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https://doi.org/10.5109/6610218

出版情報：Reports of Research Institute for Applied Mechanics. 25 (80), pp.125-132, 1978-03.
九州大学応用力学研究所

Reports of Research Institute for Applied Mechanics
Vol. XXV, No. 80, 1978

NEUTRAL WAVES IN A BAROTROPIC ZONAL JET

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This paper consideres neutral waves in a barotropic zonal jet on a beta-earth. Allowing for the 'nonlinear balance' (rather than the classical 'viscous balance') in the critical layers, an eigenvalue problem of the inviscid disturbance equation is solved numerically to determine the dispersion relations of both symmetric and antisymmetric waves. The results are presented graphically for various values of the (non-dimensional) beta-parameter $\bar{\beta}$. It is found that the modes of neutral waves are possible even in the range $\bar{\beta}>\frac{2}{3}$ where the classical linear theory admits neither neutral waves nor self-excited waves.

1. Introduction

The stability theory of a steady shear flow is mainly concerned with the behavior of small wave-like disturbances superimposed on the basic flow. In many cases of practical interest, the effect of fluid viscosity is very small and the perturbed motions can be well described by the linearized inviscid (or Rayleigh) equation almost everywhere in the flow field. (This is particularly so in the case of free shear flows with no solid boundary. [1]) As is well known, however, the Rayleigh equation is singular at the critical point where the flow velocity is equal to the wave phase veloctiy; because of this singular nature of the Rayleigh equation, its solutions cannot be determined uniquely unless an appropriate transition relation is specified at the critical point. In order to overcome the difficulty, it is necessary to examine the flow structure in the layer surrounding the critical point by allowing for additional terms ignored in the simplification. Traditionally the task has been done on the basis of the linearized viscous equation with the nonlinear effects entirely neglected (see, e.g., Lin[2]): This viscous critical-layer analysis leads to the well-known transition relation that requires the occurence of a discontinuous phase shift in the disturbance across the critical layer. More recently, however, it has been shown by Davis[3] and by Benny & Bergeron[4] that the difficulties associated with the singularity of the Rayleigh equation can be removed also by considering a nonlinear balance in the critical layer: In this case, no phase shift occurs

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across the critical layer in contrast to the result of the linear viscous analysis. Then wave-like disturbances in a shear flow must exhibit quite different characteristics according to wheather viscous or nonlinear effects predominate in the critical-layer region. In fact Benny & Bergeron demonstrated that the nonlinear critical-layer balance admits the presence of a class of neutral waves not found in the usual linear viscous analyses.

Such a nonlinear critical-layer balance may be of particular importance in large scale currents in the ocean and the atmosphere because the typical Reynolds numbers of those currents are supposed to be extremely high. Indeed, Kelly & Maslowe[5] were able to provide a new explanation for the generation of clear air turbulence (CAT) by extending the nonlinear critical-layer analysis to a stratified shear flow. In this paper we consider the case of a barotropic zonal current on a beta-plane; here it is to be noted that the inclusion of the beta-effect scarcely affects the flow structure in the critical layer (either viscous or nonlinear) since the beta-effect is only to add a constant value to the basic gradient of vorticity. The stability problem of the relevant flow has been studied within the framework of the linear viscous theory by Kuo[6], Lipps[7] and others from geophysical interest. Our objective is to supplement the earlier work on the subject by examining the characteristics of neutral waves more generally with allowance for the nonlinear critical-layer balance.

2. Basic equations and boundary conditions

Two-dimensional motions of a homogeneous fluid on a beta-plane are described by the vorticity equation

$$\frac{\partial}{\partial t}\nabla^2\Psi+\frac{\partial(\nabla^2\Psi,\ \Psi)}{\partial(x,y)}+\beta\frac{\partial\Psi}{\partial x}=\nu\nabla^4\Psi \tag{1}$$

where β and ν are respectively the beta-parameter and the kinematic viscosity; Ψ is the stream function such that the velocity components (u, v) in the x ('east') and y ('north') directions are $u=\partial\Psi/\partial x,\ v=-\partial\Psi/\partial y$.

