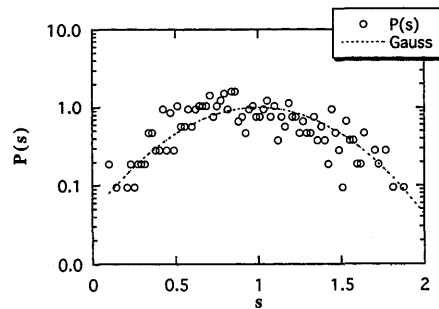


Fig. 26 The probability density function of $P_{n=3}(s)$ versus normalized temperature component, s for Rayleigh number, $R_a = 2 \times 10^4$ and Prandtl number, $P_r = 1.0$.

tial range. The circles are the data obtained from the model equation and the dotted line corresponds to the

Gaussian distribution. The probability density function does not deviate much from the Gaussian form.

Figure 27 shows the plot of the probability density function of $P_{n=11}(s)$ versus normalized temperature component, the definition of s is the same for the case of figure 26. We took the mode, $n = 11$ which is located in the dissipation range. The circles show the data from the model equation and the dotted line is the Gaussian distribution. In this case the probability density function has a tail of the temperature component. The probability density function of the shell model with a passive-scalar has been reported in ⁹⁾, which is close to the Gaussian form in the inertial range and has a tail dissipation range. Our shell model shows the similar characteristics with respect to the probability density function.

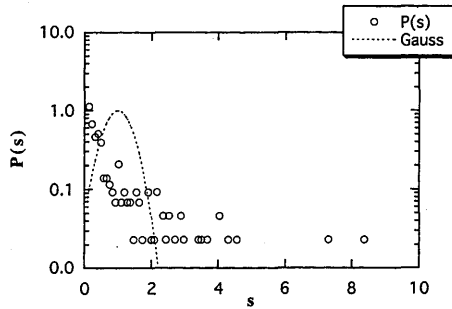


Fig. 27 The plot of the probability density function of $P_{n=11}(s)$ versus normalized temperature, s for Rayleigh number, $R_a = 2 \times 10^4$ and Prandtl number, $P_r = 1.0$.

(e) Power spectrum of total energy

The log-log plot of the power spectrum of the total energy with $P_r = 1.0$ is shown for various Rayleigh numbers in figure 28. The frequency f is defined as $f = (j - 1)\Delta f$ ($j = 1, 2, \dots$) where $\Delta f = 2.5$. The solid line shows the power spectrum for $R_a = 2 \times 10^4$, the dotted-dashed line is the case of $R_a = 10^6$ and the dotted line corresponds to the case of $R_a = 10^7$. Continuous spectra are obtained in these cases. For $R_a = 2 \times 10^4$, $R_a = 10^6$ and $R_a = 10^7$, the power spectra are fitted as $E(f) \sim f^{-3.88}$, $E(f) \sim f^{-2.45}$ and $E(f) \sim f^{-2.25}$ in $1000 < f < 2500$, respectively.

The power spectrum of the total energy with $R_a = 2 \times 10^4$ is shown for various Prandtl numbers in figure 29. The solid line corresponds to the case of $P_r = 2.0$, the dotted-dashed line to the case of $P_r = 5.0$ and the dotted line to the case of $P_r = 8.0$. For $P_r = 2.0$ and $P_r = 5.0$ the power spectra are fitted as $E(f) \sim f^{-3.15}$ and $E(f) \sim f^{-2.02}$, respectively, in the range of $1000 < f < 2500$. On the other hand, the fundamental and the harmonic frequencies are observed in the case of $P_r = 8.0$. This corresponds the fact that the fluctuation amplitude is very small and tends to converge to the

