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FUKUMA, Yoshiaki
Department of Mathematics and Physics, Faculty of Science and Technology, Kochi University

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ON THE DIMENSION OF THE GLOBAL SECTIONS OF THE ADJOINT BUNDLE FOR POLARIZED 5-FOLDS

Yoshiaki FUKUMA

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Abstract. Let (X, L) denote a polarized manifold of dimension five. This study considers the dimension of the global sections of $K_X + mL$ with $m \geq 6$. In particular, we prove that $h^0(K_X + mL) \geq \binom{m-1}{5}$ for any polarized 5-fold (X, L) with $h^0(L) > 0$. Furthermore, we also consider (X, L) with $h^0(K_X + mL) = \binom{m-1}{5}$ for some $m \geq 6$ with $h^0(L) > 0$.

1. Introduction

Let X be a smooth projective complex variety of dimension n, and let L be an ample line bundle on X. Then, (X, L) is called a *polarized manifold*. The adjoint bundle $K_X + mL$ of (X, L) plays a key role in investigating (X, L) (for example, see [**2**, Chapters 7, 9 and 11]), where K_X and m denote the canonical line bundle of X and a natural number, respectively. For example, the nefness of $K_X + mL$ has been studied by numerous authors, and as a corollary, the non-negativity of the sectional genus, $g(X, L)$, of (X, L) was obtained. In addition, we also note that numerous authors studied the base point freeness and very ampleness of adjoint bundles related to a conjecture of Fujita [**6, 19, 22, 23**].

Recently, the positivity of dimension $h^0(K_X + mL)$ has been discussed. For $m = n - 1$, Beltrametti and Sommese proposed the following conjecture [**2**, Conjecture 7.2.7].

CONJECTURE 1. (Beltrametti–Sommese) *Let* (X, L) *be a polarized manifold with* $\dim X = n \geq 3$. *Assume that* $K_X + (n-1)L$ *is nef. Then* $h^0(K_X + (n-1)L) > 0$.

For this conjecture, the following partial results have been obtained:

- In [**11**, Theorem 2.4] and [**14**, Theorem 3.1], the author proved that this conjecture is true if $n \leq 4$. (See also [**3**] and [**4**].) Besides, we also note that Andreatta and Fontanari [**1**] improved the result in [**14**].
- In [**18**, 1.2 Theorem], Höring proved that this conjecture is true if $h^0(L) > 0$.

Moreover, we have classified (X, L) for the following types in our previously conducted studies:

- Polarized 3-fold (X, L) with $h^0(K_X + 2L) \leq 2$ [**11, 13**].
- Polarized 4-fold (X, L) with $h^0(K_X + 3L) \leq 1$ [**14, 15**].

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Additionally, we also considered the case where $m \geq n$. In [**16**, Conjecture 2], we proposed the following conjecture.

CONJECTURE 2. *Let (X, L) be an n-dimensional polarized manifold with $n \geq 3$.*

(i) *Then, $h^0(K_X + mL) \geq \binom{m-1}{n}$ holds for every integer $m \geq n+1$. If equality holds for some $m \geq n+1$, then $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.*

(ii) *If $h^0(K_X + nL) = 0$, then $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.*

(iii) *If $h^0(K_X + nL) = 1$, then (X, L) can be one of the following:*
(iii.1) *$(X, L) \cong (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$;*
(iii.2) *(X, L) is a scroll over a smooth elliptic curve with $L^n = 1$.*

It has been proved that Conjecture 2 is true only for the following cases:

- The case where $n \leq 4$ [**11**, Theorem 2.5; and **16**, Theorem 3.1]. (See also [**1**, Theorem 8] for results concerning with Conjecture 2.)
- The case where $n \geq 5$ and $\dim \mathrm{Bs}|L| \leq 1$ for Conjecture 2(i) [**16**, Theorem 3.2(i)].
- The case where $n \geq 5$ and $h^0(L) > 0$ for Conjecture 2(ii) and (iii) [**16**, Theorem 3.2(ii) and (iii)].

In this study, we consider Conjecture 2(i) for the case where $n = 5$ and $h^0(L) > 0$. Consequently, we prove that Conjecture 2(i) is true for this case.

Herein, we use the customary notation in algebraic geometry.

2. Preliminaries

Notation 2.1. Let X be a projective variety of dimension n and let L be a line bundle on X. Then, we set

$$\chi(tL) = \sum_{j=0}^{n} \chi_j(X, L)\binom{t+j-1}{j}.$$

Definition 2.1. [**9**, Definition 2.1] Let X be a projective variety of dimension n and let L be a line bundle on X. For every integer i with $0 \leq i \leq n$, the *ith sectional geometric genus $g_i(X, L)$ of (X, L)* is defined as follows:

$$g_i(X, L) = (-1)^i(\chi_{n-i}(X, L) - \chi(\mathcal{O}_X)) + \sum_{j=0}^{n-i}(-1)^{n-i-j}h^{n-j}(\mathcal{O}_X).$$

Remark 2.1.

(i) Since $\chi_{n-i}(X, L) \in \mathbb{Z}$, we observe that $g_i(X, L)$ is integer by definition.

(ii) If $i = \dim X = n$, then $g_n(X, L) = h^n(\mathcal{O}_X)$.

(iii) If $i = 0$, then $g_0(X, L) = L^n$.

(iv) If $i = 1$, then $g_1(X, L) = g(X, L)$, where $g(X, L)$ denotes the sectional genus of (X, L). If X is smooth, then the sectional genus $g(X, L)$ can be given by

$$g(X, L) = 1 + \tfrac{1}{2}(K_X + (n-1)L)L^{n-1}.$$

(v) If $i = 2$, then we obtain that (see [**10**, (2.2.A)])

$$g_2(X, L) = \frac{1}{12}(K_X + (n-1)L)(K_X + (n-2)L)L^{n-2} + \frac{1}{12}c_2(X)L^{n-2} + \frac{n-3}{24}(2K_X + (n-2)L)L^{n-1} - 1 + h^1(\mathcal{O}_X).$$

(vi) If $i = 3$, then we have (see [**10**, (2.2.B)])

$$g_3(X, L) = \frac{(n-2)(n-3)^2}{48}L^n + \frac{(n-3)(3n-8)}{48}K_X L^{n-1} + \frac{n-3}{24}(K_X^2 + c_2(X))L^{n-2} + \frac{1}{24}K_X c_2(X)L^{n-3} + 1 - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X).$$

THEOREM 2.1. *Let (X, L) be a polarized manifold with* $\dim X = n$, *and let i be an integer with* $0 \le i \le n-1$. *Then*

$$g_i(X, L) = \sum_{j=0}^{n-i-1}(-1)^j\binom{n-i}{j}h^0(K_X + (n-i-j)L) + \sum_{k=0}^{n-i}(-1)^{n-i-k}h^{n-k}(\mathcal{O}_X).$$

Proof. See [**9**, Theorem 2.3]. □

Definition 2.2. [**12**, Definitions 3.1 and 3.2] Let (X, L) be a polarized manifold of dimension n.

(i) Let t be a positive integer. Then set

$$F_0(t) := h^0(K_X + tL),$$

$$F_i(t) := F_{i-1}(t+1) - F_{i-1}(t) \quad \text{for every integer } i \text{ satisfying } 1 \le i \le n.$$

(ii) For every integer i satisfying $0 \le i \le n$, the *ith Hilbert coefficient, $A_i(X, L)$, of* (X, L) is defined by $A_i(X, L) = F_{n-i}(1)$.

Remark 2.2.

