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THE ROLE OF DISPERSION RELATION IN MEASURING THE DIRECTIONAL WAVE SPECTRUM

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Conventional methods for measuring the directional wave spectrum have been classified into four categories according to their expression for wave field, and typical methods have been examined through the numerical analysis of artificial wave field with special attention to the role of dispersion relation. It has been demonstrated by numerical experiments that the dispersion relation plays an important role in measuring the directional spectrum, and it has been suggested that both the directional spectrum and the dispersion relation can be determined simultaneously by applying a new method of measurement introduced by the present author.

1. Introduction

A number of methods for measuring the directional spectrum of wind waves have been introduced by many authors and applied to actual wave fields in order to increase our understanding of the directional property of wind waves. Although detailed characteristics of those methods have been described in the original papers, discussion on the validity of the assumed use of linear dispersion relation $\omega^2=gk$ (ω is the wave frequency, k wave number, g acceleration of gravity) has been avoided consciously or unconsciously. However, the dispersion relation of surface wave, which couples the time and space scales of the wave, is likely to play an important role in measuring the directional spectrum of wind waves.

Strictly speaking, wind wave fields may be best described by "three-dimensional spectrum" ϕ (l, m, ω) where l, m, are x- and y-components of wave number vector $\boldsymbol{k}$. Since the component waves are supposed to have some kinds of dispersion relation, however, most of the wave energy may be distributed on a curved surface corresponding to the dispersion relation in (l, m, ω) space. Then the three-dimensional spectrum can be well represented by a two-dimensional spectrum along the dispersion surface. Thus, in conventional methods, the three-dimensional energy ϕ (l, m, ω) has been reduced to the two-dimensional energy (or directional spectrum) ϕ (θ, ω) on the premise that the dispersion relation is given by $\omega^2=gk$ $(k=|\boldsymbol{k}|)$.

This relation derived from the linear theory of small amplitude water wave

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has been used widely, without an experimental examination of its reliability, in both theoretical and experimental studies of wind waves. Indeed the dispersion relation for regular waves generated mechanically in a laboratory tank seems to be well represented by the linear relation (Morrison[1], 1951), but recent experimental measurements of the phase velocity of wind waves (Yefimov et al.,[2] 1972; Ramamonjiarisoa[3], 1974; Kato and Tsuruya[4], 1974) have suggested a different dispersion relation for the component waves of actual wind waves, though the technique of determining the phase velocity was not complete theoretically (see Rikiishi[10], 1977b). According to the result of Yefimov et al., for example, the dispersion relation of wind waves may be represented roughly by $\omega^2=1.1\,gk$-$1.4\,gk$ for the frequency components near the principal peak in the wave spectrum.

Now we have no ground for using the linear dispersion relation in the analysis of wind-generated waves. Then it is quite likely that the directional spectra of wind waves reported by many authors contain significant errors. Therefore it is very important to investigate possible effects of dispersion relation on the measured directional spectrum.

For this purpose, several typical methods for measuring the directional spectrum are examined in this paper by applying them to artificial wave fields in which the dispersion relation is not consistent with the assumed one. Prior to the numerical experiments of the directional spectrum analysis, characteristic feature of each method is surveyed briefly to realize the situation.

2. General review of various methods for measuring the directional wave spectrum

Methods of measuring the directional spectrum can be classified into four categories according to their expression for the wave field, because the principles of those methods are closely related to the mathematical expression for water surface.

First of all, the water surface elevation is written in the form

$$\eta(x, y, t)=\sum_{l}\sum_{m}\sum_{\omega} a_{lm\omega} \cos\,(lx+my+\omega t+\varepsilon_{lm\omega}), \tag{1}$$

where $a_{lm\omega}$ and $\varepsilon_{lm\omega}$ are amplitudes and random phases of component waves, respectively. If we were informed of the surface elevations over all a region of sea extending from 0 to A in the x direction and 0 to B in the y direction and during a time span 0 to N, crosscovariance function corresponding to the wave field (1) is given by

$$R(X, Y, \tau)=\frac{1}{ABN}\sum_{x}\sum_{y}\sum_{t} \eta(x, y, t)\ \eta(x+X, y+Y, t+\tau), \tag{2}$$

$$=\sum_{l}\sum_{m}\sum_{\omega}\frac{1}{2}a_{lm\omega}{}^2 \cos(lX+mY+\omega\tau), \tag{2$'$}$$

where X and Y are increments of distance along the x-axis and the y-axis, respectively. By taking the three-dimensional Fourier transform of (2)′, we have a three-dimensional energy spectrum $\phi(l, m, \omega)$. Practically it may be impossible to arrange a large number of wave detectors in (x, y) plane, and then we can hardly determine the three-dimensional spectrum $\phi(l, m, \omega)$.

However, the whole information on the three-dimensional spectrum is not necessarily needed, because, since water waves have some kinds of dispersion relation, the three-dimensional spectrum can be well represented by a two-dimensional (or directional) spectrum along the dispersion surface.

The first method of measuring the directional spectrum has been presented by Barber[5] (1963) on the basis of Eqs. (1) and (2)′. Assuming the ergodic property of wind wave field, Barber replaced (2) by

$$R'(X, Y, \tau)=\frac{1}{N}\sum_t \eta(x, y, t)\eta(x+X, y+Y, t+\tau). \tag{3}$$

In addition, the Fourier integral of R' with respect to X and Y is replaced by a summation over several terms because only a limited number of wave detectors are available. Consequently the directional spectrum is masked by the running average with an weighting function which is not well-behaved in many cases. (Note that the calculated spectrum has the form of three-dimension.) Mobarek[6] (1965) and Fujinawa[7] (1974) have tried to improve the situation.

The second method is based on the following expression for the surface elevation:

$$\eta(x, y, t_0)=\sum_l\sum_m a_{lm}\cos(lx+my+\varepsilon_{lm}), \tag{4}$$

where the dummy variable t_0 denotes an instantaneous time when the wave field is observed simultaneously at a region of sea extending from *0* to A and B in the x and y directions, respectively. The SWOP experiment (Cote et al.,[8] 1960) and several other methods using the optical or photographic techniques are of this type. In this case the crosscovariance function is given by

$$R(X, Y)=\frac{1}{AB}\sum_x\sum_y \eta(x, y, t_0)\ \eta(x+X, y+Y, t_0) \tag{5}$$

$$=\sum_l\sum_m\frac{1}{2}a_{lm}^2\cos(lX+mY) \tag{5$'$}$$

Thus, the two-dimensional Fourier transform of R yields the raw spectrum of two-dimension $\frac{1}{2}a_{lm}^2/\Delta l\Delta m$ in (l, m) space. The raw estimates should be averaged with an weighting function to give smoothed estimates. This averaging process can be replaced by multiplying $R(X, Y)$ by a lag window of two-dimension. It is noted here that one can not distinguish $a^2_{-l,-m}$ from a_{lm}^2, that is, one can not identify one of two plane waves propagating in the opposite directions to each other.

