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Some remarks on the behaviour of the finite element solution in nonsmooth domains

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Abstract. In this paper, we consider on the behaviour of the residual error by a smooth finite element solution for elliptic problems on nonconvex and nonsmooth domains. Against expectations, it is proved that the residual error is unbounded and actually diverges to infinity as the mesh size goes to zero. A numerical example which illustrates this phenomena will be presented for the Poisson equation on L-shaped domain using C^1-Hermite element as well as the similar results will be shown for a C^0 element with a posteriori smoothing.

Keywords. Poisson equation, Non-smooth domain, Residual error.

1 Introduction

In this paper, we consider a finite element solution u_h of the following Poisson equation.

$$\begin{cases} -\triangle u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here, Ω is a bounded polygonal domain in $\mathbb{R}^2$ and f is a function in $L^2(\Omega)$. Then, in case of $u_h \in H^2(\Omega) \cap H_0^1(\Omega)$, the residual error $\|f + \triangle u_h\|_{L^2(\Omega)}$ plays an important role in the numerical enclosure methods of solutions for nonlinear elliptic problems(see, e.g., [3], [4], [7], [5] etc.). Let S_h be a finite dimensional subspace of $H_0^1(\Omega)$ dependent on the mesh size parameter h. Usually, u_h is defined as an element of S_h such that

$$(\nabla u_h, \nabla v_h)_{L^2(\Omega)} = (f, v_h)_{L^2(\Omega)}, \qquad \forall v_h \in S_h. \tag{2}$$

In the below, we asumme that, for solutions to (1) and (2),

$$u_h \to u \ (h \to 0) \qquad \text{in } H^1(\Omega), \tag{3}$$

which would be a natural condition for usual finite element subspaces.

If S_h is a C^0 element, since the residue $f + \triangle u_h$ no longer belongs to $L^2(\Omega)$, we need some smoothing procedure to get the residual estimation[6].

In this and the next sections, we assume that S_h is a C^1 finite element. For the convex domain Ω, assuming an inverse inequality for S_h, we easily have the following estimates:

$$\begin{aligned}
\|f+\triangle u_h\|_{L^2(\Omega)} &= \|\triangle(u-u_h)\|_{L^2(\Omega)} \\
&\leq Ch^{-1}\|\nabla(u-u_h)\|_{L^2(\Omega)} \\
&\leq C\|\triangle u\|_{L^2(\Omega)} \\
&= C\|f\|_{L^2(\Omega)}, \qquad (4)
\end{aligned}$$

where C is a general constant independent of h. Hence, $\triangle u_h$ is bounded in h. There is, rather, a possibility to get some positive order estimates for $\|f+\triangle u_h\|_{L^2(\Omega)}$ in h provided that we use higher order polynomials. Therefore, we will naturally expect that $\triangle u_h$ should also be bounded even if Ω is nonconvex, because the approximation scheme (2) is equivalent to

$$(-\triangle u_h, v_h)_{L^2(\Omega)} = (f, v_h)_{L^2(\Omega)}, \qquad \forall v_h \in S_h, \qquad (5)$$

which strongly suggests that $\triangle u_h$ seems to be determined only by the function f. However, it is shown that this expectation is actually wrong. Namely, in Section 2, when u does not have H^2 smoothness, we prove that the concerning residual error is unbounded. In Section 3, we will present some computational results of these errors for the Poisson equation on L-shaped domain using C^1 Hermite functions, which confirm our theoretical assertion. Furthermore, we will show the similar result even for the case that we use some a posteriori smoothing technique with C^0 piecewise linear element. These should be interesting and rather surprising facts which is beyond our intuitive observation.

2 Unboundedness of residual error

In this section, let Ω be a nonconvex polygonal domain. Then, as well known, the weak solution u of (1) uniquely exists in $H_0^1(\Omega)$, and not necessarily belongs to $H^2(\Omega)$(see [1]). We now describe the main result in this paper.

Theorem 2.1 *Let S_h be an C^1 finite element subspace on Ω,i.e., $S_h \subset H_0^1(\Omega)\cap H^2(\Omega)$, and let u_h be a solution of (2), or equivalently defined by (5). Then, the residual error has the following property.*

$$\lim_{h\to 0}\|f+\triangle u_h\|_{L^2(\Omega)} = \infty.$$

Proof. Let assume that the set $\left\{\|\triangle u_h\|_{L^2(\Omega)}\right\}_{0<h<1}$ is bounded in $\mathbb{R}$. First, by (2) and the Poincaré inequality, we have

$$\begin{aligned}
\|\nabla u_h\|_{L^2(\Omega)}^2 &\leqq \|f\|_{L^2(\Omega)}\|u_h\|_{L^2(\Omega)} \\
&\leqq C_p\|f\|_{L^2(\Omega)}\|\nabla u_h\|_{L^2(\Omega)},
\end{aligned}$$

where C_p is a Poincaré constant. Therefore, $\left\{ \|u_h\|_{H_0^1(\Omega)} \right\}_{0<h<1}$ is also bounded.
Next, since Ω is a polygon, we have the following well known result

$$\|\triangle u_h\|_{L^2(\Omega)} = |u_h|_{H^2(\Omega)},$$

which implies that $\{u_h\}_{0<h<1}$ is a bounded set in $H^2(\Omega)$.
Thus, by the weak compactness of $H^2(\Omega)$, there exists a subsequence $\{u_{h_n}\}_{n=1}^{\infty}$ in $\{u_h\}_{0<h<1}$, which weakly converge to some $\hat{u} \in H^2(\Omega)$. By the compactness of the embedding $H^2(\Omega) \hookrightarrow H^1(\Omega)$, we have the strong convergence:

$$u_{h_n} \to \hat{u} \ (n \to \infty) \qquad \text{in } \ H^1(\Omega).$$

On the other hand, u_{h_n} converges to the solution u of (1) in $H^1(\Omega)$ by the assumption (3). Therefore, by the uniqueness of the limit, we have $u = \hat{u}$, which implies that u has to be an element of $H^2(\Omega)$. This is a contradiction. □

3 Numerical Examples

In this section, we show some numerical evidences for the actual divergent situations in the previous section, which suggest the difficulty to construct the approximate solution with convergent residual error, provided that the corresponding exact solution has no sufficient smoothness.

