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Takematsu, Masaki
Research Institute for Applied Mechanics, Kyushu University : Professor

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ON THE INSTABILITY OF AIR FLOW OVER A WATER SURFACE

By Masaki TAKEMATSU*

The linear instability of a laminar air flow over a water surface is examined anew. By numerically solving the usual eigenvalue problem for the coupled air-water system, the characteristics of both 'air' and 'water' waves are determined accurately at finite Reynolds numbers. The resulting stability characteristics of the system are not so complicated as has generally been believed, showing that the neutral stability curves presented in the early work of Lock are utterly incorrect. It is also demonstrated that the asymptotic methods of solution as developed in the Miles-Benjamin theory of wind wave generation fail to correctly predict the growth rates of 'water' waves even at fairly high Reynolds numbers. The present results compare reasonably well with experimental results of Gupta et al..

1. Introduction

The instability of laminar or 'quasi-laminar' shearing flows of air over a water surface has been of considerable interest in connection with the problem of wave generation by wind. In one of the earliest attempts, Lock[1] analysed the case of a laminar boundary-layer flow by means of ordinary asymptotic methods appropriate to large Reynolds numbers and demonstrated that for such a coupled air-water system there are two distinct types of instabilities possible, denoted by 'air waves' and 'water waves'. His calculations are incomplete, and the neutral stability curves presented are obviously incorrect. Because of the complex feature of the results, however, the Lock's analysis seems to have had the unintended effect of suggesting that stability analyses of this sort would be rather useless for practical purposes. Thus, this classical stability problem still awaits a solution despite having been a prototype of the central problem considered in the Miles-Benjamin theory of wave generation by turbulent wind.

The foundation of this wave generation theory was due to Miles[2], who considered the instability of inviscid flow with the turbulent (logarithmic) distribution of mean velocity. On the basis of the inviscid 'quasi-laminar' model, Miles proposed an approximate method of predicting the growth of 'water waves' due to turbulent wind. The question of the viscous action in the air was pursued asymptotically in later work of Miles[3),4)], Benjamin[5]

*Professor, Research Institute for Applied Mechanics, Kyushu University.

and Siegmann[6]. But so far there have been few attempts to extend the viscous analysis to lower Reynolds numbers beyond the limit of applicability of the ordinary asymptotic methods. As is well known, the asymptotic results developed in this celebrated theory failed to stand the test of comparison with experiment and the failure was attributed to the inadequacy of the 'quasi-laminar' hypothesis that the turbulent Reynolds stresses are not influenced by the wave-like disturbances. Attempts to overcome the inadequacy were then made by Davis[7], Townsend[8] and others. Although a variety of turbulence hypotheses (models) have been put forward in those attempts, the simplest way of assessing the effects of turbulence may be to rely on the constant eddy viscosity model. Indeed, Reynolds & Hussain[9] demonstrated that the simple model served unexpectedly well. Then it may be argued that the classical 'quasi-laminar' model still retains a practical significance provided an appropriate eddy viscosity is allowed for rather than the molcular viscosity (typically, the former is much larger than the latter). Unfortunately, however, such a highly viscous case has been left out of earlier treatments concerning the 'quasi-laminar' model.

In this paper we reconsider the old-aged stability problem by means of a direct numerical method. The primary objective is to illustrate the stability characterics of the two-fluid system at moderate Reynolds numbers where the ordinary asymptotic methods are not applicable. Incidentally, the present numerical results serve to correct and supplement earlier theoretical work on the subject in some essential respects. A comparison with the experiment of Gupta et al.[10] is made tentatively.

2. Statement of the stability problem

We consider a steady, parallel shear flow of a light fluid, with density ρ_1 and viscosity μ_1, over a liquid of infinite depth with density ρ_2 and viscosity μ_2. The primary flow is in the x-direction and has a velocity profile $U_1(y)$, where y is the vertical co-ordinate with the origin at the interface; also the lower fluid, dragged along by the upper fluid, has a mean motion which we denote by $U_2(y)$. Without loss of generality we can set $U_1(0)=U_2(0)=0$, so that all velocities must be taken relative to the surface current.

Our objective is to examine the stability of the stratified flow with respect to two-dimensional, small disturbances. In the following, unless otherwise noted, we work in terms of dimensionless quantities by reffering all lengths, velocities and stresses to δ (a thickness of the shear layer), $U_\infty \equiv U_1(\infty)$ and $\rho_1 U_\infty^2$, respectively. In accordance with the standard procedure, we consider a wave-like disturbace whose streamfunction has the form

$$\Psi(x, y, t)=\begin{cases} \phi_1(y)e^{i\alpha(x-ct)}; & y\geq 0 \\ \phi_2(y)e^{i\alpha(x-ct)}; & y<0 \end{cases} \tag{2.1}$$

where α is a real wave number, c is the complex wave velocity and $\phi_j(j=1,2)$ are the amplitude distributions. Then the linearized disturbance equations reduce to the Orr-Sommerfeld equations

$$(U_j-c)(\phi_j''-\alpha^2\phi_j)-U_j''\phi_j=(i\alpha R_j)^{-1}(\phi_j''''-2\alpha^2\phi_j''+\alpha^4\phi_j) \qquad (2\cdot2\text{a, b})$$

$$(j=1,2),$$

where $R_1=\dfrac{\rho_1\delta U_\infty}{\mu_1}$ and $R_2=\dfrac{\rho_2\delta U_\infty}{\mu_2}$ are the Reynolds numbers, and primes indicate differentiation with respect to y.

