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A stabilization technique for stationary flow problems

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Finite element methods with stabilization techniques for the stationary Navier–Stokes equations are studied. To solve the stationary Navier–Stokes equations, the Newton method is used. To compute the problem at each step of the nonlinear iteration, a stabilization technique is introduced. The mixed interpolation, which satisfies the inf-sup condition, with stabilized terms is also considered to investigate its computational efficiency. Numerical results show that stabilized terms improve convergences of the Newton method especially in the case of high Reynolds number as well as those of the linear solver at each step of the nonlinear iteration.

keywords: *Navier–Stokes equations, Stationary flow, Newton method, Stabilized technique*

1 Introduction

There often encounter requirements to compute what flow pattern is generated in the stationary state. With progress of computer environment and increasing demand of precise analyses, numbers of degrees of freedom of such a computation become larger. However, as far as we know, computational codes may be rare, which are efficient for large scale, stationary, and nonlinear flow problems. On the other hand, there often encounter requirements to compute convection dominated flows. When the finite element method is used, the stabilization technique is often introduced for such a computation; in case of the nonstationary Navier–Stokes equations, for example, see Hansbo and Szepessy (1990), Hughes and Brooks (1982), Tabata and Suzuki (2000), and Tezduyar et al. (1991); in case of the stationary Navier–Stokes equations, for example, see Brooks and Hughes (1982), Franca and Frey (1992) and Zhou and Feng (1993). However, as far as we know, it may be not enough to investigate what stabilization techniques are efficient for large scale, stationary, and nonlinear flow problems. From those facts, as a preliminary step of analysis of the nonlinear flow in the stationary state, we study finite element methods with stabilization techniques for the stationary Navier–Stokes equations. Moreover, it is well-known that, thanks to the stabilization technique, finite element approximations do not necessarily require the inf-sup condition. In this paper, however, the mixed interpolation, which satisfies the inf-sup condition, with stabilized terms is also considered to investigate its computational efficiency.

In this paper, to solve the stationary Navier–Stokes equations, the Newton method is used. To compute the problem at each step of the nonlinear iteration, a stabilization technique is introduced.

Numerical results show that stabilized terms improve convergences of the Newton method especially in the case of high Reynolds number as well as those of the linear solver at each step of the nonlinear iteration.

2 Formulation

Let Ω be a three-dimensional bounded domain with the Lipschitz continuous boundary Γ. We consider the stationary incompressible Navier–Stokes equations as follows:

$$
\begin{cases}
-\dfrac{1}{\rho}\nabla\cdot\sigma(u,p)+(u\cdot\nabla)\,u=\dfrac{1}{\rho}\,f & \text{in } \Omega, & \text{(1a)}\\[2mm]
\nabla\cdot u=0 & \text{in } \Omega, & \text{(1b)}\\[2mm]
u=g & \text{on } \Gamma, & \text{(1c)}
\end{cases}
$$

where $u=(u_1,u_2,u_3)^T$ is the velocity [m/s], p is the pressure [N/m^2], ρ is the density [kg/m^3], $f=(f_1,f_2,f_3)^T$ is the body force [N/m^3], $g=(g_1,g_2,g_3)^T$ is the boundary velocity [m/s], and $\sigma(u,p)$ is the stress tensor [N/m^2] defined by

$$
\sigma_{ij}(u,p)\equiv -p\delta_{ij}+2\mu D_{ij}(u),\quad D_{ij}(u)\equiv\frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\Big),\qquad i,j=1,2,3,
$$

with the Kronecker delta δ_{ij} and the kinematic viscosity μ [kg/(ms)].

