One-switch Utility Functions and Applications: a Review Article

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https://doi.org/10.15017/4796012

出版情報：経済論究. 173, pp.23-72, 2022-07-25. 九州大学大学院経済学会

One-switch Utility Functions and Applications: a Review Article

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1 Introduction

The expected utility theory could be traced back to Bernoulli in 1738 who attempted to resolve the St. Petersburg Paradox posed in 1713, which doubted if a player would be willing to pay an infinite amount of money to play a casino whose expected value is infinite and also to challenge an old idea that real-world people value random choices in accordance with the expected value of payoffs. Obviously, this is unreasonable and impossible in realistic situations. So, instead of the expected value of payoffs, the expected value of utility was introduced by Bernoulli as a solution to this paradox, whose research was republished and translated from Latin in 1954. There are two main ideas in the resolution: one is the diminishing marginal utility of money, and the other is expected utility hypothesis. More specifically, the former refers to that as a decision maker's wealth increases, the decision maker will gain correspondingly smaller benefits such as satisfaction or happiness. This implies that a player would only be willing to pay a finite amount of money to play a game. The latter shows that the valuation of a risky gamble for a player does not depend on the expected value of the gamble but depend on the expected utility of such gamble.

John von Neumann and Oskar Morgenstern (1944) formalized Bernoulli's utility theory in book *Theory of Games and Economic Behavior* by von Neumann-Morgenstern (vNM) expected utility theory which states that a rational decision maker will try to maximize the expected value of utility function while facing a gamble with risky outcomes. By using the expected utility criterion, a decision maker's preference must satisfy the axioms of expected utility theory. Based on these axioms, Bell first introduced the one-switch rule and one-switch utility functions in 1988. Furthermore, the well-known paradoxes, say, Allais Paradox and Ellsberg Paradox may happen on decision makers in real-world decision-making processes. Many researchers devoted to the resolutions of these two paradoxes through the release of one of the axioms of expected utility theory. Quiggin (1982), Tversky and Kahneman (1992), Gilboa and Schmeidler (1989) and so forth, for instance, introduced a variety of non-expected utility theory to avoid the paradoxes. In this review article, only expected utility theory will be concentrated on because it offers the basic axioms to one-switch

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utility functions.

Moreover, the expected utility theory can contribute to predict a decision maker's risk attitude. We can categorize all the decision makers into three types: risk averse, risk neutral and risk seeking. The utility functions for risk-averse and risk-seeking decision makers are concave and convex, respectively and exponential utility functions are one of the examples of them. For instance, if $u(w)=e^w$ with $w>1$, the utility function is convex; if $u(w)=e^w$ with $0<w<1$, the utility function is concave. In addition, a decision maker with linear utility function is risk neutral. However, a choice on an alternative made by a decision maker with linear or exponential utility function depends only on the increase in wealth level which is the value of a gamble obtained rather than the total wealth a decision maker owns (Liu and Koenig, 2005). This implies that linear and exponential utility functions ignore how the wealth level of a decision maker can affect his or her risk attitude because a rational decision maker is usually risk averse and tends to be risk neutral as his or her wealth level is increasingly larger.

Therefore, abiding by the axioms of expected utility theory, Bell (1988) introduced one-switch utility functions which focus on not only utility maximization but also a decision maker's preference change to a pair of alternatives will change with increase or decrease in wealth level. Meanwhile, the initial wealth level of a decision maker has to be considered in the functions of expected utility, especially utility functions for wealth.

Based on the basic properties of one-switch utility functions introduced by Bell (1988), several researchers mainly extended and enriched these properties and even tried to consider one-switch utility functions with multiattributes. Conventionally, researchers usually strive to apply utility theories in realistic situations to make the models explain the economic phenomenon accurately and efficiently or even analyze utility theories in empirical ways. Although there have been many research papers associated with the fundamental theories of one-switch utility functions and Scholz (2016) also conducted empirical research on how one of the functions works in decision-making process, there are rare papers about the applications of one-switch utility functions.

So far, there has been no review article whose topic is related to one-switch utility functions. Hence, the main objective of this review article is not to challenge the previous research but to consider the possibilities of applications of one-switch utility functions. To achieve this goal, the basic properties of one-switch utility functions were reviewed and then a comparison was made between one-switch utility functions and other standard utility functions such as the Cobb-Douglas utility function. Finally, the economics of information and multiattribute functional equations were two possible fields to apply and extend one-switch utility functions in further research.

This review article is organized as follows. Section 2 is the preamble, where we discuss how the zero-switch and one-switch utility functions are defined and Section 3 shows how we used PRISMA method to filter research journal articles; Section 4 is the comparison between the Cobb-Douglas utility function and one-switch utility functions; Section 5 is the presentation of the results by reviewing previous literatures, which contains two applications in the economics of information and functional equations with multiattributes, respectively and also how the future research in these two fields can be extended. Lastly, several conclusions were drawn in Section 6. The unique properties of one-switch utility functions are focusing not only on utility maximization but, more importantly, on the at most once change on preference; among all four types of one-switch utility functions, it is noticeable that the sumex and linear plus exponential one-switch utility functions are comparatively more easily-applicable thanks to their realistic risk attitudes of a decision maker. Furthermore, as a utility function for wealth, one-switch utility functions are more suitable to be applied into uncertain situations such as choosing gambles and deriving the Marshallian or Hicksian demand functions are meaningless and not reasonable. Finally, we also discussed how one-switch utility functions go in possible applicable research directions: the monotonicity in the economics of information and solutions for a system of functional equations.

2 Preamble

While you are willing to buy a new house, suppose that you are facing two choices, House A and House B, and you currently prefer House A to House B. Now, suppose you win a lottery which will increase your wealth by $10,000, will you change your preference from House A to House B or keep the same choice as original?

The phenomenon of preference switching can be described by one-switch utility functions. In this review article, the method, PRISMA, will be used to filter the journal articles from two major databases including Scopus and Web of Science. The one-switch rule or one-switch utility functions are different from standard utility functions whose value of utility depends on the consumption level or the value of gambles so the definitions and necessary properties of one-switch utility functions will be reviewed first.

2.1 Definition of zero-switch utility functions

Prior to the discussion of characteristics of one-switch utility functions, it is essential to assume that a decision maker is willing to be rational and abides by the six axioms of expect utility theory: completeness, transitivity, continuity, monotonicity, substitution, and reduction to simple gambles (for details, see Appendix A).

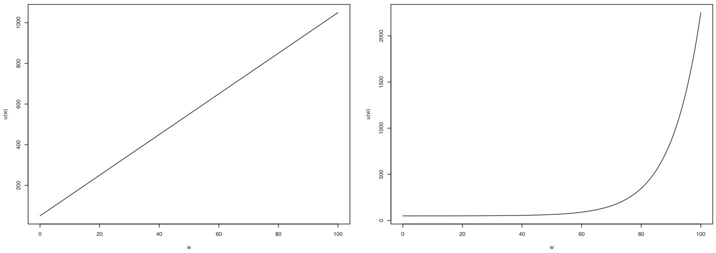
In the meantime, the linear and exponential utility functions are the only zero-switch utility functions (Bell, 1988). The only monotone utility functions satisfying the zero-switch utility should be (Abbas and Bell, 2012):

$$u(w)=aw+b, \tag{2.1.1}$$

$$u(w)=ae^{cw}+b, \tag{2.1.2}$$

where $b \neq 0$, $c \neq 0$, w is the wealth level.

Figure 1. Zero-switch Utility Functions



Left: *linear utility function* $u(w)=aw+b$, $a=10$, $b=50$.
Right: *exponential utility function* $u(w)=ae^{bw}+c$, $a=b=0.1$, $c=50$.

If a rational decision maker abides by the axioms of expected utility theory and prefers lottery A to lottery B with wealth level w, it can be interpreted as

$$E_A[u(w)]>E_B[u(w)].$$

In both zero-switch utility functions (2.1.1) and (2.1.2), with the wealth level increasing by δ to $(w+\delta)$, a decision maker will still prefer lottery A to lotter B because of monotonicity of zero-switch utility functions, which can also be seen from Figure 1.　Thus, as Pfanzag (1959) noted, the ranking of lotteries for a rational decision maker is independent to the wealth.

$$E_A[u(w+\delta)]>E_B[u(w+\delta)].$$

Or, the zero-witch utility functions (2.1.1) and (2.1.2) are also called *cardinal zero-switch* utility functions because $E_A[U(w+\delta)]-E_B[U(w+\delta)]$ does not change sign as wealth level, w, varies, which a function does *not* change sign implies that it is always positive, always negative, or always zero (Abbas and Bell, 2015).　With any changes of the wealth level, a decision maker prefers lottery A to lottery B, which implies the expected utility of lottery A is larger than that of lottery B so the difference between the expected utilities is always positive; if a decision maker prefers lottery B to lottery A, the difference between the expected utilities is always negative; and if a decision maker is indifferent to these two lotteries, the difference between the expected utilities is always zero even changes of wealth level appear.

2.2　Definition of one-switch utility functions

However, this is not the case for one-switch utility functions which attempt to maximize the expected utility of a decision maker who obeys the *one-switch rule* which assumed that the relationship between a decision maker's wealth level and the ranking of a pair of alternatives is dependent (Bell, 1988).

There are only four utility functions satisfying the one-switch rule for any choice of parameter *a, b, c* and *d* with *w* for wealth (Bell, 1988).

quadratics: $u(w)=aw^2+bw+c$,

linear plus exponential: $u(w)=aw+be^{cw}$,

linear times exponential: $u(w)=(aw+b)e^{cw}$,

the sumex: $u(w)=ae^{bw}+ce^{dw}$.

If a decision maker follows the one-switch rule, he or she may prefer lottery A to lottery B with wealth level w. And then with the increase in wealth level by δ to $(w+\delta)$, there must be *at most* once switch in the decision maker's preference. So, we can have

$$E_A[u(w)]-E_B[u(w)]>0,$$
$$E_A[u(w+\delta)]-E_B[u(w+\delta)]<0.$$

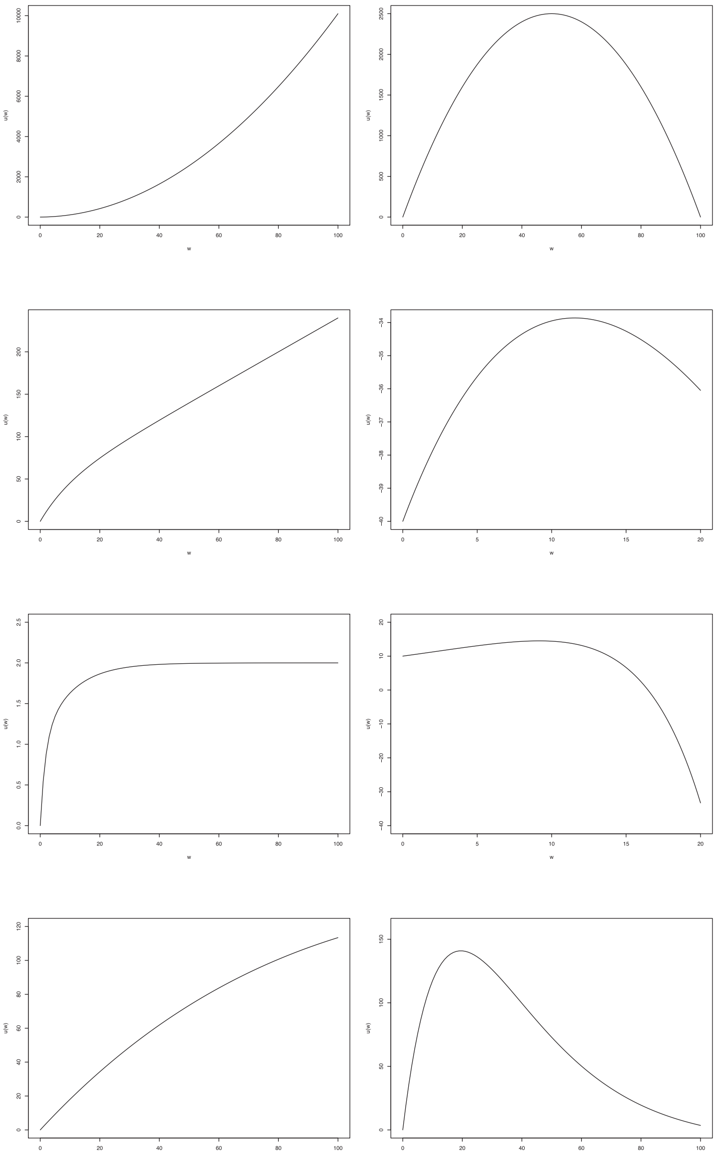
The decision maker prefers lottery A to lottery B with wealth level w. When his or her wealth level increased by δ to $(w+\delta)$, the preference will change to lottery B and will not change back to lottery again. The reason for this is that there must be a turning point of wealth level which makes a decision maker indifferent to lottery A and lottery B and the decision maker prefers A to B with the wealth level below that turning point while the decision maker prefers lottery B to lottery A with the wealth level above that turning point and will not change the preference back to lottery A anymore. Thus, the difference of expected utility between lottery A and lottery B with wealth level $(w+\delta)$ showed above will cross *0* only once.