The disturbance velocity must die out at infinity; thus

$$\phi_1'(\infty)=\alpha\phi_1(\infty)=0, \qquad (2\cdot3\text{a})$$

$$\phi_2'(-\infty)=\alpha\phi_2(-\infty)=0. \qquad (2\cdot3\text{b})$$

At the interface of the two fluids, the disturbance velocity and the tangential stress must be continuous, while the normal stress must be in equilibrium with the gravitational force (g) and the surface tension (σ); these requirements can be written, in the lenearized approach, as (cf. Lock[1])

$$\phi_1(0)=\phi_2(0), \qquad \phi_1'(0)=\phi_2'(0) \qquad (2\cdot4\text{a, b})$$

$$\phi_1''(0)+\alpha^2\phi_1(0)=\mu\{\phi_2''(0)+\alpha^2\phi_2(0)\}, \qquad (2\cdot5)$$

$$s(i\alpha R_1)^{-1}\{\phi_1'''(0)-3\alpha^2\phi_1'(0)\}+s\{c\phi_1'(0)+U_1'(0)\phi_1(0)\}+c_0^2\frac{\alpha}{c}\phi_1(0)$$

$$=(i\alpha R_2)^{-1}\{\phi_2'''(0)-3\alpha^2\phi_2'(0)\}+c\phi_2'(0)+U_2'(0)\phi_2(0), \qquad (2\cdot6)$$

where
$$s=\rho_1/\rho_2, \quad \mu=\mu_2/\mu_1,$$
$$F=g\delta/U_\infty^2, \quad W=\sigma/(\rho_2\delta U_\infty^2),$$
$$c_0^2=(1-s)F/\alpha+W\alpha,$$

F^{-1} and W^{-1} are the Froude and Weber numbers, respectively, and c_0 is the free surface wave speed.

The two forth-order differential equations (2·2a, b) together with the eight homogeneous boundary conditions (2·3a)~(2·6) constitute a conventional eigenvalue problem. The relevant eigenvalue relation then fixes the functional dependences of $c=c_r+ic_i$ on α, R_1 and R_2 with F and W as parameter. If $c_i>0$ the disturbance is unstable and grows initially with the exponential growth rate αc_i.

A simplified version of the problem

The full problem posed above can greatly be simplified by neglecting the mean flow in the lower fluid; this may be a good approximation for any gas-liquid combination. When $U_2(y)\equiv 0$, particular solutions of (2·2b) are readily obtained as

$$\phi_2 = e^{\pm\alpha y}, \qquad e^{\pm\gamma_2 y}$$

where

$$\gamma_2^2=\alpha^2-i\alpha R_2 c, \qquad \mathrm{R}[\gamma_2]>0.$$

The growing exponentials as $y\rightarrow-\infty$ must be rejected in conformity with (2·3b). Then the appropriate general solution in the lower region is

$$\phi_2(y)= A_2\ e^{+\alpha y}+B_2\ e^{+\gamma_2 y} \tag{2·7}$$

where A_2 and B_2 are unknown constants. Substituting (2·7) in (2·4a, b), (2·5), (2·6) and eliminating the constans A_2, B_2, we are led to the following two equations, which constitute boundary conditions on ϕ_1:

$$\phi_1''(0)+\omega_1\phi_1'(0)+\omega_2\phi_1(0)=0, \tag{2·8}$$

$$\phi_1'''(0)+\omega_3\phi_1'(0)+\omega_4\phi_1(0)=0, \tag{2·9}$$

with the abbreviations

$$\begin{aligned}
&\omega_1=-\mu(\alpha+\gamma_2),\\
&\omega_2=\mu\alpha\gamma_2-(\mu-1)\alpha^2,\\
&\omega_3=i\alpha R_1 c-\mu\alpha(\gamma_2-\alpha)-3\alpha^2,\\
&\omega_4=i\alpha R_1\{\frac{\alpha(c_0^2-c^2)}{cs}+U_1'(0)\}+\mu\alpha^2(\alpha+\gamma_2).
\end{aligned} \tag{2·10}$$

In general, the differential equation (2·2a) will have four linearly independent solutions. For convenience, let φ and f denote two solutions that satisfy the vanishing condition (2·3a). Then the appropriate general solution in the upper fluid region is written

$$\phi_1(y)= A_1\ \varphi(y)+\ B_1\ f(y). \tag{2·11}$$

Imposing the boundary conditions (2·8), (2·9) and equating the determinant of the coefficients A_1 and B_1 to zero, we obtain

$$\mathrm{D}(c, \alpha, R_1, R_2; c_0(\alpha))$$

$$=\begin{vmatrix} (\varphi_0''+\omega_1\varphi_0'+\omega_2\varphi_0) & (f_0''+\omega_1 f_0'+\omega_2 f_0) \\ (\varphi_0'''+\omega_3\varphi_0'+\omega_4\varphi_0) & (f_0'''+\omega_3 f_0'+\omega_4 f_0) \end{vmatrix}=0, \tag{2·12}$$

where $\varphi_0\equiv\varphi(0)$, $f_0\equiv f(0)$. This is the formal expression of the eigenvalue relation for the simplified problem. In the following, we shall be mainly concerned with this simplified version of the problem envisaging an air boundary layer over water.