(i) If $1 \le i \le n$, then $A_i(X, L)$ can be defined as follows (see [**12**, Proposition 3.2]):

$$A_i(X, L) = g_i(X, L) + g_{i-1}(X, L) - h^{i-1}(\mathcal{O}_X).$$

(ii) By employing Definition 2.2 and [**12**, Proposition 3.1(2)], we obtain the following:
 (ii.1) $A_i(X, L) \in \mathbb{Z}$ for every integer i satisfying $0 \le i \le n$;
 (ii.2) $A_0(X, L) = L^n$;
 (ii.3) $A_1(X, L) = g(X, L) + L^n - 1 \ge 0$ (see Remark 2.1(iii) and (iv));
 (ii.4) $A_n(X, L) = h^0(K_X + L)$.

(iii) By applying Remark 2.1(v) and (vi) and Remark 2.2(i), we observe that $A_2(X, L)$ and $A_3(X, L)$ are respectively given by

$$A_2(X, L) = \frac{(3n-2)(n+1)}{24}L^n + \frac{n}{4}K_X L^{n-1} + \frac{1}{12}(K_X^2 + c_2(X))L^{n-2},$$

$$A_3(X, L) = \frac{(n-2)(n^2-1)}{48}L^n + \frac{n(3n-5)}{48}K_X L^{n-1} + \frac{n-1}{24}K_X^2 L^{n-2} + \frac{1}{24}c_2(X)(K_X + (n-1)L)L^{n-3}.$$

THEOREM 2.2. *Let (X, L) be a polarized manifold of dimension n and let t be a positive integer. Then, for every integer i satisfying $0 \le i \le n$, we have*

$$F_{n-i}(t) = \sum_{j=0}^{i} \binom{t-1}{i-j} A_j(X, L).$$

Proof. See [**12**, Theorem 3.1]. □

COROLLARY 2.1. [**12**, Corollary 3.1] *Let (X, L) be a polarized manifold of dimension n, and let t be a positive integer. Then, we have*

$$h^0(K_X + tL) = \sum_{j=0}^{n} \binom{t-1}{n-j} A_j(X, L).$$

THEOREM 2.3. *Let X be a projective manifold. Then there exist smooth projective varieties X' and Y, a birational morphism $\mu : X' \to X$ and a fiber space $\phi : X' \to Y$ such that Y is not uniruled, and if $\dim X' > \dim Y$, then the general fiber of ϕ is rationally connected.*

Proof. See [**5**], [**17**] and [**21**]. □

Definition 2.3. The fiber space $\phi : X' \to Y$ in Theorem 2.3 is called the *maximal rationally connected fibration (MRC-fibration) of X*, while Y is called the *base of the MRC-fibration*.

PROPOSITION 2.1. *Let (X, A) be a polarized manifold of dimension n. Assume that $h^0(K_X + A) > 0$. Then $\Omega_X \langle A \rangle$ is generically nef.*

Proof. See [**16**, Claim 2.1]. □

PROPOSITION 2.2. *Let X be a normal projective variety, and let L and M be a line bundle on X such that $h^0(L) > 0$ and $h^0(M) > 0$, respectively. Then, $h^0(L + M) \ge h^0(L) + h^0(M) - 1$ holds.*

Proof. See [**20**, 15.6.2 Lemma]. □

3. Main result

THEOREM 3.1. *Let (X, L) be a polarized manifold of dimension five such that $h^0(L) > 0$. Then $h^0(K_X + mL) \ge \binom{m-1}{5}$ holds for every integer $m \ge 6$. If equality holds for some $m \ge 6$, then $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.*

Proof. Suppose that $K_X + 4L$ is not nef. Then (X, L) can be one of the following three types since $n = 5$ (see [**7**, Theorems (11.2) and (11.7)] and [**2**, Theorems 7.2.1, 7.2.3, 7.2.4 and Proposition 7.2.2]):

(a) $(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(1))$;
(b) $(\mathbb{Q}^5, \mathcal{O}_{\mathbb{Q}^5}(1))$;
(c) a scroll over a smooth projective curve.

In these cases, we observe from [**16**, Remark 2.4] that Theorem 3.1 holds. Therefore we can assume that $K_X + 4L$ is nef.

First, we get from Corollary 2.1 that the following equality holds:

$$h^0(K_X + mL) = \sum_{i=0}^{5} \binom{m-1}{5-j} A_j(X, L). \tag{1}$$

Since $A_5(X, L) = h^0(K_X + L)$, we have

$$A_5(X, L) \geq 0. \tag{2}$$

CLAIM 3.1. $A_4(X, L) \geq 0$.

Proof. We note that $A_4(X, L) = h^0(K_X + 2L) - h^0(K_X + L)$. If $h^0(K_X + L) = 0$, then $A_4(X, L) = h^0(K_X + 2L) \geq 0$. Thus, we may assume that $h^0(K_X + L) > 0$. It follows that $h^0(K_X + 2L) \geq h^0(K_X + L) + h^0(L) - 1$ holds by utilizing Proposition 2.2 because $h^0(L) > 0$. Hence, $A_4(X, L) = h^0(K_X + 2L) - h^0(K_X + L) \geq h^0(L) - 1 \geq 0$. Hence, this completes the assertion of Claim 3.1. □

Additionally, we also note the following.

CLAIM 3.2. $A_1(X, L) \geq 1$.

Proof. We observe from Remark 2.2(ii.3) that $A_1(X, L) \geq 0$. Assume that $A_0(X, L) = 0$. Then, $g(X, L) = 0$ and $L^5 = 1$. It follows that $(X, L) \cong (\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(1))$. This is impossible because we assumed that $K_X + 4L$ is nef. □

We split the proof of Theorem 3.1 into three cases (i)–(iii) and many sub-cases.

(i) We consider the case that $h^0(K_X + 2L) = 0$.

PROPOSITION 3.1. *If* $h^0(K_X + 2L) = 0$, *then* $h^0(K_X + mL) > \binom{m-1}{5}$ *for every integer* $m \geq 6$.

Proof. By employing equation (1), inequality (2), Claim 3.1 and Remark 2.2(ii.2), we have

$$h^0(K_X + mL) > \binom{m-1}{2} A_3(X, L) + \binom{m-1}{3} A_2(X, L).$$

Thus, to prove Proposition 3.1, it suffices to show that

$$\binom{m-1}{2} A_3(X, L) + \binom{m-1}{3} A_2(X, L) \geq 0. \tag{3}$$

In this case, $h^0(K_X + L) = 0$ holds since $h^0(L) > 0$. Therefore we have $A_3(X, L) = h^0(K_X + 3L) - 2h^0(K_X + 2L) + h^0(K_X + L) = h^0(K_X + 3L) \geq 0$. Thus, $A_2(X, L) \geq 0$ implies that inequality (3) holds. Furthermore we prove that $A_2(X, L) \geq 0$ in this case.

Let $\phi : X' \to Y$ be the MRC-fibration of X, and let Y be the base of the MRC-fibration, where X' and Y are smooth projective varieties such that X' is birational to X.

(i.1) Let us consider the case that $\dim Y \geq 3$. Then, by utilizing [**18**, Step 2 on p. 741], we have $A_2(X, L) \geq 0$ (see also [**14**, p. 350]).

(i.2) We consider the case that $\dim Y = 0$. Then, $h^i(\mathcal{O}_X) = 0$, for any $i \geq 1$. Thus, $g_2(X, L) = h^0(K_X + 3L) \geq 0$. Besides, we also note that $g_1(X, L) \geq 0 = h^1(\mathcal{O}_X)$. Hence $A_2(X, L) = g_2(X, L) + g_1(X, L) - h^1(\mathcal{O}_X) \geq 0$.

(i.3) Let us consider the case that $\dim Y = 1$. Then $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_Y)$ holds because $h^1(\mathcal{O}_F) = 0$. By applying [**8**, Theorem 1.2.1], we have $g_1(X, L) \geq h^1(\mathcal{O}_Y) = h^1(\mathcal{O}_X)$. Moreover, we get $g_2(X, L) = h^0(K_X + 3L) \geq 0$ since $h^i(\mathcal{O}_X) = 0$, for any $i \geq 2$. Consequently, $A_2(X, L) = g_2(X, L) + g_1(X, L) - h^1(\mathcal{O}_X) \geq 0$.