For further exploration of the difference between zero-switch utility functions and one-switch utility functions, Abbas (2018) showed figures (Figure 2) of four one-switch utility functions respectively with different settings of parameters, which shows that there is zero or one peak in each of those four one-switch utility functions.

The graphs of one-switch utility functions on the left panel in Figure 2 show no peaks whereas the other four graph on the right panel show one peak by different settings of parameters. Thus, Abbas (2018) interpreted that there is at most one peak in one-switch utility functions with any settings of parameters. With particular settings, quadratic and linear plus exponential one-switch utility functions can be reduced to zero-switch utility functions with no peaks given by (2.1.1) and (2.1.2). For instance, the quadratic one-switch utility function, aw^2+bw+c, can be reduced to linear utility function whose shape is same as the figure on the left panel in Figure 1 by setting $a=0$, $b=10$, and $c=50$. Similarly, the linear plus exponential one-switch utility function can also be reduced to exponential utility function as the same shape as the figure on the right panel in Figure 1 by setting $a=0$, $b=0.1$, $\gamma=-0.1$, and $c=50$. This, therefore, again, implies that zero-switch utility functions belong to one-switch utility functions.

Let us check three examples below to investigate the properties of zero- and one-switch utility functions: **Example 1** (Clemen and Reilly, 2013) A gambler with utility function $U(w)=\ln(w)$, where

Figure 2. Example of One-switch Utility Functions



Quadratic ay^2+by+c. (Left): $a=b=c=1$. (Right) $a=-1$, $b=100$, $c=0$.
Linear plus exponential $ay+be^{-\gamma y}+c$. (Left): $a=2$, $b=-40$, $\gamma=0.1$, $c=40$. (Right): $a=-1.2$, $b=-40$, $\gamma=0.06$, $c=0$.[1)]
Sumex $ae^{-by}+ce^{-dy}+f$. (Left): $a=-1$, $b=0.1$, $c=-1$, $d=0.6$, $f=2$. (Right): $a=10$, $b=-0.1$, $c=-2$, $d=-0.2$, $f=2$.
Linear times exponential $(ay+b)e^{-\gamma y}+c$. (Left): $a=2$, $b=20$, $\gamma=0.005$, $c=-20$. (Right): $a=20$, $b=10$, $\gamma=0.05$, $c=-10$.
Source: Abbas, Ali E, Foundations of Multiattribute Utility, Cambridge University Press, 2018, Chapter 18.3

1) The right-side figure of linear plus exponential one-switch utility function is not originally cited from Abbas' book but self-setting with resembling shape because the same shape of figure as the book shows is unable to be drawn based on the parameters from the book with $a=2$, $b=-40$, $\gamma=0.1$, and $c=40$.

w is total wealth, has a choice between the following two alternatives:

Two Alternatives a Gambler Facing

Alternative A	Alternative B
Win \$10,000 with probability 0.2	Win \$3,000 with probability 0.9
Win \$1,000 with probability 0.8	Lose \$2,000 with probability 0.1

If the gambler currently has the wealth level, \$2,500, then,

$$EU(A)=0.2\ln(12{,}500)+0.8\ln(3{,}500)=8.415,$$
$$EU(B)=0.9\ln(5{,}500)+0.1\ln(500)=8.373.$$

Comparing $EU(A)$ with $EU(B)$, the gambler will choose the Alternative A as it has higher expected utility.

If the gambler's wealth level increases to \$5,000, then,

$$EU(A)=0.2\ln(15{,}000)+0.8\ln(6{,}000)=8.883,$$
$$EU(B)=0.9\ln(8{,}000)+0.1\ln(3{,}000)=8.889.$$

The gambler will prefer Alternative B to Alternative A.

If the gambler's wealth level increases to \$10,000, then,

$$EU(A)=0.2\ln(20{,}000)+0.8\ln(11{,}000)=9.425,$$
$$EU(B)=0.9\ln(13{,}000)+0.1\ln(8{,}000)=9.424.$$

The gambler will change the preference back to the Alternative A again.

By taking the same manner, if the gambler's wealth level increases to a much significantly larger level, say, \$100,000, then,

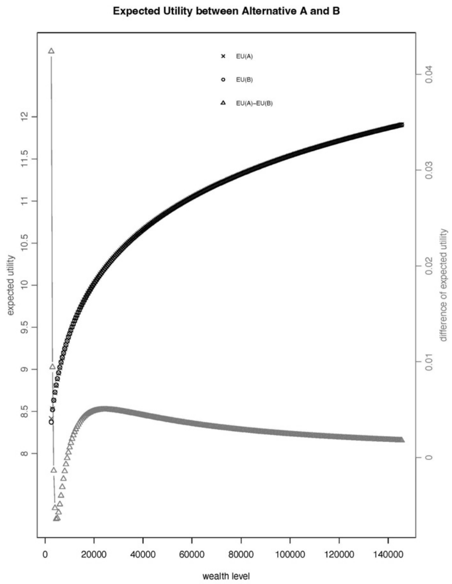
$$EU(A)=0.2\ln(110{,}000)+0.8\ln(101{,}000)=11.540,$$
$$EU(B)=0.9\ln(103{,}000)+0.1\ln(98{,}000)=11.538.$$

The gambler will keep the preference on Alternative A.

Some may be not familiar with and feel confused about the change in a gambler's preference. As we mentioned in Section 2.1 and Section 2.2, the one-switch rule stated that as wealth level increases, a decision maker's preference may change from one alternative to the other, but at most once. However, for a decision maker with zero-switch utility functions, his or her preference will not change as the wealth level increases.

As can be seen from Figure 3. (a), the difference of expected utility between two alternatives labeled by triangles is getting smaller and tends to be 0, which implies that as the decision maker's wealth increases to much larger level or even reaches to infinity, the difference of the expected utilities between two alternatives is getting closer and closer or finally the same, which means when a decision maker has infinite fortune, he or she will consider these two alternatives indifferent. Moreover, it is possible to say that $E_A[u(w+\delta_n)]-E_B[u(w+\delta_n)]\to 0$ as the wealth level goes to

Figure 3. Expected Utility Between Alternative A and B in Example 1, 2, and 3.



(a)

infinity. From the above calculating process, $E_A[u(w)]-E_B[u(w)]$ is positive when the wealth level is \$2,500; but $E_A[u(w+\delta_1)]-E_B[u(w+\delta_1)]$ is negative as the wealth level increases to \$5,000; and $E_A[u(w+\delta_2)]-E_B[u(w+\delta_2)]$ is positive again as the wealth level increases to \$100,000. In the meantime, the line labeled with cross in Figure 3. (a) crossed zero twice so this means the preference changed twice with increasingly smaller difference of expected utility between two alternatives. Thus, the utility function in this example, the logarithmic utility function, does not belong to zero-switch utility functions and any one of the Bell's one-switch utility functions.

Example 2 If a gambler has the utility function $U(w)=0.002e^{0.0001w}+10$, and the Alternative A and B keep the same as above. If a gambler currently has the wealth level, \$2,500, then,

$$EU(A)=0.2u(12{,}500)+0.8u(3{,}500)=10.0037,$$

$$EU(B)=0.9u(5{,}500)+0.1u(500)=10.0033.$$

Comparing the expected utility between Alternative A and Alternative B, the gambler will choose Alternative A.

If the gambler's wealth level increases to \$5,000, then,

$$EU(A)=0.2u(15{,}000)+0.8u(6{,}000)=10.0047,$$

$$EU(B)=0.9u(8{,}000)+0.1u(3{,}000)=10.0043.$$

Comparing the expected utility between Alternative A and Alternative B, the gambler will choose Alternative A.

If the gambler's wealth level increases to \$10,000, then,

$$EU(A)=0.2u(20{,}000)+0.8u(11{,}000)=10.0078,$$

$$EU(B)=0.9u(13{,}000)+0.1u(8{,}000)=10.0071$$

Comparing the expected utility between Alternative A and Alternative B, the gambler will still choose Alternative A.

As can be seen from the calculating process above, the decision maker does not change his or her preference on the alternative with the increasing of wealth level.

As we can see from Figure 3. (b), $E_A[u(w)]-E_B[u(w)]$, the line labeled by triangles, is always positive and experiencing an increasing trend with wealth level increases. Meanwhile, the line labeled by cross never crosses zero, which implies that there is no switching point in this decision maker's preference. Thus, this also illustrates that the utility function in this example is zero-switch utility function.

Example 3 (Clemen and Reilly, 2013) If a gambler has the utility function $U(w)=0.0003w-8.48e^{-w/2775}$ and the Alternative A and B keep the same as above. If a gambler currently has the wealth level, \$2,500, then,

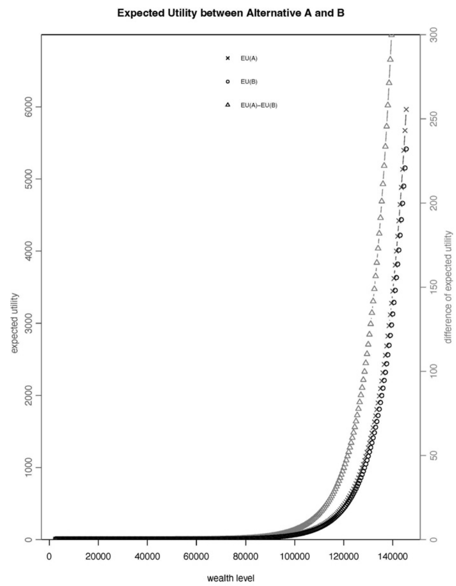
$$EU(A)=0.2\left[0.0003(12{,}500)-8.48e^{-\frac{12{,}500}{2{,}775}}\right]+0.8\left[0.0003(3{,}500)-8.48e^{-\frac{3{,}500}{2{,}775}}\right]=-0.35,$$

$$EU(B)=0.9\left[0.0003(5{,}500)-8.48e^{-\frac{5{,}500}{2{,}775}}\right]+0.1\left[0.0003(500)-8.48e^{-\frac{500}{2{,}775}}\right]=-0.26.$$

Comparing the expected utility between Alternative A and Alternative B, the gambler will choose Alternative B.

If the gambler's wealth level increases to \$5,000, then,

Figure 3. Expected Utility Between Alternative A and B in Example 1, 2, and 3.



(b)

$$EU(A)=0.2\left[0.0003(15{,}000)-8.48e^{-\frac{15{,}000}{2{,}775}}\right]+0.8\left[0.0003(6{,}000)-8.48e^{-\frac{6{,}000}{2{,}775}}\right]=1.55,$$

$$EU(B)=0.9\left[0.0003(8{,}000)-8.48e^{-\frac{8{,}000}{2{,}775}}\right]+0.1\left[0.0003(3{,}000)-8.48e^{-\frac{3{,}000}{2{,}775}}\right]=1.54.$$

Comparing the expected utility between A and B, the gambler will change the preference to the Alternative A.

If the gambler's wealth level increases to \$10,000, then,

$$EU(A)=0.2\left[0.0003(20{,}000)-8.48e^{-\frac{20{,}000}{2{,}775}}\right]+0.8\left[0.0003(11{,}000)-8.48e^{-\frac{11{,}000}{2{,}775}}\right]=3.71,$$

$$EU(B)=0.9\left[0.0003(13{,}000)-8.48e^{-\frac{13{,}000}{2{,}775}}\right]+0.1\left[0.0003(8{,}000)-8.48e^{-\frac{8{,}000}{2{,}775}}\right]=3.63.$$

Comparing the expected utility between the Alternative A and the Alternative B, the gambler will change his or her preference back to Alternative A.

By taking the same manner, if the gambler's wealth level increases to a much significantly larger level, say, \$100,000, then,

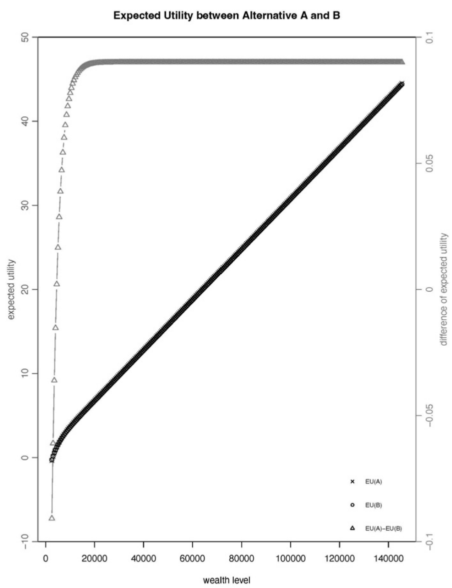
$$EU(A)=0.2\left[0.0003(110{,}000)-8.48e^{-\frac{110{,}000}{2{,}775}}\right]+0.8\left[0.0003(101{,}000)-8.48e^{-\frac{101{,}000}{2{,}775}}\right]=30.84,$$

$$EU(B)=0.9\left[0.0003(103{,}000)-8.48e^{-\frac{103{,}000}{2{,}775}}\right]+0.1\left[0.0003(98{,}000)-8.48e^{-\frac{98{,}000}{2{,}775}}\right]=30.75.$$

Comparing the expected utility between the Alternative A and the Alternative B, the gambler will still keep his or her preference on Alternative A.