Some preliminaries are arranged for the derivation of a variational formulation of (1). Let X denote $H^1(\Omega)^3$, and an affine space and function spaces are defined as follows:

$$
V(g)\equiv\big\{v\in X;\ v=g\ \text{on}\ \Gamma\big\},\qquad V\equiv V(0),\qquad Q\equiv L^2(\Omega)/\mathbb{R},
$$

where $L^2(\Omega)$ denotes the space of square summable functions in Ω, $H^1(\Omega)$ denotes the space of functions in $L^2(\Omega)$ with derivatives up to the first order, and "/$\mathbb{R}$" corresponds to determine up to an additive constant. Let a_0 be a continuous bilinear form on $X\times X$, b a continuous bilinear form on $X\times Q$, and a_1 a continuous trilinear form on $X\times X\times X$ defined by

$$
a_0(u,v)\equiv\frac{2\mu}{\rho}\int_\Omega D(u)\!:\!D(v)\ dx,\qquad b(v,q)\equiv-\frac{1}{\rho}\int_\Omega q\,\nabla\cdot v\ dx,
$$

$$
a_1(w,u,v)\equiv\frac{1}{2}\bigg\{\int_\Omega\big[(w\cdot\nabla)u\big]v\ dx-\int_\Omega\big[(w\cdot\nabla)v\big]u\ dx\bigg\},
$$

respectively. Here, the notation ":" denotes the tensor product.

The variational formulation of (1) is described as follows: find $(u,p)\in V(g)\times Q$ such that

$$
\begin{cases}
a_0(u,v)+a_1(u,u,v)+b(v,p)=\dfrac{1}{\rho}(f,v) & \text{for } v\in V, & \text{(2a)}\\[3mm]
b(u,q)=0 & \text{for } q\in Q, & \text{(2b)}
\end{cases}
$$

where $(\cdot,\cdot)$ denotes the L^2-inner product over Ω.

By application of the Newton method to (2), the kth step linearized equations become the following: find $(u^k,p^k)\in V(g)\times Q$ such that

$$
\begin{cases}
a_0(u^k,v)+a_1(u^{k-1},u^k,v)+a_1(u^k,u^{k-1},v)+b(v,p^k)\\[2mm]
\qquad\qquad\qquad =\dfrac{1}{\rho}(f,v)+a_1(u^{k-1},u^{k-1},v) & \text{for } v\in V, & \text{(3a)}\\[2mm]
b(u^k,q)=0 & \text{for } q\in Q. & \text{(3b)}
\end{cases}
$$

To avoid some intricate notations, we rewrite the linearized Navier–Stokes equations as follows: find $(u, p) \in V(g) \times Q$ such that

$$
\begin{cases}
a_0(u,v) + a_1(w,u,v) + a_1(u,w,v) + b(v,p) = (\widetilde{f}, v) & \text{for } v \in V, \quad (4\text{a}) \\
b(u,q) = 0 & \text{for } q \in Q, \quad (4\text{b})
\end{cases}
$$

where w is a given velocity [m/s]. Obviously, the equations (4) yeild (3) by substituting

$$
u^{k-1}, \quad u^k, \quad p^k, \quad \text{and} \quad \frac{1}{\rho}f + \left(u^{k-1}{\cdot}\nabla\right)u^{k-1}
$$

into w, u, p, and $\widetilde{f}$, respectively.

3 Finite element approximation

For simplicity, we confine ourselves to the tetrahedral subdivision of a polyhedral domain Ω, and to a continuous boundary velocity. Let $\{\mathscr{T}_h\}_{h\downarrow 0}$ be a uniformly regular family of decompositions of $\overline{\Omega}$ into tetrahedra, where h stands for the maximum diameter of tetrahedra and "$\overline{}$" denotes the closure.

In this paper, the P1/P1 elements are considered first:

$$
X_h \equiv \left\{ v_h \in X \cap \mathscr{C}(\overline{\Omega})^3; \ v_h|_K \in \mathscr{P}_1(K)^3 \ \text{for} \ K \in \mathscr{T}_h \right\},
$$

$$
Q_h \equiv \left\{ q_h \in Q \cap \mathscr{C}(\overline{\Omega}); \ q_h|_K \in \mathscr{P}_1(K) \ \text{for} \ K \in \mathscr{T}_h \right\},
$$

where $\mathscr{C}(\overline{\Omega})$ denotes the space of continuous functions in $\overline{\Omega}$, and for each integer $k \geq 1$, $\mathscr{P}_k(K)$ denotes the space of polynomials of degree k defined in $K \in \mathscr{T}_h$. We set

$$
V_h(g) \equiv \left\{ v_h \in X_h; \ v_h(P) = g(P) \ \text{at} \ P \in \Gamma \right\}, \qquad V_h \equiv V_h(0),
$$

where P is any nodal point on Γ.