A new method introduced by the present author (Rikiishi[9] 1977a and 1977b[10], referred to as Part I and Part II respectively) belongs to the third category. The principle of this method consists in the direct application of the discrete Fourier transform to the two-dimensional spectrum estimation. In this method the surface elevation is expressed in the form

$$\eta(x, y, t) = \sum_n \sum_m a_{nm} \cos(k_n \cos\theta_{nm} \cdot x + k_n \sin\theta_{nm} \cdot y + \omega_n t + \varepsilon_{nm}), \tag{6}$$

where the wave number k_n is not a variable but determined by the dispersion relation. Thus, unlike the expression (1), the wave field is thought of as a superposition of the spectral component waves subject to the dispersion relation exactly. When the surface elevation is measured simultaneously at positions (x_i, y_i), $i=1, 2, \cdots, M$, the amplitudes a_{nm}, $m=1, 2, \cdots, M'(M' \leqq M)$, can be determined by solving simultaneous equations as described in Part I. Then the raw directional spectrum is defined by

$$\phi_{nm} = \frac{1}{2} a_{nm}^2 / \Delta f \Delta\theta. \tag{7}$$

The ϕ_{nm} are averaged in two-dimensional space. One of remarkable points of this method is that, unlike other methods, it has nothing to do with the crosscovariance function or cross spectrum explicitly.

The fourth method of measuring the directional spectrum is based on the measurement of several physical quantities related to the wave motion, such as, surface elevation, wave slope, wave curvature, wave force, wave pressure, and water particle velocity at a fixed position (x_0, y_0). The observation of the directional spectrum of sea waves using the motion of a floating buoy is a typical of this method (Longuet-Higgins et al.,[11] 1963; Cartwrite and Smith[12], 1964; Ewing[13], 1969; Mitsuyasu et al.,[14] 1975). Other examples are the works by Nagata[15] (1964) and Suzuki[16] (1969) based on the information on the water particle velocity or the wave force, respectively. Mathematical expression for wave field in these methods may be represented by the following expression for surface elevation:

$$\eta(x_0, y_0, t) = \sum_n a_n \cos(k_n \cos\theta_n \cdot x_0 + k_n \sin\theta_n \cdot y_0 + \omega_n t + \varepsilon_n). \tag{8}$$

(Other physical quantities related to the wave motion can be also written in similar forms.) A unique point of the floating buoy technique is that the directional spectrum is expanded in a Fourier series and the Fourier coefficients are related successfully to the cross spectra obtained from those several kinds of physical information.*)

We have surveyed so far gross features of the typical methods for measuring the directional spectrum. Now it is time to proceed to examine each method specifically by applying it to artificial wave fields. In the following numerical analysis, special attention will be paid on the role of dispersion relation. However, the second method based on the simultaneous observation at an instantaneous time will not be examined, because it is free from the use of dispersion relation. Another method of the first category with a linear detectors array, suggested by Barber[17] (1959) and applied to actual wind waves by Gilchrist[18] (1966), will not be examined either, because, since the resolution of the method depends sharply and selectively on the direction of wave propagation, general discussion may not be made easily.

3. Methods developed by Barber, Mobarek and Fujinawa

We start by examining the methods developed by Barber, Mobarek, and Fujinawa. Suppose that several wave detectors are arranged in (x, y) plane at positions $\boldsymbol{x}_i=(x_i, y_i)$ $(i=\mathrm{I}, \mathrm{II}, \mathrm{III}, \cdots)$ to measure the surface elevation field. Then crosscovariance function $R'(X_j, Y_j, \tau)$ can be estimated by following the definition (3) (X_j and Y_j are x- and y-components of the distances between pairs of wave detectors). Taking the Fourier transform of (2)$'$ with respect to variable τ, we have

$$\left.\begin{aligned} C_j(X_j, Y_j, \omega) &= \sum_l\sum_m \phi(l, m, \omega)\cos(lX_j+mY_j), \\ -Q_j(X_j, Y_j, \omega) &= \sum_l\sum_m \phi(l, m, \omega)\sin(lX_j+mY_j), \end{aligned}\right\} \tag{9}$$

where C_j and Q_j are the co-spectrum and quadrature spectrum. Under the practical condition that only several wave detectors are available, Barber (1963) replaced the inverse Fourier transform of (9) by a summation:

$$\phi'(l, m, \omega) = \sum_{j=-L}^{L}\{C_j\cos(lX_j+mY_j) - Q_j\sin(lX_j+mY_j)\}, \tag{10}$$

where $(X_{-j}, Y_{-j})=(-X_j, -Y_j)$, $(C_{-j}, Q_{-j})=(C_j, -Q_j)$. Clearly Eq. (10) gives a three-dimensional spectrum as a function of l, m and ω. Then Barber assumed additionally the dispersion relation

$$\omega^2=gk, \tag{11}$$

*) In the original paper of Longuet-Higgins et al. (1963), the wave field was expressed by integral representation which allowed continuous angular spreading of spectrum. In this case, however, the Fourier coefficients of directional spectrum expansion may not be related exactly to the cross spectra (refer to Section 4). In a strict sense, therefore, the wave field expression of the form (8) is considered to correspond to the floating buoy technique.

where $k=|\boldsymbol{k}|$ and $\boldsymbol{k}=(l, m)=(k\cos\theta,\ k\sin\theta)$, and obtained the two-dimensional spectrum

$$\phi_1(\omega,\theta)=\sum_{j=-L}^{L}\{C_j\cos(\frac{\omega^2}{g}\cos\theta\cdot X_j+\frac{\omega^2}{g}\sin\theta\cdot Y_j)$$

$$-Q_j\sin(\frac{\omega^2}{g}\cos\theta\cdot X_j+\frac{\omega^2}{g}\sin\theta\cdot Y_j)\}. \tag{12}$$

The spectrum $\phi'(l, m, \omega)$ and $\phi_1(\omega,\ \theta)$ do not give true spectrum but its convolution with averaging functions

$$G(l,\ m)=1+2\sum_{j=1}^{L}\cos(lX_j+mY_j), \tag{13}$$

and

$$H(\omega,\theta)=1+2\sum_{j=1}^{L}\cos k(\cos\theta\cdot X_j+\sin\theta\cdot Y_j), \tag{14}$$

respectively [see Eqs. (16) and (17)]. The functions G and H are not necessarily good-natured, and then unrealistic directional spectrum is obtained in many cases.

