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Finite Element Analysis for Stationary Incompressible Viscous Flows Using Balancing Domain Decomposition

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We have constructed a parallel algorithm of the Balancing Domain Decomposition (BDD) in structural analysis, and succeeded in solving problems with over 100 million degrees of freedom. The results are released as an open source software, named ADVENTURE_Solid, from the ADVENTURE project. In this study, as an application of the parallel BDD algorithm of ADVENTURE_Solid to other fields, we apply it to a stationary Stokes solver. The effectiveness is shown with the consideration of applications to Navier-Stokes solvers.

1. INTRODUCTION

Recently, the simulation which uses large-scale models becomes possible and it seems that demands for several hundred million degrees of freedom (DOF) models will increase in the future. To solve the stationary Navier-Stokes problem with large-scale models, most iterative solution methods suffer from poor convergence. In this study, in order to improve convergence of stationary Navier-Stokes solvers, we intend to apply the Balancing Domain Decomposition (BDD) preconditioner to the solvers. The BDD preconditioner for the substructuring method was proposed by Mandel[1)] for thermal and elastic problems, and has been extended to a class of saddle point problems and the mixed formulation of liner elasticity[2)]. Implementation of the BDD to incompressible Stokes equations has been shown by Pavarino and Widlund[3)] or Goldfeld[4)].

We have constructed a parallel algorithm of the BDD in structural analysis, and succeeded in solving problems with over 100 million DOF[5,6)]. The results are released an open source software, named ADVENTURE_Solid, from the ADVENTURE project[7)]. The ADVENTURE_Solid, which solves structural problems, has functions of the BDD preconditioner and a modified BDD preconditioner for purpose of memory shortage, called BDD-DIAG[8,9)]. With functions of BDD preconditioners in ADVENTURE_Solid and minor modifications, we construct a BDD preconditioner for incompressible viscous flow problems.

We begin to examine convergence of our method by solving Stokes problems. Next, as a challenging, using the similar method to Stokes problems, we attempt to implement BDD to stationary Navier-Stokes problems, which are linearized by the Newton method. Then, stabilized finite element method (see Franca and Fray[10)] or Franca et al.[11)] etc) is introduced for each step of the Newton method[12)]. Our BDD algorithm is applied to the above linear system. Numerical results of cavity flow problems are demonstrated for Stokes problems to show the effectiveness of the algorithm. Though applications to Navier-Stokes problems are also shown, its effectiveness remains a future problem.

This paper is organized as follows. In section 2, the formulations of the Stokes problem and the Navier-Stokes problem are described. To discretize these problems, we describe the stabilized finite element method in section 3. The substructuring (or Domain Decomposition Method (DDM)) process is explained in section 4. In section 5, the BDD preconditioner and the BDD-DIAG preconditioner are introduced. In section 6, numerical results are reported, and, finally, in section 7, we state conclusions of this paper.

2. FORMULATIONS

Let Ω be a three-dimensional polygonal domain with the boundary $\partial\Omega$. We consider the stationary incompressible Navier-Stokes equations as follows:

$$\begin{cases} -\frac{1}{\rho}\nabla\cdot\sigma(u,p)+(u\cdot\nabla)u=\frac{1}{\rho}f & \text{in } \Omega, \quad (1a)\\ \nabla\cdot u = 0 & \text{in } \Omega, \quad (1b)\\ u = g & \text{on } \partial\Omega, \quad (1c)\end{cases}$$

where $u=(u_1,u_2,u_3)$ is the velocity [m/s], p is the pressure [N/m^2], ρ is the density [kg/m^3], $f=(f_1,f_2,f_3)$ is the body force [N/m^3], $g=(g_1,g_2,g_3)$ is the boundary velocity [m/s], and $\sigma(u,p)$ is the stress tensor [N/m^2] defined by

$$\sigma_{ij}(u,p)\equiv -p\delta_{ij}+2\mu D_{ij}(u),\quad D_{ij}\equiv\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right),\quad i,j=1,2,3,$$

with the Kronecker delta δ_{ij} and the viscosity μ [kg/ms].