As in Franca and Frey (1992), an approximate problem of (4) with stabilized terms are introduced as follows: find $(u_h, p_h) \in V_h(g) \times Q_h$ such that, for $(v_h, q_h) \in V_h \times Q_h$,

$$
a_0(u_h, v_h) + a_1(w_h, u_h, v_h) + a_1(u_h, w_h, v_h) + b(v_h, p_h) + b(u_h, q_h)
$$

$$
+ \sum_{K \in \mathscr{T}_h} \left\{ \tau_K \left((w_h{\cdot}\nabla)u_h + (u_h{\cdot}\nabla)w_h + \frac{1}{\rho}\nabla p_h, \right. \right.
$$

$$
\left. \left. (w_h{\cdot}\nabla)v_h + (v_h{\cdot}\nabla)w_h - \frac{1}{\rho}\nabla q_h \right)_K + \delta_K(\nabla{\cdot}u_h, \nabla{\cdot}v_h)_K \right\}
$$

$$
= (\widetilde{f}, v_h) + \sum_{K \in \mathscr{T}_h} \tau_K \left(\widetilde{f}, (w_h{\cdot}\nabla)v_h + (v_h{\cdot}\nabla)w_h - \frac{1}{\rho}\nabla q_h \right)_K \qquad (5)
$$

where $(\,\cdot\,, \cdot\,)_K$ denotes the L^2-inner product over K. The stabilized parameters τ_K and δ_K are defined by

$$
\tau_K \equiv \min\left\{ \frac{h_K}{2\,\|w\|_\infty}, \ \frac{\rho\,h_K^2}{24\mu} \right\}, \qquad \delta_K \equiv \min\left\{ \frac{\lambda\rho h_K^2\|w\|_\infty^2}{12\mu}, \ \lambda h_K\|w\|_\infty \right\}, \qquad (6)
$$

where λ denotes a positive constant, $\|w\|_\infty$ denotes the maximum norm of w in K, h_K denotes the diameter of K.
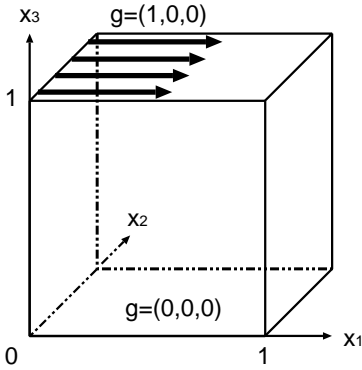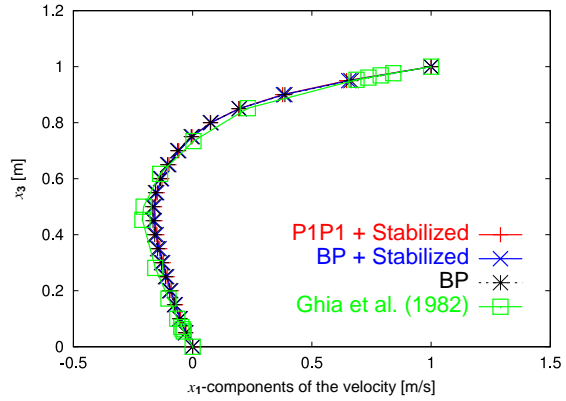
Figure 1: A cavity flow problem.

Figure 2: The x_1-components of the velocity along the line $x_1 = 0.5$ and $x_2 = 0.5$.