In order to overcome the difficulty, Mobarek[6] (1965) discarded the inverse Fourier transform of (9) and resorted to the least squares method: the discrete energy distribution was determined so that it might give the most consistent cross-spectrum field. On substituting $(l,\ m)=(k\cos\theta,\ k\sin\theta)$ and $(X_j, Y_j)=(D_j\cos\varphi_j,\ D_j\sin\varphi_j)$ into (9), Mobarek derived the following equations:

$$\left.\begin{aligned} C_j &= \sum_{i=1}^{L'}E_i\cos(k\,D_j\cos(\varphi_j-\theta_i)), &&(j=0,\cdots,L),\\ Q_j &= -\sum_{i=1}^{L'}E_i\sin(kD_j\cos(\varphi_j-\theta_i)), &&(j=1,\cdots,L),\end{aligned}\right\} \tag{15}$$

where E_i's are the discrete energy spectrum, L the number of pairs of wave detectors, L' the number of the directional line spectrum, and $k=\omega^2/g$. Since L' is set to be less than L, E_i's can be determined by the least squares method. In our numerical experiments described later, the following six sets of five angles have been adopted for θ_i:

1) $\theta_i=-85°+(i-1)\times30°$ $(i=1,\cdots,5)$
2) $\theta_i=-80°+(i-1)\times30°$ (〃)
3) $\theta_i=-75°+(i-1)\times30°$ (〃)
4) $\theta_i=-70°+(i-1)\times30°$ (〃)
5) $\theta_i=-65°+(i-1)\times30°$ (〃)
6) $\theta_i=-60°+(i-1)\times30°$ (〃)

Note that θ_i's must be predetermined so that they may cover the mean wave direction of propagation.

Another endeavor to improve the Barber method was made by Fujinawa[7] (1974), attempting to derive a deconvolution spectrum $\phi_2(\omega, \theta')$ from the integral equation

$$\phi_1(\omega, \theta) = \int_{-\pi}^{\pi} H(\omega, \theta, \theta') \ \phi_2(\omega, \theta') \ d\theta', \tag{16}$$

where

$$H(\omega, \theta, \theta') = 1 + 2\sum_{j=1}^{L} \cos \ \frac{\omega^2}{g}\{(\cos\theta - \cos\theta')X_j + (\sin\theta - \sin\theta')Y_j\}. \tag{17}$$

To solve the integral equation, Fujınawa expanded the functions ϕ_1, ϕ_2 and H in finite Fourier series as follows:

$$H(\theta, \theta', \omega) = \sum_{n=-L'}^{L'} \sum_{m=-L'}^{L'} \{a_{mn}\cos(m\theta + n\theta') + b_{mn}\sin(m\theta + n\theta')\} \tag{18}$$

$$\phi_1(\theta, \omega) = \sum_{n=-L'}^{L'} (c_n \cos n\theta + d_n \sin n\theta) \tag{19}$$

$$\phi_2(\omega, \theta) = \sum_{n=-L'}^{L'} (u_n \cos n\theta + v_n \sin n\theta) \tag{20}$$

The coefficients (a_{mn}, b_{mn}) and (c_n, d_n) can be obtained by the numerical integration of Eqs. (17) and (12), respectively. Then (u_n, v_n) can be determined by solving the simultaneous equations

$$\begin{pmatrix} \{a_{mn}\} & \{b_{mn}\} \\ \{b_{mn}\} & \{-a_{mn}\} \end{pmatrix} \begin{pmatrix} \{u_n\} \\ \{v_n\} \end{pmatrix} = \frac{1}{2\pi} \begin{pmatrix} \{c_n\} \\ \{d_n\} \end{pmatrix}, \tag{21}$$

$$(m, n = -L', \cdots, 0, \cdots L').$$

The method by Fujinawa is similar to the floating buoy technique in that the directional spectrum is expanded in a Fourier series [Eq. (20)]. However, while the Fourier coefficients are determined from theoretical consideration in the floating buoy technique, those of the Fujinawa method are given by the solution of simultaneous equations [Eq. (21)]. In determining the Fourier coefficients (u_n, v_n) practically, one is confronted with several significant difficulties resulting from the truncation of Fourier expansion [Eqs. (18) and (19)] and from the process of solution of simultaneous equations.

Now we proceed to examine the above methods numerically by applying them to artificial wave fields. We consider here two cases of four wave detectors arranged in a symmetrical star (Fig. 1a) or an asymmetrical one (Fig. 1b). In the symmetrical arrangement, four wave detectors are placed

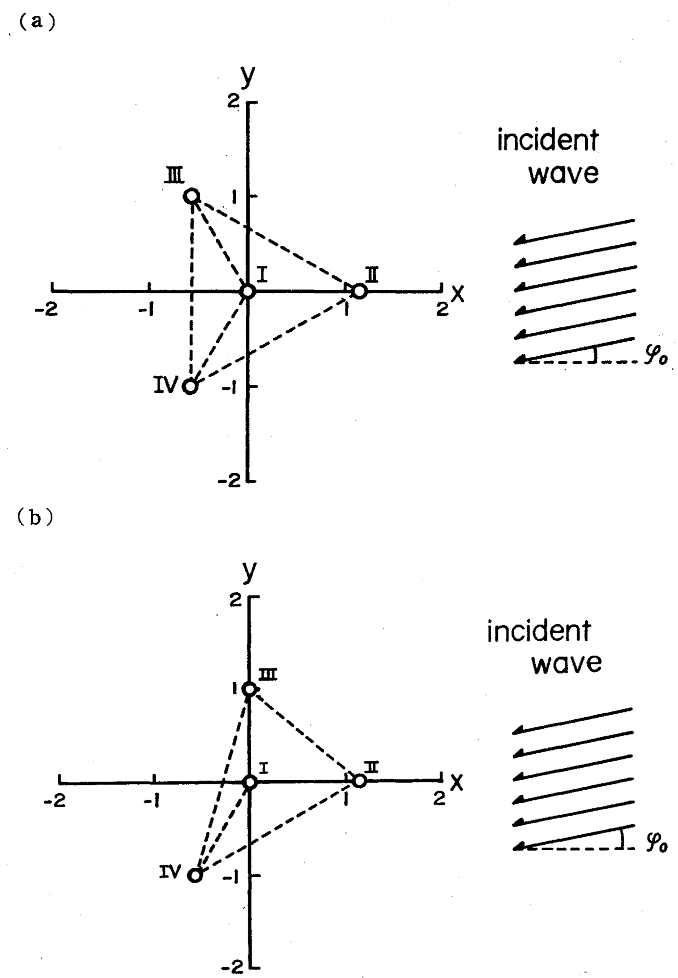
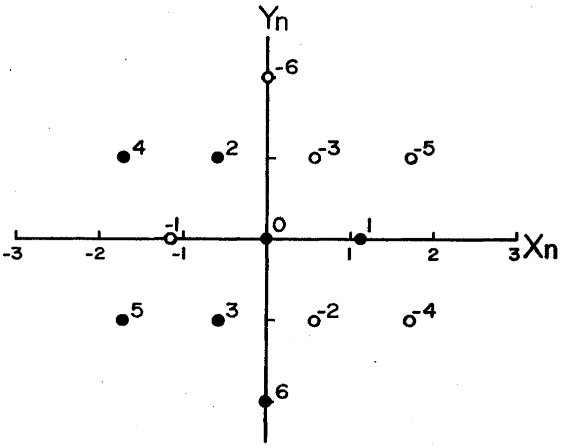