As can be seen from this example, there is solely one (and at most one) switching point on this

Figure 3. Expected Utility Between Alternative A and B in Example 1, 2, and 3.



(c)

decision maker's preference and after the switching point the preference will make no change even the wealth level is getting increasingly larger. This can also be illustrated by Figure 3. (c), which shows the line labeled by triangles, $E_A[u(w)]-E_B[u(w)]$, cross zero only once and keeps a static level with wealth level goes to much larger level. Thus, the utility function in this example belongs to one of the one-switch utility functions.

By comparing these three utility functions from the example above, we can notice that how large the expected utility is determines which alternative a decision maker will select and that how many times (the maximum times) a decision maker's preference will change determines the types of utility functions. In Example 2, the decision maker with exponential utility function does not change his or her preference between two alternatives with the increasing wealth level so it is called zero-switch utility function whereas the decision maker with linear plus exponential utility function changes his or her preference at most once in Example 3 so it belongs to the one-switch utility functions. In Example 1, the decision maker has logarithmic utility function whose preference on Alternative A and Alternative B switched twice so we may name it as two-switch utility function, but it is evident that the logarithmic utility function does not belong to the categories of zero- or one-switch utility functions. In the meantime, there might be three-switch, four-switch or even n-switch utility functions depending on how many times a decision maker's preference will switch at most between two alternatives with increasing wealth level. In this review article, however, we will only focus on zero-switch and one-switch utility function and especially how one-switch utility functions work and can be applied in other fields.

After discussing the basic characterizations of zero- and one-switch utility functions, we will change our attention to what journal articles will be included about this topic via PRISMA in next subsection.

3 Relevant Studies covered by this literature review

The PRISMA (Moher et al., 2009) is one of good methods to review research articles. By constructing the flow chart with PRISMA, we can clearly determine how many and what journal papers are closely associated with the topic of one-switch utility functions. Moreover, based on the citations of these journal papers, the scale of searching about one-switch utility functions can be extended, which finally contributes to build a research network about this topic.

Literature search

The following databases were searched in Dec. 2021: Scopus and Web of Science. No language, publication period or region restrictions were applied.

Eligibility criteria for including studies

Titles, abstracts, introduction and even conclusions were considered to assess papers which are eligible for our research.　The criteria include: 1). papers include the term *one-switch utility*; 2). the topic is associated with theoretical or mathematical economics; 3). papers introduced characteristics of one-switch utility functions or applied one-switch utility functions.　In the first place, the records contain the term, *one-switch utility*, were identified from the sources of Scopus and Web of Science. After removing the duplicates, 42 records were identified.　In the second place, the records above were screened by reviewing the abstract among those 42 records and only 21 records whose research topics are associated with one-switch utility functions were left.　Last but not least, the full-text articles were assessed and the articles whose topics are not in the economic field or do not introduce the characteristics about one-switch utility functions were omitted.　Thus, 14 articles left after filtering.　The flow chart in Figure 4 shows the explicit process.

Results

A total of 14 included studies were listed in the Table 1 which lists and categorizes the filtering

Figure 4. PRISMA Flow Chart

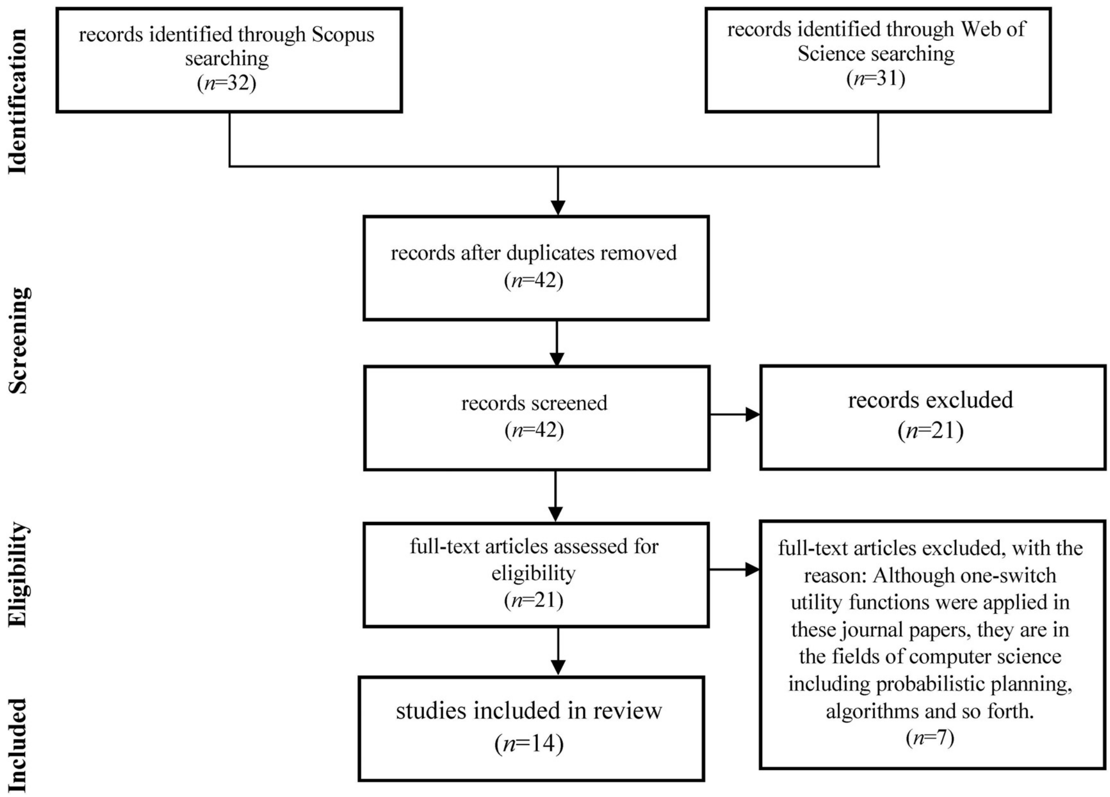


Table 1. Literatures Included in Review

	Authors	Year	Article Title
1	Abbas, A.E., Bell, D.E.	2012	One-switch conditions for multiattribute utility functions
2	Abbas, A.E., Bell, D.E.	2011	One-switch independence for multiattribute utility functions
3	Abbas, A.E., Bell, D.E.	2015	Ordinal one-switch utility functions
4	Abbas, A.E., Chudziak, J.	2013	One-switch utility functions with annuity payments
5	Bakir, Niyazi Onur; Klutke, Georgia-Ann	2011	Information and preference reversals in lotteries
6	Bell, D.E., Fishburn, P.C.	2001	Strong one-switch utility
7	Bell, D.E., Fishburn, P.C.	2000	Utility functions for wealth
8	BELL, D. E	1988	One-switch utility functions and a measure of risk
9	Chudziak, J.	2012	On a class of one-switch multiattribute utility functions
10	Denuit, M.M., Eeckhoudt, L., Schlesinger, H.	2013	When Ross meets Bell: The linex utility function
11	Sandvik, Bjorn; Thorlund-Petersen, Lars	2010	Sensitivity analysis of risk tolerance
12	Scholz, M.	2016	A note on the power of multiattribute one-switch utility functions
13	Tsetlin, I., Winkler, R.L.	2012	Multiattribute one-switch utility
14	Tsetlin, I., Winkler, R.L.	2009	Multiattribute utility satisfying a preference for combining good with bad

results about one-switch utility functions by using PRISMA.

While considering the eligibility of the records, I only concentrated on the research direction in economics. Liu and Koenig (2005, 2006a, 2006b, 2008)'s, Lin et al. (2012)'s and Zeng et al. (2014)'s research was excluded after reviewing abstract, introduction, conclusion or even full texts. They mainly concentrated on the planning and algorithm in computer science based on Markov Decision Processes even though one-switch utility functions were applied in the analytic process. In this review article, only two directions will be discussed: multiattributes utility functions and information economics (see Figure 5). In our analytical process, several literatures are not on the list of Table 1, but they are closely associated with or make contributions to the one-switch utility functions.

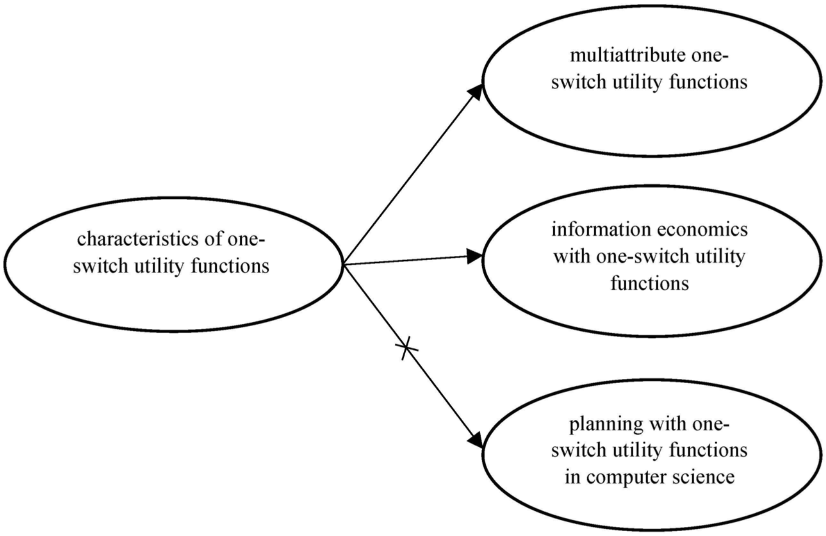
4 One-switch utility functions v.s. Cobb-Douglas utility function

Arrow et al. (1961) introduced Constant Elasticity of Substitution (CES) production functions which include three main functions: linear function, Leontief function and the Cobb-Douglas utility function. These three functions have infinite, zero, and a unit elasticity of substitution between production factors. The Cobb-Douglas utility function describes how much output two or more inputs in a production process can make and the typical inputs are capital and labor. Two-goods Cobb-Douglas utility function will be considered here to make a comparison.

4.1 Demand Functions

Firstly, we will check Marshallian demand functions, indirect utility function, Hicksian demand functions and expenditure function for the Cobb-Douglas utility function (for details, see Appendix B).

Figure 5. Research Focuses in this Review Article



A two-goods Cobb-Douglas utility function, $u(x_1,\ x_2)=x_1^{\alpha_1}\cdot x_2^{\alpha_2}$, with $\alpha_1+\alpha_2=1$, will be considered as an example. Here, x_1, and x_2 are the amounts of Good 1 and Good 2 distributed into production process, respectively; the total budget or wealth a producer owns is w, which is assumed to be equal to total costs of Good 1 and Good 2, $(P_1\cdot x_1+P_2\cdot x_2)$ with the prices, P_1 for Good 1 and P_2, for Good 2.

The Marshallian demand functions are

$$d_1(P_1,\ P_2,\ w)=x_1^*=\frac{w\left(\dfrac{\alpha_1}{\alpha_1+\alpha_2}\right)}{P_1}=\frac{w\cdot\alpha_1}{P_1},$$

$$d_2(P_1,\ P_2,\ w)=x_2^*=\frac{w\left(\dfrac{\alpha_2}{\alpha_1+\alpha_2}\right)}{P_2}=\frac{w\cdot\alpha_2}{P_2}\ ;$$

The indirect utility function is

$$v=v(P_1,\ P_2,\ w)=\left(\frac{\alpha_1}{P_1(\alpha_1+\alpha_2)}\right)^{\alpha_1}\cdot\left(\frac{\alpha_2}{P_2(\alpha_1+\alpha_2)}\right)^{\alpha_2}\cdot w=\left(\frac{\alpha_1}{P_1}\right)^{\alpha_1}\cdot\left(\frac{\alpha_2}{P_2}\right)^{\alpha_2}\cdot w\ ;$$

The Hicksian demand functions are

$$h_1(P_1,\ P_2,\ U^0)=U^0\left(\frac{P_2}{P_1}\cdot\frac{\alpha_1}{\alpha_2}\right)^{\alpha_2},$$

$$h_2(P_1,\ P_2,\ U^0)=U^0\left(\frac{P_1}{P_2}\cdot\frac{\alpha_2}{\alpha_1}\right)^{\alpha_1}\ ;$$

The expenditure function is

$$e(P_1,\ P_2,\ U^0)=U^0\cdot\left(\frac{P_1}{\alpha_1}\right)^{\alpha_1}\cdot\left(\frac{P_2}{\alpha_2}\right)^{\alpha_2}.$$

The Marshallian demand functions and indirect utility function are trying to maximize the utility for a given budget set; and the Hicksian demand functions and expenditure function are trying to minimize costs of reaching a satisfaction level. Thus, the Marshallian demand functions are telling how the quantity of demanded of a good varies with its price; as can be seen from the Marshallian demand functions for both Good 1 and Good 2, the wealth shares spent on various commodities or capital or and labor are constant given by α_1 and α_2. The indirect utility function is showing the most utility an agent can gain with price of a good and wealth, w; the Hicksian demand functions are indicating the cheapest consumption bundle which price of a good and the quantity demanded of it can achieve a given utility level; lastly, the expenditure function is about the minimum amount of money an agent has to pay to get utility with price of goods.