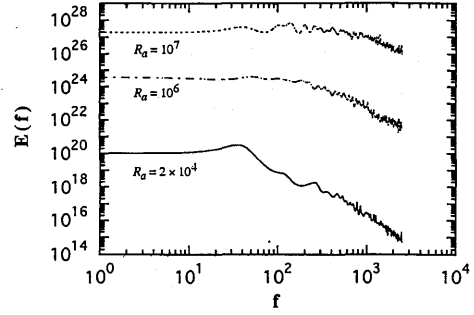


Fig. 28 The log-log plot of power spectra of the total energy for the fixed Prandtl number, $P_r = 1.0$. The solid line: $R_a = 2 \times 10^4$, the dotted-dashed line: $R_a = 10^6$ and the dotted line: $R_a = 10^7$.

fixed point.

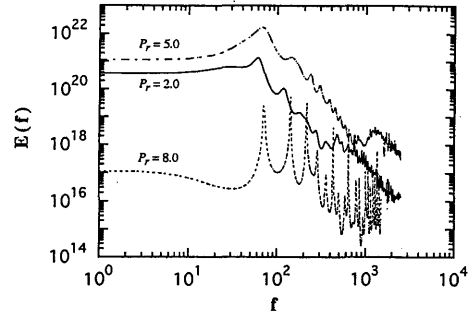


Fig. 29 The log-log plot of power spectra of the total energy for the fixed Rayleigh number, $R_a = 2.0 \times 10^4$. The solid line: $P_r = 2.0$, the dotted-dashed line: $P_r = 5.0$ and the dotted line: $P_r = 8.0$.

3.4 Analysis of intermittency by shell models for Rayleigh-Bénard turbulence and by GOY shell model

In this section, we analyze an intermittency of the model equations with respect to the scaling exponent of higher order structure function. The number of shell is fixed for $N = 20$ and a constant C is fixed for $C = 10^{-2}$ in each case. In our model the p th order structure functions of u_n and p_n are defined from (33) by

$$S_p(n) = \langle |u_n|^p \rangle, \quad (106)$$

and

$$S'_p(n) = \langle |p_n|^p \rangle, \quad (107)$$

respectively. In the inertial range, the scaling exponents, ζ_p and ζ'_p are defined from (34) by

$$S_p(n) \propto k_n^{-\zeta_p}, \quad (108)$$

and

$$S'_p(n) \propto k_n^{-\zeta'_p}. \quad (109)$$

The degree of intermittency is evaluated by (hyper-) flatness. The flatness is defined from (17) by

$$F_6(n) = \frac{S_6(n)}{S_2(n)} \quad (110)$$

and

$$F'_6(n) = \frac{S'_6(n)}{S'_2(n)}. \quad (111)$$

From (108) and (109) the flatness is represented by

$$F_6(n) \propto k_n^\mu \quad (112)$$

and

$$F'_6(n) \propto k_n^{\mu'} \quad (113)$$

where $\mu = -\zeta_6 + 3\zeta_2$ and $\mu' = -\zeta'_6 + 3\zeta'_2$. The system with an intermittency has the k_n dependence and the flatness diverges in the limit of $k_n \rightarrow \infty$. On the other hand, the flatness converges to a finite value in the limit of $k_n \rightarrow \infty$ for a non-intermittent system.

Firstly, the Rayleigh number dependence of the intermittency is investigated. Figure 30 shows the scaling exponents of $S_p(n)$ by the nearest neighbour real component model(I) for the fixed Prandtl number, $P_r = 1.0$. Cases with various Rayleigh numbers are plotted. The circles correspond to the exponents for $R_a = 2 \times 10^4$, the squares to the exponents for $R_a = 2 \times 10^5$ and the solid line to the K41 prediction ($\zeta_p = p/3$). The exponents