3. Asymptotic analysis

For later discussion it seems convenient to give here a brief description of the asymptotic analysis and results for large or infinite Reynolds numbers; this is also necessary to prepare 'initial guesses' for the numerical solution.

The asymptotic solutions of the Orr-Sommerfeld equation for large Reynolds numbers are available in standard text (see, e. g., Lin[11]). One of the required solutions (say, the 'non-viscous' solution) is found in the form

$$\varphi=\varphi^{(0)}(y)+(\alpha R_1)^{-1}\varphi^{(1)}(0)+\cdots\cdots, \tag{3·1}$$

where the first approximation $\varphi^{(0)}$ is the solution of the Rayleigh equation that is bounded as $y\to\infty$; i. e.

$$(U_1-c)(\varphi^{(0)''}-\alpha^2\varphi^{(0)})-U_1''\varphi^{(0)}=0, \tag{3·2a}$$

$$\varphi^{(0)'}(\infty)+\alpha\varphi^{(0)}(\infty)=0. \tag{3·2b}$$

The remaining solution is found in the form

$$f=\exp[-(\alpha R_1)^{1/2}Q(y)]\cdot\{(U_1-c)^{-5/4}+(\alpha R_1)^{1/2}h_1(y)+\}\cdots\cdots, \tag{3·3}$$

where $Q(y)=\int_{y_c}^{y}\{i(U_1-c)\}^{1/2}dy,\quad U_1(y_c)=c.$

These solutions are valid in certain domains of the complex y-plane excluding the immediate vicinity of the critical point $(y=y_c)$ where they are singular. In that vicinity, the rapidly varying solution f is well approximated by the Airry function solution

$$f=(\varepsilon U_c')^{-5/4}\{\int_\infty^z A_i(z)dz+0(\varepsilon)\}, \tag{3·4}$$

where $z=(y-y_c)/\varepsilon,\quad \varepsilon=(i\alpha R_1U_c')^{-1/3}$ and $U_c'=U'(y_c)$.

An improved approximation to φ that is valid near the critical point is also available (Holstein), but the approximation is too complicated to be used in any actual calculations.

We now confine our attention to a non-viscous type of instabilities as exemplified by the 'water waves'. In general, such instabilities have non-zero c and α in the inviscid limit, so that the inequality $|c|(\alpha R_1)^{1/3}\gg1$ may be fulfilled for sufficiently large values of the Reynolds number. This justifies the use of the asymptotic solutions (3·1) and (3·3) so far as the eigenvalue problem is concerned. Using (3·1) and (3·3) in (2·12), we can easily obtain an asymptotic expansion of the determinant. Equating the leading term of the expansion to zero, we obtain

$$D_0 = c^2 - c_0^2 - s\frac{c}{\alpha}\{U_1'(0) - \alpha c g(0)\} = 0, \tag{3·5}$$

where the function g is defined by

$$g(y) = -\varphi^{(0)\prime}(y) / \alpha \ \ \varphi^{(0)}(y). \tag{3·6}$$

This is equivalent to the inviscid eigenvalue equation obtained by Miles[2] for 'water waves'. When $s \ll 1$ as is the case for the air-water system, (3·5) can be approximately solved for $c(=c_r + ic_i)$ as

$$c_r = c_0, \tag{3·7}$$

$$\zeta_a \equiv \alpha c_i = -\frac{1}{2} s\alpha c_r \mathrm{I}_m[g(0)]. \tag{3·8}$$

In his original inviscid treatment, Miles took account of viscous dissipation in the water to evaluate an effective growth rate: The result can be placed in the form

$$\zeta = -\frac{1}{2} s\alpha c_r \mathrm{I}_m[g(0)] - \frac{2\alpha^2}{R_2}, \tag{3·9}$$

$$(R_1/R_2 = \mu s)$$

in the present notation. The question of viscous action in the air has been considered in some detail by Benjamin[5] and others using the Airy function solution for f. An important conclusion is that air viscosity acts quite differently depending on the value of the parameter $z_c = y_c(\alpha R_1 U_c')^{1/3}$: That is,

a) in case $z_c > 2.3$, air viscosity acts as damping but the effect is insignificant as compared with the inviscid effect ζ_a;

b) in case $0 < z_c < 2.3$, the viscous action produces energy transfer toward the 'water waves' and enhances wavegrowth significantly.

In particular the rule a) implies that the inviscid equation (3·9) may provide a good approximation to the growth rate of 'water waves' even at finite Reynolds numbers, provided $z_c > 2.3$. The inviscid function $g(0)$ appearing in (3·9) can easily be computed by a numerical method (see Appendix).