The utility functions relevant to demand theory are associated with quantities consumed while the vNM utility function concentrates on outcomes of risky decision (Willig, 1977). Thus, with the Cobb-Douglas utility function, we consider the amount of a good which was distributed into production process in order to maximize the output of the good. In other words, the Cobb-Douglas utility function shows the relationship between output of a good and input required to produce that good. However, this is not the case for one-switch utility functions, which are the functions stating that an individual's utility is dependent on his or her wealth level. This is also a point which makes one-switch utility functions differ from some functions commonly used in research because like Cobb-Douglas utility function, many utility functions attempt to maximize individual's utility by the goods he or she consumes and their amounts. This is the reason why the Marshallian demand functions were constructed for two goods, x_1, and x_2, to obtain maximum output or utility or why Hicksian demand functions were constructed to obtain minimum costs of two goods based on a given output or utility level.

Next, the Marshallian demand functions, indirect utility function, Hicksian demand functions and expenditure function for two-attribute one-switch utility functions to explore the differences from Cobb-Douglas utility function. Also, these four functions will be resolved for one-switch utility functions to check if we can make the same explanation for one-switch utility functions as we did for Cobb-Douglas utility function. When the one-switch utility functions were extended to multiattribute one-switch utility functions with ith attributes, they should follow:

Theorem (Tsetlin and Winkler, 2012) *A utility function* $\mathrm{u}(\mathbf{x})$ *satisfies the one-switch property if and only if it belongs to one of the following families with outcomes* $\mathbf{x}=(x_1, \ldots, x_N)\in[\underline{\mathbf{x}}, \overline{\mathbf{x}}]$, $-\infty\leq\underline{\mathbf{x}}\leq\overline{\mathbf{x}}\leq\infty$, *for* N *real-valued attributes* $x_1, \ldots, x_N$*:*

1. *quadratics:* $\mathrm{u}(\mathbf{x})=q(c_1x_1+\cdots+c_Nx_N)^2+d_1x_1+\cdots+d_Nx_N$,

   where $q=\pm1$ *and* $c_i\geq0$ *for* $i=1, \ldots, N$ *or* $c_i\leq0$ *for* $i=1, \ldots, N$;

2. *linear plus exponential:* $\mathrm{u}(\mathbf{x})=c_1x_1+\cdots+c_Nx_N+be^{d_1x_1+\cdots+d_Nx_N}$,

   where $d_i\geq0$ *for* $i=1, \ldots, N$ *or* $d_i\leq0$ *for* $i=1, \ldots, N$;

3. *the sumex:* $\mathrm{u}(\mathbf{x})=ae^{c_1x_1+\cdots+c_Nx_N}+be^{d_1x_1+\cdots+d_Nx_N}$,

where $d_i \geq c_i$ *for* $i=1, \ldots, N$;

4. *linear times exponential:* $\mathrm{u}(\mathbf{x})=(c_1x_1+\cdots+c_Nx_N+b)e^{d_1x_1+\cdots+d_Nx_N}$,

where $c_i \geq 0$ *for* $i=1, \ldots, N$ *or* $c_i \leq 0$ for $i=1, \ldots, N$.

In order to make clear easy comparison between one-switch utility functions and the Cobb-Douglas utility function, two-attributes one-switch utility functions will be considered here based on this theorem.

So, suppose $N=2$, we have:

1. *quadratics*: $u(x_1, x_2)=q(c_1x_1+c_2x_2)^2+d_1x_1+d_2x_2$,
2. *linear plus exponential*: $u(x_1, x_2)=c_1x_1+c_2x_2+be^{d_1x_1+d_2x_2}$,
3. *linear times exponential*: $u(x_1, x_2)=(c_1x_1+c_2x_2+b)e^{d_1x_1+d_2x_2}$,
4. *sumex*: $u(x_1, x_2)=ae^{c_1x_1+c_2x_2}+be^{d_1x_1+d_2x_2}$.

Based on the definitions and properties mentioned in Section 2.2, one-switch utility functions are the utility functions for wealth. In single attribute one-switch utility function $u(w)$, w can be thought of as a decision maker's initial wealth level when money is the only attribute. When this decision maker made a decision on a lottery, say, $\tilde{x}$, his or her wealth level will be $(w+\tilde{x})$. In the above theorem (Tsetlin and Winkler, 2012), the utility function u was defined as the utility for increments to an initial wealth so there might be δ_i to make lotteries increase or decrease. When this decision maker made a decision on a lottery, say, $\tilde{\mathbf{x}}$, with increment δ_i on the amount of the ith attribute, his or her wealth level or lotteries $\tilde{\mathbf{x}}$ will be $(\tilde{\mathbf{x}}+\delta)$. In the same manner, the wealth level for lotteries $\tilde{\mathbf{y}}$ will be $(\tilde{\mathbf{y}}+\boldsymbol{\delta})$. Being a one-switch utility function, u should satisfy that $E[u(\tilde{\mathbf{x}}+\boldsymbol{\delta})-u(\tilde{\mathbf{y}}+\boldsymbol{\delta})]$ should change sign no more than once. The reason for this is that one-switch utility functions only allow a decision maker's preference changing at most once with increasing in wealth.

For these four one-switch utility functions with $N=2$, the lotteries, x_1, and x_2, are not the amount of goods as in Cobb-Douglas utility function but usually the value of lotteries and there is not price for these two lotteries which are usually counted as monetary values so we cannot get prices and the budget owned by a decision maker to maximize the utility. Thus, with these characteristics here, it is nearly impossible to derive Marshallian and Hicksian demand functions for one-switch utility functions.

However, when we take the Marshallian and Hicksian demand functions for a utility function, such utility functions are usually about consumer's choice with certainty such as the Cobb-Douglas utility function mentioned above. We can resolve the utility maximization problem with given income and prices and also the expenditure minimization on all goods by setting fixed level of utility. Thus, it will be unreasonable and meaningless to check these demand functions for utility functions under uncertainty.

However, if we reduce two-attribute one-switch utility functions into the utility functions with certainty. That is, the lotteries, x_1 and x_2, will not be the payoffs of gambles but the amount of goods; meanwhile, the total budget a producer owns, W, is still equal to $(P_1x_1+P_2x_2)$, where P_1 and P_2 are prices for Good 1 and Good 2, respectively. Based on the modified two-attribute one-switch utility functions above, the Marshallian demand functions, indirect utility function, Hicksian demand functions and expenditure function for one-switch utility functions are summarized in Table 2 (for details, see Appendix C).

Without the modifications on the original one-switch utility functions, the Marshallian demand functions, indirect utility function, Hicksian demand functions and expenditure function cannot be derived. Under this condition, we can make an account that a consumer of two goods faces positive prices and has a positive income with one-switch utility functions can maximize his or her utility by Marshallian demand functions or minimize his or her costs by Hicksian demand functions as shown in Table 2.

Overall, comparing between one-switch utility functions and the Cobb-Douglas utility function, we can note that the utility functions concerning uncertainties such as Bernoulli utility functions or wealth utility functions are not suitable to derive the Marshallian and Hicksian demand functions. However, for the utility functions concerning consumer's problem to seek a non-negative consumption bundle, deriving the Marshallian and Hicksian demand functions is reasonable and meaningful. For example, a CES utility function ($u(x_1, x_2)=(x_1^\rho+x_2^\rho)^{1/\rho}$, where $0\neq\rho<1$.), the Cobb-Douglas utility function ($u(x_1, x_2)=Ax_1^\alpha x_2^{1-\alpha}$, $0<\alpha<1$, *and* $A>0$), the Stone-Gear utility function ($u(\mathbf{x})=\prod_{i=1}^n(x_i-a_i)^{b_i}$, where $b_i\geq 0$ and $\sum_{i=1}^n b_i=1$.) and so forth were cited as examples to derive demand function in the book (Jehle and Philip, 2011) and also contributed to applications of utility functions.

4.2 Risk Aversion

On the other hand, we can also check risk attitudes of a decision maker with one-switch utility functions while he or she is making a selection. In the first place, it is commonly known that there are three kinds of decision makers: risk-lover, risk-averse and risk neutral decision makers. For a risk averse decision maker, the shape of utility function is concave, which implies that with each additional unit of wealth, the utility of wealth is increasing with a decreasing rate. So, such utility function is also called *diminishing marginal utility*. In this case, for a utility function $u(w)$, $u'(w)>0$, and $u''(w)<0$. Conversely, the shape of utility function of a risk loving decision maker is convex, which implies that with each additional unit of wealth, the utility of wealth is always increasing with an increasing rate, so this utility function is also called *increasing marginal utility*. In this case, for a utility function $u(w)$, $u'(w)>0$, and $u''(w)>0$. Lastly, a decision maker with linear utility function is risk neutral, which implies that the decision maker is indifferent between taking a gamble and not taking a gamble. The marginal utility of wealth is constant to such decision maker so each additional

Table 2. Marshallian Demand Functions, Indirect Utility Function, Hicksian Demand Functions, and Expenditure Function for One-switch Utility Functions