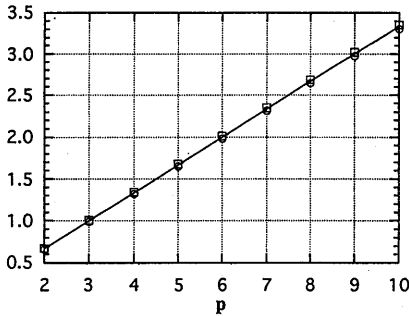


Fig. 30 The scaling exponent of $S_p(n)$ obtained by the nearest neighbour real component model(I) for fixed Prandtl number, $P_r = 1.0$.

obtained by this model are in a good agreement with the K41 prediction. The deviation from the K41 prediction indicates an intermittency. Therefore, this model does not contain an intermittent nature, in other words, does not have a chaotic attractor. The flatnesses of $u_n(k_n)$ are $F_6(k_n) \propto k_n^{3.0 \times 10^{-5}}$ in the case of $R_a = 2 \times 10^4$ and $R_a = 2 \times 10^5$. The values of flatnesses are nearly unity. When $k_n \rightarrow \infty$, the values of flatnesses become nearly constant of the order of unity. It is confirmed that this model does not have an intermittency from the aspect of the convergence of flatness.

Figure 31 is the plot of the scaling exponents of $S_p(n)$ obtained by the nearest and second nearest neighbour complex component model(II). The solid line corresponds to the K41 prediction. Circles and squares show the exponents with $R_a = 2 \times 10^4$ and $R_a = 2 \times 10^5$, respectively. The dotted line is the result from the GOY model²⁴⁾. The scaling exponents are found to be different from the K41 prediction. The flatnesses are analyzed and we have $F_6(k_n) \propto k_n^{0.27}$ for $R_a = 2 \times 10^4$ and $F_6(k_n) \propto k_n^{0.23}$ for $R_a = 2 \times 10^5$, respectively. For GOY model $F_6(k_n) \propto k_n^{0.36}$ is reported. The flatness diverges in the limit of $k_n \rightarrow \infty$. The flatness of velocity field diverges slower than that of the GOY model does.

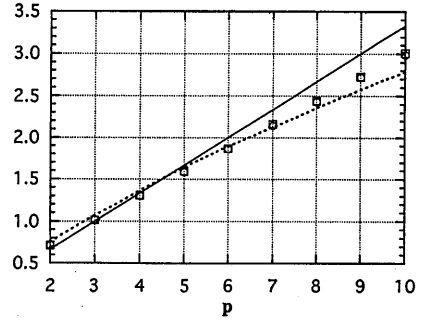


Fig. 31 The plot of the scaling exponent obtained of $S_p(n)$ obtained by the nearest and the second nearest neighbour complex component model(II).

Figure 32 shows the plot of the scaling exponents of $S'_p(n)$ by the nearest and the second nearest complex component model(II) for the fixed Prandtl number, $P_r = 1.0$. The circles correspond to the exponents for $R_a = 2 \times 10^4$, the squares to the exponents for $R_a = 2 \times 10^5$ and the dotted line to the exponents⁹⁾. The scaling exponents are different from the K41 pre-

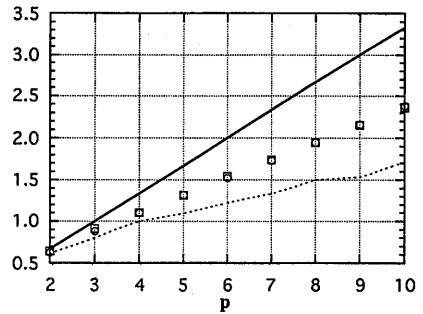


Fig. 32 The plot of the scaling exponent of $S'_p(n)$ obtained by the nearest and the second nearest neighbour complex component model(II) for the fixed Prandtl number, $P_r = 1.0$.

diction. The flatnesses are obtained as $F'_6(k_n) \propto k_n^{0.35}$ for $R_a = 2 \times 10^4$ and $F'_6(k_n) \propto k_n^{0.37}$ for $R_a = 2 \times 10^5$, respectively. The results should be compared to the one

$F'_6(k_n) \propto k_n^{0.59}$ obtained by the GOY model combined with the passive scalar field. The flatness diverges in each case when $k_n \rightarrow \infty$, however the Rayleigh number dependence of flatness is not clearly observed. The flatness of the temperature field also diverges slower than that of the GOY model does.