Second, the Bercovier–Pironneau (BP) elements are also considered as finite elements; see Bercovier and Pironneau (1979):

$$X_h \equiv \left\{ v_h \in X \cap \mathscr{C}(\overline{\Omega})^3;\ v_h|_K \in \mathscr{P}_1(K)^3 \text{ for } K \in \mathscr{T}_{h/2} \right\},$$

$$Q_h \equiv \left\{ q_h \in Q \cap \mathscr{C}(\overline{\Omega});\ q_h|_K \in \mathscr{P}_1(K) \text{ for } K \in \mathscr{T}_h \right\},$$

where $\mathscr{T}_{h/2}$ is a union of tetrahedra consisting of sub-tetrahedra obtained by decomposing each element $K \in \mathscr{T}_h$ into eight sub-tetrahedra, see Kanayama et al. (1989). In case of BP elements, "stabilized" terms are considered in the following way although they are slightly different from the method stated above: the summations in stabilized terms of (5) are executed over K belonging to $\mathscr{T}_{h/2}$ instead of $\mathscr{T}_h$.

4   Numerical examples

The cavity flow problem is considered as numerical examples; see Figure 1. For related results, see Ghia et al. (1982) and Kanayama and Toshigami (1988). The domain is a unit cube. On the top of the cube, $g = (1, 0, 0)\,[\text{m/s}]$ is given as flows along the x_1-axis, and the fixed boundary conditions are imposed on the other boundaries. The pressure is imposed 1.0 $[\text{N/m}^2]$ in the center of the cube. The body force is not loaded. The density ρ is set to be 1.0 $[\text{kg/m}^3]$. Throughout this section, the representative velocity U and the representative length D are set to be 1.0 $[\text{m/s}]$ and 1.0 $[\text{m}]$, respectively. Therefore the Reynolds number Re becomes $\rho U D / \mu = 1/\mu$.

The domain Ω is decomposed into a union of tetrahedra. The flow field is approximated by BP and P1/P1 elements. It is noted that BP elements are considered over $\mathscr{T}_h$, and that P1/P1 elements over $\mathscr{T}_{h/2}$. In the following computations, Ω is divided into $6 \times 10 \times 10 \times 10$ tetrahedra: in BP elements, the number of elements and degrees of freedom are 6,000 and 29,114, respectively; in P1/P1 elements, the number of elements and degrees of freedom are 48,000 and 37,044, respectively. As in Section 3, the Newton method are used for the nonlinear iteration in P1/P1 elements with stabilized terms (P1/P1 + Stabilized), BP elements with stabilized terms (BP + Stabilized), and BP elements without stabilized terms (BP: τ_K and δ_K is set to be 0). Throughout this section, λ is set to be 1.0. The initial value of the nonlinear iteraion is the finite element solution of the corresponding Stokes problem. The nonlinear iteration is stopped when the relative residual norm $\left\| \mathbf{F}(\mathbf{u}^{n+1}) - \mathbf{K}(\mathbf{u}^{n+1})\mathbf{u}^{n+1} \right\|_2 / \left\| \mathbf{F}(\mathbf{u}^0) \right\|_2$ becomes smaller than 1.0×10^{-7}. Here $\mathbf{u}^n$ denotes the solution vector at the nth step, $\mathbf{K}$ the resultant coefficient matrix, $\mathbf{F}$ the resultant given vector, and $\| \,.\, \|_2$ the Euclidean norm. In the Stokes equation for the initial condition, the resultant linear equations are solved by the Conjugate Gradient method

(CG), where the shifted incomplete LDL^T factorization is used as the preconditioner. Here the shifted value is set to be 1.05. The CG iteration is stopped when the relative residual norm $\left\|\mathbf{F} - \mathbf{K}\mathbf{u}^0\right\|_2 / \left\|\mathbf{F}\right\|_2$ becomes smaller than 1.0×10^{-12}. In each step of the nonlinear iteration, the resultant linear equations are solved by BiCGSTAB(L); see Gerard and Diederik (1993). Here the shifted incomplete LDU factorization is used as the preconditioner, where the degree L and the shifted value are set to be 10 and 1.05, respectively. The BiCGSTAB(L) iteration is stopped when the relative residual norm $\left\|\mathbf{F}(\mathbf{u}^n) - \mathbf{K}(\mathbf{u}^n)\mathbf{u}^{n+1}\right\|_2 / \left\|\mathbf{F}(\mathbf{u}^n)\right\|_2$ becomes smaller than 1.0×10^{-12}. Computation of the model was performed on Compaq Alpha 700MHz with 1 CPU.