	Marshallian Demand Functions	Indirect Utility Function	Hicksian Demand Functions	Expenditure Function
1. quadratics	$d_1=\frac{w\cdot(c_2^2P_1-c_1c_2P_2)-\frac{d_1P_2^2-d_2P_1P_2}{2q}}{(c_1P_2-c_2P_1)^2}$ $d_2=\frac{w\cdot(c_1^2P_2-c_1c_2P_1)-\frac{d_1P_1P_2-d_2P_1^2}{2q}}{(c_1P_2-c_2P_1)^2}$	$v=q\left(c_1\cdot\frac{w\cdot(c_2^2P_1-c_1c_2P_2)-\frac{d_1P_2^2-d_2P_1P_2}{2q}}{(c_1P_2-c_2P_1)^2}+c_2\cdot\frac{w\cdot(c_1^2P_2-c_1c_2P_1)-\frac{d_1P_1P_2-d_2P_1^2}{2q}}{(c_1P_2-c_2P_1)^2}\right)^2$ $+d_1\cdot\frac{w\cdot(c_2^2P_1-c_1c_2P_2)-\frac{d_1P_2^2-d_2P_1P_2}{2q}}{(c_1P_2-c_2P_1)^2}+d_2\cdot\frac{w\cdot(c_1^2P_2-c_1c_2P_1)-\frac{d_1P_1P_2-d_2P_1^2}{2q}}{(c_1P_2-c_2P_1)^2}$	$h_1=\frac{U^0c_2-qc_2\left(\frac{P_2d_1-P_1d_2}{2qP_1c_2-2qP_2c_1}\right)^2-\frac{P_2d_1d_2-P_1d_2^2}{2qP_1c_2-2qP_2c_1}}{d_1c_2-d_2c_1}$ $h_2=\frac{U^0c_1-qc_1\left(\frac{P_2d_1-P_1d_2}{2qP_1c_2-2qP_2c_1}\right)^2-\frac{P_2d_1^2-P_1d_1d_2}{2qP_1c_2-2qP_2c_1}}{d_2c_1-d_1c_2}$	$e(P_1,\ P_2,\ U^0)=\frac{U^0c_1-qc_1\left(\frac{P_2d_1-P_1d_2}{2qP_1c_2-2qP_2c_1}\right)^2-\frac{P_2d_1^2-P_1d_1d_2}{2qP_1c_2-2qP_2c_1}}{d_2c_1-d_1c_2}$
2. linear plus exponential	$d_1=\frac{wd_2-P_2\ln\left(\frac{P_1c_2-P_2c_1}{P_2bd_1-P_1bd_2}\right)}{P_1d_2-P_2d_1}$ $d_2=\frac{wd_1-P_1\ln\left(\frac{P_1c_2-P_2c_1}{P_2bd_1-P_1bd_2}\right)}{P_2d_1-P_1d_2}$	$v=c_1\cdot\frac{wd_2-P_2\ln\left(\frac{P_1c_2-P_2c_1}{P_2bd_1-P_1bd_2}\right)}{P_1d_2-P_2d_1}+c_2\cdot\frac{wd_1-P_1\ln\left(\frac{P_1c_2-P_2c_1}{P_2bd_1-P_1bd_2}\right)}{P_2d_1-P_1d_2}$ $+be^{d_1\cdot\frac{wd_2-P_2\ln\left(\frac{P_1c_2-P_2c_1}{P_2bd_1-P_1bd_2}\right)}{P_1d_2-P_2d_1}+d_2\cdot\frac{wd_1-P_1\ln\left(\frac{P_1c_2-P_2c_1}{P_2bd_1-P_1bd_2}\right)}{P_2d_1-P_1d_2}}$	$h_1=\frac{U^0-\frac{c_2}{d_2}\cdot\ln\left(\frac{P_2c_1-P_1c_2}{P_1bd_2-P_2bd_1}\right)-be^{\ln\left(\frac{P_2c_1-P_1c_2}{P_1bd_2-P_2bd_1}\right)}}{c_1-\frac{c_2d_1}{d_2}}$ $h_2=\frac{U^0-\frac{c_1}{d_1}\cdot\ln\left(\frac{P_2c_1-P_1c_2}{P_1bd_2-P_2bd_1}\right)-be^{\ln\left(\frac{P_2c_1-P_1c_2}{P_1bd_2-P_2bd_1}\right)}}{c_2-\frac{c_1d_2}{d_1}}$	$e(P_1,\ P_2,\ U^0)=P_1\cdot\frac{U^0-\frac{c_2}{d_2}\cdot\ln\left(\frac{P_2c_1-P_1c_2}{P_1bd_2-P_2bd_1}\right)-be^{\ln\left(\frac{P_2c_1-P_1c_2}{P_1bd_2-P_2bd_1}\right)}}{c_1-\frac{c_2d_1}{d_2}}$ $+P_2\cdot\frac{U^0-\frac{c_1}{d_1}\cdot\ln\left(\frac{P_2c_1-P_1c_2}{P_1bd_2-P_2bd_1}\right)-be^{\ln\left(\frac{P_2c_1-P_1c_2}{P_1bd_2-P_2bd_1}\right)}}{c_2-\frac{c_1d_2}{d_1}}$
3. linear times exponential	$d_1=\frac{wc_2-P_2\left[\frac{P_1c_2-P_2c_1}{P_2d_1-P_1d_2}-b\right]}{P_1c_2-P_2c_1}$ $d_2=\frac{wc_1-P_1\left[\frac{P_1c_2-P_2c_1}{P_2d_1-P_1d_2}-b\right]}{P_2c_1-P_1c_2}$	$v=\left(c_1\cdot\frac{wc_2-P_2\left[\frac{P_1c_2-P_2c_1}{P_2d_1-P_1d_2}-b\right]}{P_1c_2-P_2c_1}+c_2\cdot\frac{wc_1-P_1\left[\frac{P_1c_2-P_2c_1}{P_2d_1-P_1d_2}-b\right]}{P_2c_1-P_1c_2}\right)$ $\cdot e^{d_1\cdot\frac{wc_2-P_2\left[\frac{P_1c_2-P_2c_1}{P_2d_1-P_1d_2}-b\right]}{P_1c_2-P_2c_1}+d_2\cdot\frac{wc_1-P_1\left[\frac{P_1c_2-P_2c_1}{P_2d_1-P_1d_2}-b\right]}{P_2c_1-P_1c_2}}$	$h_1=\frac{c_2\cdot\ln\left(U^0\cdot\frac{P_2d_1-P_1d_2}{P_1c_2-P_2c_1}\right)-\frac{d_2(P_1c_2-P_2c_1)}{P_2d_1-P_1d_2}+d_2b}{d_1c_2-d_2c_1}$ $h_2=\frac{c_1\cdot\ln\left(U^0\cdot\frac{P_2d_1-P_1d_2}{P_1c_1+P_2c_1}\right)-\frac{d_1(P_1c_2-P_2c_1)}{P_2d_1-P_1d_2}+d_1b}{d_2c_1-d_1c_2}$	$e(P_1,\ P_2,\ U^0)=P_1\cdot\frac{c_2\cdot\ln\left(U^0\cdot\frac{P_2d_1-P_1d_2}{P_1c_2-P_2c_1}\right)-\frac{d_2(P_1c_2-P_2c_1)}{P_2d_1-P_1d_2}+d_2b}{d_1c_2-d_2c_1}$ $+P_2\cdot\frac{c_1\cdot\ln\left(U^0\cdot\frac{P_2d_1-P_1d_2}{P_1c_1+P_2c_1}\right)-\frac{d_1(P_1c_2-P_2c_1)}{P_2d_1-P_1d_2}+d_1b}{d_2c_1-d_1c_2}$
4. Sumex	$d_1=\frac{w(c_2-d_2)-P_2\cdot\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{P_1(c_2-d_2)+P_2(d_1-c_1)}$ $d_2=\frac{w(c_1-d_1)-P_1\cdot\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{P_1(d_2-c_2)+P_2(c_1-d_1)}$	$v=ae^{c_1\cdot\frac{w(c_2-d_2)-P_2\cdot\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{P_1(c_2-d_2)+P_2(d_1-c_1)}+c_2\cdot\frac{w(c_1-d_1)-P_1\cdot\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{P_1(d_2-c_2)+P_2(c_1-d_1)}}$ $+be^{d_1\cdot\frac{w(c_2-d_2)-P_2\cdot\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{P_1(c_2-d_2)+P_2(d_1-c_1)}+d_2\cdot\frac{w(c_1-d_1)-P_1\cdot\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{P_1(d_2-c_2)+P_2(c_1-d_1)}}$	$h_1=\ln\left(\frac{U^0}{ae^{\frac{c_2\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_2-d_2}}+be^{\frac{d_2\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_2-d_2}}}\right)\cdot\frac{c_2-d_2}{x_1(c_2d_1-c_1d_2)}$ $h_2=\ln\left(\frac{U^0}{ae^{\frac{c_1\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_1-d_1}}+be^{\frac{d_1\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_1-d_1}}}\right)\cdot\frac{c_1-d_1}{x_2(c_1d_2-c_2d_1)}$	$e(P_1,\ P_2,\ U^0)=P_1\cdot\ln\left(\frac{U^0}{ae^{\frac{c_2\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_2-d_2}}+be^{\frac{d_2\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_2-d_2}}}\right)\cdot\frac{c_2-d_2}{x_1(c_2d_1-c_1d_2)}$ $+P_2\cdot\ln\left(\frac{U^0}{ae^{\frac{c_1\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_1-d_1}}+be^{\frac{d_1\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_1-d_1}}}\right)\cdot\frac{c_1-d_1}{x_2(c_1d_2-c_2d_1)}$

unit of wealth will add the same utility of wealth to this decision maker.

While considering the risk aversion to a decision maker, a decision maker's risk attitude can be determined by the Arrow-Pratt measure of absolute risk aversion (Pratt, 1964; Arrow, 1971).

$$ARA_u(w) = -\frac{u''(w)}{u'(w)},$$

where $u(\cdot)$ is the utility function derived from w, the wealth of a decision maker.

Risk aversion needs $ARA_u(w)$ to be positive, and a larger value of $ARA_u(w)$ indicates a more risk averse decision maker. Meanwhile, $ARA_u(w)$ should be monotonically decreasing if a decision maker is decreasingly risk averse.

The risk aversions for each of one-switch utility functions are:

1. *quadratics*: $u(w) = aw^2 + bw + c$, $u'(w) = 2aw + b$, $u''(w) = 2a$,

$$ARA_{quadratics}(w) = -\frac{2a}{2aw+b}\ ;$$

2. *the sumex*: $u(w) = ae^{bw} + ce^{dw}$, $u'(w) = abe^{bw} + cde^{dw}$, $u''(w) = ab^2e^{bw} + cd^2e^{dw}$,

$$ARA_{sumex}(w) = -\frac{ab^2e^{bw} + cd^2e^{dw}}{abe^{bw} + cde^{dw}}\ ;$$

3. *linear times exponential*: $u(w) = (aw+b)e^{cw}$, $u'(w) = (a+bc+acw)e^{cw}$, $u''(w) = (2ac + bc^2 + ac^2w)e^{cw}$,

$$ARA_{linear\ times\ exponential}(w) = -c - \frac{ac}{(a+bc+acw)}\ ;$$

4. *linear plus exponential*: $u(w) = aw + be^{cw}$, $u'(w) = a + bce^{cw}$, $u''(w) = bc^2e^{cw}$,

$$ARA_{linear\ plus\ exponential}(w) = -\frac{bc^2e^{cw}}{a + bce^{cw}}.$$

Although it is possible to obtain the absolute risk aversion for all these four one-switch utility functions, we cannot ignore if the risk aversion is realistic and reasonable for a decision maker. The $ARA_u'(w)$ for quadratics and linear times exponential is $\frac{4a^2}{(2aw+b)^2}$ and $\frac{a^2c^2}{(a+bc+acw)^2}$, respectively, which are surely positive. Thus, Bell (1988) noted the quadratic and the linear times exponential one-switch utility functions have increasing risk aversion with any values of parameter a, b, and c. However, increasing risk aversion is an unreasonable behavior for a rational decision maker who is likely to be more nervous, anxious, and careful to a gamble with the wealth level increases. This is by no means a realistic situation in decision-making process because most decision makers are willing to face a challenge of risk in order to gain more wealth back in a bet while his or her wealth level is high. Therefore, some research was conducted with the other two: sumex and linear plus exponential one-switch utility functions due to the limitation of quadratic and linear times exponential one-switch utility functions in application.

In addition, the $ARA'_u(w)$ for linear plus exponential one-switch utility function is $-\frac{abc^3e^{-cw}}{(ae^{-cw}+bc)^2}$.

As Bell (1988) noted, the necessary and sufficient conditions for the parameters are $a>0$, $b<0$, and $c<0$, which can make the utility function decreasingly risk averse. In the meantime, the sumex one-switch utility functions can be both increasingly and decreasingly risk averse. In order to make the functions more suitable to realistic situations, Bell (1988) put more constraints including decreasing risk aversion on a decision maker at all wealth level and the risk neutrality when the decision maker is extremely rich in Proposition 3 so the updated format of linear plus exponential one-switch utility function is $u(w)=w-be^{-cw}$ with positive b and c. Meanwhile, the condition 'risk consistency' in Proposition 6 (Bell, 1988) implies that with increase in wealth level, the newly preferred lottery always has a higher mean under the situation of preference switch (Bell and Fishburn, 2001). Thus, the feasible utility function, $u(w)=aw-be^{-cw}$ where $a\geq 0$ and b and c are positive, was introduced by Bell and Fishburn (2001) as a new term, strong one-switch utility function.

In addition, because of the requirement of being reasonable to one-switch utility functions such as the decreasing risk aversion of a decision maker, Denuit, Eeckhoudt and Schlesigner (2013) applied linear plus exponential one-switch utility function (called linex utility function) with the notion of stronger decreasing absolute risk aversion (Ross DARA); similarly, these advantages were also mentioned by Gelles and Mitchell (1999) to explore broadly DARA. Furthermore, both Scholz (2016) and Bell (1988) cited Schlaifer (1971)'s research to emphasize the convenient assessment of the sumex one-switch utility function.

Tsetlin and Winkler (2012) also showed the advantage of the sumex one-switch utility function which multiattribute one-switch utility functions could be approximated by the sumex type of one-switch utility functions, even in multiattribute case so if one utility function satisfies the one-switch rule, it can be approximated by the sumex one-switch utility function. With sufficient basic characteristics of the sumex functions, Scholz (2016) is the only one researcher who investigated if the perceivable preferential switches can be accurately modelled by the sumex functions via experimental study.

Based on the discussion about the characteristics of one-switch utility functions above, it is evident that most researchers select the sumex and linear plus exponential one-switch utility functions in research to make the utility functions of a decision maker more reasonable, rational, and easily applicable. In most cases, the applicability of one-switch utility functions is preferred to be limited to the sumex and linear plus exponential functions.