Next, the Prandtl number dependence of intermittency is investigated. Figure 33 shows the plot of the scaling exponents of $S_p(n)$ obtained by the nearest and the second nearest neighbour complex component model(II) for the fixed Rayleigh number, $R_a = 2 \times 10^4$. Cases of various Prandtl numbers are plotted. The white circles represent the exponents for $P_r = 1.0$, the squares the exponents for $P_r = 2.0$, the triangles the exponents for $P_r = 5.0$ and the black circles correspond to the exponents for $P_r = 8.0$. The dotted line shows the results from the GOY model. The flatnesses are $F_6(k_n) \propto k_n^{0.23}$ for $P_r = 2.0$, $F_6(k_n) \propto k_n^{0.12}$ for $P_r = 5.0$ and $F_6(k_n) \propto k_n^{0.09}$ for $P_r = 8.0$. In the cases with $P_r = 1.0$, $P_r = 2.0$ and $P_r = 5.0$, the flatnesses diverge when $k_n \rightarrow \infty$. The intermittency is observed for these parameters. However for $P_r = 8.0$, it converges unity in the limit of $k_n \rightarrow \infty$ because the exponent of flatness is very small. The disappearance of intermittency is also confirmed by the maximum Lyapunov exponent as is shown in figure 25.

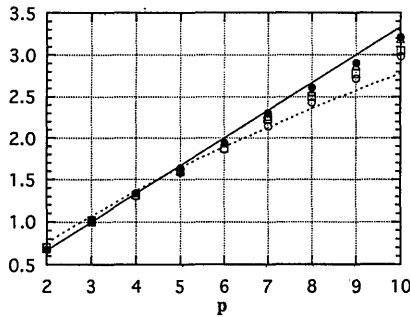


Fig. 33 The plot of the scaling exponent of $S_p(n)$ by the nearest and the second nearest neighbour complex component model(II) for the fixed Rayleigh number, $R_a = 2 \times 10^4$.

Figure 34 shows the exponents of $S'_p(n)$ obtained by the nearest and the second nearest neighbour complex component model(II) for the fixed Rayleigh number $R_a = 2.0 \times 10^4$. The circles correspond to the exponents for $P_r = 1.0$, the squares to the exponents for $P_r = 2.0$, the triangles to the exponents for $P_r = 5.0$, the black circles to the exponents for $P_r = 8.0$ and the solid line to the K41 prediction. The dotted line shows the exponents in the case with a passive scalar field⁹⁾. The flatnesses are $F'_6(k_n) \propto k_n^{0.35}$ for $P_r = 1.0$, $F'_6(k_n) \propto k_n^{0.35}$ for $P_r = 2.0$, $F'_6(k_n) \propto k_n^{0.05}$ for $P_r = 5.0$ and $F'_6(k_n) \propto k_n^{2.0 \times 10^{-3}}$ for $P_r = 8.0$. The flatness of

the GOY model is given by $F'_6(k_n) \propto k_n^{0.59}$. The disappearance of intermittency is observed as in figure 33.

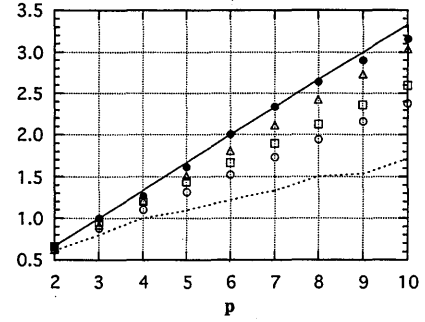


Fig. 34 The exponent of $S'_p(n)$ by the nearest and the second nearest neighbour complex component model(II) for the fixed Rayleigh number $R_a = 2.0 \times 10^4$.

