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Tatsuno, Masakazu
Research Institute for Applied Mechanics, Kyushu University : Research Associate

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SECONDARY FLOW AROUND AN OSCILLATING ELLIPTIC CYLINDER

By Masakazu TATSUNO*

The secondary flow generated by an elliptic cylinder oscillating harmonically in a viscous, incompressible fluid is investigated theoretically and experimentally, which is inclined its major axis at an arbitrary angle to the direction of oscillation.

The positions of stagnation points where the streaming is toward the cylinder shift on the surface of the body depending on the values of k and α, where k is the ratio of the length of a minor axis to that of a major one of the elliptic cylinder and α the angle between the major axis and the direction of oscillation. Moreover, both the direction of the flow toward the cylinder and that of the flow away from the cylinder also depend upon k and α. The flow pattern of the streaming has symmetry with respect to a turn of π around the center axis of the cylinder.

Key words: Secondary flow, Oscillating cylinder, Inclined elliptic cylinder

1. Introduction

It is well known that when a bluff body is oscillated in a viscous flow, the secondary flow is generated around the body owing to the non-linear effect of the fluid flow.

Schlichting[1] was the first to investigate theoretically the steady streaming around a bluff body. He applied the solution of the boundary layer equation about an oscillating infinite flat plate to a circular cylinder for $\varepsilon^2 \ll R_s \ll 1$, where R_s is the Reynolds number of the steady streaming and ε the ratio of the amplitude of oscillation to the body dimension. In this case, the flow field consists of the boundary layer adjacent to the body and the main flow outside. The steady stream induced in the boundary layer by the action of the Reynolds stresses drives a steady streaming in the main field.

After that, many investigations have been done in detail of the streaming around a circular cylinder both theoretically and experimentally by Westervelt[2], Holtsmark et al.[3], Stuart[4], Riley[5] and Bertelsen et al.[6].

* Research Associate, Research Institute for Applied Mechanics, Kyushu University

Basing on the past works, the flow pattern of the steady streaming around a circular cylinder is symmetric with respect to both the axis of oscillation and the axis perpendicular to it, and the flow is directed away from the body along the axis of oscillation.

Now, Tamada and Miyagi[7] investigated the streaming around a Joukowsky profile oscillating parallel to its chord in simple harmonic motion. Its section was asymmetric with respect to the axis perpendicular to that of oscillation. They reported that a uniform flow was induced at infinity. Thus, the steady streaming undergoes an essential change if the cylinder section is asymmetric. Then, a problem is brought up subsequent to their investigation. What changes do the streaming undergo when the cylinder section is symmetric but its geometrical arrangement is asymmetric with respect to the direction of oscillation? With regard to this question, this work studies the case of an elliptic cylinder. It is symmetric with respect to its own two axes, the major one and the minor. When the major axis, however, is inclined at an angle of α to the axis of oscillation, the arrangement of it is asymmetric with respect to both axes of oscillation and perpendicular to it. The theoretical investigation is performed assuming the same condition that Schlichting applied to the flow around a circular cylinder.

2. Foundation for Theoretical Analysis

In this paper, we suppose that a cylindrical body performs small amplitude oscillations with velocity $U_\infty \cos \omega t$ in a viscous fluid otherwise at rest, where U_∞ is the typical velocity and ω^{-1} the typical time.

In the first place, we describe the outline of the fundamental theory for the streaming based upon that developed by Schlichting, Davidson & Riley[8] and Tamada & Miyagi.

Supposing the co-ordinate system $\boldsymbol{x}=(x, y)$ is fixed on the body and the x-axis is parallel to the direction of oscillation, the fundamental equation and the boundary condition satisfied by the non-dimensional stream function ψ may be described as follows:

$$\left.\begin{aligned} &\frac{\partial}{\partial t}(\nabla^2\psi)-\varepsilon\frac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)}=\frac{\varepsilon^2}{R_s}\nabla^4\psi, \\ &\varepsilon=s/d=U_\infty/\omega d, \quad R_s=s^2/\delta^2=U_\infty^2/\omega\nu, \quad \delta=\sqrt{\nu/\omega}, \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} &\boldsymbol{v}=0 \quad \text{on the body surface,} \\ &\psi\sim y e^{it} \quad \text{as } |\boldsymbol{x}|\to\infty, \end{aligned}\right\} \tag{2}$$

where s is the oscillation amplitude, d the representative length of the body, ν the kinematic viscosity, R_s the Reynolds number for the steady streaming and δ the boundary layer thickness present at the body surface. The real part of any complex expression is to be understood.