(i.4) Let us consider the case that $\dim Y = 2$. Here we note that

$$g_1(X, L) = 1 + \tfrac{1}{2}(K_X + 4L)L^4, \tag{4}$$

$$g_2(X, L) = h^0(K_X + 3L) + h^2(\mathcal{O}_X). \tag{5}$$

By utilizing (4) and (5), we have

$$\begin{aligned} A_2(X, L) &= g_2(X, L) + g_1(X, L) - h^1(\mathcal{O}_X) \\ &= h^0(K_X + 3L) + h^2(\mathcal{O}_X) - h^1(\mathcal{O}_X) + 1 + \tfrac{1}{2}(K_X + 4L)L^4 \\ &\geq h^0(K_X + 3L) + h^2(\mathcal{O}_Y) - h^1(\mathcal{O}_X) + 1 + \tfrac{1}{2}(K_X + 4L)L^4 \\ &= h^0(K_X + 3L) + \chi(\mathcal{O}_Y) + \tfrac{1}{2}(K_X + 4L)L^4. \end{aligned}$$

Since Y is not uniruled, we have $\kappa(Y) \geq 0$ and $\chi(\mathcal{O}_Y) \geq 0$. Hence we obtain that $A_2(X, L) \geq 0$.

It follows from the above argument that $A_2(X, L) \geq 0$ and inequality (3) holds for the case where $h^0(K_X + 2L) = 0$. This completes the proof of Proposition 3.1. □

(ii) We consider the case that $h^0(K_X + 2L) > 0$ and $h^0(K_X + L) = 0$ and prove the following proposition.

PROPOSITION 3.2. *If $h^0(K_X + 2L) > 0$ and $h^0(K_X + L) = 0$, then*

$$h^0(K_X + mL) > \binom{m-1}{5} \text{ for every integer } m \geq 6.$$

Proof. First we note that

$$\begin{aligned} &\binom{m-1}{2} A_3(X, L) + \binom{m-1}{3} A_2(X, L) + \binom{m-1}{4} A_1(X, L) \\ &\quad = \frac{(m-1)(m-2)}{6}\left(3A_3(X, L) + (m-3)A_2(X, L) + \frac{(m-3)(m-4)}{4} A_1(X, L)\right). \end{aligned}$$

To prove this proposition, it suffices to show that

$$3A_3(X, L) + (m-3)A_2(X, L) + \frac{(m-3)(m-4)}{4} A_1(X, L) > 0. \tag{6}$$

Using Proposition 2.1, we observe from $h^0(K_X + 2L) > 0$ that $\Omega_X\langle 2L\rangle$ is generically nef. Additionally, we note that $K_X + 10L$ is nef by applying the adjunction theory. Hence, by employing [**18**, 2.11 Corollary] we have

$$c_2(X)H_1H_2H_3 \geq -(8K_XL + 40L^2)H_1H_2H_3 \tag{7}$$

for every nef line bundles H_1, H_2 and H_3. Therefore, we get the following by utilizing inequality (7):

$$\begin{aligned}
&3A_3(X, L) + (m-3)A_2(X, L) + \frac{(m-3)(m-4)}{4}A_1(X, L) \\
&= \frac{3m^2-8m+15}{4}L^5 + \frac{m^2+3m+7}{8}K_XL^4 + \frac{m+3}{12}K_X^2L^3 \\
&\quad + \frac{3K_X+(2m+6)L}{24}c_2(X)L^2 \\
&\geq \frac{m-9}{12}K_X^2L^3 + \frac{3m^2-7m-147}{24}K_XL^4 + \frac{18m^2-128m-150}{24}L^5 \\
&= \frac{m-9}{24}\left(2K_X^2L^3 + (3m+20)K_XL^4 + (18m+34)L^5\right) + \frac{1}{24}(33K_X+156L)L^4 \\
&= \frac{m-9}{24}\left((2K_X+(3m+16)L)(K_X+2L)L^3 + (12m+2)L^5\right) \\
&\quad + \frac{1}{24}(33K_X+156L)L^4. \qquad (8)
\end{aligned}$$

(ii.1) If $m \geq 9$, then inequality (6) holds by utilizing (8) because $K_X + 4L$ is nef.

(ii.2) Assume that $m = 6$. Then, since $h^0(L) > 0$ and [**18**, 1.2 Theorem], we have $h^0(K_X + 6L) \geq 1 = \binom{6-1}{5}$. If $h^0(K_X + 6L) = 1$, then $h^0(K_X + mL) = 1$, for $m = 4, 5, 6$, using $h^0(L) > 0$ and [**18**, 1.2 Theorem]. Thus, by applying Corollary 2.1, we have

$$\left.\begin{aligned}
&A_5(X, L) + 3A_4(X, L) + 3A_3(X, L) + A_2(X, L) = 1, \\
&A_5(X, L) + 4A_4(X, L) + 6A_3(X, L) + 4A_2(X, L) + A_1(X, L) = 1, \\
&A_5(X, L) + 5A_4(X, L) + 10A_3(X, L) + 10A_2(X, L) + 5A_1(X, L) + A_0(X, L) = 1.
\end{aligned}\right\} \qquad (9)$$

Since $A_5(X, L) = h^0(K_X + L) = 0$, we observe from (9) that $5A_4(X, L) = 3 - 3A_0(X, L) - 5A_1(X, L) \leq -5A_1(X, L) \leq -15$, which implies that $A_4(X, L) \leq -3$. However, this is impossible because $0 < h^0(K_X + 2L) = A_5(X, L) + A_4(X, L) = A_4(X, L)$. Therefore, we obtain that $h^0(K_X + 6L) > 1 = \binom{6-1}{5}$.

(ii.3) We consider the case that $m = 7$. First we prove the following claim.

CLAIM 3.3. *In this case, we may assume that* $A_1(X, L) \geq 3$.

Proof. Using Claim 3.2 we have $A_1(X, L) \geq 1$. Assume that $A_1(X, L) = 1$. Then, since $A_1(X, L) = g(X, L) + L^5 - 1$, we have $g(X, L) \leq 1$. Moreover, by the classification of (X, L) with $g(X, L) \leq 1$ (see [**7**, Theorems (12.1) and (12.3)]), we observe that $h^0(K_X + 2L) = 0$ in this case. This contradicts the assumption.

Assume that $A_1(X, L) = 2$. Then we have $2 = A_1(X, L) = g(X, L) + L^5 - 1$. Since $L^5 \geq 1$, we get $g(X, L) \leq 2$. If $g(X, L) \leq 1$, then $(K_X + 4L)L^4 \leq 0$. However, this is impossible because $h^0(K_X + 2L) > 0$ and L is ample. Therefore $g(X, L) = 2$ and $L^5 = 1$. Consequently, we observe that $(K_X + 2L)L^4 = 0$ because $g(X, L) = 1 + \frac{1}{2}(K_X + 4L)L^4$. Since $h^0(K_X + 2L) > 0$ and L is ample, we have $K_X + 2L = \mathcal{O}_X$ and $h^0(K_X + 2L) = 1$.