Figure 35 is the plot of the exponents of flatnesses, μ and μ' , versus the Rayleigh number for the fixed Prandtl number, $P_r = 1.0$. The circles correspond to the exponents, μ and squares to the exponents, μ' . The Rayleigh number dependence of the exponent of flatness is found to be weak. The exponent, μ' is larger than that of μ .

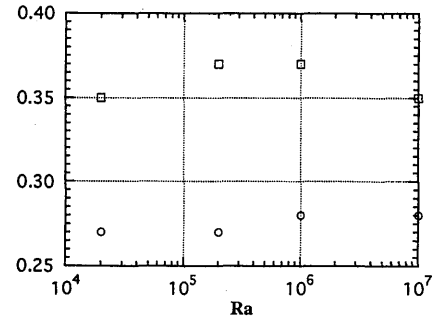


Fig. 35 The plot of the exponents of flatnesses, μ and μ' , versus Rayleigh number for the fixed Prandtl number, $P_r = 1.0$.

Figure 36 is the plot of the exponents of the flatnesses, μ and μ' , versus the Rayleigh number for the fixed Rayleigh number, $R_a = 2.0 \times 10^4$. The circles correspond to the exponents, μ and squares to the exponents, μ' . The exponents of both variables decrease with the increase of the Prandtl number. The exponent, μ' decreases more rapidly with the increase of the Prandtl number.

From these calculations, it is confirmed that the nearest neighbour real component model(I) does not show an intermittency, and the nearest and the second nearest neighbour complex component model(II) shows it. The Prandtl number dependence of the intermittency is a newly obtained by our model. The original GOY model does not contain such a nature.

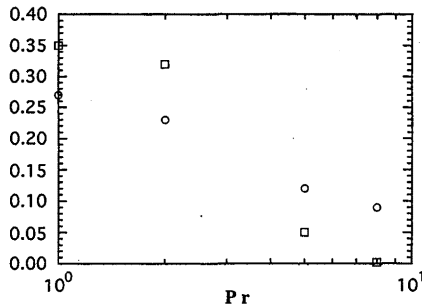


Fig. 36 The plot of the exponents of flatnesses, μ and μ' , versus Prandtl number for the fixed Rayleigh number, $P_a = 2.0 \times 10^4$.

4. Summary and Discussions

In this thesis, two new shell models, which are extended to describe a turbulence driven by the thermal instability are introduced and analyzed. One is the nearest neighbour real component model(I) and other is the nearest and the second nearest neighbour complex component model(II). Numerical simulations of two new shell models are performed. Parameter surveys are done with respect to the Rayleigh number, R_a and the Prandtl number, P_r .

For the nearest neighbour real component model(I), the time averaged wavenumber spectra of variables, $u_n(k)$ and $p_n(k)$ show the scaling laws, $u_n(k) \propto k_n^{-1/3}$ and $p_n(k) \propto k_n^{-1/3}$, respectively in the inertial range. The Kolmogorov type scaling law is obtained. The total energy converges to a constant value in the nonlinear phase for arbitrary values of the Rayleigh number and the Prandtl number, the fact indicates that this model does not have a chaotic attractor.

For the nearest and the second nearest neighbour complex component model(II), the time averaged wavenumber spectra of variables, $u_n(k)$ and $p_n(k)$ are examined. They show the scaling like the Kolmogorov type scaling law as is obtained in the nearest neighbour real component model(I). The mean value of the total energy and its variance increase with the increase of the Rayleigh number. The ratio of the fluctuation amplitude to the mean value of the total energy is 20% ~ 90%, which shows a chaotic nature contained in this model. The maximum Lyapunov exponents are positive and the Rayleigh number dependence is found to be $\sim R_a^{0.53}$. This result indicates that a chaotic nature exists in the system. The power spectrum of the total energy is found to be a smooth function and has a fractional power law. This is another evidence that the system contains the chaos. The ratio of the variance to the mean value of total energy decreases with the increase of the Prandtl number. This indicates that the system moves from the chaotic state to the stationary state. The maximum

Lyapunov exponent becomes nearly 0 for $P_r = 8.0$ and peaks of the fundamental and the harmonic frequencies in the power spectrum are observed. The nearest and the second nearest neighbour complex component model(II) shows both a chaotic and a stationary state, the appearance depends on the Prandtl number. This result is firstly pointed out by this thesis.