We now consider a steady parallel shear flow with velocity components $(U(y), 0)$ and allow the basic flow to be perturbed by a wave-like disturbance propagating in the x-direction with a phase speed c. Let ε be a measure of the disturbance amplitude. The total stream function is then written as

$$\Psi=\int^{y}(U-c)\,dy+\varepsilon\psi(x,y) \tag{2}$$

where use has been made of a reference frame moving with the disturbance so that the total flow appears steady. The governing equation for the perturbation stream function ψ is readily obtained from (1) by substituting (2) and noting that $U(y)$ itself is (or is assumed to be) a solution of (1). In non-dimensional form, this reads

$$(U-c)\frac{\partial}{\partial x}\nabla^2\psi-(U''-\bar{\beta})\frac{\partial\psi}{\partial x}+\varepsilon\frac{\partial(\nabla^2\psi,\psi)}{\partial(x,y)}=\bar{\nu}\nabla^4\psi, \tag{3}$$

where the prime denotes differentiation with respect to y. $\bar{\beta}$ and $\bar{\nu}$ are the non-dimensional counterparts of β and ν; i.e. $\bar{\beta}=\beta L^2/U_0$, $\bar{\nu}=\nu/(U_0L)$ with L and U_0 as the length and velocity scales, respectively. When ε and $\bar{\nu}$ are both small, the full nonlinear equation (3) may be approximated by the inviscid linear equation

$$(U-c)\frac{\partial}{\partial x}\nabla^2\psi-(U''-\bar{\beta})\frac{\partial\psi}{\partial x}=0, \tag{4}$$

although it must be kept in mind that the approximation is not valid near the critical layer (or layers) where $U=c$ and the simplified equation exhibits singularity. Despite the non-uniformity of the approximation, Eq. (4) provides a good basis for a wide class of stability problems including that of free shear flows with no solid boundary. In what follows, we will be primarily concerned with a boundary value problem of the inviscid equation.

Eq. (4) is linear and admits periodic solutions of the form

$$\psi(x,y)=\phi(y)\,e^{i\alpha x}, \tag{5}$$

where α is the wave number and the amplitude function ϕ satisfies the ordinary differential equation

$$(U-c)\,(\phi''-\alpha^2\phi)-g\cdot\phi=0 \tag{6}$$

with the abbreviation $g(y)=U''(y)-\bar{\beta}$. When $U(y)$ is symmetric in y, as in the present case, it suffices to consider the symmetric and antisymmetric disturbances separately in the semi-infinite region, say, $y\leq 0$. The relevant boundary conditions at $y=0$ are

$$\begin{cases}\phi'(0)=0 & \text{(for the symmetric case)} \quad (7a)\\ \phi(0)=0 & \text{(for the antisymmetric case).} \quad (7b)\end{cases}$$

At infinity we require that all disturbance velocities vanish, i.e., that $\phi',\phi\to 0$ as $|y|\to\infty$. For the basic flows such that both U and U'' decay exponentially as $|y|\to\infty$, the requirement at infinity is ensured by imposing the single condition

$$\phi'/\phi\to\gamma \quad \text{as} \quad y\to-\infty, \tag{8}$$

where

$$\gamma=(\alpha^2+\bar{\beta}/c)^{1/2};\qquad \mathrm{Re}[\gamma]>0.$$

Let ϕ_1 and ϕ_2 be two linearly independent solutions of Eq. (6) that satisfy a certain transition relation across the critical layer. The general solution can be written as

$$\phi = A_1 \phi_1 + A_2 \phi_2, \tag{9}$$

where A_1 and A_2 are arbitrary constants. Substituting (9) into (7a) and (8), we are led to the eigenvalue relation for the symmetric case

$$F_s(\alpha, c; \bar{\beta}) = \begin{vmatrix} \phi_1'(0) & \phi_2'(0) \\ \phi_1'(y_\infty) - \gamma\phi_1(y_\infty) & \phi_2'(y_\infty) - \gamma\phi_2(y_\infty) \end{vmatrix} = 0 \tag{10}$$

where y_∞ is a sufficiently large (negative) value of y. From (10) we can determine c as a function of α with $\bar{\beta}$ as a parameter. For the antisymmetric case the first row of the determinant must be replaced by $[\phi_1(0) \ \phi_2(0)]$ in accordance with the condition (7b). The resulting determinant will be denoted by F_a .