Thus, we obtain the following by applying Remark 2.2(ii.3):

$$A_5(X, L) = h^0(K_X + L) = 0, \tag{10}$$

$$A_4(X, L) = h^0(K_X + 2L) - h^0(K_X + L) = 1. \tag{11}$$

Since

$$0 \leq h^0(K_X + 3L) - h^0(K_X + 2L) = A_4(X, L) + A_3(X, L) = 1 + A_3(X, L),$$

we have

$$A_3(X, L) \geq -1. \tag{12}$$

Next, we calculate $A_2(X, L)$. First, we calculate $g_2(X, L)$. Here, we note that $h^i(\mathcal{O}_X) = 0$, for every integer $i \geq 1$ since $K_X = -2L$. Then, by applying Theorem 2.1, we have

$$\begin{aligned} g_2(X, L) &= h^0(K_X + 3L) - 3h^0(K_X + 2L) + 3h^0(K_X + L) - \sum_{k=0}^{3}(-1)^{3-k}h^{5-k}(\mathcal{O}_X) \\ &= h^0(K_X + 3L) - 3. \end{aligned}$$

Since $h^0(L) > 0$ and $h^0(K_X + 2L) = 1$, we have $h^0(K_X + 3L) \geq 1$. Hence, $g_2(X, L) \geq -2$. Thus, we get

$$A_2(X, L) = g_2(X, L) + g_1(X, L) - h^1(\mathcal{O}_X) \geq 0. \tag{13}$$

Since $A_1(X, L) = 2$ and $A_0(X, L) = L^5 = 1$, we observe from equation (1), equation (11), inequality (12) and inequality (13) that

$$\begin{aligned} h^0(K_X + 7L) &= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L) + 20A_2(X, L) + 15A_1(X, L) \\ &\quad + 6A_0(X, L) \\ &\geq 0 + 6 - 15 + 0 + 30 + 6 \\ &= 27 > \binom{6}{5}. \end{aligned}$$

Therefore we get the assertion of Claim 3.3. □

If $h^0(K_X + pL) \geq 7$, for some integer p that satisfies $1 \leq p \leq 6$, then $h^0(K_X + 7L) \geq h^0(K_X + 6L) + h^0(L) - 1 \geq 7 > \binom{7-1}{5}$. Thus, we may assume that

$$h^0(K_X + pL) \leq 6 \quad \text{for any integer } p \text{ with } 1 \leq p \leq 6. \tag{14}$$

Hence, we obtain the following using equation (10):

$$\begin{aligned} 6 &\geq h^0(K_X + 3L) \\ &= A_5(X, L) + 2A_4(X, L) + A_3(X, L) = 2A_4(X, L) + A_3(X, L). \end{aligned}$$

Therefore,

$$A_3(X, L) \leq 6 - 2A_4(X, L). \tag{15}$$

Now, we note that

$$0 \leq h^0(L) - 1 \leq h^0(K_X + 3L) - h^0(K_X + 2L) = A_4(X, L) + A_3(X, L). \tag{16}$$

Moreover, we get

$$\begin{aligned}1 \le h^0(K_X + 4L) &= A_5(X, L) + 3A_4(X, L) + 3A_3(X, L) + A_2(X, L)\\ &= 3A_4(X, L) + 3A_3(X, L) + A_2(X, L).\end{aligned}$$

Hence, inequality (15) gives

$$\begin{aligned}A_2(X, L) &\ge\ - 3A_4(X, L) - 3A_3(X, L) + 1\\ &\ge\ - 3A_4(X, L) + 3(2A_4(X, L) - 6) + 1\\ &= 3A_4(X, L) - 17. \qquad (17)\end{aligned}$$

We note that the following hold by employing inequality (14):

$$1 \le h^0(K_X + 2L) = A_5(X, L) + A_4(X, L) \le 6.$$

Thus, we observe from equation (10) that

$$1 \le A_4(X, L) \le 6.$$

(ii.3.1) Let us consider the case that $A_4(X, L) = 6$. Since $A_4(X, L) + A_3(X, L) \ge 0$ by applying inequality (16), we have $A_3(X, L) \ge -6$. Moreover, inequality (17) gives that $A_2(X, L) \ge 1$. Hence, using Claim 3.3, we observe that

$$\begin{aligned}h^0(K_X + 7L) &= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L)\\ &\quad + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L)\\ &\ge 36 - 90 + 20 + 45 + 6 = 17.\end{aligned}$$

Hence the assertion holds.

(ii.3.2) We consider the case that $A_4(X, L) = 5$. Since $A_4(X, L) + A_3(X, L) \ge 0$ by employing inequality (16), we have $A_3(X, L) \ge -5$. However, $A_3(X, L) \le -4$ by utilizing inequality (15). Hence $A_3(X, L) = -5$ or -4. Moreover, using inequality (17), we obtain $A_2(X, L) \ge -2$.

It follows that

$$\begin{aligned}h^0(K_X + 7L) &= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L)\\ &\quad + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L)\\ &\ge 30 - 75 + 20A_2(X, L) + 45 + 6\\ &= 6 + 20A_2(X, L).\end{aligned}$$

Hence, if $A_2(X, L) \ge 1$, then we obtain the assertion. Thus, we may assume that $A_2(X, L) \le 0$. Inequality (17) gives that $A_2(X, L) \ge -2$. Therefore, we have $A_2(X, L) =$

0, −1 or −2. Furthermore

$$
\begin{aligned}
1 &\le h^0(K_X+5L)\\
&= A_5(X,L)+4A_4(X,L)+6A_3(X,L)+4A_2(X,L)+A_1(X,L)\\
&= \begin{cases}
-18+A_1(X,L), & \text{if } (A_3(X,L),A_2(X,L))=(-5,-2),\\
-14+A_1(X,L), & \text{if } (A_3(X,L),A_2(X,L))=(-5,-1),\\
-10+A_1(X,L), & \text{if } (A_3(X,L),A_2(X,L))=(-5,0),\\
-12+A_1(X,L), & \text{if } (A_3(X,L),A_2(X,L))=(-4,-2),\\
-8+A_1(X,L), & \text{if } (A_3(X,L),A_2(X,L))=(-4,-1),\\
-4+A_1(X,L), & \text{if } (A_3(X,L),A_2(X,L))=(-4,0).
\end{cases}
\end{aligned}
$$

Consequently, we observe that

$$
A_1(X,L)\ge \begin{cases}
19, & \text{if } (A_3(X,L),A_2(X,L))=(-5,-2),\\
15, & \text{if } (A_3(X,L),A_2(X,L))=(-5,-1),\\
11, & \text{if } (A_3(X,L),A_2(X,L))=(-5,0),\\
13, & \text{if } (A_3(X,L),A_2(X,L))=(-4,-2),\\
9, & \text{if } (A_3(X,L),A_2(X,L))=(-4,-1),\\
5, & \text{if } (A_3(X,L),A_2(X,L))=(-4,0).
\end{cases}
$$

Hence, we get the assertion for the case that $A_4(X,L)=5$.

(ii.3.3) We consider the case that $A_4(X,L)=4$. In this case, we get $A_3(X,L)\ge -4$ by applying inequality (16). However, we obtain $A_3(X,L)\le -2$ by employing inequality (15). Thus, we have $(A_4(X,L),A_3(X,L))=(4,-4)$, $(4,-3)$ or $(4,-2)$.

(ii.3.3.1) We now consider the case that $(A_4(X,L),A_3(X,L))=(4,-4)$. Then, we note that

$$A_2(X,L)\ge -3A_4(X,L)-3A_3(X,L)+1=1$$

using inequality (17). Thus, we obtain that

$$
\begin{aligned}
&h^0(K_X+7L)\\
&\quad= A_5(X,L)+6A_4(X,L)+15A_3(X,L)+20A_2(X,L)+15A_1(X,L)+6A_0(X,L)\\
&\quad\ge 35.
\end{aligned}
$$

Thus, we get the assertion.