We also investigated the property of an intermittency characteristic. In the nearest neighbour real component model(I), the scaling exponents of both variables, $u_n(k)$ and $p_n(k)$ show the good agreement with the K41 prediction and their flatnesses converge to unity in the limit $k \rightarrow \infty$. This fact indicates that the nearest neighbour real component model(I) does not contain the characteristic of an intermittency. In the nearest and the second nearest neighbour complex component model(II) the scaling exponent is different from the K41 prediction. The clear dependence on the Rayleigh number is not observed from the calculation of flatness. The dependence on the Prandtl number is newly obtained, which is the decreasing function of the Prandtl number. When $P_r \rightarrow 8.0$, the scaling exponent approaches the K41 prediction. This may correspond to the disappearance of an intermittency. The exponent for the flatness in our model is found to be smaller than that in the GOY model with the passive scalar field. From our calculations, we confirm the phase difference between the shell (i.e., complex variables) are important for existing a chaos and an intermittency.

We also derive the nearest neighbour complex component model(III). The model equation and its results of numerical calculation is shown in Appendix A3.. However we do not successfully obtain the steady state spectrum of this model yet. The analysis of this model is left for a future work.

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Appendix

A1. Equations for Rayleigh-Bénard turbulence

We assume that the fluid layer of height h under the gravity acceleration g is submitted to a temperature gradient $\beta = \Delta T/h > 0$, where ΔT is the temperature difference between the location considered and the bottom (subscript "b") plate. Under this assumption, the equation of motion and the equation of heat conduction are as follows:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{v} = -\nabla \left(\frac{p}{\rho_0} \right) - \{1 + (T_b - T)\alpha\} g \hat{\mathbf{z}} + \nu \Delta \mathbf{v} \quad (\text{A1})$$

and

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \kappa \Delta T, \quad (\text{A2})$$

where $\hat{\mathbf{z}}$ is a unit vector in the z direction, ν is the kinematic viscosity, α is the thermal expansion coefficient and κ is the coefficient of thermometric conductivity.

Let there be no motion. The initial state is, therefore, one in which

$$\mathbf{v} = 0, T = T(z). \quad (\text{A3})$$

When no motions are present, the hydrodynamical equations require only that pressure distribution is governed by the equation

$$\frac{\partial p}{\partial z} = -g\rho \quad (\text{A4})$$

and the temperature distribution is governed by

$$\nabla^2 T = 0 \quad (\text{A5})$$

where

$$\rho = \rho_b \{1 + (T_b - T)\} \quad (\text{A6})$$

and ρ_b is the density at the bottom plate.

The solution of equation (A5) appropriate to the problem on hand is

$$T = T_b - \beta z. \quad (\text{A7})$$

The corresponding distribution of the density is given by

$$\rho = \rho_b (1 + \beta z). \quad (\text{A8})$$

With this expression for ρ , equation (A4) can be integrated to give

$$p = p_b - g\rho_b \left(z + \frac{1}{2}\alpha\beta z^2\right). \quad (\text{A9})$$

Let the initial state described by equations (A3), (A7), (A8), and (A9) be slightly perturbed. Let $\mathbf{v}$ denote the velocity in the perturbed state and let the altered temperature distribution be

$$\hat{T} = T_b - \beta z + \theta. \quad (\text{A10})$$

Finally, let $\hat{p}$ denote the change in the pressure distribution. We shall now obtain the form of the equations in this system;