Figure 2 shows distributions of the x_1-component of the velocity along the line $x_1 = 0.5$ and $x_2 = 0.5$ in "P1/P1 + Stabilized", "BP + Stabilized", and "BP". μ is set to be 1.0×10^{-2} [kg/(ms)], that is $Re = 100$. The computed results relatively agree with results in Ghia et al. (1982).

Figures 3 and 4 show residual norms versus the number of iterations in "P1/P1 + Stabilized", "BP + Stabilized", and "BP". Figure 3 illustrates the case of CG for the initial condition. Figure 4 illustrates the case of BiCGSTAB(10) at the first step of the nonlinear iteration. μ is set to be 1.0×10^{-2} [kg/(ms)], that is $Re = 100$. Results show that the number of iterations are reduced in "BP + Stabilized". Tables 1 and 2 show the number of iterations and CPU time for the Stokes problem and at the first step of the nonlinear iteration, respectively. From Tables 1 and 2, "BP + Stabilized" produces the lowest computational cost than the others; its CPU time is reduced by $14 \sim 40$ %.

Figures 5 and 6 compare residual norms of BiCGSTAB(10) at the first step of the nonlinear iteration in "BP" and "BP + Stabilized", respectively. μ is set to be 1.0×10^{-2}, 2.5×10^{-3}, and 1.0×10^{-3} [kg/(ms)], that is $Re = 100$, 400, and 1000. From Figure 5, high Reynolds number causes that convergence becomes difficult. On the other hand, Figure 6 shows that such difficulty was overcome thanks to stabilized terms. In $Re = 1000$, stabilized terms reduce the number of iterations to half.

Figures 7 and 8 show residual norms of the nonlinear iteration in "BP" and "BP + Stabilized", respectively. μ is set to be 2.5×10^{-3}, 1.25×10^{-3}, and 1.0×10^{-3} [kg/(ms)], that is $Re = 400$, 800, and 1000. From Figure 7, high Reynolds number causes that convergence becomes difficult without stabilized terms. Especially, the residual diverges in $Re = 1000$. On the other hand, from Figure 8, the nonlinear iteration using stabilized terms converges even in $Re = 1000$.

Table 1: Number of iterations and CPU time for the Stokes problem.

	P1/P1 + Stabilized	BP + Stabilized	BP
Num. of iterations	128	97	120
CPU time [s]	32	19	22

Table 2: Number of iterations and CPU time at the first step of the nonlinear iteration.

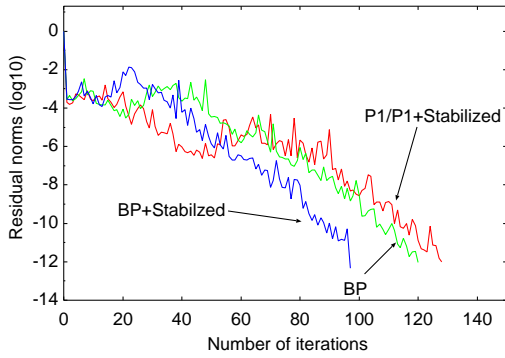|	P1/P1 + Stabilized	BP + Stabilized	BP
Num. of iterations	69	65	88
CPU time [s]	42	32	37

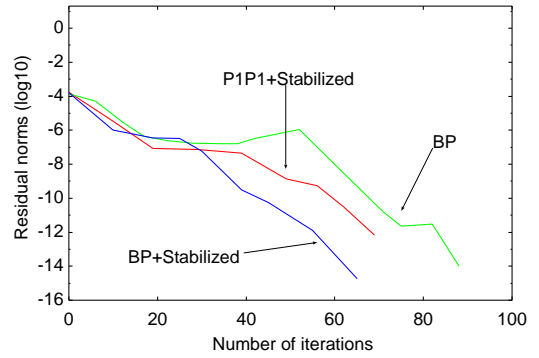Figure 3: The profile of residual norms of CG for the Stokes problem.