Fig. 1. Array of four wave detectors in a symmetric star (a) or an asymmetric star (b). Four detectors of array (a) are placed at the center and three vertices of an equilateral triangle with a length of $2\,m$ for each side. Only the position of detector Ⅲ of array (b) is different from that of array (a).

at the center and three vertices of an equilateral triangle with a length of $2\,m$ for each side. In the asymmetrical arrangement, only the position of detector Ⅲ is different from the symmetrical one. Crosscovariances are defined for seven pairs of wave detectors: (0) Ⅰ-Ⅰ, (1) Ⅰ-Ⅱ, (2) Ⅰ-Ⅲ, (3) Ⅰ-Ⅳ, (4) Ⅱ-Ⅲ, (5) Ⅱ-Ⅳ, (6) Ⅲ-Ⅳ. (The zero-th pair Ⅰ-Ⅰ can be replaced by one of pairs Ⅱ-Ⅱ, Ⅲ-Ⅲ and Ⅳ-Ⅳ on the assumption that the

(a)



(b)

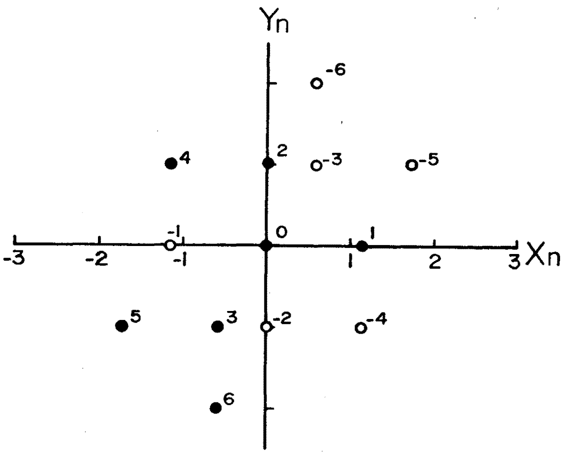


Fig. 2. Cartesian representation of distances between pairs of wave detectors for the symmetric array (a) or asymmetric array (b).

wave field is uniform.) Therefore the parameter L becomes six. In our numerical experiments the parameter L' is set to be five for the Mobarek and Barber methods. The distances between the positions of those pairs of wave detectors have heen numbered in Fig. 2a for the symmetric arrangement and in Fig. 2b for the asymmetric one. The corresponding cross spectra, *i.e.* the Fourier transform version of crosscovariances, are defined at

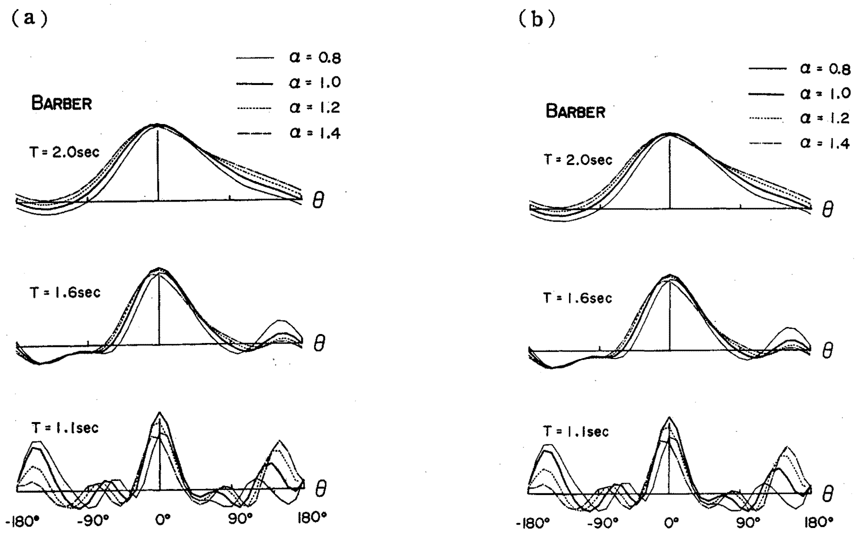


Fig. 3. Directional spectra obtained by the Barber method with the symmetric array (a) or asymmetric array (b) of four wave detectors. The artificial wave fields are given by Eq. (22) for $\varphi_0=0°$ and for wave periods $T=2.0$, 1.6, 1.1 *sec*. The vertical lines represent the input line spectra. (Values in arbitrary scale.)

those points.

Artificial wave fields to be analyzed are generated by a digital computer as follows:

$$\eta_i(x_i, y_i, t)=\cos(k_n \cos\varphi_0\cdot x_i+k_n \sin\varphi_0\cdot y_i+\omega_n t+\varepsilon_n)$$
$$(t=0, 1, \cdots, N-1;\ i=\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}) \tag{22}$$

where $N=128$, $\omega_n=2\pi n/N$, $\varepsilon_n=0°$, $n=4$-20 (wave period $T=0.8$-$4.0\,sec$), $\varphi_0=0$ and sampling interval is $1/8\,sec$. Eq. (22) represents uni-directional wave fields with respect to frequency ω_n. For the reasons that the uni-directional wave field may be considered to be the most appropriate field for examining the characteristics of each method and that the substitution of Eq. (3) for Eq. (2) is not valid in the case of artificial multi-directional wave field, we shall limit ourself here to the analysis of uni-directional wave fields. The wave number k_n has been related with the frequency ω_n by

$$\omega_n{}^2=\alpha g k_n, \qquad (0.6\leqq\alpha\leqq1.6), \tag{23}$$

whereas the numerical analysis is carried out on the premise that $\omega_n{}^2=gk_n$ [Eq. (11)]. Thus, the parameter α represents the difference of the real dispersion relation [(23)] to the assumed one [(11)].