We assume that $\varepsilon \ll 1$ throughout the present study, and seek a solution of (1) in the form

$$\psi = \psi_0 + \varepsilon \psi_1 + \cdots\cdots . \tag{3}$$

Furthermore we now assume that $\varepsilon^2/R_s = (\nu/\omega)/d^2 \ll 1$, that is, the thickness of the boundary layer at the body surface is assumed to be small in comparison with the representative length of the body. In that case, (1) becomes in the main flow as follows:

$$\nabla^2 \psi_0 = 0 . \tag{4}$$

Expanding the solution of (4) close to the body surface, we may write in the form

$$\psi_0 \simeq n V(s) \mathrm{e}^{it} \qquad n \simeq 0 , \tag{5}$$

where $V(s)$ is the surface velocity distribution of the potential flow. Here, the orthogonal co-ordinate system (s, n) is adopted in the neighborhood of the body for the convenience sake, where s and n are distances measured respectively along and normal to the body surface coinciding with $n=0$.

On the other hand, equation of (1) becomes

$$\frac{\partial^3 \Psi}{\partial t \partial n^2} = \frac{\varepsilon^2}{R_s} \nabla^4 \Psi, \tag{6}$$

in the boundary layer. We may seek a solution of (6) as follows:

$$\Psi = \Psi_0 + \varepsilon \Psi_1 + \cdots\cdots . \tag{7}$$

The term Ψ_0 must match with (5) on the edge of the boundary layer, and then we obtain the well-known Stokes solution:

$$\Psi_0 = \frac{\sqrt{2}\,\varepsilon}{R_s^{1/2}} V(s) \left\{ \eta - \frac{1-i}{2} \left(1 - \mathrm{e}^{-(1+i)\eta}\right) \right\} \mathrm{e}^{it}, \quad \eta = \frac{1}{\sqrt{2}} \frac{R_s^{1/2}}{\varepsilon} n . \tag{8}$$

In the following step, in order to seek the second approximation of Ψ with respect to ε, we substitute (8) into (1). Consequently we may obtain the steady part in the form:

$$-\frac{\varepsilon^2}{R_s} \cdot \frac{\partial^4 \Psi_1^{(s)}}{\partial n^4} = \varepsilon \left[\frac{\partial(\Psi_0, \partial^2 \Psi_0 / \partial n^2)}{\partial(s, n)} \right]^{(s)}, \tag{9}$$

where the superscript (s) indicates the steady part of each quantity. The solution of (9) satisfying the boundary conditions (2) was given by Schlichting as follows:

$$\Psi_1^{(s)} = \frac{1}{2} \frac{\mathrm{d} V^2}{\mathrm{d} s} \chi_1^{(s)}(\eta) . \tag{10}$$

From this solution, under the condition of $\eta \to \infty$, the next solution may be obtained:

$$\frac{\partial \Psi_1^{(s)}}{\partial n} \sim -\frac{3}{8}\frac{\mathrm{d}V^2}{\mathrm{d}s} . \tag{11}$$

This solution indicates that a steady streaming exists along the edge of the boundary layer. Then, this solution must match with the velocity of the main flow at $n \simeq 0$. Consequently, if we term this velocity the slip velocity, say u_s, the boundary conditions of the flow dominating the main flow field are

$$u_s = \left(\frac{\partial \Psi_1^{(s)}}{\partial n}\right)_{\eta\to\infty} = \left(\frac{\partial \psi_1^{(s)}}{\partial n}\right)_{n\to 0} = -\frac{3}{8}\frac{\mathrm{d}V^2}{\mathrm{d}s} , \qquad \left(\frac{\partial \psi_1^{(s)}}{\partial s}\right)_{n\to 0} = 0 . \tag{12}$$

Provided that $R_s \ll 1$, the second approximation of the equation satisfied by $\psi_1^{(s)}$ in the main flow is obtained as follows from (1) and (3):

$$\nabla^4 \psi_1^{(s)} = 0. \tag{13}$$

This is the Stokes equation and it proves to be induced the steady streaming of Stokes type in the main flow.