(ii.3.3.2) We consider the case that $(A_4(X,L),A_3(X,L))=(4,-3)$. Then we note that

$$A_2(X,L)\ge -3A_4(X,L)-3A_3(X,L)+1=-2 \tag{18}$$

by applying inequality (17). Here we assume that $7>h^0(K_X+7L)$. Then, using equation (10), Claim 3.3 and assumptions, we have

$$
\begin{aligned}
7 &> h^0(K_X+7L)\\
&= A_5(X,L)+6A_4(X,L)+15A_3(X,L)+20A_2(X,L)+15A_1(X,L)+6A_0(X,L)\\
&\ge 20A_2(X,L)+30.
\end{aligned}
$$

Thus, we get

$$A_2(X, L) \leq -1. \tag{19}$$

By utilizing inequalities (18) and (19), we have $A_2(X, L) = -1$ or -2. Therefore, it follows that

$$\begin{aligned}1 &\leq h^0(K_X + 5L)\\ &= A_5(X, L) + 4A_4(X, L) + 6A_3(X, L) + 4A_2(X, L) + A_1(X, L)\\ &= \begin{cases}-6 + A_1(X, L), & \text{if } A_2(X, L) = -1,\\ -10 + A_1(X, L), & \text{if } A_2(X, L) = -2.\end{cases}\end{aligned}$$

Thus, we have

$$A_1(X, L) \geq \begin{cases}7, & \text{if } A_2(X, L) = -1,\\ 11, & \text{if } A_2(X, L) = -2.\end{cases}$$

Consequently, we obtain that

$$\begin{aligned}h^0(K_X + 7L) &= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L)\\ &\quad + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L)\\ &\geq \begin{cases}24 - 45 - 20 + 105 + 6 > 7, & \text{if } A_2(X, L) = -1,\\ 24 - 45 - 40 + 105 + 6 > 7, & \text{if } A_2(X, L) = -2.\end{cases}\end{aligned}$$

This contradicts the assumption that $h^0(K_X + 7L) < 7$.

(ii.3.3.3) We consider the case that $(A_4(X, L), A_3(X, L)) = (4, -2)$. Then, we note that

$$A_2(X, L) \geq -3A_4(X, L) - 3A_3(X, L) + 1 = -5 \tag{20}$$

by employing inequality (17). Here, we assume that $7 > h^0(K_X + 7L)$. Then, using equation (10), Claim 3.3 and assumptions, we obtain that

$$\begin{aligned}7 &> h^0(K_X + 7L)\\ &= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L) + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L)\\ &\geq 20A_2(X, L) + 45.\end{aligned}$$

Therefore we get

$$A_2(X, L) \leq -1. \tag{21}$$

By applying inequalities (20) and (21), we have $-5 \leq A_2(X, L) \leq -1$.

(ii.3.3.3.1) Assume that $A_2(X, L) = -1$. Then, using equation (10), Claim 3.3 and assumptions, we obtain that

$$\begin{aligned}h^0(K_X + 7L) &= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L)\\ &\quad + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L)\\ &\geq 24 - 30 - 20 + 45 + 6 > 6.\end{aligned}$$

This contradicts the assumption that $h^0(K_X + 7L) < 7$.

(ii.3.3.3.2) Assume that $A_2(X, L) = -2$. Then, it is not difficult to see that

$$\begin{aligned}1 &\le h^0(K_X + 5L)\\ &= A_5(X, L) + 4A_4(X, L) + 6A_3(X, L) + 4A_2(X, L) + A_1(X, L)\\ &= -4 + A_1(X, L).\end{aligned}$$

Thus, we get $A_1(X, L) \ge 5$. Therefore, it follows that

$$\begin{aligned}h^0(K_X + 7L) &= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L)\\ &\quad + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L)\\ &\ge 24 - 30 - 40 + 75 + 6 > 6.\end{aligned}$$

This also contradicts the assumption that $h^0(K_X + 7L) \le 7$.

(ii.3.3.3.3) Assume that $-5 \le A_2(X, L) \le -3$. Since $A_2(X, L) \le -3$, one can easily see that

$$\begin{aligned}1 &\le h^0(K_X + 5L)\\ &= A_5(X, L) + 4A_4(X, L) + 6A_3(X, L) + 4A_2(X, L) + A_1(X, L)\\ &\le -8 + A_1(X, L).\end{aligned}$$

Thus, we have $A_1(X, L) \ge 9$. Therefore, it follows that

$$\begin{aligned}h^0(K_X + 7L) &= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L)\\ &\quad + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L)\\ &\ge 24 - 30 - 100 + 135 + 6 > 6.\end{aligned}$$

This also contradicts the assumption that $h^0(K_X + 7L) < 7$.

(ii.3.4) Let us consider the case that $A_4(X, L) = 3$. This case yields that $A_3(X, L) \ge -3$ by utilizing inequality (16). However, we have $A_3(X, L) \le 0$ by applying inequality (15). Hence, we have $(A_4(X, L), A_3(X, L)) = (3, -3)$, $(3, -2)$, $(3, -1)$ or $(3, 0)$.

(ii.3.4.1) We consider the case that $(A_4(X, L), A_3(X, L)) = (3, -3)$. Then, we note that

$$A_2(X, L) \ge -3A_4(X, L) - 3A_3(X, L) + 1 = 1 \tag{22}$$

by utilizing inequality (17). Thus, we observe that

$$\begin{aligned}&h^0(K_X + 7L)\\ &\quad\ge A_5(X, L) + 6A_4(X, L) + 15A_3(X, L) + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L)\\ &\quad\ge 44.\end{aligned}$$

Consequently, we obtain the assertion.

(ii.3.4.2) Let us consider the case that $(A_4(X, L), A_3(X, L)) = (3, -2)$. Then, using inequality (17) we obtain that

$$A_2(X, L) \ge -3A_4(X, L) - 3A_3(X, L) + 1 = -2. \tag{23}$$

Here, we assume that $7 > h^0(K_X + 7L)$. Then, by applying equation (10), Claim 3.3 and assumptions, it follows that

$$\begin{aligned}7 &> h^0(K_X + 7L)\\ &= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L) + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L)\\ &\geq 20A_2(X, L) + 39.\end{aligned}$$

This implies that

$$A_2(X, L) \leq -1. \tag{24}$$

By employing inequalities (23) and (24), we have $A_2(X, L) = -1$ or -2. Then, we get

$$\begin{aligned}1 &\leq h^0(K_X + 5L)\\ &= A_5(X, L) + 4A_4(X, L) + 6A_3(X, L) + 4A_2(X, L) + A_1(X, L)\\ &= \begin{cases} -4 + A_1(X, L), & \text{if } A_2(X, L) = -1,\\ -8 + A_1(X, L), & \text{if } A_2(X, L) = -2.\end{cases}\end{aligned}$$

This yields that $A_1(X, L) \geq 5$. Therefore, we have

$$\begin{aligned}h^0(K_X + 7L) &= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L)\\ &\quad + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L)\\ &\geq \begin{cases} 18 - 30 - 20 + 75 + 6 > 7, & \text{if } A_2(X, L) = -1,\\ 18 - 30 - 40 + 135 + 6 > 7, & \text{if } A_2(X, L) = -2.\end{cases}\end{aligned}$$

This contradicts the assumption that $h^0(K_X + 7L) < 7$.

(ii.3.4.3) We consider the case that $(A_4(X, L), A_3(X, L)) = (3, -1)$. Then, we observe that

$$A_2(X, L) \geq -3A_4(X, L) - 3A_3(X, L) + 1 = -5 \tag{25}$$

by applying inequality (17). Here, we assume that $7 > h^0(K_X + 7L)$. Then, using equation (10), Claim 3.3 and assumptions, we have

$$\begin{aligned}7 &> h^0(K_X + 7L)\\ &= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L) + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L)\\ &\geq 20A_2(X, L) + 54.\end{aligned}$$

It follows that

$$A_2(X, L) \leq -2. \tag{26}$$

By using inequalities (25) and (26), we observe that $-5 \leq A_2(X, L) \leq -2$.