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \left(\frac{\hat{p}}{\rho_0}\right) + g\alpha\theta \hat{\mathbf{z}} + \nu \Delta \mathbf{v}, \quad (\text{A11})$$

$$\frac{\partial \theta}{\partial t} + \mathbf{v} \cdot \nabla \theta = \beta \mathbf{v} \cdot \hat{\mathbf{z}} + \kappa \Delta \theta \quad (\text{A12})$$

and the velocity field remains, of course, solenoidal:

$$\nabla \cdot \mathbf{v} = 0. \quad (\text{A13})$$

Equation (A11) and (A12) describe the Rayleigh-Bénard turbulence. With these equations, the model equation should be established.

A2. The maximum Lyapunov exponent

In this section, we show the procedure to calculate a Lyapunov exponent. Lyapunov exponent by the nearest and the second nearest neighbour complex component model is evaluated as follows.

Linearizing (72) and (73) we obtain the equations:

$$\begin{aligned} \frac{d\delta u_n}{dt} = & i(k_n \delta u_{n+1} u_{n+2} + k_n u_{n+1} \delta u_{n+2} \\ & - k_{n-2} \delta u_{n-1} u_{n+1} - k_{n-2} u_{n-1} \delta u_{n+1} \\ & - k_{n-3} \delta u_{n-1} u_{n-2} - k_{n-3} u_{n-1} \delta u_{n-2})^* \\ & + P_\tau \delta p_n - P_\tau k_n^2 \delta u_n \end{aligned} \quad (\text{A14})$$

and

$$\begin{aligned} \frac{d\delta p_n}{dt} = & i[k_{n-1}(\delta u_{n-1} p_{n+1} + u_{n-1} \delta p_{n+1} \\ & - \delta u_{n+1} p_{n-1} + u_{n+1} \delta p_{n-1}) \\ & - k_{n-3}(\delta u_{n-2} p_{n-1} + u_{n-2} \delta p_{n-1} \\ & + \delta u_{n-1} p_{n-2} + u_{n-1} \delta p_{n-2}) \\ & + k_{n+2}(\delta u_{n+1} p_{n+2} + u_{n+1} \delta p_{n+2} \\ & + \delta u_{n+2} p_{n+1} + u_{n+2} \delta p_{n+1})]^* \\ & + R_a \delta u_n - k_n^2 \delta p_n. \end{aligned} \quad (\text{A15})$$

This system has $4N$ Lyapunov exponents. We define the eigenvector given by

$$\mathbf{v}^{(k)} = \begin{pmatrix} \delta u_1^r \\ \delta u_1^i \\ \vdots \\ \delta u_N^r \\ \delta u_N^i \\ \delta p_1^r \\ \delta p_1^i \\ \vdots \\ \delta p_N^r \\ \delta p_N^i \end{pmatrix} \quad (\text{A16})$$

where $k = 1, 2, \dots, N$ and superscript 'r' and 'i' represent the real part and the imaginary part, respectively. We integrate (A14) and (A15) with the $4N$ initial conditions given by

$$\begin{aligned} \mathbf{v}_0^{(1)} &= (1, 0, \dots, 0), \mathbf{v}_0^{(2)} = (0, 1, \dots, 0), \dots, \\ \mathbf{v}_0^{(4N)} &= (0, 0, \dots, 1). \end{aligned} \quad (\text{A17})$$