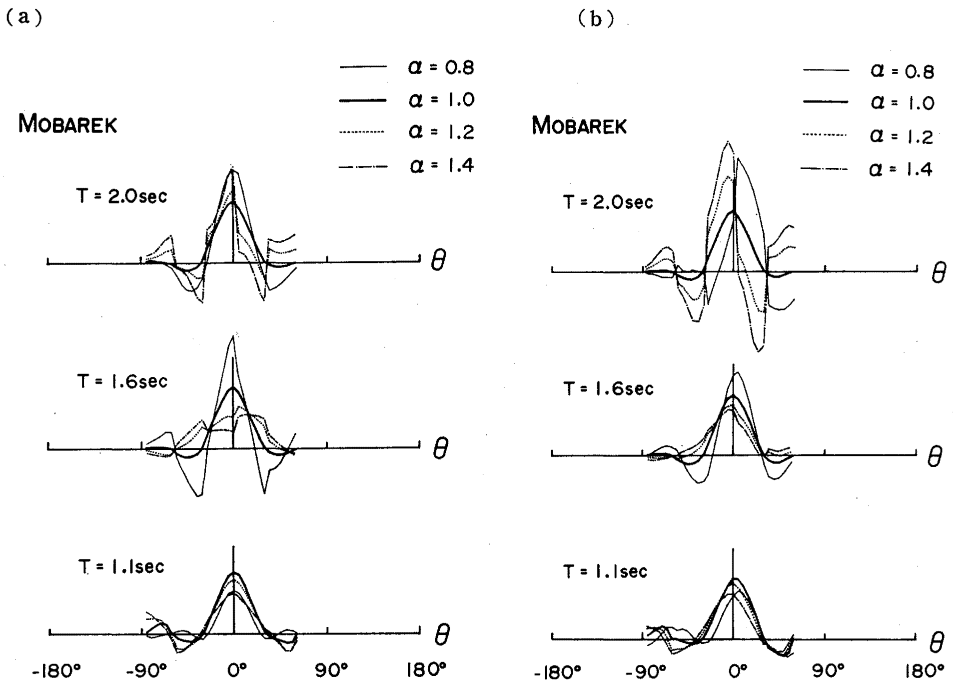


Fig. 4. Directional spectra obtained by the Mobarek method with the asymmetric array (a) or asymmetric array (b) of four wave detectors. The artificial wave fields are given by Eq. (22) for $\varphi_0=0°$ and for wave periods $T=2.0$, 1.6, 1.1 *sec*. The vertical lines represent the input line spectra. (Values in arbitrary scale.)

Some of the results of numerical experiments with the symmetric arrangement of wave detectors have been shown in Figs. 3a, 4a, and 5a for Barber, Mobarek, and Fujinawa methods, respectively. One can see in these figures that the directional spectrum is highly dependent on the dispersion relation: when the real and assumed dispersion relations are equal to each other ($\alpha=1.0$) the true direction of propagation can be detected, but when this is not the case spurious and negative estimates appear. The spurious estimates increase with α deviating from 1.0. The directional spectrum by Fujinawa method is very sensitive to the dispersion relation, and shows a larger α-dependency than other two methods.

The directional spectrum is also dependent on the frequency. When the wave length under consideration is much longer or shorter than the scale of wave detectors array, the true direction of propagation cannot be detected even in the case of $\alpha=1.0$. Apart from the accuracy of the directional

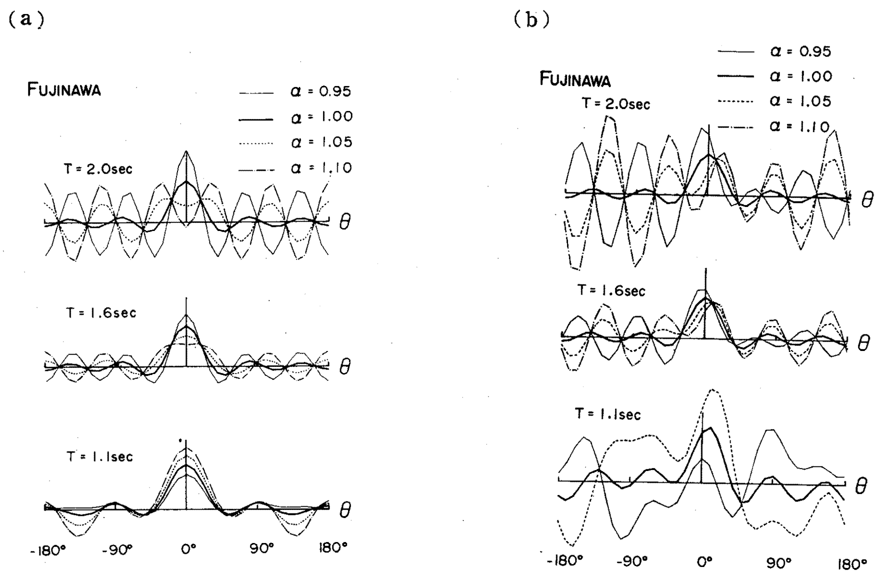


Fig. 5. Directional spectra obtained by the Fujinawa method with the symmetric array (a) or asymmetric array (b) of four wave detectors. The artificial wave fields are given by Eq. (22) for $\varphi_0=0°$ and for wave periods $T=2.0$, 1.6, 1.1 sec. The vertical lines represent the input line spectra. (Values in arbitary scale.)

spectrum, the true direction can be detected in the frequency range of 0.35-0.80 Hz in Barber method, 0.25-1.25 Hz in Mobarek method and 0.35-0.90 Hz in Fujinawa method. The range becomes narrower with α deviating from 1.0.

For the asymmetric arrangement of wave detectors, the results of numerical experiments have been shown in Figs. 3b, 4b, 5b. In this case it is important to point out that the calculated directional spectra are asymmetric with respect to wave direction, and that the obtained mean wave direction deviates from the true one with α deviating from 1.0. To detect the true wave direction with accuracy, therefore, the wave detectors should be arranged preferably in a symmetrical array.