The behavior of the modified Tollmien-Schlichting waves (i.e. the 'air waves') has been examined asymptotically by Miles[4]. The eigenvalue equation presented in his paper is much more complicated than the corresponding one for a shear flow over a rigid wall, but an examination indicates that the resulting eigenvalues may differ only moderately from those of the pure Tollmien-Schlichting waves over a rigid wall. In fact, the experiment of Gupta et al.[10] showed that this is true in the case of air flow over water. Then it seems adequate to use the results of the rigid-wall case as 'initial guesses' in the numerical analysis concerning 'air waves'.

Finally we note that the classical Kelvin-Helmholtz equation is easily retrieved from (3・5) by letting $\alpha\to 0$. When α is small, $g(0)$ may be approximated by the first few terms of the expansion (A・3) in the appendix. Substituting (A・3) in (3・5), we have

$$c^2-c_0^2+s(1-c)^2\{1+\alpha G(c)+O(\alpha^2)\}=0, \tag{3・10}$$

where G is a complex function of c. Then, letting $\alpha\to 0$, we are led to the same equation as derived from the Kelvin-Helmholtz model: That is

$$c^2-c_0^2+s(1-c)^2=0. \tag{3・11}$$

4. Numerical method of solution

We now proceed to give an outline of the numerical method employed in the present investigation. The method is quite straightforward, being to numerically integrate (2・2a), use the integrated values in (2・12) and seek zeros of the determinant by trial and error.

In the freestream region outside the boundary layer the admissible solutions of (2・2a) can be found explicitly as

$$\varphi(y)=Ae^{-\alpha y}, \quad f(y)=Be^{-\gamma y} \tag{4・1a, b}$$

where $\quad \gamma=[\alpha^2+i\alpha R_1(1-c)]^{1/2}, \quad \mathrm{R}[\gamma]>0,$

A and B being arbitrary constant multiples. Therefore, we need only to continue the solutions through the boundary-layer region. The matching conditions at the edge of the boundary layer ($y=1$) follow from (4・1a, b) as

$$(\varphi, \varphi', \varphi'', \varphi''')=(1, -\alpha, \alpha^2, -\alpha^3), \tag{4・2a}$$

$$(f, f', f'', f''')=(1, -\gamma, \gamma^2, -\gamma^3) \tag{4・2b}$$

with the choice $A=e^{\alpha}, B=e^{\gamma}$. Starting from the initial values specified by (4・2a, b), we integrate (2・2a) to the interface ($y=0$) by a Runge-Kutta method for a forth-order differential equation. In doing so, it must be kept in mind that the slowly growing solution φ is soon predominated by a parasitic round-off error proportional to the rapidly growing solution f. We must therefore 'filter out' the harmful error properly in order to ensure the continued linear independence of the two solutions. While this can be done in a variety of ways, we employ the Gram-Schmidt orthonormalization scheme suggested by Wazzan et al.[12]. Trials indicate that it suffices to apply the purification scheme only at every three steps for the Reynolds number range of our interest.

We assume that the required eigenvalue c is already known approximately for an arbitrary choice of α and R_1. Then we can choose an appropriate grid in the complex c-plane so that the true eigenvalue may be included in the range. Over the grid we evaluate values of the determinant (2•12), where the boundary values of the solutions and of their first three derivatives are computed by the aforementioned procedure. If the grid size is sufficiently small, the eigenvalue can be determined, by interporation, as the zero of the complex determinant (2•12). Repeating the search procedure, c_r and c_i are determined as functions of α for prescribed values of R_1, F and W. The particular values of c_r and α for $c_i=0$ (i. e. for the neutral case) are then determined graphically.

5. Numerical results and discussion

By the foregoing numerical method stability computations were carried out for the case of a laminar boundary-layer flow of air over water. For all computations, the values of the physical constants involved were fixed as follows:

$$\begin{cases} \nu_1=0.145, & \nu_2=0.0114, \\ \rho_1=1.0, & \rho_2=1.23\times10^{-3}, \quad \sigma=73, \end{cases}$$

all in c. g. s. units, and then

$$s=1.23\times10^{-3}, \quad \mu=64.0.$$

The mean velocity profile in the air was taken as

$$U_1(y)=\begin{cases} 2y-2y^3+y^4, & 0\leqq y\leqq 1, \\ 1 \quad , & y>1 \end{cases} \tag{5•1}$$

This is in fact a useful approximation to the boundary layer profile calculated by Lock[13]. In passing we note that if the boundary layer starts from a certain point upstream, say from $\bar{x}=0$, a nominal thickness of the boundary layer is given by

$$\delta = 5.7\sqrt{\frac{\nu_1\bar{x}}{U_\infty}} \tag{5•2}$$

where $\bar{x}$ is the actual distance in the flow direction.