Some preliminaries are arranged for the derivation of a variational formulation of (1). Let X denote $H^1(\Omega)^3$. An affine space and function spaces are defined as follows:

$$V(g)\equiv\{v\in X;\quad v=g \ \text{ on } \partial\Omega\},\quad V\equiv V(0),\quad Q\equiv\left\{q\in L^2(\Omega);\int_\Omega q\ dx=0\right\},$$

where $L^2(\Omega)$ denotes the space of square summable functions in $\Omega, H^1(\Omega)$ is the space of functions in $L^2(\Omega)$ with derivatives up to the first order. Let a_0 be a continuous bilinear form on $X\times X$, b a continuous bilinear form on $X\times Q$, a_1 a continuous trilinear form on $X\times X\times X$, respectively defined by

$$a_0(u,v)\equiv\int_\Omega\frac{2\mu}{\rho}D(u):D(v)\,dx,\quad b(v,q)\equiv-\int_\Omega\frac{1}{\rho}q\nabla\cdot v\,dx,$$

$$a_1(w,u,v)\equiv\frac{1}{2}\left\{\int_\Omega[(w\cdot\nabla)u]\,v\,dx-\int_\Omega[(w\cdot\nabla)v]\,u\,dx\right\}.$$

Here, the notation ":" denotes the tensor product.

The variational formulation of the Navier-Stokes problem (1) is described as follows:
Find $(u,p)\in V(g)\times Q$ such that

$$\begin{cases} a_0(u,v)+a_1(u,u,v)+b(v,p)=(\frac{1}{\rho}f,v) & \text{for } v\in V, \quad (2a)\\ b(u,q)=0 & \text{for } q\in Q. \quad (2b)\end{cases}$$

Here, $(\cdot,\cdot)$ denotes the L^2-inner product over Ω.

The variational formulation of the Stokes problem, for which the convective term is removed from the Navier-Stokes problem, is described as follows:
Find $(u,p)\in V(g)\times Q$ such that

$$\begin{cases} a_0(u,v)+b(v,p)=(\frac{1}{\rho}f,v) & \text{for } v\in V, \quad (3a)\\ b(u,q)=0 & \text{for } q\in Q. \quad (3b)\end{cases}$$

In order to linearize (2), we apply the Newton method. The (k+1)th step linearized problems of (2) becomes the following:

Find $(u^{k+1}, p^{k+1}) \in V(g) \times Q$ such that

$$\begin{cases} a_0(u^{k+1}, v) + a_1(u^k, u^{k+1}, v) + a_1(u^{k+1}, u^k, v) + b(v, p^{k+1}) \\ \qquad\qquad = (\frac{1}{\rho} f, v) + a_1(u^k, u^k, v) \quad \text{for } v \in V, & \text{(4a)} \\ b(u^{k+1}, q) \qquad = 0 \qquad\qquad \text{for } q \in Q. & \text{(4b)} \end{cases}$$

3. STABILIZED METHODS

Let us consider approximations of above formulations. We confine ourselves to the tetrahedral division of a polyhedral domain Ω, and to a continuous boundary velocity. Let T_h be a regular tetrahedral division family, where h stands for the maximum diameter of tetrahedral elements, and $P_m(K)$ be the usual polynomial spaces on $K \in T_h$, for each integer $m \geq 0$.

The finite element spaces X_h and Q_h for the approximations of X and Q, respectively, are then defined as

$$X_h \equiv \left\{ v_h \in X \cap C(\overline{\Omega})^3; \quad v_h|_K \in P_1(K)^3 \quad \forall K \in T_h \right\},$$
$$Q_h \equiv \left\{ v_h \in Q \cap C(\overline{\Omega}); \quad q_h|_K \in P_1(K) \quad \forall K \in T_h \right\},$$

where $C(\overline{\Omega})$ denotes the space of continuous functions in the closure of the domain Ω.
Let $V_h(g)$ denote the approximation of $V(g)$. Namely we set

$$V_h(g) \equiv \left\{ v_h \in X_h; \quad v_h(\mathrm{P}) = g(\mathrm{P}) \quad \text{at} \quad \mathrm{P} \in \partial\Omega \right\},$$

where P is any nodal point on $\partial\Omega$. Set $V_h \equiv V_h(0)$.
As in Franca et al.[10), 11)], with stabilized terms, an approximate problem of (3) is introduced as follows:
Find $(u_h, p_h) \in V_h(g) \times Q_h$ such that, for $(v_h, q_h) \in V_h \times Q_h$,

$$\begin{aligned} a_0(u_h, v_h) + b(v_h, p_h) + b(u_h, q_h) + \sum_{K \in T_h} \tau_K \left(\frac{1}{\rho} \nabla p_h \,,\, -\nabla \frac{1}{\rho} q_h \right)_K \\ = \left(\frac{1}{\rho} f, v_h\right) + \sum_{K \in T_h} \tau_K \left(\frac{1}{\rho} f \,,\, -\nabla \frac{1}{\rho} q_h \right)_K. \end{aligned} \tag{5}$$