We see in Figs. 3, 4 and 5 that the dispersion relation of wind waves plays an important role in measuring the directional spectrum. These results suggest that, under the practical condition that the real dispersion relation is not known to us, the calculated directional spectrum on the premise Eq. (11) may contain a lot of estimation error. To obtain the accurate directional spectrum, the real dispersion relation should be estimated with an error less than roughly $\pm 20\%$ for Barber method, $\pm 15\%$ for Mobarek method, and $\pm 5\%$ for Fujinawa method.

4. The floating buoy technique

The method of measurement using the motion of a floating buoy [Longuet-Higgins et al.[11] (1963), Cartwright et al.[12] (1964) and others] has been the most successful technique in obtaining the information concerning the directional energy distribution of wind waves. The theory of this method is constructed on the simultaneous measurements of vertical acceleration of surface displacement η_{tt}, two components of wave slope η_x and η_y, and three components of wave curvature η_{xx}, η_{yy} and η_{xy} (subscripts x and y denote differential operators).

In this technique the directional spectrum $\phi(\omega, \theta)$ is expanded in a Fourier series:

$$\phi(\omega, \theta) = \phi(\omega)[\frac{1}{2} + (u_1 \cos \theta + v_1 \sin \theta) + (u_2 \cos 2\theta + v_2 \sin 2\theta) + (u_3 \cos 3\theta + v_3 \sin 3\theta) + (u_4 \cos 4\theta + v_4 \sin 4\theta) + \cdots], \tag{24}$$

where $\phi(\omega)$ is the frequency spectrum. If we have the records of η_1 $(=\eta_{tt})$, η_2 $(=\eta_x)$, η_3 $(=\eta_y)$, η_4 $(=\eta_{xx})$, η_5 $(=\eta_{yy})$, η_6 $(=\eta_{xy})$, then we can obtain the Co-spectrum C_{ij} and the Quardrature-spectrum Q_{ij} from the records η_i and η_j. The above Fourier coefficients are related successfully with those cross spectra as follows [see Longuet-Higgins et al.[11] (1963) and Cartwright et al.[12] (1964)]:

$$\left.\begin{aligned}
u_1 &= \frac{Q_{12}}{C_{11}}, & v_1 &= \frac{Q_{13}}{C_{11}}, \\
u_2 &= \frac{C_{22}-C_{33}}{C_{22}+C_{33}}, & v_2 &= \frac{2C_{23}}{C_{22}+C_{33}}, \\
u_3 &= \frac{3Q_{25}-Q_{24}}{C_{14}+C_{15}}, & v_3 &= \frac{Q_{35}-3Q_{34}}{C_{14}+C_{15}}, \\
u_4 &= \frac{C_{44}-6C_{45}+C_{55}}{C_{44}+2C_{45}+C_{55}}, & v_4 &= \frac{4(C_{46}-C_{56})}{C_{44}+2C_{45}+C_{55}}.
\end{aligned}\right\} \tag{25}$$

Now let us see the role of dispersion relation by analyzing again the artificial uni-directional wave field. Artificial records for the cloverleaf buoy technique have been generated as

$$\left.\begin{aligned}
\eta_1 &= \eta_{tt} = -a_n \omega_n^2 \cos(\omega_n t + \varepsilon_n), \\
\eta_2 &= \eta_x = -a_n k_n \cos \varphi_n \sin(\omega_n t + \varepsilon_n), \\
\eta_3 &= \eta_y = -a_n k_n \sin \varphi_n \sin(\omega_n t + \varepsilon_n),
\end{aligned}\right\} \tag{26}$$
$$\vdots$$

where $\varphi_n = 0°$, $\varepsilon_n = 0°$, $\omega_n = 3.14$ *rad/sec*, $a_n = 0.1m$, and the wave number

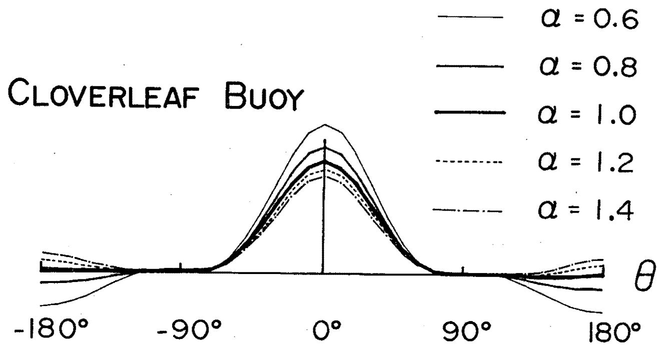


Fig. 6. Directional spectra obtained by the cloverleaf buoy technique. The artificial wave fields are given by Eq. (26) for $\varphi_0=0°$. (Values in arbitrary scale.)

k_n has been given by $k_n=\omega_n^2/\alpha g$, $0.6\leq\alpha\leq1.6$. Then the coefficients u_n and v_n have been calculated to the fourth order by Eq. (25), and weighted by a function $W_n(W_1=8/9,\ W_2=28/45,\ W_3=56/165,\ W_4=14/99)$ to obtain smoothed spectrum. Finally the directional spectrum has been calculated by Eq. (24).

The results of numerical experiment (Fig. 6) indicate that general form of the calculated spectrum is not dependent so much on the dispersion relation as compared with the other methods. However, the angular distribution of wave energy widens apparently with increasing α. Unless the true dispersion relation is known beforehand, therefore, detailed discussion on the concentration of wave energy in the dominant direction of propagation may not be made without ambiguity.

5. New method based on the direct Fourier transform

The new method of measuring the directional spectrum, introduced by the present author, is marked by its simple mathematical expression for wave field and by the use of the discrete Fourier transform technique. Theory of the method has been described in detail and the use of the method has been demonstrated in Part I and Part II. This method also requires to know the dispersion relation beforehand. Therefore we should examine carefully the role of dispersion relation in the method.