Fig. 1 shows a typical example of neutral stability curves (c_r vs. R_1) obtained with F and W fixed. For comparison the neutral stability condition of free surface waves ('water waves') predicted by Mile's inviscid theory, i. e. by (3•9), is also included. The solid curve for the T-S ('air wave') mode is the well-known neutral curve for the pure Tollmien-Schlichting waves in the boundary layer over a rigid wall: It is to be noted that the corresponding neutral points for the air-water system under consideration fall approximately on the solid curve, in agreement with the experimental

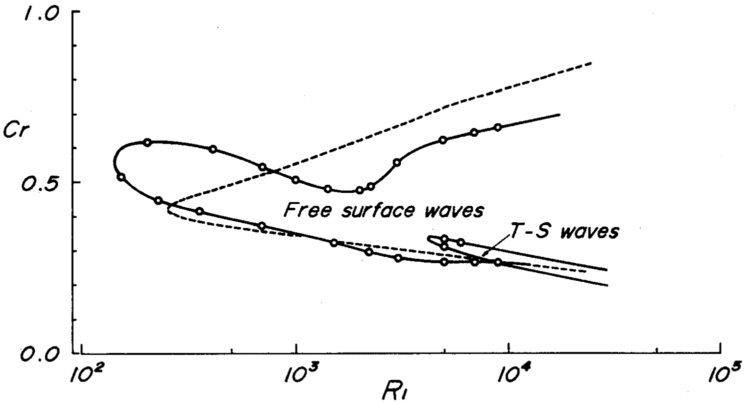

Fig. 1 Neutral stability curves ($F=0.24$, $W=0$); Dashed curve, predicted by (3•9) with exact numerical values of $g(0)$.

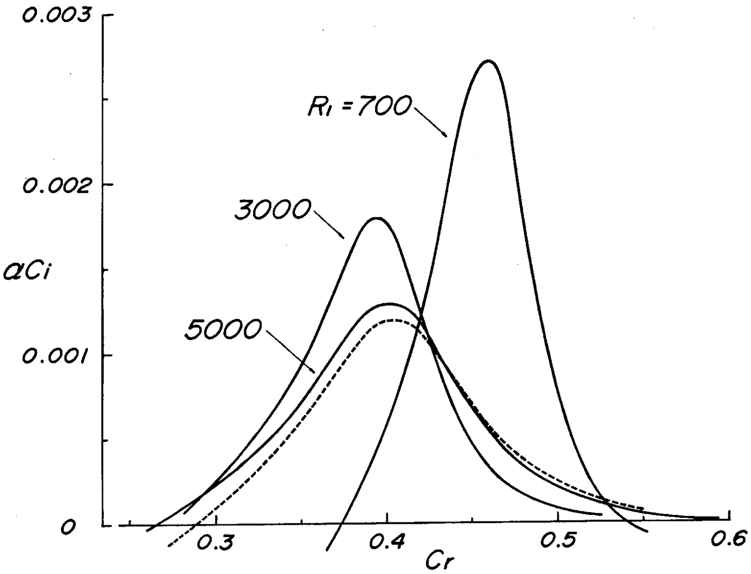

Fig. 2 Growth rates of 'water waves' at various Reynolds numbers ($F=0.24$, $W=0$); Dashed curve, predicted by (3•9) with exact $g(0)$ and $R_1=5\times10^3$

observation by Gupta et al.[10]. Partial cross plots of Fig. 1 at various Reynolds numbers are shown in Fig. 2, together with the approximate results at $R_1=5\times10^3$ predicted by (3•9) (the corresponding approximate curves for $R_1=3\times10^3$ and 7×10^2 differ only insignificantly from the one for

$R_1=5\times10^3$ and are omitted in the figure). Here we note that the values of z_c for the most unstable waves at $R_1=5\times10^3$, 3×10^3 and 7×10^3 are about 5, 4 and 3 respectively; they are all larger than the critical value $z_c=2.3$ concerning the viscous effects in the air (see §3). Then, it might be expected that in the range of Reynolds numbers of interest asymptotic results predicted by Mile's original theory (or its revised version) should agree tolerably with exact numerical results. However, the present computation indicates that this is not always the case: Indeed a fair agreement is attained in the range $R>5\times10^3$ with $z_c\simeq5$, but the asymptotic approximation considerably underpredicts the growth rate of 'water waves' at smaller Reynolds numbers ($R_1<3\times10^3$, say) yet with $z_c>2.3$.

In the interest of checking Lock's analyses[1] as well as of making a direct comparison with the experimental results of Gupta et al.[10], similar computations were carried out for various values of $\bar{x}$ with U_∞ fixed. (In this case, δ is given by (5•2) and R_1, W and F vary with $\bar{x}$ accordingly.)

Fig. 3 is a wavelengh vs. $\bar{x}$ representation of the neutral stability curves for $U_\infty=100$cm/sec. In the figure the neutral curve for 'air waves', which are possible only far downstream, is not included; for $U_\infty=340$, however, they occur at $\bar{x}$ as low as 200. As is now obvious from Figs. 1 and 3, the true neutral curves of the two-fluid system, though double-looped, are fairly simple at variance with the earlier calculations by Lock[1] (see Fig. 9). No doubt, Lock's neutral curves are not correct in any way. It is remarkable that this has been overlooked for more than 20 years!