By the Gram-Schmidt method we normalize the $4N$ linear solutions, $\mathbf{v}^{(k)}, k = 1, \dots, N$ under the constant intervals;

$$\begin{aligned} \mathbf{v}'^{(1)} &= \frac{\mathbf{v}^{(1)}}{\|\mathbf{v}^{(1)}\|}, \\ \mathbf{v}'^{(2)} &= \frac{\mathbf{v}^{(2)} - (\mathbf{v}^{(2)} \cdot \mathbf{v}'^{(1)})\mathbf{v}'^{(1)}}{\|\mathbf{v}^{(2)} - (\mathbf{v}^{(2)} \cdot \mathbf{v}'^{(1)})\mathbf{v}'^{(1)}\|}, \\ \mathbf{v}'^{(2N)} &= \frac{\mathbf{v}^{(2N)} - (\mathbf{v}^{(2N)} \cdot \mathbf{v}'^{(2N-1)})\mathbf{v}'^{(2N-1)} \dots (\mathbf{v}^{(2N)} \cdot \mathbf{v}'^{(1)})\mathbf{v}'^{(1)}}{\|\mathbf{v}^{(2N)} - (\mathbf{v}^{(2N)} \cdot \mathbf{v}'^{(2N-1)})\mathbf{v}'^{(2N-1)} \dots (\mathbf{v}^{(2N)} \cdot \mathbf{v}'^{(1)})\mathbf{v}'^{(1)}\|} \end{aligned}$$

where $\|\dots\|$ represents the norm. By these normalizations, we obtain the increments of the solutions, $\epsilon_k = \|\mathbf{v}^{(k)}(t)\|$ in the orthogonal direction in the phase space. A local Lyapunov exponent, λ_i is defined as the local growth rate of an error, ϵ_k measured over the finite time interval, $[t_0, t]$;

$$\lambda_i(t) = \frac{\log\{\epsilon_i(t)/\epsilon_i(t_0)\}}{t - t_0}, \quad (\text{A18})$$

where $i = 1, \dots, 4N$. In the limit of $t \rightarrow \infty$, we obtain the Lyapunov exponent, λ_i . The maximum value of λ_i is defined as the maximum Lyapunov exponent.

A3. The nearest neighbour complex component model

We can derive the nearest neighbour complex component model(III) according to the treatment discussed in section 2.4. The model equation is as follows;

$$\frac{\partial u_n}{\partial t} = ik_n[u_{n-1}^*u_{n-1} - 2u_n^*u_{n+1}] - P_r k_n^2 u_n + P_r p_n \quad (\text{A19})$$

and

$$\frac{\partial p_n}{\partial t} = ik_n[u_{n-1}^*p_{n-1} - 2u_n^*p_{n+1}] - k_n^2 p_n + R_a u_n \quad (\text{A20})$$

where $k_n = C \cdot 2^{n-1}$, $1 \leq n \leq N$, R_a is the Rayleigh number and P_r is the Prandtl number.

We performed the numerical simulation with $C = 1.0$, $N = 13$, $R_a = 2 \times 10^4$ and $P_r = 1.0$. Figure C.1 shows the temporal evolution of the spectrum of $u_n(k)$ with $R_a = 2 \times 10^4$, $P_r = 1.0$, $C = 1.0$ and $N = 13$. The dotted line corresponds to the spectrum at $t = 0.3$, the solid line to that at $t = 0.4$ and the dotted-dashed line to that at $t = 0.8$. According to the evolution of fields, the inertial range becomes narrower and finally, a numerically unstable situation occurs. We could not obtain yet the steady state spectrum for this model.

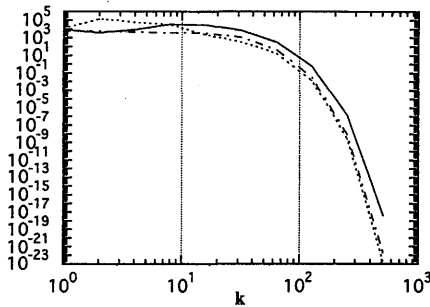


Fig. A1 The instantaneous wavenumber spectrum of $u_n(k)$ by the nearest neighbour complex component model(III). The solid line is $t = 0.3$, the dotted-dashed line $t = 0.4$ and the dotted line shows $t = 0.8$.