We are concerned here with Array I (see Part I) which consists of tweleve wave gauges distributed around a circle of radius r at equal intervals, and with multi-directional wave fields. Artificial wave fields have been generated with an electric computer as follows:

$$\eta_i(t) = \sum_{m=1}^{12} A_m \cos\{k_n r(\cos\theta_i \cos\theta_{nm} + \sin\theta_i \sin\theta_{nm}) + \omega_n t + \varepsilon_m\} \tag{27}$$

$$(t=0, 1, \cdots, N-1; i=1, 2, \cdots, 12)$$

where ε_m are random phases, $N=128$, $n=27$ or 32, $\omega_n=2\pi n/N$, $A_m=\sqrt{2}\sin^8(\theta_{nm}/2)$, $\theta_i=2\pi i/12$, and $k_n=\omega_n^2/\alpha g$ $(0.7 \leqq \alpha \leqq 1.5)$. The directional analysis of these fields have been carried out on the assumption $\omega_n^2=gk_n$. The dimensionless number λ which represents the relative scale of wave gauge array to wave length is given in this case by $\lambda=rk_n/\pi=r\omega_n^2/\pi\alpha g=\lambda_0/\alpha$, where $\lambda_0=r\omega_n^2/\pi g$.

Results of analysis have been shown in Figs. 7a and 7b for the cases of $\lambda_0=1.039$ $(n=27)$ and 1.459 $(n=32)$, respectively. One can see in the

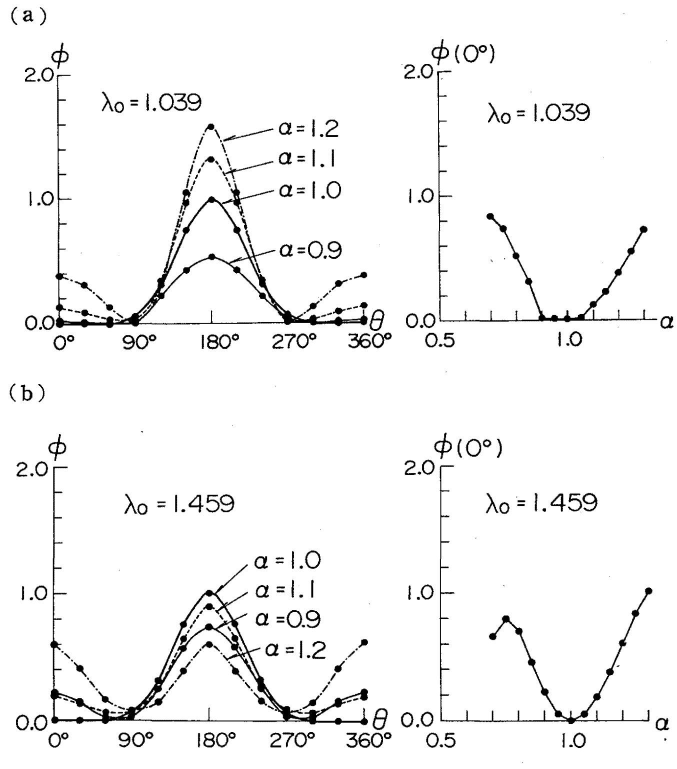


Fig. 7. Directional spectra obtained by the new method. The wave field are given artificially by Eq. (27) with various α values for the cases of $\lambda_0=1.039$ (a) or $\lambda_0=1.459$ (b). In the right of the figures spurious estimates for the direction opposite to the mean wave direction have been plotted as a function of α.

figures that the directional spectrum is much affected by the dispersion relation: when real and assumed dispersion relations are not equal to each other, *i. e.* when $\alpha \neq 1.0$, the integrated estimate of directional spectrum along the direction deviates from the true one, and more over a spurious peak of spectrum appears at directions opposite to the mean wave direction. In the right of Figs. 7a and 7b, these spurious estimates for the direction 0° (for which no energy is expected to appear) have been shown as a function of α. It is found naturally that the spurious spectrum becomes nearly zero or minimum when $\alpha=1.0$. (Although the minimum at $\alpha=1.0$ in Fig. 7a seems not to show a clear distinction, it becomes conspicuous when the values are presented in logarithmic scale.) The same is true for other cases of arbitrary λ_0 values. This is because, while estimation errors in amplitude of component wave [a_{nm} in Eq. (6)] resulting from the use of incorrect dispersion relation may be positive or negative, errors in directional spectrum [or a_{nm}^2 in Eq. (7)] are absolutely positive. This feature of α-dependency of the calculated directional spectrum is very important in determining the real or reasonable dispersion relation of actual wind waves.

6. Note on the ergodic hypothesis* for wave field

As has been described in Section 3, the methods of measuring the directional spectrum developed by Barber, Mobarek and Fujinawa are based essentially on the measurement of crosscovariance function. In measuring the crosscovariances, they have replaced the averaging process over time and space by that over time alone on the assumption that the wave field under consideration is ergodic. When the wave field is not uniform in space, however, the assumption of the ergodicity cannot be allowed. In general, the artificial multi-directional wave field generated by a digital computer is not uniform spatially, and therefore not ergodic. This is why we have confined ourself to the analysis of the uni-directional wave field in Section 3.

The method using the motion of a floating buoy is much the same as above in that it cannot be applied to the artificial multi-directional wave field. This is because the basic relation (25) does not hold generally in the case of artificial multi-directional wave field. In this respect, the use of the relation (25) may correspond to assuming the ergodic property.

The new method based on the direct Fourier tranform technique, on the other hand, is not concerned with the crosscovariance function. Accordingly the method can analyze the artificial multi-directional wave field as well as the actual random field. In this respect, the method is considered to have some advantages over the others.

* The term "ergodic" or "ergodicity" is used in this paper to represent the same concept that Barber[5] (1963) described.

Here it may be of some interest to analyze the artificial multi-directional wave field through the conventional methods. As examples, we consider bi-directional wave fields represented by

$$\begin{aligned}\eta_i(x_i, y_i, t) = a_1 \cos(k_n \cos\varphi_1 \cdot x_i + \sin\varphi_1 \cdot y_i + \omega_n t + \varepsilon_1) \\ + a_2 \cos(k_n \cos\varphi_2 \cdot x_i + \sin\varphi_2 \cdot y_i + \omega_n t + \varepsilon_2) \\ (t = 0, 1, \cdots, N\text{-}1; i = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV})\end{aligned} \tag{28}$$

where $N=128$, $\omega_n = 2\pi n/N$, $n=10$, $\omega_n{}^2 = g k_n$, $\varepsilon_1 = \varepsilon_2 = 0$ and various values for a_1, a_2, φ_1, φ_2. The directional spectrum has been calculated by using the symmetric array through the procedures described in Sections 3 and 4, and has been shown in Fig. 8. One can see from the figure that the true directions of propagation are not detected. If we use the same values as above for a_1, a_2, φ_1 and φ_2 but different values for ε_1 and ε_2, then the obtained directional spectrum differs from the above one.