(ii.3.4.3.1) Assume that $-2 \leq A_2(X, L) \leq -1$. Then, by employing equation (10), Claim 3.3 and assumptions, we have

$$\begin{aligned}h^0(K_X + 7L) &= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L)\\ &\quad + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L)\\ &\geq 18 - 15 - 40 + 45 + 6 > 7.\end{aligned}$$

(ii.3.4.3.2) Assume that $A_2(X, L) = -3$ (respectively -4 or -5). Then, we get

$$\begin{aligned}
1 &\le h^0(K_X + 5L) \\
&= A_5(X, L) + 4A_4(X, L) + 6A_3(X, L) + 4A_2(X, L) + A_1(X, L) \\
&= \begin{cases} -6 + A_1(X, L), & \text{if } A_2(X, L) = -3, \\ -10 + A_1(X, L), & \text{if } A_2(X, L) = -4, \\ -14 + A_1(X, L), & \text{if } A_2(X, L) = -5. \end{cases}
\end{aligned}$$

Thus, we have

$$A_1(X, L) \ge \begin{cases} 7, & \text{if } A_2(X, L) = -3, \\ 11, & \text{if } A_2(X, L) = -4, \\ 15, & \text{if } A_2(X, L) = -5. \end{cases}$$

Therefore, it follows that

$$\begin{aligned}
h^0(K_X + 7L) &= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L) \\
&\quad + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L) \\
&> 6.
\end{aligned}$$

This also contradicts the assumption that $h^0(K_X + 7L) < 7$.

(ii.3.4.4) Let us consider the case that $(A_4(X, L), A_3(X, L)) = (3, 0)$. Then, we observe that

$$A_2(X, L) \ge -3A_4(X, L) - 3A_3(X, L) + 1 = -8 \tag{27}$$

by utilizing inequality (17). Here, we assume that $7 > h^0(K_X + 7L)$. Then, using (10), Claim 3.3 and assumptions, it follows that

$$\begin{aligned}
7 &> h^0(K_X + 7L) \\
&= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L) + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L) \\
&\ge 20A_2(X, L) + 69.
\end{aligned}$$

Thus, we get

$$A_2(X, L) \le -3. \tag{28}$$

By employing inequalities (27) and (28), we have $-8 \le A_2(X, L) \le -3$.

(ii.3.4.4.1) Assume that $A_2(X, L) = -4$ (respectively -5, -6, -7 or -8). Then, it follows that

$$\begin{aligned}
1 &\le h^0(K_X + 5L) \\
&= A_5(X, L) + 4A_4(X, L) + 6A_3(X, L) + 4A_2(X, L) + A_1(X, L) \\
&= \begin{cases} -4 + A_1(X, L), & \text{if } A_2(X, L) = -4, \\ -8 + A_1(X, L), & \text{if } A_2(X, L) = -5, \\ -12 + A_1(X, L), & \text{if } A_2(X, L) = -6, \\ -16 + A_1(X, L), & \text{if } A_2(X, L) = -7, \\ -20 + A_1(X, L), & \text{if } A_2(X, L) = -8. \end{cases}
\end{aligned}$$

Thus, we have

$$A_1(X, L) \geq \begin{cases} 5, & \text{if } A_2(X, L) = -4, \\ 9, & \text{if } A_2(X, L) = -5, \\ 13, & \text{if } A_2(X, L) = -6, \\ 17, & \text{if } A_2(X, L) = -7, \\ 21, & \text{if } A_2(X, L) = -8. \end{cases}$$

Therefore, one can see that

$$\begin{aligned} h^0(K_X + 7L) &= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L) \\ &\quad + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L) \\ &> 6. \end{aligned}$$

This also contradicts the assumption that $h^0(K_X + 7L) < 7$.

(ii.3.4.4.2) Assume that $A_2(X, L) = -3$. Then, using equation (10), Claim 3.3 and assumptions, we observe that

$$\begin{aligned} h^0(K_X + 7L) &= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L) \\ &\quad + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L) \\ &\geq 18 - 60 + 45 + 6 > 7. \end{aligned}$$

(ii.3.5) We consider the case that $A_4(X, L) = 2$. This case yields that $A_3(X, L) \geq -2$ by utilizing inequality (16). However, we get $A_3(X, L) \leq 2$ by applying inequality (15). Thus, we have

$$(A_4(X, L), A_3(X, L)) = (2, -2), (2, -1), (2, 0), (2, 1) \text{ or } (2, 2).$$

(ii.3.5.1) We consider the case that $(A_4(X, L), A_3(X, L)) = (2, -2)$ or $(2, -1)$. Then, we observe that

$$A_2(X, L) \geq -3A_4(X, L) - 3A_3(X, L) + 1 = \begin{cases} 1, & \text{if } A_3(X, L) = -2, \\ -2, & \text{if } A_3(X, L) = -1 \end{cases}$$

by employing inequality (17). Thus, we have

$$\begin{aligned} & h^0(K_X + 7L) \\ & \quad \geq A_5(X, L) + 6A_4(X, L) + 15A_3(X, L) + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L) \\ & \quad \geq \begin{cases} 44, & \text{if } A_3(X, L) = -2, \\ 8, & \text{if } A_3(X, L) = -1. \end{cases} \end{aligned}$$

Hence, we get the assertion.

(ii.3.5.2) Let us consider the case that $(A_4(X, L), A_3(X, L)) = (2, 0)$, $(2, 1)$ or $(2, 2)$. Then we note that

$$\begin{aligned} h^0(K_X + 3L) &= A_5(X, L) + 2A_4(X, L) + A_3(X, L) \\ &= 2A_4(X, L) + 3A_3(X, L) \\ &= \begin{cases} 4, & \text{if } A_3(X, L) = 0, \\ 5, & \text{if } A_3(X, L) = 1, \\ 6, & \text{if } A_3(X, L) = 2. \end{cases} \end{aligned} \tag{29}$$

Since $h^0(L) > 0$, it follows from equation (29) that

$$h^0(K_X + 4L) \geq \begin{cases} 4, & \text{if } A_3(X, L) = 0, \\ 5, & \text{if } A_3(X, L) = 1, \\ 6, & \text{if } A_3(X, L) = 2. \end{cases} \tag{30}$$

Since

$$h^0(K_X + 4L) = A_5(X, L) + 3A_4(X, L) + 3A_3(X, L) + A_2(X, L),$$

inequality (30) gives that

$$A_2(X, L) \geq \begin{cases} -2, & \text{if } A_3(X, L) = 0, \\ -4, & \text{if } A_3(X, L) = 1, \\ -6, & \text{if } A_3(X, L) = 2. \end{cases}$$

Here, we assume that $7 > h^0(K_X + 7L)$. Then, by employing equation (10), Claim 3.3 and assumptions, we observe that

$$\begin{aligned} 7 &> h^0(K_X + 7L) \\ &= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L) + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L) \\ &\geq \begin{cases} 20A_2(X, L) + 63, & \text{if } A_3(X, L) = 0, \\ 20A_2(X, L) + 78, & \text{if } A_3(X, L) = 1, \\ 20A_2(X, L) + 93, & \text{if } A_3(X, L) = 2. \end{cases} \end{aligned}$$

It follows that

$$A_2(X, L) \leq \begin{cases} -2, & \text{if } A_3(X, L) = 0, \\ -4, & \text{if } A_3(X, L) = 1, \\ -5, & \text{if } A_3(X, L) = 2. \end{cases}$$

(ii.3.5.2.1) Assume that $(A_4(X, L), A_3(X, L), A_2(X, L)) = (2, 0, -2)$ (respectively $(2, 1, -4)$). Since $h^0(L) > 0$, equation (29) gives that

$$h^0(K_X + 5L) \geq \begin{cases} 4, & \text{if } (A_3(X, L), A_2(X, L)) = (0, -2), \\ 5, & \text{if } (A_3(X, L), A_2(X, L)) = (1, -4). \end{cases} \tag{31}$$

Meanwhile, we observe that

$$h^0(K_X + 5L) = A_5(X, L) + 4A_4(X, L) + 6A_3(X, L) \\ + 4A_2(X, L) + A_1(X, L). \tag{32}$$

Therefore, we observe from inequality (31) and (32) that

$$A_1(X, L) \geq \begin{cases} 4, & \text{if } (A_3(X, L), A_2(X, L)) = (0, -2), \\ 7, & \text{if } (A_3(X, L), A_2(X, L)) = (1, -4). \end{cases}$$

Therefore, it follows that

$$\begin{aligned} h^0(K_X + 7L) &= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L) \\ &\quad + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L) \\ &> 7. \end{aligned}$$

This contradicts the assumption that $h^0(K_X + 7L) < 7$.