The cross plots of Fig. 3 at $\bar{x}=150$ and 200 are shown in Fig. 4, in

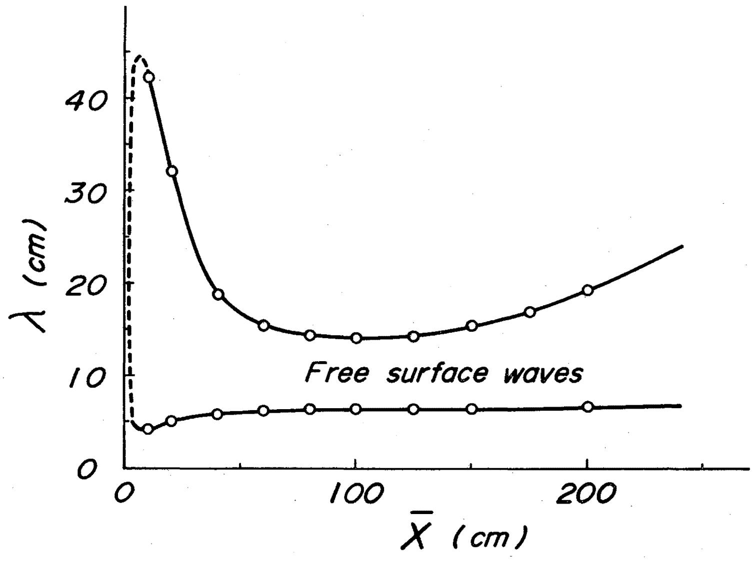

Fig. 3 Neutral stability curve ($U_\infty=100$ cm/sec).

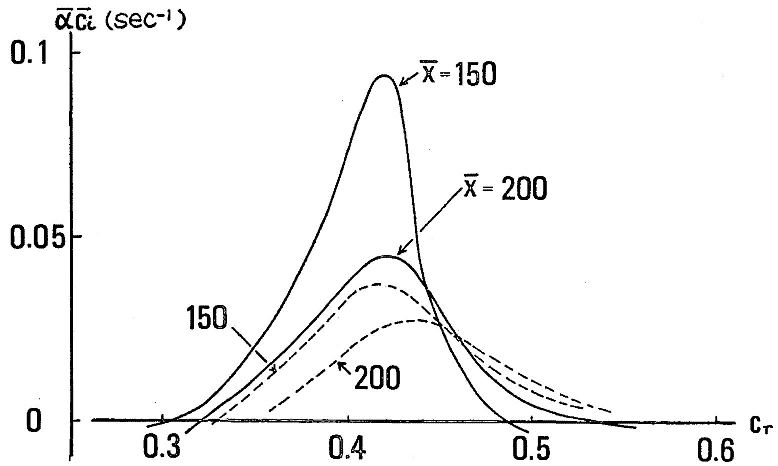

Fig. 4 Growth rates of 'water waves' at $\bar{x}=150$, 200 cm ($U_\infty=100$cm/sec); Dashed curves, (3•9).

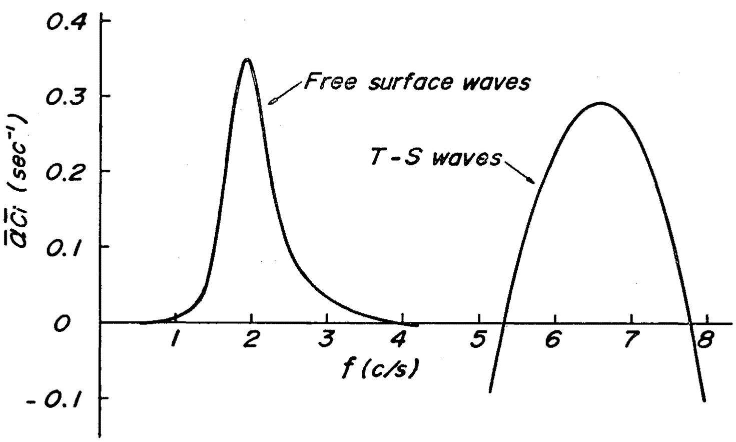

Fig. 5 Growth rates of 'air waves' and 'water waves' at $\bar{x}=300$ cm ($U_\infty=340$ cm/sec).

comparison with the approximate results given by (3•9). We note that $R_1>1.5\times10^3$ for $\bar{x}>100$, and that $z_c>2.3$ is true for most eigenvalues of interest here. Nevertheless, the approximate values again stand far from the exact numerical values. Thus we have good reason to infer that the asymptotic methods developed in the Miles-Benjamin theory are not always adequate even under conditions where they seem to valid.

Fig. 5 is intended to illustrate that unstable 'air waves' generally have a different frequency band from that of unstable 'water waves'. Therefore it seems unlikely that resonant interaction between the two distinct modes occurs actually.

Gupta et al.[10] measured the spatial growth of small wavy disturbances excited artificially by a vibrating ribbon at an upstream potition ($\bar{x}_0$) in the air-water system. Invoking Gaster's theorem[14], it may be easily seen that the *r. m. s.* amplitudes (u') of those disturbances are approximately given by

$$\log_{10}\frac{u'}{u'_0}=\frac{1}{2.3\bar{c}_g}\int_{\bar{x}_0}^{\bar{x}}\bar{\alpha}\bar{c}_i d\bar{x} \tag{5·3}$$