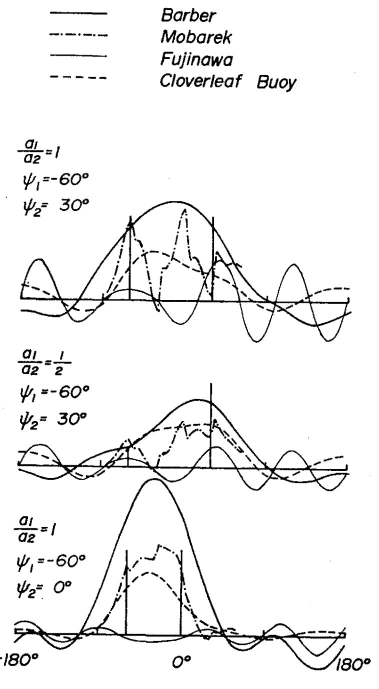


Fig. 8. Directional Spectra obtained by Barber, Mobarek, Fujinawa methods and the cloverleaf buoy technique. The wave fields are given by Eq. (28) with various values for a_1, a_2, φ_1 and φ_2. The vertical lines in the figure represent the input line spectra. (Values in arbitrary scale.)

In general, however, those results from unrealistic artificial wave fields may not serve as a good reference for the analysis of the actual wave field. In other words, it does not follow from the above numerical experiments that the conventional methods based on the assumption of the ergodicity are of no use in the directional analysis of actual wind waves.

Now it is necessary to examine whether or not the assumption of ergodicity is reasonable for real wind waves. Although full information on the three-dimensional wave field can hardly be obtained in practice, it is possible to make an approximate examination by using simultaneous wave records at several positions: if two or more crosscovariance functions obtained by Eq. (3) are coincident with each other, one may suppose the assumption to be acceptable. Along this line we have analyzed simultaneous records of wind waves in a experimental tank which were measured originally for the purpose of obtaining the directional spectrum and phase velocity of laboratory wind waves (refer to Part II). In Figs. 9a and 9b pairs of measured crosscovariance functions $R'(X, Y, \tau)$ are shown for the cases of various values X and Y. Figure 9a corresponds to the wave field of a uni-modal directional spectrum, and Fig. 9b a bi-modal directional spectrum (refer to Part II). One can see in the figures that the coincidences between pairs

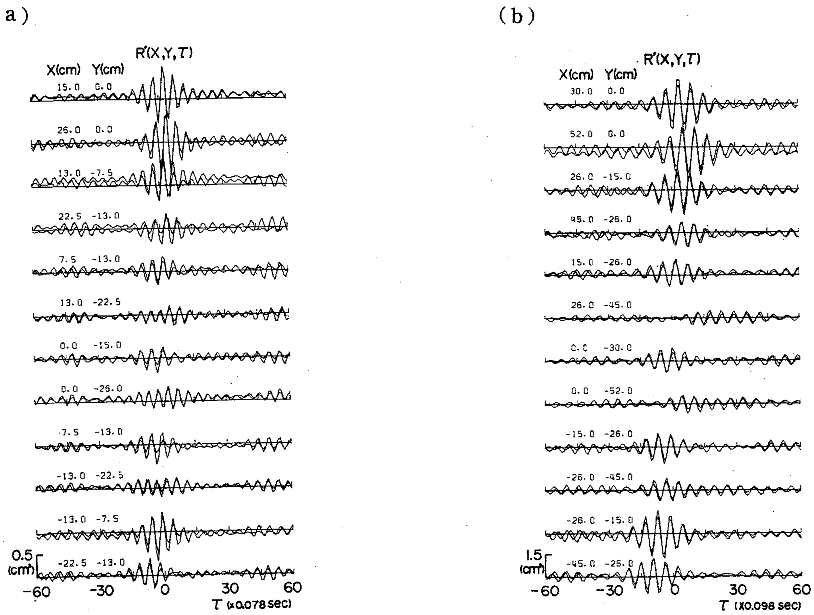


Fig. 9. Pairs of measured crosscovariance function $R'(X, Y, \tau)$ for various X and Y. The wave records under the analysis correspond to the wave fields of a unimodal angular distribution (a) or a bimodal distribution (b) of wave energy.

of crosscovariance functions are satisfactory. Therefore it might be concluded that the assumption of ergodicity is generally acceptable in the wind wave field whether the wave energy is distributed in a uni-modal distribution or in a bi-modal distribution.

7. Concluding remarks

Through the numerical analysis of artificial wave fields, it has turned out that the dispersion relation plays an important role in estimating the directional spectrum: if incorrect dispersion relation is used for the analysis, then obtained directional spectrum is distorted considerably and shows a spurious distribution. In order to obtain reliable estimates, therefore, real dispersion relation should be predetermined with accuracy. Until the present, however, we have not been possessed of a way of estimating the dispersion relation or phase velocity* for actual wind waves. Although some experimental studies have been reported on the phase velocity of spectral components along a predetermined direction (Yefimov et al., 1972; Ramamonjiarisoa, 1974; Kato and Tsuruya[4], 1974), their results may contain significant errors (about 10-20%) because they have not taken the effect of directional spectrum into account in measuring the phase velocity. The directional spectrum and phase velocity (or dispersion relation) are related with each other so closely that neither of them may be estimated independently of the other.

By applying the new method presented by the author, however, it is possible to determine both the directional spectrum and dispersion relation simultaneously with a good accuracy. This is because most of unreasonable estimates obtained by the method can be ascribed to the use of incorrect dispersion relation. In other words, since the use of the improper dispersion relation always yields a spurious peak of directional sepctrum at the opposite direction to the mean wave direction, one can determine inductively the reasonable dispersion relation and directional spectrum by the condition that the spurious estimates should become minimum (for detailed discussion see Parts I and II).

Other conventional methods of measuring the directional spectrum also have been examined whether or not they can determine the consistent dispersion relation (and corresponding directional spectrum) for actual wind waves. The methods using the optical and photographic techniques are beside the subject of consideration, because they have nothing to do with dispersion relation. The methods by Barber, Mobarek and Longuet-Higgins et al. also may not determine the dispersion relation with accuracy

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*) Dispersion relation and phase velocity may be considered to have a same physical conception.

because they are quite insensitive to the dispersion relation. Although the method by Fujinawa is much sensitive to the dispersion relation, unreasonable estimates obtained can not be ascribed to the use of incorrect dispersion relation alone because the method has several other factors in methodological error.

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