If we work with complex co-ordinates $(z, \bar{z})$ and (u_1, v_1) as the velocity of steady streaming in order to solve the equation (13) subject to (12), the next relation is defined,

$$W \equiv u_1 - i v_1 = 2i \frac{\partial \psi_1^{(s)}}{\partial z} . \tag{14}$$

Then, the general solution of W satisfying the Stokes equation of (13) is obtained as follows:

$$W = f'(z) + \bar{z} g'(z) - \bar{g}(\bar{z}), \tag{15}$$

where $f(z)$ and $g(z)$ are arbitrary analytic functions regular in the flow field.

3. Application to Elliptic Cylinder

In order to apply the aforementioned theory for an arbitrary bluff cylinder to an elliptic cylinder of semi-major and semi-minor axes a and b respectively, the next transformation is introduced:

$$z = \mathrm{e}^{-i\alpha}(\zeta + m\,\zeta^{-1}) , \tag{16}$$

which transforms the region outside an ellipse in the z-plane to the region outside a unit circle in the ζ-plane, where $(a+b)/2 = 1$, $m \equiv (1-k)/(1+k)$, $k = b/a$ and $-\alpha$ the angle between the major axis of the ellipse and x-axis.

In this case, the boundary condition (12) of the flow becomes

$$W = -\frac{3}{8}\frac{\mathrm{d}V^2}{\mathrm{d}\zeta} \cdot \frac{\mathrm{d}\zeta}{\mathrm{d}z} \qquad \text{on} \quad \zeta\bar{\zeta} = 1. \tag{17}$$

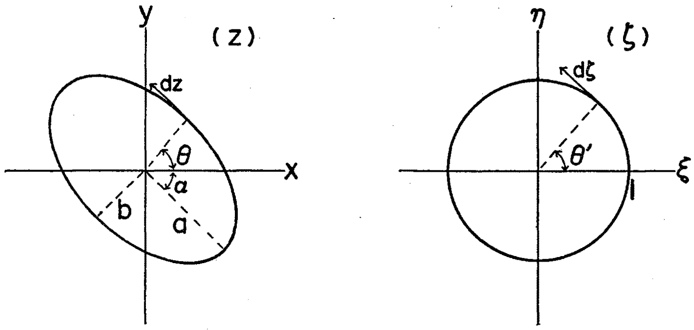


Fig. 1 Ellipse in z-plane and unit circle in ζ-plane.

Substituting (17) into (15), the next equation is obtained:

$$\frac{\mathrm{d}f(\zeta)}{\mathrm{d}\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}z}+\bar{z}\left(\frac{1}{\zeta}\right)\frac{\mathrm{d}g(\zeta)}{\mathrm{d}\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}z}-\bar{g}\left(\frac{1}{\zeta}\right)=-\frac{3}{8}\frac{\mathrm{d}V^2}{\mathrm{d}\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}z}\ , \tag{18}$$

on $\zeta\bar{\zeta}=1$ in the ζ-plane, and it holds for arbitrary quantity of ζ by analytic continuation.

The complex velocity potential F of the irrotational flow past the unit circle with unit x-velocity at infinity in the ζ-plane may be written as

$$F=\mathrm{e}^{-i\alpha}\zeta+\mathrm{e}^{i\alpha}\zeta^{-1}\,. \tag{19}$$

Then the surface velocity squared V^2 is obtained as follows:

$$V^2=\frac{(\mathrm{e}^{-i\alpha}\zeta^2-\mathrm{e}^{i\alpha})^2}{(\zeta^2-m)(m\zeta^2-1)}\ . \tag{20}$$

Now, we continue to calculate the stream function of the steady streaming for a special case of $\alpha=\pi/4$. Substiuting (20) into (18) we may obtain the following equation,

$$\bar{g}\left(\frac{1}{\zeta}\right)+\frac{3\sqrt{2}}{8}\left\{\frac{(1-i)m^4+2(1+i)m^3-(1-i)m^2}{(1-m^2)^3}\frac{\zeta}{m\zeta^2-1}\right.$$

$$\left.-\frac{(1-i)m^2+2(1+i)m-(1-i)}{(1-m^2)^2}\frac{\zeta}{(m\zeta^2-1)^2}\right\}$$

$$=\frac{1+i}{\sqrt{2}}\frac{\zeta^2}{\zeta^2-m}\frac{\mathrm{d}f(\zeta)}{\mathrm{d}\zeta}+i\frac{\zeta(1+m\zeta^2)}{\zeta^2-m}\frac{\mathrm{d}g(\zeta)}{\mathrm{d}\zeta}$$