(ii.3.5.2.2) Assume that $(A_4(X, L), A_3(X, L), A_2(X, L)) = (2, 2, -5)$ (respectively $(2, 2, -6)$). Since $h^0(L) > 0$, equation (29) gives that

$$h^0(K_X + 5L) \geq 6. \tag{33}$$

Hence, we see from inequality (33) and (32) that

$$A_1(X, L) \geq \begin{cases} 6, & \text{if } A_2(X, L) = -5, \\ 10, & \text{if } A_2(X, L) = -6. \end{cases}$$

Therefore, we get

$$\begin{aligned} h^0(K_X + 7L) &= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L) \\ &\quad + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L) \\ &> 7. \end{aligned}$$

This contradicts the assumption that $h^0(K_X + 7L) < 7$.

(ii.3.6) We consider the case that $A_4(X, L) = 1$. Using inequality (16), this case yields that $A_3(X, L) \geq -1$. However, we have $A_3(X, L) \leq 4$ by employing inequality (15). Thus, one can see that

$$(A_4(X, L), A_3(X, L)) = (1, -1), (1, 0), (1, 1), (1, 2), (1, 3) \text{ or } (1, 4).$$

(ii.3.6.1) We consider the case that $(A_4(X, L), A_3(X, L)) = (1, -1)$ (respectively $(1, 0)$, $(1, 1)$). Then, we observe that

$$\begin{aligned} h^0(K_X + 3L) &= A_5(X, L) + 2A_4(X, L) + A_3(X, L) \\ &= 2A_4(X, L) + 3A_3(X, L) \\ &= \begin{cases} 1, & \text{if } A_3(X, L) = -1, \\ 2, & \text{if } A_3(X, L) = 0, \\ 3, & \text{if } A_3(X, L) = 1. \end{cases} \end{aligned} \tag{34}$$

Since $h^0(L) > 0$, it follows from (34) that

$$h^0(K_X + 4L) \geq \begin{cases} 1, & \text{if } A_3(X, L) = -1, \\ 2, & \text{if } A_3(X, L) = 0, \\ 3, & \text{if } A_3(X, L) = 1. \end{cases} \tag{35}$$

Besides, since

$$h^0(K_X + 4L) = A_5(X, L) + 3A_4(X, L) + 3A_3(X, L) + A_2(X, L),$$

we have the following by employing inequality (35):

$$A_2(X, L) \geq \begin{cases} 1, & \text{if } A_3(X, L) = -1, \\ -1, & \text{if } A_3(X, L) = 0, \\ -3, & \text{if } A_3(X, L) = 1. \end{cases}$$

Therefore, it follows that

$$\begin{aligned} h^0(K_X + 7L) &= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L) \\ &\quad + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L) \\ &> 7. \end{aligned}$$

(ii.3.6.2) Let us consider the case that $(A_4(X, L), A_3(X, L)) = (1, 2)$. Then, we observe that

$$\begin{aligned} h^0(K_X + 3L) &= A_5(X, L) + 2A_4(X, L) + A_3(X, L) \\ &= 2A_4(X, L) + 3A_3(X, L) \\ &= 4. \end{aligned} \tag{36}$$

Since $h^0(L) > 0$, we get from (36) that

$$h^0(K_X + 4L) \geq 4. \tag{37}$$

Since

$$h^0(K_X + 4L) = A_5(X, L) + 3A_4(X, L) + 3A_3(X, L) + A_2(X, L),$$

we have $A_2(X, L) \geq -5$ by utilizing inequality (37).

Here, we assume that $7 > h^0(K_X + 7L)$. Then, using (10), Claim 3.3 and assumptions, we get

$$\begin{aligned} 7 &> h^0(K_X + 7L) \\ &= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L) + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L) \\ &\geq 20A_2(X, L) + 87. \end{aligned}$$

It follows that

$$A_2(X, L) \leq -5.$$

Therefore, we have $A_2(X, L) = -5$.

Assume that $A_2(X, L) = -5$. Since $h^0(L) > 0$, we observe from inequality (37) that

$$h^0(K_X + 5L) \geq 4. \tag{38}$$

Thus, we get from inequality (38) and (32) that $A_1(X, L) \geq 8$. Therefore, it follows that

$$\begin{aligned} h^0(K_X + 7L) &= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L) \\ &\quad + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L) \\ &> 7. \end{aligned}$$

This contradicts the assumption that $h^0(K_X + 7L) < 7$.

(ii.3.6.3) We consider the case that $(A_4(X, L), A_3(X, L)) = (1, 3)$. Then, we obtain that

$$\begin{aligned} h^0(K_X + 3L) &= A_5(X, L) + 2A_4(X, L) + A_3(X, L) \\ &= 2A_4(X, L) + 3A_3(X, L) \\ &= 5. \end{aligned} \tag{39}$$

Using $h^0(L) > 0$, we obtain from (39) that

$$h^0(K_X + 4L) \geq 5. \tag{40}$$

Since

$$h^0(K_X + 4L) = A_5(X, L) + 3A_4(X, L) + 3A_3(X, L) + A_2(X, L),$$

we have $A_2(X, L) \geq -7$ by applying inequality (40).

Here, we assume that $7 > h^0(K_X + 7L)$. Then, by employing equation (10), Claim 3.3 and assumptions, we have

$$\begin{aligned} 7 &> h^0(K_X + 7L) \\ &= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L) + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L) \\ &\geq 20A_2(X, L) + 102. \end{aligned}$$

It follows that

$$A_2(X, L) \leq -5.$$

Therefore, $A_2(X, L) = -5, -6$ or -7. Consequently, one can see that

$$\begin{aligned} h^0(K_X + 4L) &= A_5(X, L) + 3A_4(X, L) + 3A_3(X, L) + A_2(X, L) \\ &= \begin{cases} 7, & \text{if } A_2(X, L) = -5, \\ 6, & \text{if } A_2(X, L) = -6, \\ 5, & \text{if } A_2(X, L) = -7. \end{cases} \end{aligned} \tag{41}$$

By employing $h^0(L) > 0$, we obtain from equation (41) that

$$h^0(K_X + 5L) \geq \begin{cases} 7, & \text{if } A_2(X, L) = -5, \\ 6, & \text{if } A_2(X, L) = -6, \\ 5, & \text{if } A_2(X, L) = -7. \end{cases} \tag{42}$$

Hence, we get from inequality (42) and (32) that

$$A_1(X, L) \geq \begin{cases} 5, & \text{if } A_2(X, L) = -5, \\ 8, & \text{if } A_2(X, L) = -6, \\ 11, & \text{if } A_2(X, L) = -7. \end{cases}$$

Therefore, we obtain that

$$\begin{aligned} h^0(K_X + 7L) &= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L) \\ &\quad + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L) \\ &> 7. \end{aligned}$$

This contradicts the assumption that $h^0(K_X + 7L) < 7$.