Figure 4: The profile of residual norms of BiCGSTAB(10) at the first step of the nonlinear iteration.
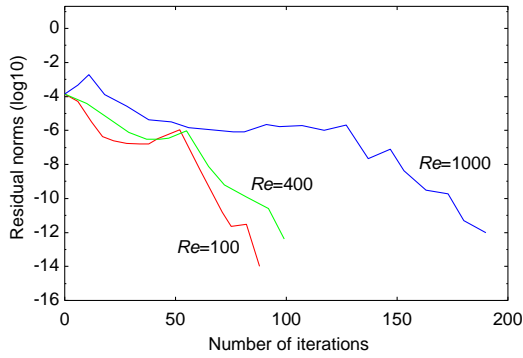

Figure 5: The profile of residual norms of BiCGSTAB(10) in "BP" at the first step of the nonlinear iteration.
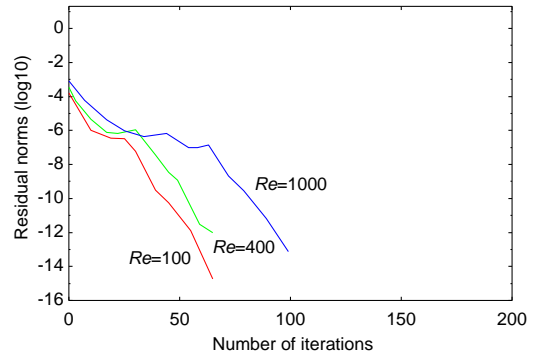

Figure 6: The profile of residual norms of BiCGSTAB(10) in "BP + Stabilized" at the first step of the nonlinear iteration .
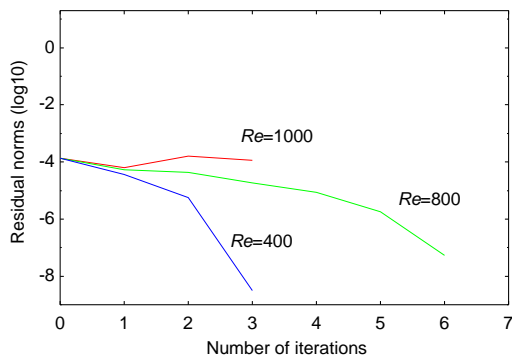

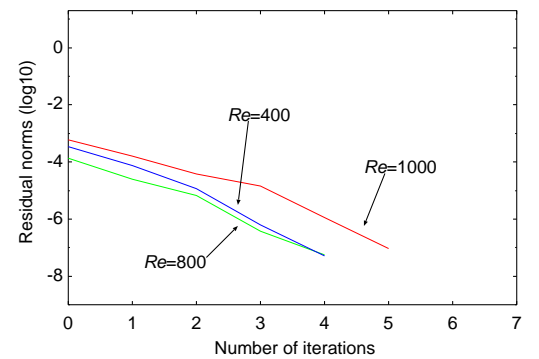Figure 7: The profile of residual norms of the nonlinear iteration in "BP".

Figure 8: The profile of residual norms of the nonlinear iteraion in "BP + Stabilized".

5   Concluding Remarks

We have considered the stationary incompressible Navier–Stokes equations, and have introduced the Newton method and a finite element method with a stabilization technique at each step of

the nonlinear iteration. P1/P1 elements with stabilized terms, BP elements with stabilized terms, and BP elements without stabilized terms have been adopted. Numerical results have shown that BiCGSTAB(L) in BP elements with stabilized terms converged faster than the others at each step of the nonlinear iteration. In addition, stabilized terms have improved the convergence of the Newton method as well as that of BiCGSTAB(L).

We are planning to compute larger scale problems using iterative domain decomposition method.

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