$$+\frac{3\sqrt{2}}{8}\left\{\frac{(1-i)m^3+2(1+i)m^2-(1-i)m}{(1-m^2)^3}\frac{\zeta}{\zeta^2-m}\right.$$

$$+\frac{(1-i)m^2-2(1+i)m-(1-i)}{1-m^2}\frac{\zeta}{(\zeta^2-m)^2}$$

$$+\frac{(1-i)m^3-2(1+i)m^2-(1-i)m}{1-m^2}\frac{\zeta}{(\zeta^2-m)^3}\Big\}\ . \tag{21}$$

Since $0<m<1$, the left hand side of (21) is regular in $|\zeta|\leq 1$ and the right hand side is regular in $|\zeta|\geq 1$. Each side represents an identical function and proves to be constant. In this work the generality isn't lost supposing that the constant is equal to zero. As the result, the following two analytic functions may be found.

$$g(\zeta)=\frac{3\sqrt{2}}{8}\Big\{\frac{(1+i)m^4+2(1-i)m^3-(1+i)m^2}{(1-m^2)^3}\frac{\zeta}{\zeta^2-m}$$

$$+\frac{(1+i)m^2+2(1-i)m-(1+i)}{(1-m^2)^2}\frac{\zeta^3}{(\zeta^2-m)^2}\Big\}\ ,$$

$$f(\zeta)=-\frac{3\,im}{2(1-m^2)^2}\log\zeta+\frac{3(m^4-2m^2-1)(im^2+2m-i)}{4(1-m^2)^3}\frac{1}{\zeta^2-m}$$

$$-\frac{3m(1+m^2)(im^2+2m-i)}{4(1-m^2)^2}\frac{1}{(\zeta^2-m)^2}\ . \tag{22}$$

The velocity field may be obtained from the following equation.

$$u_1-iv_1=\frac{df(\zeta)}{d\zeta}\frac{d\zeta}{dz}+\bar{z}\Big(\frac{1}{\zeta}\Big)\frac{dg(\zeta)}{d\zeta}\frac{d\zeta}{dz}-\bar{g}\Big(\frac{1}{\zeta}\Big)\ . \tag{23}$$

Substituting (22) into (23), it will be evident that (23) has no constant term. The stream function of the steady streaming is derived by integrating (14) as follows:

$$\psi_1^{(s)}=\mathrm{I_m}\Big[-\frac{3\,im}{2(1-m^2)^2}\log\zeta+\frac{3(m^4-2m^2-1)(im^2+2m-i)}{4(1-m^2)^3}\frac{1}{\zeta^2-m}$$

$$-\frac{3m(1+m^2)(im^2+2m-i)}{4(1-m^2)^2}\frac{1}{(\zeta^2-m)^2}$$

$$+\frac{3(1+i)}{8}\Big(\bar{\zeta}+\frac{m}{\bar{\zeta}}\Big)\Big\{\frac{(1+i)m^4+2(1-i)m^3-(1+i)m^2}{(1-m^2)^3}\frac{\zeta}{\zeta^2-m}$$

$$+\frac{(1+i)m^2+2(1-i)m-(1+i)}{(1-m^2)^2}\frac{\zeta^3}{(\zeta^2-m)^2}\Big\}\Big]\ , \tag{24}$$

where Im means to take imaginary part.

4. Experimental Method

The experiments were carried out with the same apparatus that the author used in the previous experiments on the streaming around some kinds of oscillating cylinders. Figure 2 shows the essential elements of the apparatus used in this work. A loudspeaker was used to oscillate the test body. Elliptic cylinders of two different sizes ($k=0.336$, $d=0.835$ cm and $k=0.670$, $d=0.466$ cm) were made of wood, where $d=(a+b)$. A thin elliptic cylinder was made of metal and values of k and d were 0.093 and 0.585 cm respectively. In the experiment, the major axis is inclined at an angle of $\pi/4$ to the axis of oscillation.

The aluminium dust was suspended uniformly in the fluid, which was a mixture of water and glycerin. A stroboscope or a projector was used for illumination of a plane perpendicular to the axis of the cylinder.