Similarly an approximate problem of (4), for which u, w, p and $\tilde{f}$ respectively denote u^{k+1}, u^k, p^{k+1} and $(1/\rho) f + (u^k \cdot \nabla) u^k$, is introduced as follows:
Find $(u_h, p_h) \in V_h(g) \times Q_h$ such that, for $(v_h, q_h) \in V_h \times Q_h$,

$$\begin{aligned} & a_0(u_h, v_h) + a_1(w_h, u_h, v_h) + a_1(u_h, w_h, v_h) + b(v_h, p_h) + b(u_h, q_h) \\ & + \sum_{K \in T_h} \left\{ \begin{aligned} & \tau_K \left((w_h \cdot \nabla) u_h + (u_h \cdot \nabla) w_h + \frac{1}{\rho} \nabla p_h \,,\, (w_h \cdot \nabla) v_h + (v_h \cdot \nabla) w_h - \frac{1}{\rho} \nabla q_h \right)_K \\ & + \delta_K (\nabla \cdot u_h \,,\, \nabla \cdot v_h)_K \end{aligned} \right\} \\ & = (\tilde{f}, v_h) + \sum_{K \in T_h} \tau_K \left(\tilde{f} \,,\, (w_h \cdot \nabla) v_h + (v_h \cdot \nabla) w_h - \frac{1}{\rho} \nabla q_h \right)_K \end{aligned} \tag{6}$$

where $(\cdot,\cdot)_K$ denotes the L^2-innner product over K. The stabilized parameters τ_K and δ_K are defined by

$$\tau_K \equiv \begin{cases} \dfrac{\rho h_k^2}{24\mu}, & \text{in (5)}, \\ \min\left\{ \dfrac{h_K}{2\|w\|_\infty} \,,\, \dfrac{\rho h_k^2}{24\mu} \right\}, & \text{in (6)}, \end{cases} \qquad \delta_K \equiv \min\left\{ \frac{\lambda \rho h_K^2 \|w\|_\infty^2}{12\mu} \,,\, \lambda h_K \|w\|_\infty \right\},$$

where λ denotes a positive constant, $\|w\|_\infty$ denotes the maximum norm of w in K, and h_K denotes the diameter of K.

4. SUBSTRUCTURING

The domain Ω is decomposed into open, nonoverlapping subdomains $\Omega^{(i)}$. The interface Γ is defined as

$$\Gamma \equiv \bigcup_{i=1}^{N} \Gamma^{(i)}, \quad \Gamma^{(i)} \equiv \partial\Omega^{(i)} \setminus \partial\Omega \ , \quad N \text{ stands for the number of subdomains}.$$

Then we can write $\underline{\mathbf{u}}$: the vector of unknowns as

$$\underline{\mathbf{u}} = \begin{bmatrix} \underline{\mathbf{u}}_I \\ \hline \underline{\mathbf{u}}_\Gamma \end{bmatrix} = \begin{bmatrix} \mathbf{u}_I \\ \mathbf{p}_I \\ \hline \mathbf{u}_\Gamma \\ \mathbf{p}_\Gamma \end{bmatrix} \quad \begin{matrix} \text{interior velocities} \\ \text{interior pressures} \\ \text{interface velocities} \\ \text{interface pressures} \end{matrix}$$

and using the same permutation, $\mathbf{K}$: the discrete matrix of (5) or (6) and $\mathbf{f}$: the vector of the right hand side, can be written as

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{I\Gamma} \\ \mathbf{K}_{\Gamma I} & \mathbf{K}_{\Gamma\Gamma} \end{bmatrix} \ , \quad \mathbf{f} = \begin{bmatrix} \mathbf{f}_I \\ \mathbf{f}_\Gamma \end{bmatrix}.$$