On the other hand, we have found an optimal value of inputs to make output of Cobb-Douglas utility function be maximized but here we will also attempt to consider how a decision maker's risk attitude is with Cobb-Douglas utility function. However, the Cobb-Douglas utility function is usually considered under certainties as what we also did in the subsection of demand functions. So, the concepts of variables in Cobb-Douglas utility function need to be adjusted to make the utility function more reasonable. In the meantime, while considering the Arrow-Pratt measure of risk aversion, we traditionally use wealth, w, as an argument for the function. However, James Cox and Vjollca Sadiraj (2004) considered the Arrow-Pratt measure of risk aversion in a more flexible way by using wealth and income for vNM utility function. Thus, if we borrow this idea that applying wealth w and income y into the Cobb-Douglas utility function, would be more reasonable to make a further explanation after obtaining the absolute risk aversion. The Cobb-Douglas utility function, $u(x_1, x_2)=x_1^{\alpha_1}\cdot x_2^{\alpha_2}$, with $\alpha_1+\alpha_2=1$, can be transformed to $u(y, w)=y^{\alpha_1}\cdot w^{\alpha_2}$, in which the independence between the income and wealth is assumed. As James Cox and Vjollca Sadiraj (2004) considered, wealth belongs to a decision maker's total assets, so it will bet set as a fixed value. So, the absolute risk aversion respect to income with fixed value of wealth will be:

$$ARA_u(y, w)=-\frac{u''(y, w)}{u'(y, w)}=-\frac{\alpha_1(\alpha_1-1)w^{\alpha_2}y^{\alpha_1-2}}{\alpha_1 w^{\alpha_2}y^{\alpha_1-1}}=\frac{\alpha_2}{y}.$$

In the Cobb-Douglas utility function, α_2 is positive and income is usually positive as well, so $ARA'_u(y, w)=-\frac{\alpha_2}{y^2}$ is negative, which means that the decision maker is decreasingly risk averse. This implies that as a decision maker with the Cobb-Douglas utility function owning fixed wealth obtains more incomes, he or she is willing to face more uncertainties and bear more risks. This is quite reasonable behavior to a rational decision maker because most individuals will have much more strongly risk-bearing minds and make selections with higher risks when they own more money.

Comparing the Cobb-Douglas utility function with one-switch utility functions, we can recognize that the former is trying to resolve the amount of input which can make maximum output or utility, or the minimum amount of input given fixed utility whereas the latter is about making a selection under uncertainties. Thus, most researchers usually do not tend to place the Cobb-Douglas utility function into uncertain conditions (even though Faro (2013) extend the Cobb-Douglas utility function into uncertainty) or one-switch utility functions into certain circumstances.

Last but not least, as can be seen from the risk aversion, ARA, for each one-switch utility functions, the value of risk aversion depends on the wealth w while in an example will be discussed below in Section 5.1, the utility function $u(x)=1-e^{-cx}$ has constant risk aversion c, which implies the risk aversion of a decision maker is independent of the payoff or wealth. Thus, comparatively speaking, one-switch utility functions are more complicated to apply due to the dependence while considered the problems or applications associated with risk aversion.

5 Discussions on Further Applications

In addition to the research which contributed to the characterization of one-switch utility functions mentioned in Section 2, we also made a comparison between one-switch utility functions and the Cobb-

Douglas utility function in Section 4. Furthermore, researchers attempted to consider how to apply one-switch utility functions in different fields of studies, albeit theoretical. In this section, we will solely discuss the applications in economics of information and functional equations with multiattributes.

5.1 Economics of Information

Most researches about utility functions were conducted in order to make accounts and explanations about how a decision maker reacts in the context with uncertainty and risk. Under the uncertain circumstances, a decision maker may ask him/herself, 'What can I do to reduce the risk in my selection process?' The answer is information acquisition. As Lawrence (2012) defined in his book, '. . . *information* is defined as any stimulus that has changed the recipient's knowledge, that is, that has changed the recipient's probability distribution over a well-described set of states.' Prior to making decisions, decision makers may be able to acquire information which is helpful to making decisions so that the risks could be avoided under uncertain conditions. Thus, the economic value of information or even the economics of information is a topic worth exploring.

In this research topic, there are many previous literatures exploring the relationship between the value of information and a decision maker's risk attitude. The measure of the value of information, however, are various with different utility functions to a decision maker in a range of researches. A term, *the expected value of perfect information* (EVPI), answers the question, 'If I were able to find out exactly which state will occur before I choose the optimal action, what do I expected that ability to be worth?' Mehrez and Stulman (1982) studied the distributional properties of the EVPI which is a linear function of standard deviation so as to reject or accept an investment for risk neutral decision makers. Moreover, Mehrez (1985) also attempted to explore the bound on the maximum amount of money which a decision maker with a constant mean of risk aversion is willing to pay to completely reduce the uncertainty and while facing projects with nonpositive expected monetary value, a risk neutral decision maker is willing to pay as much money for perfect information as a risk averse decision maker did.

The relationship between risk aversion and the demand of information attracted researchers to pay more attention on. Some may ask what kind of relation is between risk aversion and the demand of information. The monotonicity between the value of information and degree of risk aversion was the main focused point we should carry out. It is commonly known that risk averse decision makers are willing to gather more information to reduce the risks in their decision making so there will be monotonicity in the relationship between the value of information and risk aversion, which implies that the more risk averse a decision maker is, the higher demand of information is. However, the opposite condition shows that the demand of information decreases with risk aversion because seeking information is also a risk behavior and the information gathered by a decision maker might include useless information or the information not helpful in decision making process. Because a risk averse decision maker is not willing to bear more risks, they may be unwilling to gather that information which may be able to reduce the risks in their decision-making process later so the relationship is not monotonic anymore.

Willinger (1989) concluded that with constant absolute risk aversion (CARA) utility function, the expected value of information which measures the decision maker's willingness to pay for information will increase with risk averse if and only if that information can reduce the expected risk which is associated with decision maker's optimal actions. Furthermore, some authors also considered how different types of information may have impact on the relationship with risk aversion. For instance, Nadiminti et al. (1996) explored how the methods of payments for information, including costless information which a decision maker does not need to make any payments for it, ex-ante payment which a decision maker makes a payment before making decision and contingent payment which a decision maker will make a payment for information after he or she realized the payoff, impact the relationship between the value of information and risk aversion. In Nadiminti et al. (1996)'s research, CARA utility function was applied again due to the independence between decision makers' risk attitude and payoff.

With further development in this research topic, there are also several measures of value of information such as buying price, selling price, uncertain equivalent, expected utility increases and probability price (LaValle, 1968). So, applying different measures to derive the value of information will have impact on the monotonicity of the relationship between the value of information and degree of risk aversion. Thus, some researchers attempted to check the monotonicity of the relationship between the value of information and risk aversion under different measures of the value of information. Two measures of the value of information most commonly applied in researches are selling price and buying price measures (for details, see Appendix D). For instance, Bakır (2015) illustrated that there is a monotonic relationship between the selling price of information and risk aversion. However, there are also exceptions that more risk aversion does not necessarily generate a greater perfect information value (Eechhoudt and Godfroid, 2000). Bakır and Klutke (2014) compared the buying price of information for risk-averse decision makers with that for risk-neutral decision makers in two-action decision problems which are a setting of the initial decision being to reject or accept the information.

Returning our story to one-switch utility functions, we may ask how one-switch utility functions work in this research topic. Evaluating the measures of value of information, Bakır and Klutke (2011) discussed the conditions under which the methods of expected utility increase, the selling price method and the buying price method make an agreement in decision-making situations with *linear-plus-exponential* one-switch utility function, which is the only function which a decision maker keeps risk consistent as he or she gets wealthier. In addition, the demand of information before making decisions is relatively significant to the utility functions depending on the initial wealth levels. The case of *quadratic* one-switch utility function was considered by Abbas et al. (2013), who state that the value of information is monotonic to the risk aversion only in quadratic utility function. Their research applied buying price of information and also the information was also divided into perfect information and partition information. As we mentioned above, there are four types of one-switch utility functions so for further extension Bakır (2017) checked the monotonicity with special conditions in the *sumex* and *linear times exponential* one-switch utility functions, which shows that there is no monotonicity for linear times exponential utility functions but monotonicity for the sumex with some constraints.

As utility functions for wealth, one-switch utility functions are affected by the wealth of a decision maker. In the meantime, we notice that one-switch utility functions are more suitable to resolve problems under uncertainties. The acquisition of information is a common belief and a practical means of reducing risks under uncertainties in a decision-making process. The value of information may influence the level of decision maker's wealth and even the utilities rather than impacts on any impacts on consumption or cost in some utility functions. Thus, checking the monotonicity of the relationship between risk aversion and the value of information with one-switch utility functions is a possible and feasible research direction. However, as we have mentioned above, there were many researches about how the monotonicity of the relationship is between risk aversion and the value of information with one-switch utility functions, especially applying the measures of buying price, selling price, and expected utilities being the values of information. Therefore, in my point of view, the next step of research for one-switch utility functions in this field should not emphasize too much on how the different measures of the value of information influence the monotonicity but concentrate on how different types of information impact the monotonicity of relationship between the value of information and risk aversion.

While Bakır (2017) supplemented the research of Abbas et al. (2013) with one-switch functions to some extent, the gap in the research of Bakır (2017) is that the author solely considered perfect information. Compared with perfect information, the partition information is rather realistic case in decision making process because the information purchased by decision makers cannot completely eliminate all the uncertainty as Abbas et al. (2013) defined. Furthermore, the information is categorized by methods of payment into costly information which includes ex-ante payment and contingent payment and costless information in research of Nadiminti et al. (1996) so it is possible to conduct research about how the monotonicity of relationship between the risk aversion and the value of information will differ based on costless and costly information for decision makers with one-switch utility functions.

5.2 Mutiattributes

5.2.1 Independence

In Section 2, the basic concepts of one-switch utility functions with single attribute were discussed but researchers also conducted research with multiattribute conditions. Prior to the introduction of application and further research about one-switch utility functions with multiattributes, a concept which should not be ignored is *independence*.

While making a gamble with two or more attributes (multiattributes), a decision maker must take the independence into account. As Abbas and Bell (2011) cited, Keeney and Raiffa (1976) introduced *utility independence* in their research which states that if an attribute X is utility independent of attribute Y, then the preferences for lotteries over X do not depend on the value of Y. Or, it is also referred to as *zero-switch independence* in Abbas and Bell (2011) which emphasizes the unchanged preferences for attribute X as attribute Y varies. Abbas and Bell (2011) pointed out an example that a young person may consider various factors such as the enjoyment of a particular house and the affordable price of a house which can be represented by two attributes, Home Quality (X) and Wealth (Y), respectively while he or she is willing to buy a new house. What we should consider first is if or not X is utility independent of Y. In this case, it is evident that the decision maker will prefer the house with higher quality whose price is reasonably higher than the one with lower quality if the decision maker has higher wealth level. This implies that the decision maker has to select the lower-quality house rather than the higher one depending on the wealth he or she owns. However, there must exist a turning point of wealth level which can make this decision maker to select the lower-quality house below it or select the higher quality one above it.

Thus, Abbas and Bell (2011) introduced a new independent assumption, *one-switch independence*, based on the one-switch property to make the multiattribute utility functions be decomposed into pieces more easily in order to fill the research gap:

An attribute X exhibits one-switch independence from another attribute Y, written as X 1S Y, if preference between any pair of gambles on X can switch at most once as the level of Y increases.

One point we can gain from the definitions of zero-switch and one-switch independence is that the zero-switch independence is included in the conditions of one-switch independence. In the two attributes case, we may consider the lotteries over X with the fixed value of y so y can be treated as a parameter and the four two-attribute one-switch utility functions will be (Abbas and Bell, 2012):

1. *quadratics*: $u(x, y) = ayx^2 + byx + cy$,
2. *linear plus exponential*: $u(x, y) = ayx + bye^{cyx} + dy$,
3. *linear times exponential*: $u(x, y) = (ayx + by)e^{cyx} + dy$,
4. *sumex*: $u(x, y) = aye^{cyx} + bye^{dyx} + fy$.

Some may ask, '*why do we need to consider the one-switch independence?*' Abbas and Bell (2011) answered this question that one-switch independence could be treated as a 'backup' when utility independence does not hold between variables. Moreover, if one-switch independence does not hold for multiattributes, two-switch, three-switch, or even n-switch independence can be considered further.

Last but not least, based on the zero-switch independence and one-switch independence, several notions of independence were also introduced later, say strong zero-switch independence, strong one-switch independence, contextual one-switch independence, complete one-switch independence, mutual one-switch independence (Abbas and Bell, 2012), ordinal zero-switch independence and ordinal one-switch independence (Abbas and Bell, 2015).

5.2.2 Functional Equations

Next, we will consider multiattribute one-switch utility functions with functional equations, albeit mathematical and theoretical. In this subsection, there are two main parts including the approximation of one-switch utility functions with multiattributes and looking for solutions for functional equations. In functional equations, the unknowns are not the normal variables as we recognized in the standard utility functions but functions which are variables in functional equations so most researchers try to search solutions to the functional equations or look for what conditions or constraints the functional equations should satisfy. In other words, the solutions and variables for functional equations are still functions rather than real numbers (Leigh-Lancaster, 2006). When we attempt to use functional equations as an efficient tool to build models for a phenomenon, the functional equations can precisely describe the quantitative relationship for one or more equations in a model, combing with the constraints (Aczél, Falmagne, and Luce, 2000). Aczél (1966) considered functional equations into many fields, including geometry, physics, probability theory and so forth but did not cover too much in the field of economics so Eichhnorn (1978) introduced how functional equations can be applied in economics with mathematical assumptions by formulate and solve functional equations.