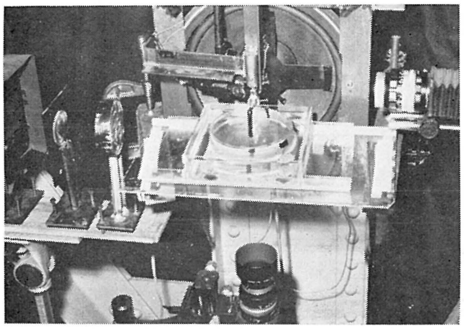


Fig. 2 Experimental apparatus.

5. Results and Discussion

The surface of the unit circle in the ζ-plane may be expressed $\zeta=e^{i\theta'}$, and there exists a relationship of $\theta'-\theta=\alpha$ between the angle θ' of a point on the unit circle and the angle θ of the corresponding point on the elliptic cylinder surface in the z-plane.

The slip velocity is obtained for elliptic cylinders as

$$u_s=\frac{3(1+k)^3\sin\theta\{(1+k^2)\cos\theta-(1-k^2)\cos(\theta+2\alpha)\}}{16\{k^2\cos^2(\theta+\alpha)+\sin^2(\theta+\alpha)\}^{5/2}}\ . \tag{25}$$

In Fig. 3, u_s is shown for some values of k at $\alpha=\pi/4$, wher $k=1$ corresponds to the case of a circular cylinder and $k=0$ that of a flat plat plate. From (25) positions of stagnation points on the surface of the body are obtained as follows:

$$\theta_1=0,\ \pi \quad \text{and} \quad \theta_2=\arctan\left\{-\frac{(1+k^2)-(1-k^2)\cos 2\alpha}{(1-k^2)\sin 2\alpha}\right\}\ . \tag{26}$$

The steady streaming generated around the cylinder is toward that at θ_2 and away from that at θ_1. The value of θ_1 is independent of k and α. Namely, the positions of θ_1 agree with the points where x-axis intersects the surface of the cylinder. As will be seen from Fig. 7, which will be shown later, the stagnation points for θ_1 actually exist nearly

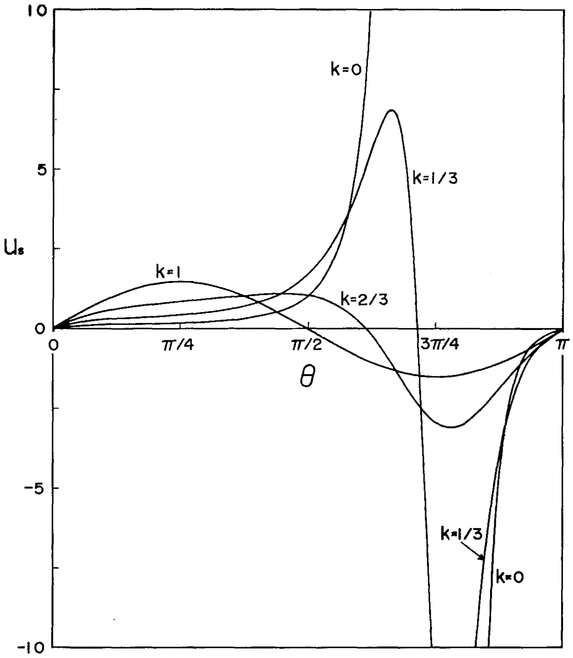


Fig. 3 Slip velocity distribution.

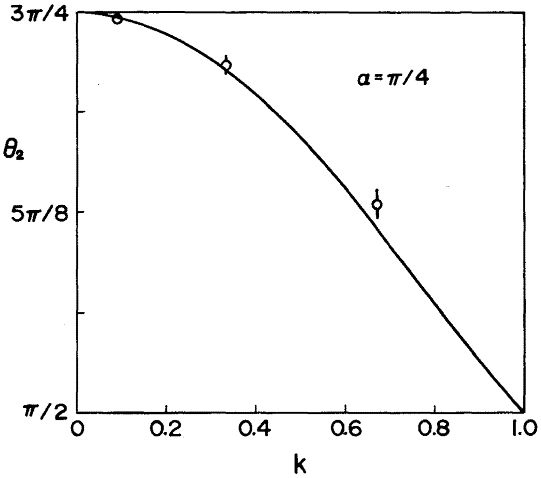


Fig. 4 Relationship between θ_2 and k.
——: theory, o : experiment.

on the x-axis in the neighborhood of the cylinder surface independently of k. Figure 4 shows the relationship between θ_2 and k at $\alpha=\pi/4$.