Next, let $\mathbf{N}^{(i)}$ denote the 0-1 matrix that maps vectors corresponding to $\Omega^{(i)}$ into global vectors corresponding to the domain Ω. We can decompose as follows:

$$\mathbf{N}^{(i)} = \begin{bmatrix} \mathbf{N}_I^{(i)} & 0 \\ 0 & \mathbf{N}_\Gamma^{(i)} \end{bmatrix}, \qquad \mathbf{K} = \sum_{i=1}^{N} \mathbf{N}^{(i)} \mathbf{K}^{(i)} \mathbf{N}^{(i)T} = \sum_{i=1}^{N} \mathbf{N}^{(i)} \begin{bmatrix} \mathbf{K}_{II}^{(i)} & \mathbf{K}_{I\Gamma}^{(i)} \\ \mathbf{K}_{\Gamma I}^{(i)} & \mathbf{K}_{\Gamma\Gamma}^{(i)} \end{bmatrix} \mathbf{N}^{(i)T},$$

$$\underline{\mathbf{u}}^{(i)} = \begin{bmatrix} \underline{\mathbf{u}}_I^{(i)} \\ \underline{\mathbf{u}}_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{N}_I^{(i)} & 0 \\ 0 & \mathbf{N}_\Gamma^{(i)} \end{bmatrix}^T \begin{bmatrix} \underline{\mathbf{u}}_I \\ \underline{\mathbf{u}}_\Gamma \end{bmatrix} = \mathbf{N}^{(i)T} \underline{\mathbf{u}}, \quad \mathbf{f}^{(i)} = \begin{bmatrix} \mathbf{f}_I^{(i)} \\ \mathbf{f}_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{N}_I^{(i)} & 0 \\ 0 & \mathbf{N}_\Gamma^{(i)} \end{bmatrix}^T \begin{bmatrix} \mathbf{f}_I \\ \mathbf{f}_\Gamma \end{bmatrix} = \mathbf{N}^{(i)T} \mathbf{f},$$

where $\mathbf{K}^{(i)}$ is the local discrete matrix of (5) or (6) corresponding to $\Omega^{(i)}$, and $\underline{\mathbf{u}}^{(i)}$ is the vector of DOF corresponding to all tetrahedral elements in $\overline{\Omega}^{(i)}$. After eliminating the interior unknowns $\underline{\mathbf{u}}_I$, the system $\mathbf{K}\underline{\mathbf{u}} = \mathbf{f}$ becomes

$$\mathbf{S}\underline{\mathbf{u}}_\Gamma = \mathbf{g}, \tag{7a}$$

where

$$\mathbf{S} = \mathbf{K}_{\Gamma\Gamma} - \mathbf{K}_{\Gamma I}\mathbf{K}_{II}^{-1}\mathbf{K}_{I\Gamma}, \quad \mathbf{g} = \mathbf{f}_\Gamma - \mathbf{K}_{\Gamma I}\mathbf{K}_{II}^{-1}\mathbf{f}_I.$$

$\mathbf{S}$ is usually referred to as the Schur complement which is the assembly of the local Schur complements $\mathbf{S}^{(i)}$ associated with $\Omega^{(i)}$ as follows:

$$\mathbf{S} = \sum_{i=1}^{N} \mathbf{N}_\Gamma^{(i)} \mathbf{S}^{(i)} \mathbf{N}_\Gamma^{(i)T}, \qquad \mathbf{S}^{(i)} = \mathbf{K}_{\Gamma\Gamma}^{(i)} - \mathbf{K}_{\Gamma I}^{(i)} \mathbf{K}_{II}^{(i)-1} \mathbf{K}_{I\Gamma}^{(i)}.$$

After solving the reduced system (7a) , then we obtain unknowns $\underline{\mathbf{u}}_I^{(i)}$ as follows

$$\underline{\mathbf{u}}_I^{(i)} = \mathbf{K}_{II}^{(i)-1}(\mathbf{f}_I^{(i)} - \mathbf{K}_{I\Gamma}^{(i)} \underline{\mathbf{u}}_\Gamma^{(i)}), \quad i = 1, \dots, N. \tag{7b}$$

In order to solve the system (7a), the preconditioned Conjugate Gradient (CG) method is employed for the Stokes problem, or the preconditioned GPBiCG[13] is employed for the Navier-Stokes problem. On the other hand, the system (7b) is solved by the skyline Gaussian elimination. In the next section, we describe the algorithm of the BDD.