in terms the timewise growth rate $\bar{\alpha}\bar{c}_i$. Here u'_0 is the *r. m. s.* amplitude at $\bar{x}_0$ and $\bar{c}_g$ is the group velocity: For 'water waves' we can take as $\bar{c}_g \simeq \bar{c}_r/2$. The (timewise) growth rates of 'water waves' were computed in particular detail for the cases $U_\infty=100$ and 340. By way of example the numerical results for $U_\infty=340$ are shown in Fig. 6 as fuctions of $\bar{x}$ with wave frequency $f=\bar{\alpha}\bar{c}_r/(2\pi)$ as parameter. Integrating those results over $\bar{x}$ by Simpson's rule, the logarithms of u' were determined according to (5·3) and are presented in Figs. 7 and 8. In particular the computed results in Fig. 7 can be compared with the experimental results of Gupta et al.[10] for, among other cases, $U_0=11.3$ (ft/sec), which are shown in Fig. 10. (Note that subtracting the water surface velocity $U_w=0.176$ ft/sec from the free stream velocity $U_0=11.3$ ft/sec gives 339 cm/sec.) It may be seen that the computed curves fit the observed points fairly well; agreement is particularly good for near neutral cases. From those observed results, Gupta et al. concluded that 'the higher the excitation frequency, the weaker the wave growth'. However, Figs. 7 and 8 indicate that this may not necessarily be the case.

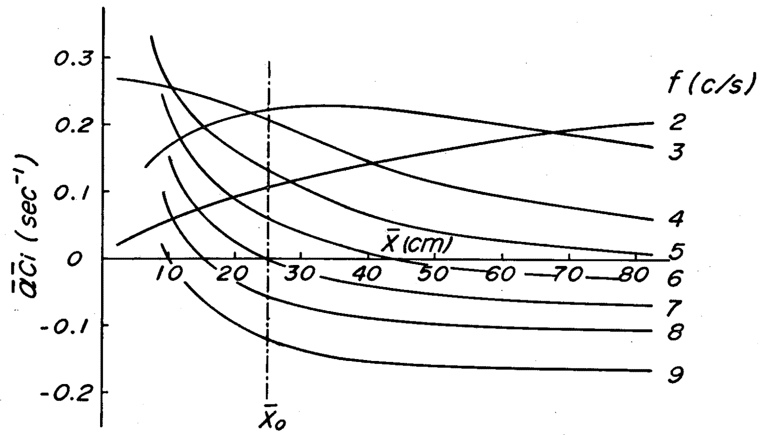

Fig. 6 Growth rates vs. downstream distance $\bar{x}$ ($U_\infty=340$ cm/sec).

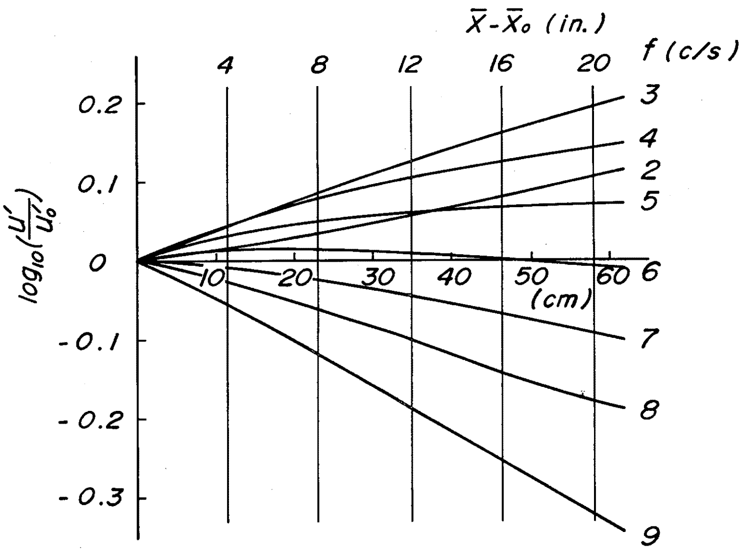

Fig. 7 Root-mean-square u' amplitude vs. $\bar{x}$ with wave frequency as parameter (U_∞=340 cm/sec, $\bar{x}_0$=25 cm).

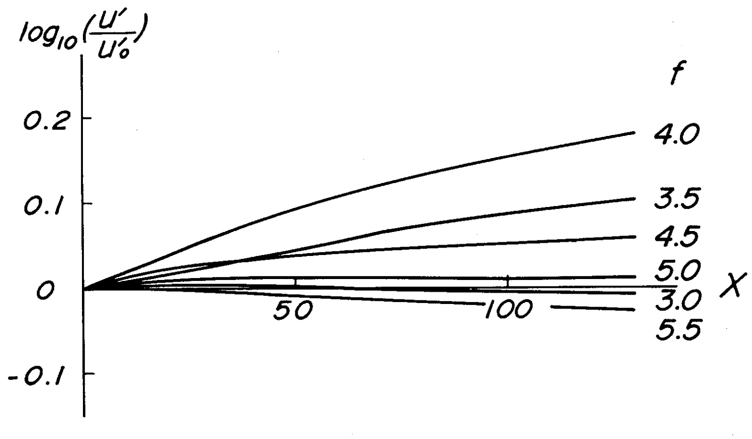

Fig. 8 u' vs. $\bar{x}_0$ (U_∞=100 cm/sec, $\bar{x}$=15 cm).