(ii.3.6.4) Let us consider the case that $(A_4(X, L), A_3(X, L)) = (1, 4)$. Then, we observe that

$$\begin{aligned} h^0(K_X + 3L) &= A_5(X, L) + 2A_4(X, L) + A_3(X, L) \\ &= 2A_4(X, L) + 3A_3(X, L) \\ &= 6. \end{aligned} \tag{43}$$

Since $h^0(L) > 0$, it follows from (43) that

$$h^0(K_X + 4L) \geq 6. \tag{44}$$

Meanwhile, since

$$h^0(K_X + 4L) = A_5(X, L) + 3A_4(X, L) + 3A_3(X, L) + A_2(X, L),$$

we have $A_2(X, L) \geq -9$ by applying inequality (44).

Here, we assume that $7 > h^0(K_X + 7L)$. Then, using equation (10), Claim 3.3 and assumptions, we have

$$\begin{aligned} 7 &> h^0(K_X + 7L) \\ &= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L) + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L) \\ &\geq 20A_2(X, L) + 117. \end{aligned}$$

It follows that

$$A_2(X, L) \leq -6.$$

Therefore $A_2(X, L) = -6, -7, -8$ or -9.

Assume that $A_2(X, L) = -6$ (respectively $-7, -8, -9$). Then, we get

$$\begin{aligned} h^0(K_X + 4L) &= A_5(X, L) + 3A_4(X, L) + 3A_3(X, L) + A_2(X, L) \\ &= \begin{cases} 9, & \text{if } A_2(X, L) = -6, \\ 8, & \text{if } A_2(X, L) = -7, \\ 7, & \text{if } A_2(X, L) = -8, \\ 6, & \text{if } A_2(X, L) = -9. \end{cases} \end{aligned} \tag{45}$$

Since $h^0(L) > 0$, we obtain from equation (45) that

$$h^0(K_X + 5L) \geq \begin{cases} 9, & \text{if } A_2(X, L) = -6, \\ 8, & \text{if } A_2(X, L) = -7, \\ 7, & \text{if } A_2(X, L) = -8, \\ 6, & \text{if } A_2(X, L) = -9. \end{cases} \tag{46}$$

Thus, we see from inequality (46) and (32) that

$$A_1(X, L) \geq \begin{cases} 5, & \text{if } A_2(X, L) = -6, \\ 8, & \text{if } A_2(X, L) = -7, \\ 11, & \text{if } A_2(X, L) = -8, \\ 14, & \text{if } A_2(X, L) = -9. \end{cases}$$

Therefore, we have

$$\begin{aligned} h^0(K_X + 7L) &= A_5(X, L) + 6A_4(X, L) + 15A_3(X, L) \\ &\quad + 20A_2(X, L) + 15A_1(X, L) + 6A_0(X, L) \\ &> 7. \end{aligned}$$

This contradicts the assumption that $h^0(K_X + 7L) < 7$.

(ii.4) We consider the case that $m = 8$. Assume that $h^0(K_X + 8L) < 22$. Since $h^0(K_X + 7L) \geq 7$, we have $h^0(K_X + 8L) - h^0(K_X + 7L) < 15$. Moreover,

$$\begin{aligned} &h^0(K_X + 8L) - h^0(K_X + 7L) \\ &\quad = A_4(X, L) + 6A_3(X, L) + 15A_2(X, L) + 20A_1(X, L) + 15A_0(X, L). \end{aligned}$$

Therefore, we get

$$A_4(X, L) + 6A_3(X, L) + 15A_2(X, L) + 20A_1(X, L) + 15(A_0(X, L) - 1) < 0. \tag{47}$$

Next, we evaluate the left-hand side of this inequality. First, we note that $A_4(X, L) \geq 0$. Moreover, we observe that

$$\begin{aligned} &6A_3(X, L) + 15A_2(X, L) + 20A_1(X, L) + 15(A_0(X, L) - 1) \\ &\quad = 9L^5 + \tfrac{25}{4}K_X L^4 + K_X^2 L^3 + \tfrac{1}{4}c_2(X)(K_X + 4L)L^2 + \tfrac{195}{4}L^5 \\ &\qquad + \tfrac{75}{4}K_X L^4 + \tfrac{5}{4}(K_X^2 + c_2(X))L^3 + 10K_X L^4 + 60L^5 + 15L^5 - 15 \\ &\quad = \tfrac{471}{4}L^5 + 35K_X L^4 + \tfrac{9}{4}K_X^2 L^3 + \tfrac{1}{4}c_2(X)(K_X + 9L)L^2 + 15L^5 - 15. \end{aligned} \tag{48}$$

The application of inequality (7) gives that

$$\begin{aligned} \tfrac{1}{4}c_2(X)(K_X + 9L)L^2 &\geq -\tfrac{1}{4}(K_X + 9L)L^2(8K_X L + 40L^2) \\ &= -2K_X^2 L^3 - 28K_X L^4 - 90L^5. \end{aligned} \tag{49}$$

Thus, by employing equation (48) and inequality (49), we have

$$\begin{aligned}
&6A_3(X, L) + 15A_2(X, L) + 20A_1(X, L) + 15(A_0(X, L) - 1) \\
&\geq \tfrac{1}{4}K_X^2L^3 + 7K_XL^4 + \tfrac{111}{4}L^5 + 15(L^5 - 1) \\
&= \tfrac{1}{4}(K_X + 26L)(K_X + 2L) + \tfrac{59}{4}L^5 + 15(L^5 - 1) \\
&\geq 1.
\end{aligned}$$

This contradicts inequality (47). Hence, we obtain the assertion of Proposition 3.2. □

(iii) Let us consider the case that $h^0(K_X + L) > 0$. If we can show that $3A_3(X, L) + (m - 3)A_2(X, L) \geq 0$ using Claim 3.2 and inequality (3), then we have that $h^0(K_X + mL) > \binom{m-1}{5}$ holds. By employing Remark 2.2(iii), we get

$$\begin{aligned}
&3A_3(X, L) + (m - 3)A_2(X, L) \\
&\quad = \frac{13m - 21}{4}L^5 + \frac{10m - 5}{8}K_XL^4 + \frac{m + 3}{12}K_X^2L^3 \\
&\quad\quad + \frac{3K_X + (2m + 6)L}{24}c_2(X)L^2.
\end{aligned} \tag{50}$$

However, using Proposition 2.1 and the assumption that $h^0(K_X + L) > 0$, we infer that $\Omega_X\langle L\rangle$ is generically nef. Furthermore, we observe that $K_X + 5L$ is ample since we assumed that $K_X + 4L$ is nef. Hence, by utilizing the generical nefness of $\Omega_X\langle L\rangle$ and [**18**, 2.11 Corollary], we have

$$\begin{aligned}
&\frac{3K_X + (2m + 6)L}{24}c_2(X)L^2 \\
&\quad \geq -\frac{1}{24}(12K_X^2 + (8m + 54)K_XL + (20m + 60)L^2)L^3.
\end{aligned} \tag{51}$$

Thus, by applying (50) and inequality (51) we get

$$\begin{aligned}
&3A_3(X, L) + (m - 3)A_2(X, L) \\
&\geq \frac{m - 3}{12}K_X^2L^3 + \frac{22m - 69}{24}K_XL^4 + \frac{58m - 186}{24}L^5 \\
&= \frac{m - 3}{24}L^3\left(2K_X^2L^3 + \frac{22m - 69}{m - 3}K_XL + \frac{58m - 186}{m - 3}L^2\right) \\
&= \frac{m - 3}{24}L^3\left((2K_X + 14L)(K_X + 3L) + \frac{2m - 9}{m - 3}(K_X + 8L)L + \frac{2m + 12}{m - 3}L^2\right) \\
&\geq 0.
\end{aligned}$$

Therefore, we obtain that $h^0(K_X + mL) > \binom{m-1}{5}$ for case (iii).

Hence, using cases (i), (ii) and (iii), we get the assertion of Theorem 3.1. □

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Yoshiaki Fukuma
Department of Mathematics and Physics
Faculty of Science and Technology
Kochi University
Akebono-cho, Kochi 780-8520
Japan
(E-mail: fukuma@kochi-u.ac.jp)