5. BDD PRECONDITIONERS

BDD is proposed by Mandel[1] for thermal and elastic problems. This method is formulated as the extension and generalization of the Neumann-Neumann preconditoner[14], which is an earlier technique for preconditioners.

By considering a coarse space, the BDD compensates a weak point of the Neumann-Neumann preconditioner whose convergence is worse. In the first part of this section, we consider only the case when $\mathbf{S}$ is positive definite.

The BDD preconditioner for symmetric positive definite systems is described as follows[1]:

$$\mathbf{M}_{\mathrm{BDD}}^{-1} = \mathbf{Q}_H + (\mathbf{I} - \mathbf{Q}_H\mathbf{S})\mathbf{T}_{\mathrm{NN}}(\mathbf{I} - \mathbf{S}\mathbf{Q}_H), \tag{8}$$

where $\mathbf{T}_{\mathrm{NN}}$ denotes the Neumann-Neumann preconditoner, and $\mathbf{Q}_H$ is the coarse operator which satisfies $\mathbf{Q}_H\mathbf{S} = \mathbf{P}$; $\mathbf{P}$ is the $\mathbf{S}$-orthogonal projection onto the coarse space.
The coarse operator $\mathbf{Q}_H$ is defined as follows:

$$\mathbf{Q}_H = \mathbf{R}_W \mathbf{S}_W^{-1} \mathbf{R}_W^T,$$

where $\mathbf{S}_W$ is called the coarse matrix defined as

$$\mathbf{S}_W = \mathbf{R}_W^T \mathbf{S}\, \mathbf{R}_W,$$

and the coarse space W is defined and the matrix $\mathbf{R}_W$ is described as

$$W = \mathrm{span}\left\{ \mathbf{N}_\Gamma^{(i)}\mathbf{D}^{(i)}u^{(i)} \mid i = 1, \cdots, N \right\}, \;\; u^{(i)} \in \mathrm{Range}\,\mathbf{Z}^{(i)},$$

$$\mathbf{R}_W = \left[\mathbf{N}_\Gamma^{(1)}\mathbf{D}^{(1)}\mathbf{Z}^{(1)}, \ldots, \mathbf{N}_\Gamma^{(N)}\mathbf{D}^{(N)}\mathbf{Z}^{(N)} \right].$$

In the above, $\mathbf{Z}^{(i)}$ is a key matrix to construct the coarse space, and $\mathbf{D}^{(i)}$ is the weight matrix corresponding to a subdmain $\Omega^{(i)}$, which satisfies the following:

$$\sum_{i=1}^{N} \mathbf{N}_\Gamma^{(i)}\mathbf{D}^{(i)}\mathbf{N}_\Gamma^{(i)T} = I \;, \; I \text{ is the unit matrix.}$$

The definition of $\mathbf{D}^{(i)}$ has some patterns[4], and here, for simplicity, our choice of $\mathbf{D}^{(i)}$ is a diagonal matrix with diagonal elements being the reciprocal of the number of subdomains with which the degree of freedom is associated. In the BDD algorithm, the matrices $\mathbf{Z}^{(i)}$ satisfy

$$\mathrm{Null}\,\mathbf{S}^{(i)} \subset \mathrm{Range}\,\mathbf{Z}^{(i)}, \quad i = 1, \ldots, N. \tag{9}$$

In the structural analysis, as one of the choices of $\mathbf{Z}^{(i)}$, it is constructed from rigid body motions at any nodal point on the subdomain interface $\Gamma^{(i)}$,

$$\mathbf{Z}^{(i)} = \sum_{\mathrm{P}} \mathbf{B}_{\mathrm{P}}^{(i)} \overline{\mathbf{Z}}_{\mathrm{P}}^{(i)}, \quad \mathbf{B}_{\mathrm{P}}^{(i)} : \mathrm{P} \to \Gamma^{(i)}, \quad \mathrm{P} : \text{any nodal point on } \Gamma^{(i)},$$

and

$$\overline{\mathbf{Z}}_{\mathrm{P}}^{(i)} = \begin{bmatrix} 1 & 0 & 0 & 0 & x_3 & -x_2 \\ 0 & 1 & 0 & -x_3 & 0 & x_1 \\ 0 & 0 & 1 & x_2 & -x_1 & 0 \end{bmatrix}, \tag{10}$$

where (x_1, x_2, x_3) is the coordinate of the nodal point P, and $\mathbf{B}_{\mathrm{P}}^{(i)}$ is a 0-1 matrix that maps P to $\Gamma^{(i)}$.