Solutions of the Rayleigh equation

In order to actually solve the eigenvalue problem for a prescribed velocity profile $U(y)$, it is necessary to find two particular solutions of Eq. (6) with due attention to the singular behaviour of the equation at the critical point $(y=y_c)$. In the neighborhood of the regular singular point, as is well known, two mutually independent solutions can be found by the method of Frobenius: The first few terms of the Frobenius expansions are

$$\begin{cases} \phi_1 = (y-y_c) + \dfrac{g_c}{2U_c'}(y-y_c)^2 + \cdots\cdots & \text{(11a)} \\ \phi_2 = 1 + \dfrac{1}{2} b_2 (y-y_c)^2 + \cdots\cdots + \dfrac{g_c}{U_c'} \cdot \phi_1 \cdot \log(y-y_c) & \text{(11b)} \end{cases}$$

where $b_2 = \alpha^2 + \dfrac{U_c'''}{U_c'} - \dfrac{g_c}{2(U_2')^2}(U_c'' + 3g_c)$, etc., and a subscript c distinguishes the value of the quantity at the critical point. Note that the solution ϕ_2 has a logarithmic branch point at $y=y_c$ so long as $g_c \neq 0$. The proper branch of the multivalued solution must specified by a transition relation resulting from a detailed analysis of the critical-layer region.

As mentioned in the introduction, two alternative transition relations are now available depending on the relative magnitude of the two parameters ε and $\bar{\nu}$ (more precisely, $\varepsilon^{1/2}$ and $\bar{\nu}^{1/3}$).

(a) In the case $\bar{\nu}^{1/3} \gg \varepsilon^{1/2}$, the full equation (3) can be approximated in all essential respects by the linear viscous equation with the nonlinear terms neglected. In particular near the critical point, the important balace is between $U_c'(y-y_c)\partial^3\psi/\partial x\partial y^2$ and $\bar{\nu}\partial^4\psi/\partial y^4$, which together with (5) leads to an Airy equation. In order for the singular solution ϕ_2 to match with one of the Airy-function solutions, it is required that

$$\log(y-y_c)=\begin{cases}\log|y-y_c| \ ; \ y>y_c \\ \log|y-y_c| \ \mp i\pi, \ (U'_c \gtrless 0) \ ; \ y<y_c.\end{cases} \tag{12}$$

This is the familiar transition relation that has been employed in most existing stability analyses. An important conclusion drawn from this transition relation is that (non-trivial) neutral waves in a symmetric jet must satisfy the following condition[1]:

$$g_c \equiv U''_c(y_c)-\bar{\beta}=0, \quad U(y_c)-c=0. \tag{13}$$

(b) In the other extreme case $\bar{\nu}^{1/3} \ll \varepsilon^{1/2}$, it is the nonlinear terms that provide the important balance in the critical layer. The flow structure in such a nonlinear critical layer has been analyzed by Benny & Bergeron[4] for the non-rotating ($\bar{\beta}=0$) case. The resulting 'inner' solution around the critical point indicates that the logarithmic term involved in the 'outer' solution ϕ_2 must be taken as

$$\log(y-y_c)=\log|y-y_c| \ ; \quad y \gtrless y_c \tag{14}$$

irrespective of the sign of U'_c. Their nonlinear critical layer analysis can be carried over to the rotating ($\bar{\beta} \neq 0$) case with minor modifications. Here it suffices only to state that the inclusion of the beta-effect does not affect the transition relation (14).

Finally we note that the Frobenius expansions (11) are not useful except in the vicinity of the critical layer (because of their slow convergence). For the purpose of evaluating the values of the solutions far from the critical point, it is then necessary to extend the Frobenius solutions by a direct numerical integration of the Rayleigh equation.

3. Neutral waves in a jet

We now solve the inviscid eigenvalue problem for a symmetric jet on a beta-plane, with special interest in the class of neutral waves that are subject to the new transition relation (14) across the critical layer. To be specific, the basic flow is assumed to be a westerly Bickley jet, whose velocity distribution is represented in nondimensional form as

$$U(y) = \operatorname{sech}^2 y. \tag{15}$$

Our main task is to find particular combinations of c and α that satisfy the eigenvalue relation (10) for a prescribed value of $\bar{\beta}$. The search for zeros of the determinant must be made by a trial-and-error numerical method because the determinant generally admits no explicit representation; in fact the end values of the solutions (ϕ_1, ϕ_2), involved in the determinant, are given only numerically.