Stability characteristics at moderate and small Reynolds numbers may necessarily depend on the form of non-slip condition at the air-water interface. To examine its possible influence, partial checks were carried out by tentatively using an alternative non-slip condition $\phi_1'(0)+U_1'(0)\phi_1(0)/c=\phi_2'(0)$ instead of (2·4b). It should be emphasized that all of the foregoing main conclusions are essentially unaffected by the choice of the two alternative

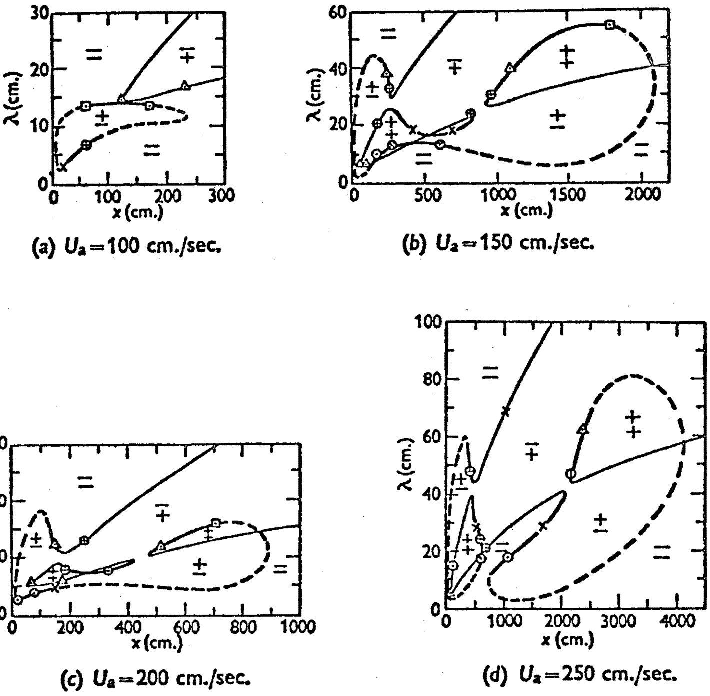

Fig. 9 Neutral stabilty curves presented by Lock; Thin solid curve, 'air waves'; Thick solid curve (dashed part, conjectural), 'water waves'.

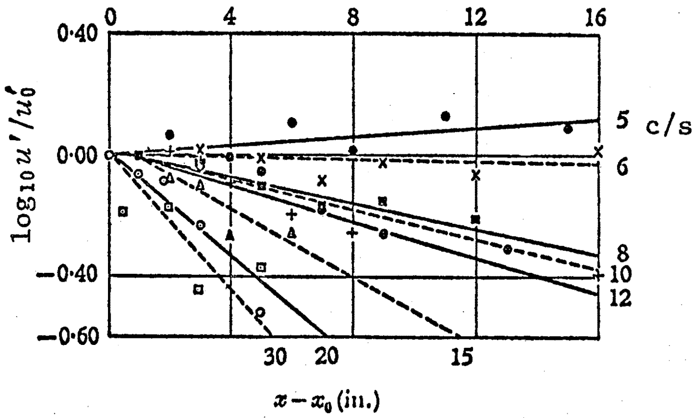

Fig. 10 Measured u' vs. $\bar{x}$ with wave generator frequency as parameter —after Gupta et al. ($U_\infty \simeq 111$ ft/sec, $\bar{x}_0 = 10.5$ in).

boundary conditions: A comparison with the experimental results of Gupta et al. seems to indicate that the non-slip condition (2•4b) is preferable in practice to the alternative one.

6. Concluding remarks

The stability characteristics of a laminar air flow over deep water were examined numerically in some detail. The resulting numerical solutions demonstrate, among other things, that the usual asymptotic methods of solution are rather misleading, so far as 'water waves' are concerned, even under conditions when the methods seem to be applicable. Why this is so is a point deserving some clarification. In this respect it is to be noted that the asymptotic results are particularly inaccurate for relatively small z_c, say for $z_c<3$. This seems to imply the possible importance of the so-called viscous correction to the non-viscous solution (φ in the present notation).

The author is indepted to Miss S. Koide for preparing the manuscript. The computations were performed on the FACOM-48 machine at the Computor Room of Research Institute for Applied Mechanics.

Appendix

Applying the transformation (3•6) to (3•2), we have

$$g'+\alpha(1-g^2)+\frac{U_1''}{\alpha(U_1-c)}=0, \tag{A•1}$$

$$g(\infty)=1. \tag{A•2}$$

The most efficient way of evaluating $g(0)$ accurately is to integrate (A•1) by a standard Runge-Kutta down to $y=0$ along an appropriate path, with (A•2) as the 'initial' condition. For small values of α, it is often useful to obtain the solution of (A•1) in powers of α. In particular, the end value of the solution is found in the form

$$\alpha g(0)=\frac{U_1'(0)}{c}+\alpha\frac{(1-c)^2}{c^2}$$

$$+\alpha^2\frac{(1-c)^2}{c^2}\int_0^\infty\left\{\left(\frac{U_1-c}{1-c}\right)^2-\left(\frac{1-c}{U_1-c}\right)^2\right\}dy+O(\alpha^3). \tag{A•3}$$

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