Though there are two stagnation points on the surface corresponding to θ_2 and $\theta_2+\pi$, only the value in the secon quadrant may be shown hereinafter. For a circular cylinder ($k=1$), the value of θ_2 is always equal to $\pi/2$, and the flow field is symmetric with respect to both the axis of oscillation and that perpendicular to it as well known in past works. The position of the stagnation point for θ_2, however, moves on the surface as the value of k becomes less than 1, and the flow field gets to be asymmetric with respect to any axes. In the case of a flat plate ($k=0$), the point for θ_2 is situated in each edge and the streaming flows along the flat plate. In Fig. 4, the measured values are plotted simultaneously. According to the author's previous work[9], the stagnation points on the cylinder surface of the secondary flow in the boundary layer are independent of the Reynolds number. Therefore, it may be considered that the stagnation points of the main flow, which exist on the edge of the boundary layer, are also independent of the Reynolds number.

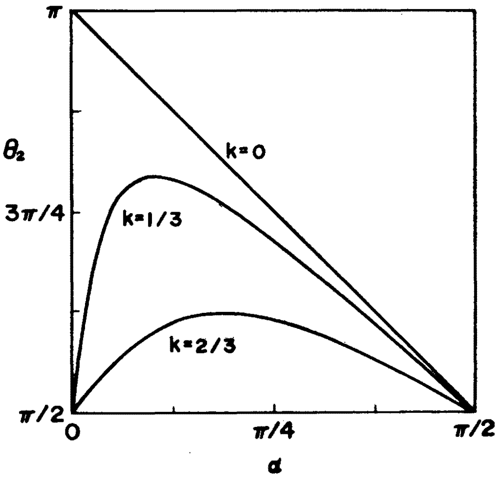
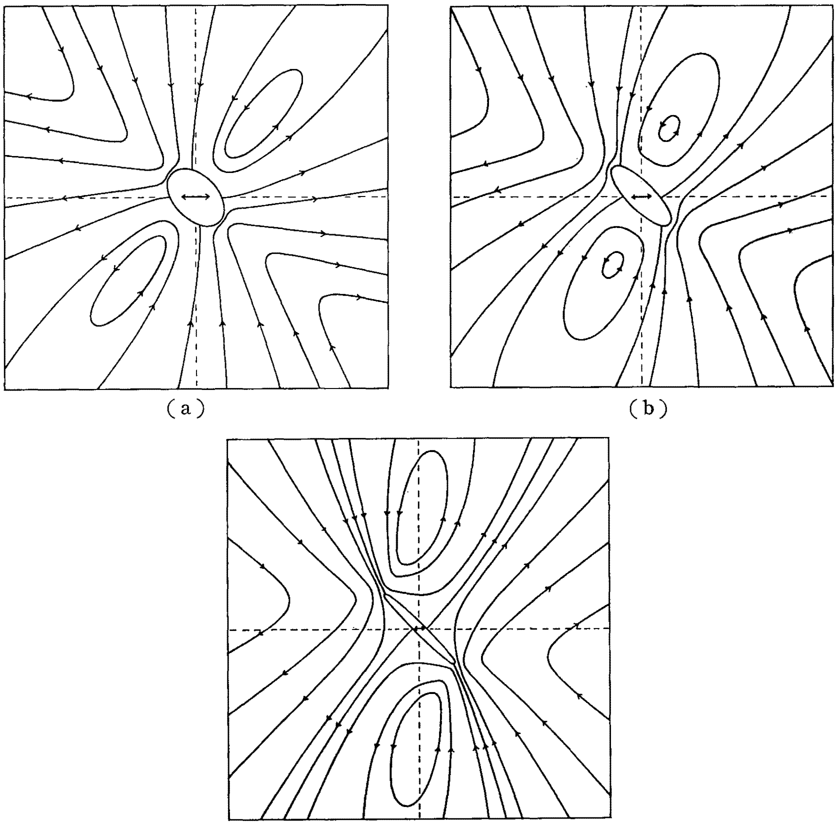


Fig. 5 Relationship between θ_2 and α.

The relationship between θ_2 and α is shown in Fig. 5. In the case of $\alpha=0$ the major axis of the cylinder coincides with x-axis and when $\alpha=\pi/2$ that coincides with y-axis. The value of θ_2 grows large rapidly with increasing the value of α, and after that it gradually approaches to $\pi/2$, namely the stagnation point moves toward the tip of the major axis. As mentioned above, the stagnation points of the flow toward a flat plate exist always at each edge, but we cannot obtain an accurate flow field by means of this analysis for $\alpha=0$, in that case the flat plate oscillates in its own plane.