The outline of the numerical procedure employed here is as follows: i) for an assigned value of α choose a set of discrete values of c according to an initial guess, ii) integrate the Rayleigh equation toward $y=0$ and $y=y_\infty$ from the critical-layer region by a Runge-Kutta-Gill method, where the 'initial conditions' are provided at $y=y_c\pm\delta$ ($\delta\ll1$) by the Frobenius expansions

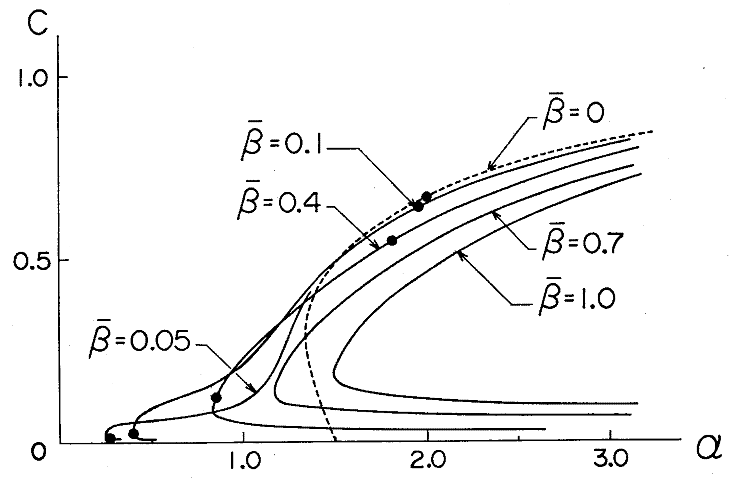

Fig. 1 Dispersion relations of the symmetric neutral waves. (● Lipps).

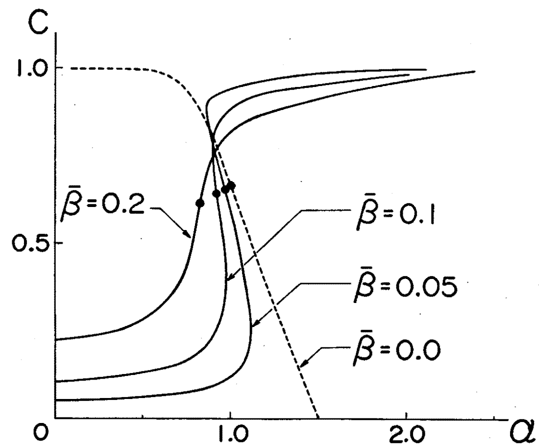

Fig. 2 Dispersion relations of the antissymmetric neutral waves. (● Lipps).

(11) with the transition relation (14), iii) evaluate the determinant value using the integrated values $\phi_1'(0), \phi_2(y_\infty)$, etc., and iv) repeat the above procedures adjusting c properly until a zero of the determinant is located with sufficient accuracy. The dispersion relations thus determined are shown in Fig. 1 and Fig. 2 for various values of $\bar{\beta}$.

In the classical work of Lipps, use was made of the other transition relation (12). In this treatment the phase velocities of possible neutral waves are readily determined from the condition (13) in a closed form; such neutral waves exhibit no singular nature at the critical points, and the associated eigenvalue problem can be solved exactly to determine their wave-number α. The neutral stability points thus determined from the classical transition relation are also included in the figures. Here it should be noted that such regular neutral waves can exist only for small values of $\bar{\beta}$; i.e., for $\bar{\beta}<\frac{2}{3}$ in the symmetric case, and for $\bar{\beta}<\frac{1}{2}$ in the antisymmetric case.

As may be seen from the figures, the two alternative transition relations (or, critical-layer balances) lead to quite different results. In particular the following two differences are noteworthy:

i) The nonlinear critical-layer balance admits a continuous spectrum of neutral waves for each value of $\bar{\beta}$, while there are at most only two neutral waves possible for each $\bar{\beta}$ if the viscous transition relation is considered.

ii) The neutral waves with nonlinear critical-layer balance are possible for any value of $\bar{\beta}$ unlike the regular neutral waves. (As stated above, the latter can exist only for limited values of $\bar{\beta}$.)

Finally it should be noted that an additional mode of neutral waves with negative c are possible in the basic flow, irrespective of the values of $\bar{\beta}$, if the vanishing conditions at infinity are relaxed.

4. Concluding remarks

We have examined the characteristics of the mode of neutral waves with nonlinear critical layers, in the particular case of a westerly Bickley jet on a beta-earth. An important result is that the mode of neutral waves are possible even in the range of the parameter $\bar{\beta}$ where neither self-excited waves nor neutral waves can exist according to the classical linear theory; the situation may be the same in the csae of a easterly jet. This seems to indicate that those nonlinear neutral waves have no connection with amplified disturbances of the usual linear stability theory. The eigenvalues presented here and the corresponding eigenfunctions may be useful in the theory of solitary eddies[8] in a large-scale sheard current.

The author is indepted to Miss S. Koide for preparing the manuscript. The computations were performed on the FACOM 230-48 machine at the Computer Room of Research Institute for Applied Mechanics.

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(Received November 26, 1977)