With consideration of domain and rule, a given function is a solution which should also be a function to a functional equation, which implies that for values of the independent variable across its domain, a given function satisfies the functional equations (Leigh-Lancaster, 2006).

Let's begin the story with a theorem introduced by Abbas which illustrated how the one-switch utility functions work in shift transformation with functional equations.

Theorem (Abbas, 2018, Chapter 18) *A utility function, U, satisfies one-switch rule in preference for all gambles when a shift transformation $g(w)=w+\delta$ is applied to the outcomes of gambles when the δ is increasing if and only if*

$$U(w+\delta)=k_0(\delta)+k_1(\delta)[f_1(w)+\phi(\delta)f_2(w)], \tag{5.2.1}$$

where $k_1(\delta)$ does not change the sign, and $\phi(\delta)$ is monotonic function.

The equation (5.2.1) can be expanded to

$$U(w+\delta)=k_0(\delta)+k_1(\delta)f_1(w)+k_2(\delta)f_2(w), \tag{5.2.2}$$

where $k_2(\delta)=\phi(\delta)k_1(\delta)$.

Abbas (2018) summarized that four one-switch forms are the solutions of deriving a differential equation via Taylor expansion by Bell (1988). All those four one-switch utility functions can satisfy the functional equations in this Theorem. Abbas (2018), however, solely proved the *linear plus exponential* one-switch utility function, $U(w)=aw+be^{-\gamma w}+c$, which satisfies the functional equation below:

$$U(w+\delta)=f_1(w)+\phi(\delta)\cdot f_2(w)+k_0(\delta), \tag{5.2.3}$$

where $k_1(\delta)=1$, $\phi(\delta)=b(e^{-\gamma\delta}-1)$, $k_0(\delta)=a\delta$.

In the same manner, we will enrich the gaps for the other three one-switch utility functions here (for details, see Appendix E).

The *sumex* one-switch utility function, $U(w)=ae^{-\lambda w}+be^{-\gamma w}+c$, satisfies the functional equation:

$$U(w+\delta)=f_1(w)\cdot k_1(\delta)+k_1(\delta)\cdot f_2(w)\cdot\phi(\delta)+k_0(\delta), \tag{5.2.4}$$

where $f_1(w)=ae^{-\lambda w}$, $k_1(\delta)=e^{-\lambda\delta}$, $f_2(w)=be^{-\gamma w}$, $\phi(\delta)=e^{(\lambda-\gamma)\delta}$, and $k_0(\delta)=c$.

The *quadratic* one-switch utility function, $U(w)=aw^2+bw+c$, satisfies the functional equation:

$$U(w+\delta)=f_1(w)+\phi(\delta)\cdot f_2(w)+k_0(\delta), \tag{5.2.5}$$

where $f_1(w)=(aw^2+bw+c)$, $k_1(\delta)=1$, $\phi(\delta)=\delta$, $f_2(w)=(2aw+b)$, and $k_0(\delta)=a\delta^2$.

The *linear times exponential* one-switch utility function, $U(w)=(aw+b)e^{-\gamma w}+c$, satisfies the functional equation:

$$U(w+\delta)=f_1(w)\cdot k_1(\delta)+k_1(\delta)\cdot f_2(w)\cdot\phi(\delta)+k_0(\delta), \tag{5.2.6}$$

where $f_1(w)=aw\cdot e^{-\gamma w}$, $k_1(\delta)=e^{-\gamma\delta}$, $f_2(w)=e^{-\gamma w}$, $\phi(\delta)=(a\delta+b)$, and $k_0(\delta)=c$.

As can be seen from the above, the functional equations are one of good method to check if one utility function satisfies one-switch rule by checking the sufficiency of functional equations rather than substituting real numbers, say, initial wealth and gambles, to check the preference switches in the utility functions.

On the other hand, if we set $k_2(\delta)f_2(w)$ to be equal to zero in equation (5.2.2), the equation will be reduced to the functional equations for zero-switch utility functions.

$$U(w+\delta)=k_0(\delta)+k_1(\delta)f_1(w), \tag{5.2.7}$$

where $k_1(\delta)=1$ for a linear utility function; $k_1(\delta)=e^{\gamma\delta}$ for an exponential utility function.

Similarly, Pfanzag (1959) also characterized the linear and exponential utility functions by functional equations which are same as the equation (5.2.3). Furthermore, considering both one-switch utility functions and zero-switching utility functions simultaneously, Abbas and Bell (2012) generalized that one-switch utility functions must satisfy the system of functional equations with wealth level, w, increment in wealth, z and functions, U, K, M, W, L, W, k, l :

$$U(w+z)=K(z)U(w)+M(z)W(w)+L(z), \tag{5.2.8}$$

$$W(w+z)=k(z)W(w)+l(z), \tag{5.2.9}$$

which are also cited as the basis in Chudziak (2012)'s research on the multiattribute case. Obviously, equation (5.2.9) is same as equation (5.2.7) as a functional equation for zero-switch utility functions. The system of functional equations constructed by equations (5.2.8) and (5.2.9) includes zero-switch utility function, $W(w)$, which implies zero-switch utility functions belong to one-switch utility functions as we mentioned in Section 2.1.

With the properties of one-switch utility functions in functional equations mentioned above, especially the two functional equations (5.2.8) and (5.2.9) which one-switch utility functions ought to satisfy (Abbas and Bell, 2012), another extension of this topic introduced by Abbas and Chudziak (2013) is to extend one-switch utility functions with functional equations under the multi-period and uncertain cash flows with multiattributes as the decision maker's wealth level which represented by the annuity payment increases.

Abbas et al. (2009) has resolved the functional forms of multiattribute utility functions under zero-switch utility preferences between multi-period cash flows as the decision maker's initial wealth level increases by his or her annuity payments which were paid z amount of money in each period. Moreover, the case with one-switch utility functions was considered in the Abbas and Chudziak's paper. The one-switch utility functions over uncertain n-period cash flows as the decision maker's wealth level increase. In the meantime, the wealth level was represented by the annuity payment that pays an equal amount z for n continuous periods. In the paper, the functional forms of multiple attribute utility functions were considered to maximize the preference with one-switch rule. Therefore, Abbas and Chudziak (2013) figured out the continuous non-constant solutions for the system functional equations with one-switch utility functions with multiattributes.

$$U(w_1+z,\ \ldots,\ w_n+z)=K(z)U(w_1,\ \ldots,\ w_n)+M(z)W(w_1,\ \ldots,\ w_n)+L(z), \tag{5.2.10}$$

$$W(w_1+z,\ \ldots,\ w_n+z)=k(z)W(w_1,\ \ldots,\ w_n)+l(z). \tag{5.2.11}$$

Thus, the assumption that the decision maker's annuity payment, z, is constrained by an inflation parameter, λ, will be made for further research. In terms of the parameter, λ, it will also be changed with exponential for n successive periods. So, the functional equations with adjusted inflation will be:

$$U(w_1+\lambda^1 z,\ \ldots,\ w_n+\lambda^n z)=K(\lambda^1 z,\ \ldots,\ \lambda^n z)U(w_1,\ \ldots,\ w_n)+M(\lambda^1 z,\ \ldots,\ \lambda^n z)W(w_1,\ \ldots,\ w_n)+L(\lambda^1 z,\ \ldots,\ \lambda^n z), \tag{5.2.12}$$

$$W(w_1+\lambda^1 z,\ \ldots,\ w_n+\lambda^n z)=k(\lambda^1 z,\ \ldots,\ \lambda^n z)W(w_1,\ \ldots,\ w_n)+l(\lambda^1 z,\ \ldots,\ \lambda^n z). \tag{5.2.13}$$

By putting different constrains on the assumptions of the functional equation (5.2.10) and (5.2.11) in Abbas and Chudziak's work (2013), it is likely to obtain the general solutions satisfying both functional equations (5.2.12) and (5.2.13) above with discounting λ^n, $n=1,\ 2,\ \ldots,\ n$.

In order to make more contribution on the research of functional equations (5.2.12) and (5.2.13) into extended economic fields, it is essential to reduce this system of functional equations into simpler equations whose format will be able to be applied and calculated more simply. So, the system of functional equations below can be explored, then,

$$U(w_1+z_1, \ldots, w_n+z_n)=K(z_1, \ldots, z_n)U(w_1, \ldots, w_n)+M(z_1, \ldots, z_n)$$
$$W(w_1, \ldots, w_n)+L(z_1, \ldots, z_n), \tag{5.2.14}$$
$$W(w_1+z_1, \ldots, w_n+z_n)=k(z_1, \ldots, z_n)W(w_1, \ldots, w_n)+l(z_1, \ldots, z_n). \tag{5.2.15}$$

The general solutions, with nonconstant W, continuous at a point, of the functional equation (5.2.15) on an arbitrary nonempty open subset of $\mathbb{R}^n$, has been obtained and proved by Proposition 2.1 (Abbas, Aczél and Chudziak, 2009).

Proposition 2.1 *Assume that D is a nonempty connected open subset of $\mathbb{R}^n$, k, l map D into $\mathbb{R}^n$, and $W: D \to \mathbb{R}$ is nonconstant and continuous at a point. Then a triple (k, l, W) satisfies equation (5.2.15) if and only if one of the following two situations occurs.*

1. *There exist constants $a_1, \ldots, a_n \in \mathbb{R}$ with $\sum_{j=1}^{n}|a_j| \neq 0$, $A \in \mathbb{R}\backslash\{0\}$ and $C \in \mathbb{R}$ such that*

$$\begin{aligned} &W(x_1, \ldots, x_n)=Ae^{\sum_{j=1}^{n}a_jx_j}+C, \\ &k(z_1, \ldots, z_n)=e^{\sum_{j=1}^{n}a_jz_j}, \\ &l(z_1, \ldots, z_n)=C(1-e^{\sum_{j=1}^{n}a_jz_j}). \end{aligned} \tag{5.2.16}$$

2. *There exist constants $a_1, \ldots, a_n \in \mathbb{R}$ with $\sum_{j=1}^{n}|a_j| \neq 0$ and $B \in \mathbb{R}$ such that*

$$\begin{aligned} &W(x_1, \ldots, x_n)=\textstyle\sum_{j=1}^{n}a_jx_j+B, \\ &k(z_1, \ldots, z_n)=1, \\ &l(z_1, \ldots, z_n)=\textstyle\sum_{j=1}^{n}a_jz_j. \end{aligned} \tag{5.2.17}$$

Thus, based on the existing solutions for the equation (5.2.15), the further research direction could be the exploration of conditions or constraints which should be added on the solutions (5.2.16) and (5.2.17) to find general solutions which can satisfy a system of functional equations for both equation (5.2.14) and equation (5.2.15). The reason for this is that the proposition 2.1 only shows the sufficient conditions or constraints for functions W, k, and l. However, the conditions for other functions, U, K, M, and L, in equation (5.2.14) should be considered based on the conditions in proposition 2.1 to find general conditions which can make the whole system of functional equations satisfy.

Although the term, annuity payment, was applied in the research by Abbas and Chudziak (2013), the main procedures are still mathematical and theoretical rather than practical or empirical the authors only borrowed annuity payment, z, as a constant increment in a decision maker's wealth level. Therefore, finding general solutions for functional equations with different constraints or conditions can be one of research fields associated with one-switch utility functions. Thus, Abbas and Chudziak (2013) left a further gap at the end of the research, equation (5.2.12) and equation (5.2.13), which underscore that more constraints should be put on the annuity payment such as inflation rate to make the research more realistic.

6 Conclusions

Due to the potentials of one-switch utility functions, we have provided basic characterization of zero-switch and one-switch utility functions at beginning of this review paper, and then made comparison between the Cobb-Douglas utility function and one-switch utility functions by deriving demand functions and risk aversions. Finally, two applicable fields and further research directions were considered with one-switch utility functions. Thus, there are several conclusions we can draw from the discussion in this review article:

Conclusion 1: *As a utility function of wealth, one-switch utility functions are focusing not only on utility maximization but, more importantly, on the at most once change on preference.*

Conclusion 2: *The sumex and linear plus exponential utility functions are comparatively more easily-applicable among four one-switch utility functions in further research because of more realistic risk attitude of a decision maker.*

Conclusion 3: *Unlike the Cobb-Douglas utility function, one-switch utility functions are not suitable and reasonable to derive the Marshallian and Hicksian demand functions.*

Conclusion 4: *How different categories of information, costless or costly, have impacts on the relationship between the value of information and risk aversion in one-switch utility functions could be considered in further research.*

Conclusion 5: *Searching for general solution to a system of functional equations to one-switch utility functions with different constraints, albeit theoretical and mathematical, could fill the gap in previous studies.*

Through the characteristics of one-switch utility functions mentioned in Section 2, it is obvious that one-switch utility functions is so unique that they are different from the utility functions we might recognized before. It is one-switch rule that plays a crucial part in the decision-making process of a decision maker. For the research exploring the exact turning point of the preference to a decision maker via empirical studies, the unique property of at-most-once change of preference in one-switch utility functions should be considered carefully. Furthermore, based on the previous literatures and our discussion for all the four one-switch utility functions, quadratic and linear times exponential one-switch utility functions are not reasonable in real-world situations due to the increasing risk aversion to a decision maker whereas the other two, the sumex and linear plus exponential one-switch utility functions, are comparatively more reasonable and there have been several previous researches conducted based on such two utility functions as we have already discussed in previous sections. Thus, it will still be potentially feasible to carry out future tasks based on these two one-switch utility functions.