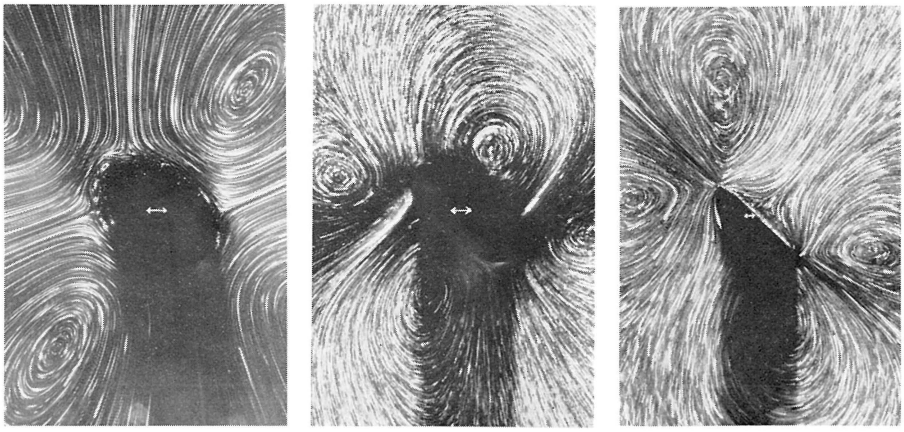
Flow patterns calculated by (24) corresponding to $k=2/3$, 1/3 and



(a) (b) (c)

Fig. 6 Flow patterns at $\alpha=\pi/4$.
(a) $k=2/3$ (b) $k=1/3$ (c) $k=0.093$

0.093 are shown in Fig. 6, and photographs of flow patterns taken in experiments generally equivalent to each in Fig. 6 are shown in Fig. 7. The thin layer surrounding the cylinder, which may be seen in Fig. 7 (a), is the inner circulatory streaming in the boundary layer. This thin layer, however, is unable to be seen in Fig. 6 since it is assumed to be considerably thin in the theory. In the case of the calculation, as the work is performed in the flow field extended infinitely, the stream lines of a pair of vortices are not closed in the figures. On the other hand, those of all four vortices in each photograph are closed owing to the wall of a container of a fluid in the experiments. As will be seen from Fig. 6 and 7, although the stagnation points exist at 0 and π normally, the direction of the flow away from the cylinder depends upon the value of k. The angle between the direction of its flow and x-axis (the axis



(a) (b) (c)

Fig. 7 Flow patterns at $\alpha=\pi/4$.

(a) $k=0.670$, $f=50.2$ Hz, $\varepsilon=0.0335$, $R_s=0.88$, $(\nu/\omega)/d^2=1.3\times10^{-3}$.
(b) $k=0.336$, $f=10.0$ Hz, $\varepsilon=0.0116$, $R_s=0.70$, $(\nu/\omega)/d^2=1.9\times10^{-4}$.
(c) $k=0.093$, $f=40.1$ Hz, $\varepsilon=0.0188$, $R_s=0.60$, $(\nu/\omega)/d^2=5.8\times10^{-4}$.

of oscillation) grows large with decreasing the value of k. In particular, its angle is about $\pi/4$ for $k=0.093$, that is, the flow runs away from the very thin cylinder in the direction almost perpendicular to the cylinder surface. Similarly, both the position of the stagnation point and the direction of the flow toward the cylinder depend upon the value of k. The flow pattern is symmetric with respect to a turn of π around the center axis of the cylinder. The result of the calculation and that of the experimental observation agree well in these characteristics.

6. Conclusion

In this study, an elliptic cylinder is arranged asymmetrically with respect to two axes of oscillation and perpendicular to it. The flow field is different from that of a circular cylinder. The flow pattern has no axes of symmetry but is symmetric with respect to a turn of π around the center axis of the cylinder. Furthermore, remarkable differences exist in the directions of flows toward and away from the cylinder. The angle between the direction of the flow away from the cylinder and the axis of oscillation grows large with decreasing the value of k. These directions of flows also depend upon the value of α. The positions of stagnation points of the flow away from the cylinder are independent of k and α, but those of the flows toward the cylinder shift on the surface depending on k and α.

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