3.1 Smooth basis

We considered (1) with $f \equiv 1$ and Ω as a L-shaped domain in Fig. 1. We used the bi-cubic Hermite function as the basis of S_h with a uniform mesh in Fig.1. In this case S_h is a subspace of $H^2(\Omega)$.

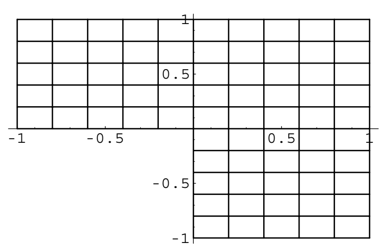

Figure 1: domain Ω (mesh size $h = 1/5$)

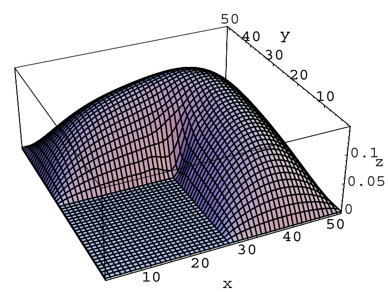

Figure 2: approximate solution u_h

Fig.3 shows the computed results of residual errors, in which the horizontal axis means mesh size h, the vertical axis residual error $\|1 + \triangle u_h\|_{L^2(\Omega)}$ with logarithmic scale. By considering the results in Fig.3, we obtained the residual error with negative order in h, i.e., approximately $1.23h^{-0.33}$, which confirms us the divergence property proved in Theorem 2.1.

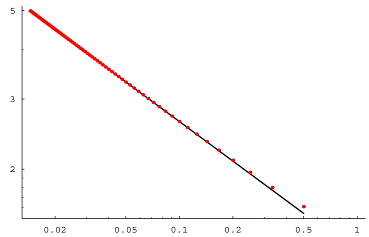

Figure 3: residual error $\|1+\triangle u_h\|_{L^2(\Omega)}$

3.2 A posteriori smoothing by piecewise linear element

In [6], some a posteriori smoothing techniques were used to get the residual error for the C^0 element. We considered the same problem in the previous subsection by using the piecewise bilinear polynomial functions for the same mesh. Naturally, the finite element solution u_h of (2) is almost same contour as Fig.2. Since the direct calculation $\triangle u_h$ is not possible, some smoothing procedure are taken as in [6]. Namely, we provided a piecewise bilinear finite element subspace S_h^* of $H^1(\Omega)$ which is constituted by adding the bases corresponding the boundary nodes to S_h. And define the vector function $\overline{u}_h$, denoted as $\overline{\nabla} u_h$, which means a smoothing of ∇u_h in $S_h^* \times S_h^*$ by

$$(\overline{u}_h, \overline{v}_h)_{L^2(\Omega)^2} = (\nabla u_h, \overline{v}_h)_{L^2(\Omega)^2}, \qquad \forall \overline{v}_h \in S_h^* \times S_h^*. \tag{6}$$

Then define $\overline{\triangle} u_h \equiv \mathrm{div}\, \overline{\nabla} u_h$.

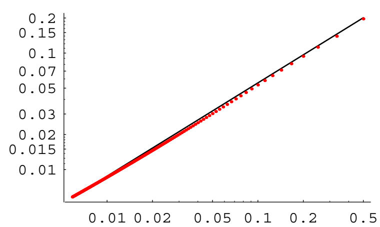

Figure 4: smoothing error $\left\|\overline{\nabla} u_h - \nabla u_h\right\|_{L^2(\Omega)^2}$

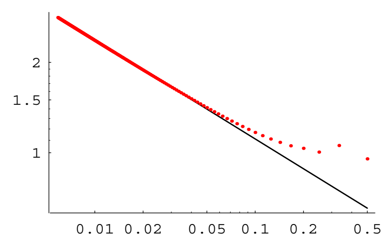

Figure 5: residual error $\left\|1+\overline{\triangle} u_h\right\|_{L^2(\Omega)}$

The smoothing error $\left\|\overline{\nabla} u_h - \nabla u_h\right\|_{L^2(\Omega)^2}$ and the residual error $\left\|1+\overline{\triangle} u_h\right\|_{L^2(\Omega)}$ are shown in Fig.4 and in Fig.5, respectively, with the same scale as before. According to these computations, we observed that $\left\|\overline{\nabla} u_h - \nabla u_h\right\|_{L^2(\Omega)^2} \approx 0.35 h^{0.80}$ and $\left\|1+\overline{\triangle} u_h\right\|_{L^2(\Omega)} \approx 0.52 h^{-0.33}$. These results suggest that it should be not possible to improve the residual error by this kind of a posteriori smoothing.

4 Concluding remarks

By Theorem 2.1 and our numerical experiments above, we could say that

1. When we use smooth approximate method by C^1 element it is proved that we can't constitute sufficient approximation to Δu.

2. Even if we take a smoothing method by some a posteriori techniques for the C^0 element, it could not be possible to improve the approximate property for Δu.

3. As an alternative approach, the singular function method, e.g., [2], might have the desired property for this problem, although it should be a little bit of unusual finite element method.

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