The discrete systems of (5) and (6) employ the stabilized method as described previously. In this study, we again make use of the matrix (10) for the stationary incompressible viscous flow problems. Then, the matrix $\overline{\mathbf{Z}}_{\mathrm{P}}^{(i)}$ is used to construct a new $\mathbf{Z}_{\mathrm{P}}^{(i)}$ as follows:

$$\mathbf{Z}_{\mathrm{P}}^{(i)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & x_3 & -x_2 \\ 0 & 1 & 0 & 0 & -x_3 & 0 & x_1 \\ 0 & 0 & 1 & 0 & x_2 & -x_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{11}$$

The matrix (11) is similarly used to construct $\mathbf{Z}^{(i)}$ for the flow problems.

Next, we express the Neumann-Neumann preconditioner $\mathbf{T}_{\mathrm{NN}}$ as follows:

$$\mathbf{T}_{\mathrm{NN}} = \sum_{i=1}^{N} \mathbf{N}_{\Gamma}^{(i)} \mathbf{D}^{(i)} \mathbf{S}^{(i)\dagger} \mathbf{D}^{(i)T} \mathbf{N}_{\Gamma}^{(i)T} , \tag{12}$$

where $\mathbf{S}^{(i)\dagger}$ is the pseudo-inverse of $\mathbf{S}^{(i)}$, for which the inverse of regularized $\mathbf{S}^{(i)}$ is used in this paper as follows:

$$\mathbf{S}^{(i)\dagger} \fallingdotseq [\, \mathbf{S}^{(i)} + \alpha_v I_v + \alpha_p I_p \,]^{-1}, \;\; \alpha_v, \alpha_p > 0 : \text{const.}$$

In the above, I_v or I_p is a 0-1 diagonal matrix whose elements corresponding to velocity or pressure, respectively. α_v and α_p are given such that :

$\alpha_v = 10^{-n_v} \times \max$ (absolute values of diagonal components in $\mathbf{K}_{\Gamma\Gamma}^{(i)}$ corresponding to velocity),

$\alpha_p = 10^{-n_p} \times \max$ (absolute values of diagonal components in $\mathbf{K}_{\Gamma\Gamma}^{(i)}$ corresponding to perssure), n_v, n_p : arbitary numbers.

Finally we also consider the BDD-DIAG preconditioner[8], which employs the diagonal scaling preconditioner $\mathbf{T}_{\mathrm{DIAG}}$ instead of the Neumann-Neumann preconditioner $\mathbf{T}_{\mathrm{NN}}$. $\mathbf{T}_{\mathrm{DIAG}}$ is an easy preconditioner such that:

$$\mathbf{T}_{\mathrm{DIAG}} = \sum_{i=1}^{N} \mathbf{N}_{\Gamma}^{(i)} (\mathrm{diag}\mathbf{K}_{\Gamma\Gamma}^{(i)})^{-1} \mathbf{N}_{\Gamma}^{(i)T}, \tag{13}$$

where $\mathrm{diag}\mathbf{A}$ is a diagonal matrix whose components are diagonal components of the matrix $\mathbf{A}$.
Then, BDD-DIAG is described as follows:

$$\mathbf{M}_{\text{BDD-DIAG}}^{-1} = \mathbf{Q}_H + (\mathbf{I} - \mathbf{Q}_H \mathbf{S}) \mathbf{T}_{\mathrm{DIAG}} (\mathbf{I} - \mathbf{S}\mathbf{Q}_H). \tag{14}$$

6. NUMERICAL RESULTS

For implementation of BDD, we make use of parallel BDD functions of ADVENTURE_Solid[7], which is the structural analysis module of the ADVENTURE project[7], using the $\mathbf{Z}_{\mathrm{P}}^{(i)}$ matrix.

For the cavity flow problem, results of numerical experiments are reported to test the convergence of BDD preconditioners to the Stokes problem. We compare BDD and BDD-DIAG preconditioners with the diagonal scaling preconditioner $\mathbf{T}_{\mathrm{DIAG}}$ in (13), denoted as DIAG, the Neumann-Neumann preconditioner $\mathbf{T}_{\mathrm{NN}}$ in (12), denoted as NN, and non-preconditioned DDM, denoted as DDM.

The cavity flow model is as follows (see Fig.1). The domain is $1.0[\mathrm{m}] \times 1.0[\mathrm{m}] \times 1.0[\mathrm{m}]$ unit cube with the Dirichlet boundary condition such that uniform flows along the x_1-axis are imposed on the upper-boundary, and on the other boundaries, all velocity components are set to be zeroes. In addition, the pressure in the center of the cube is set to be zero. The body force is not loaded. The Reynolds number is set to be 1,000.

The cube is decomposed into tetrahedra. The number of DOF in the domain Ω is 1,000,188, and on the interface Γ is 561,432, respectively. The number of subdomains is 3,900. Fig. 2 shows the tetrahedral mesh and domain decomposition of the model. Parallel computation was performed on Pentium 4 (3.2GHz) with 6 processors. We set $n_v = n_p = 2$.

Fig. 3 shows the convergence histories of the CG method, and Table 1 shows details of comparison. We judge convergence using the maximum norm of the residual at the *i*-th iteration $\|\mathbf{g}^{(i)}\|_\infty$ as follows:

$$\|\mathbf{g}^{(i)}\|_\infty \,/\, \|\mathbf{g}^{(0)}\|_\infty \;\le\; 1.0 \times 10^{-6}.$$

From these results, BDD and BDD-DIAG show improvement of convergence with respect to iteration counts and total computational time compared with DIAG that is used as a general preconditioner, and comparison between the Neumann-Neumann preconditioner (NN) and them shows that choice of the coarse space is suitable. While BDD requires more amount of memory than DIAG, BDD-DIAG does not so much.

Next, several mesh sizes for the cavity flow problem shown in Fig.1, were employed to investigate dependence of convergence on the number of subdomains. The details of these models are shown in Table 2. All models have the same Reynolds number 1,000, and DOF in any subdomain is unified with near 250. Table 3 shows numbers of iterations for DIAG, BDD and BDD-DIAG to require. BDD and BDD-DIAG are less sensitive to the number of subdmains than DIAG.

Finally, we attempt to implement BDD to the stationary Navier-Stokes problem. We apply the similar preconditioners, which make use of BDD (8) and BDD-DIAG (14), to non-symmetric linearized equations (6). We set in this case, $n_v = 2$, $n_p = 5$. Here, modification should be done for memory locations and solvers. By this simple modification, we succeed in analyzing the cavity flow model with 0.2 million DOF. Fig.4 gives the convergence histories of the Newton method by BDD and BDD-DIAG. Fig.5 shows the convergence of GPBiCG method at the second Newton step by DIAG, BDD and BDD-DIAG. Consequently, our BDD algorithm shows effective convergence for Navier-Stokes problems, while the size of application problems is up to 0.2 million DOF. The BDD algorithm should be given special consideration to non-symmetric equations.

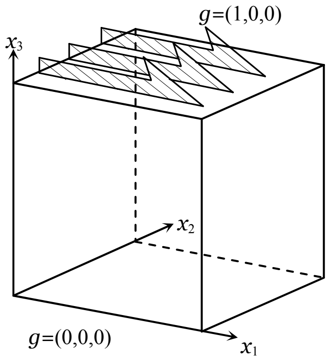

Fig.1 The cavity flow problem.

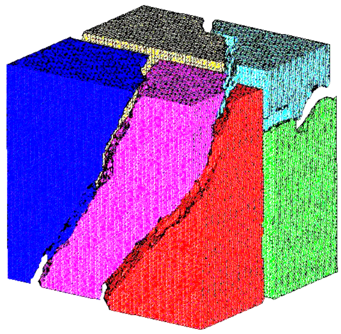
Fig.2 A 1 million DOF model.

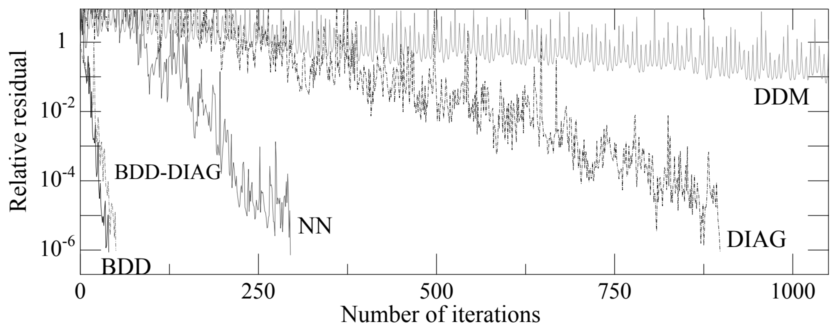

Fig.3 Histories of residual norms for the 1 million DOF model.

Table 1 Comparison of preconditioners for the 1 million DOF model.

Items \ Preconditioners	DDM	DIAG	NN	BDD	BDD-DIAG
Num. of iterations	5,855	899	296	41	51
Time for pre. * [sec]	0	0	20	510	487
Time for iter. ** [sec]	5,304	822	499	133	126
Total time [sec]	5,304	822	519	643	613
Used memory [Mbyte]	3,318	3,323	5,748	6,689	4,344

* Computational time for preconditioning
** Computational time for the iterative procedure of CG method

Table 2 Details of each model.

Model	DOF in Ω	DOF on Γ	Subdomains*
model 10	97,556	49,010	390
model 20	202,612	106,181	780
model 50	530,612	293,956	2,160
model 100	1,000,118	558,850	3,840
model 150	1,550,068	869,160	5,700

* Number of subdomains

Table 3 Dependency on numbers of subdomains.

Model	DIAG	BDD	BDD-DIAG
model 10	434	39	51
model 20	635	40	49
model 50	976	40	49
model 100	932	46	52
model 150	1,112	46	48

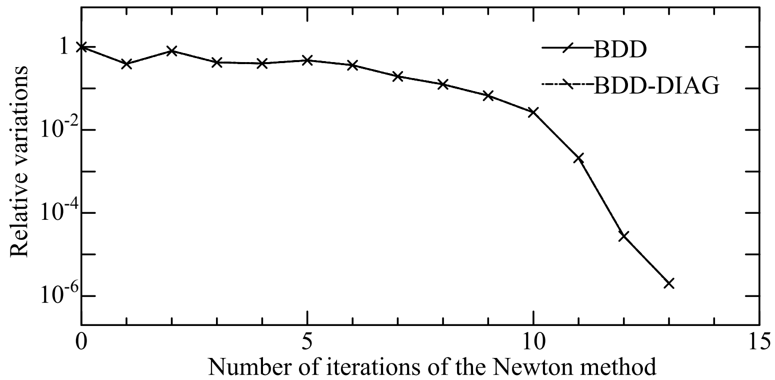

Fig.4 History of the Newton method for the 0.2 million DOF model.

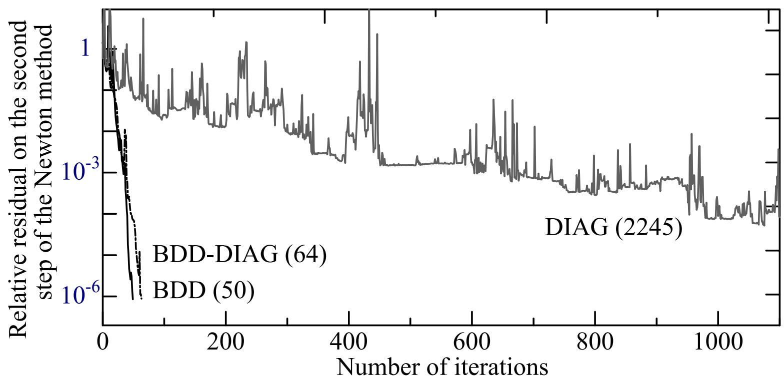

Fig.5 Convergence of the GPBiCG on the second Newton step for the 0.2 million DOF model.

7. CONCLUSIONS

We have considered the stationary incompressible Stokes and Navier-Stokes problems, and have introduced a domain decomposition method with the stabilization technique to compute the resultant linear problem. In this paper we have implemented the BDD algorithm in ADVENTURE_Solid to incompressible viscous flow problems with a suitable modification. For Stokes problems, numerical results of the cavity flow model show improvement of convergence, while for Navier-Stokes problems, we have got results up to 0.2 million DOF. The future work is to form non-symmetric BDD algorithms, and to challenge higher DOF for Navier-Stokes problems.

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