Among these four one-switch utility functions, a decision maker's preference is dependent of his or her wealth level. Thus, compared with Cobb-Douglas utility function, one-switch utility functions contribute to how the change of wealth level with lotteries impacts a decision maker's preference rather than the maximization of output with particular inputs. In other words, unlike the Cobb-Douglas utility function, which is usually applied with certainty, one-switch utility functions are more suitable to be applied in uncertain situations. Deriving consumption bundles to maximize utility for a given wealth or minimize costs to reach given level utility is not what one-switch utility functions concern so they might be not efficiently applicable and not meaningful to derive demand functions.

This review article also covered current research of applications of one-switch utility functions. We omit the applications in computer science, however, and only included the topics in the fields of the economics of information and functional equations. The topic in the economics of information is usually about the relationship between the value of information and risk aversion, say, monotonicity. However, there are still research gaps needed to be filled in this topic about types of information such as costless or costly information. Or, if there is also a monotonic relationship between risk aversion and the value of information which is measured by other methods which differ from the previous measures used to a decision maker with one-switch utility functions. Additionally, resolving the general solutions for the system of functional equations with different constraints or conditions is another point we should focus on. The idea with annuity payment in functional equations with one-switch utility functions has been already explored by Abbas and Chudziak (2013) but at conclusion part, they pointed out a possible further research direction, albeit mathematical, that what general solutions we can obtain for the system of functional equations with one-switch utility functions by the consideration of inflation rate or discounting for the annuity payment. This review article fails to take into account the impacts of one-switch utility functions in the field of computer science. The omission of this is because the main concentration of my personal research is on the economics. However, it is essential to consider how one-switch utility functions work in the field of computer science, especially associated with the process of making selections if one plans to determine the applicability of the functions in other research directions.

With strong theoretical foundations for one-switch utility functions, we should encourage deeper and broader research in the topics associated with one-switch utility functions; it is common to consider, for example, how they can be applied to address practical economic problems.

Acknowledgement

This work was supported by JST SPRING, Grant Number JPMJSP2136.

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Appendix A. *Six Axioms of Expected Utility Theory* (Jehle and Philip, 2011).

Consider preference $\succcurlyeq$ on the set $\mathbb{C}$, such that it satisfies:

Axiom 1. (*Completeness*) For any two gambles, x, and $y \in \mathbb{C}$, either $x \succcurlyeq y$, or $x \preccurlyeq y$ or both $(x \sim y)$.

Axiom 2. (*Transitivity*) For any three gambles, x, y, and $z \in \mathbb{C}$, if $x \succcurlyeq y$, and $y \succcurlyeq z$, then $x \succcurlyeq z$.

Axiom 3. (*Continuity*) For any three gambles, x, y, and z with $x \succcurlyeq y \succcurlyeq z$, then there exists a probability α, such that $y \sim (\alpha \cdot x + (1-\alpha)z)$.

Axiom 4. (*Monotonicity*) Say gamble x is preferred to y. Then, for another lottery z, and probability $\alpha \in (0,\ 1]$, $\alpha \cdot x + (1-\alpha)z \succcurlyeq \alpha \cdot y + (1-\alpha)z$.

Axiom 5. (*Substitution*) If $x = (\alpha_1 \cdot x^1,\ \ldots,\ \alpha_k \cdot x^k)$ and $y = (\alpha_1 \cdot y^1,\ \ldots,\ \alpha_k \cdot y^k)$ are in $\mathbb{C}$, and if $x^i \sim y^i$ for every i, then $x \sim y$.

Axiom 6. (*Reduction to Simple Gambles*) Let a compounded gamble $x = (\alpha_1 \cdot x_1,\ \ldots,\ \alpha_k \cdot x_k)$ induce a simple gamble $y \in \mathbb{C}$, then $x \sim y$.

Appendix B. *Demand Functions for the Cobb-Douglas Utility Function*

Detailed calculating procedures for Marshallian demand functions, indirect utility function, Hicksian demand functions and expenditure function for the Cobb-Douglas utility function, $u(x_1,\ x_2) = x_1^{\alpha_1} \cdot x_2^{\alpha_2}$, with $\alpha_1 + \alpha_2 = 1$.

The Marshallian demand functions:

We try to maximize $u(x_1,\ x_2) = x_1^{\alpha_1} \cdot x_2^{\alpha_2}$,

s.t. $w = P_1 x_1 + P_2 x_2$;

$\mathcal{L} = x_1^{\alpha_1} x_2^{\alpha_2} + \lambda(w - P_1 x_1 - P_2 x_2)$

F.O.C.

$$\frac{d\mathcal{L}}{dx_1} = \alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2} - \lambda P_1 = 0,$$

$$\frac{d\mathcal{L}}{dx_2} = \alpha_2 x_2^{\alpha_2 - 1} x_1^{\alpha_1} - \lambda P_2 = 0,$$

$$\frac{d\mathcal{L}}{d\lambda} = w - P_1 x_1 - P_2 x_2 = 0.$$

So, $\dfrac{P_1}{P_2} = \dfrac{\alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2}}{\alpha_2 x_2^{\alpha_2 - 1} x_1^{\alpha_1}} = \dfrac{\alpha_1}{\alpha_2}\ \dfrac{x_2}{x_1}$.

$$P_1 x_1 = \frac{\alpha_1 x_2 P_2}{\alpha_2}\ ;\ P_2 x_2 = \frac{\alpha_2 x_1 P_1}{\alpha_1}.$$

$$w = P_1 x_1 + \frac{\alpha_2 x_1 P_1}{\alpha_1} = P_1 x_1\left(\frac{\alpha_1 + \alpha_2}{\alpha_1}\right)\ ;\ w = P_2 x_2 + \frac{\alpha_1 x_2 P_2}{\alpha_2} = P_2 x_2\left(\frac{\alpha_1 + \alpha_2}{\alpha_2}\right).$$

Because $\alpha_1 + \alpha_2 = 1$,

$$d_1(P_1,\ P_2,\ w) = x_1^* = \frac{w}{P_1\left(\dfrac{\alpha_1 + \alpha_2}{\alpha_1}\right)} = \frac{w\left(\dfrac{\alpha_1}{\alpha_1 + \alpha_2}\right)}{P_1} = \frac{w\alpha_1}{P_1}\ ;$$

$$d_2(P_1,\ P_2,\ w)=x_2^*=\frac{w}{P_2\left(\frac{\alpha_1+\alpha_2}{\alpha_2}\right)}=\frac{w\left(\frac{\alpha_2}{\alpha_1+\alpha_2}\right)}{P_2}=\frac{w\alpha_2}{P_2}.$$

Substituting demands function into objective function, gives us **indirect utility function**:

$$v=v(P_1,\ P_2,\ w)=(x_1^*)^{\alpha_1}\cdot(x_2^*)^{\alpha_2}=\left(\frac{w\left(\frac{\alpha_1}{\alpha_1+\alpha_2}\right)}{P_1}\right)^{\alpha_1}\cdot\left(\frac{w\left(\frac{\alpha_2}{\alpha_1+\alpha_2}\right)}{P_2}\right)^{\alpha_2}=\left(\frac{\alpha_1}{P_1(\alpha_1+\alpha_2)}\right)^{\alpha_1}\cdot\left(\frac{\alpha_2}{P_2(\alpha_1+\alpha_2)}\right)^{\alpha_2}\cdot$$

$$w=\left(\frac{\alpha_1}{P_1}\right)^{\alpha_1}\cdot\left(\frac{\alpha_2}{P_2}\right)^{\alpha_2}\cdot w.$$

Hicksian demand functions

We try to minimize $w=P_1x_1+P_2x_2$

s.t. $U^0=x_1^{\alpha_1}x_2^{\alpha_2}$;

$\mathcal{L}=P_1x_1+P_2x_2+\lambda(U^0-x_1^{\alpha_1}x_2^{\alpha_2})$.

F.O.C.

$$\frac{d\mathcal{L}}{dx_1}=P_1-\lambda(\alpha_1x_1^{\alpha_1-1}x_2^{\alpha_2})=0,$$

$$\frac{d\mathcal{L}}{dx_2}=P_2-\lambda(\alpha_2x_2^{\alpha_2-1}x_1^{\alpha_1})=0,$$

$$\frac{d\mathcal{L}}{d\lambda}=U^0-x_1^{\alpha_1}x_2^{\alpha_2}=0.$$

So, $P_1=\lambda(\alpha_1x_1^{\alpha_1-1}x_2^{\alpha_2})$; $P_2=\lambda(\alpha_2x_2^{\alpha_2-1}x_1^{\alpha_1})$.

$$\frac{P_1}{P_2}=\frac{\alpha_1x_1^{\alpha_1-1}x_2^{\alpha_2}}{\alpha_2x_2^{\alpha_2-1}x_1^{\alpha_1}}=\frac{\alpha_1x_2}{\alpha_2x_1},$$

$x_1^*=\frac{\alpha_1x_2}{\alpha_2}\cdot\frac{P_2}{P_1}$, and $x_2^*=\frac{\alpha_2x_1}{\alpha_1}\cdot\frac{P_1}{P_2}$.

So, $U^0=\left(\frac{\alpha_1x_2}{\alpha_2}\cdot\frac{P_2}{P_1}\right)^{\alpha_1}x_2^{\alpha_2}$, or $U^0=x_1^{\alpha_1}\left(\frac{\alpha_2x_1}{\alpha_1}\cdot\frac{P_1}{P_2}\right)^{\alpha_2}$.

because $\alpha_1+\alpha_2=1$,

$$h_1(P_1,\ P_2,\ U^0)=x_1=U^0\left(\frac{P_2}{P_1}\cdot\frac{\alpha_1}{\alpha_2}\right)^{\alpha_2};\ h_2(P_1,\ P_2,\ U^0)=x_2=U^0\left(\frac{P_1}{P_2}\cdot\frac{\alpha_2}{\alpha_1}\right)^{\alpha_1}.$$

Expenditure Function

substituting Hicksian demand functions back into objective function to get expenditure function.

$$e(P_1,\ P_2,\ U^0)=P_1h_1+P_2h_2=P_1\left[U^0\left(\frac{P_2}{P_1}\cdot\frac{\alpha_1}{\alpha_2}\right)^{\alpha_2}\right]+P_2\left[U^0\left(\frac{P_1}{P_2}\cdot\frac{\alpha_2}{\alpha_1}\right)^{\alpha_1}\right]$$

$$=U^0\left[\left(\frac{P_2}{P_1}\cdot\frac{\alpha_1}{\alpha_2}\right)^{\alpha_2}+P_2\left(\frac{P_1}{P_2}\cdot\frac{\alpha_2}{\alpha_1}\right)^{\alpha_1}\right]$$

$$=U^0\left[\frac{P_1(P_2\alpha_1)^{\alpha_2}}{(P_1\alpha_2)^{\alpha_2}}+\frac{P_2(P_1\alpha_2)^{\alpha_1}}{(P_2\alpha_1)^{\alpha_1}}\right]$$

$=U^0[P_1(P_2\alpha_1)^{\alpha_2}(P_1\alpha_2)^{-\alpha_2}+P_2(P_1\alpha_2)^{\alpha_1}(P_2\alpha_1)^{-\alpha_1}]$, because $\alpha_1+\alpha_2=1$,

$=U^0[P_1^{1-\alpha_2}P_2^{\alpha_2}\alpha_1^{\alpha_2}\alpha_2^{-\alpha_2}+P_2^{1-\alpha_1}P_1^{\alpha_1}\alpha_2^{\alpha_1}\alpha_1^{-\alpha_1}]$

$=U^0[P_1^{\alpha_1}P_2^{\alpha_2}\alpha_1^{\alpha_2}\alpha_2^{-\alpha_2}+P_2^{\alpha_2}P_1^{\alpha_1}\alpha_2^{\alpha_1}\alpha_1^{-\alpha_1}]$

$=U^0[P_1^{\alpha_1}P_2^{\alpha_2}(\alpha_1^{\alpha_2}\alpha_2^{-\alpha_2}+\alpha_2^{\alpha_1}\alpha_1^{-\alpha_1})]$

$=U^0\left[P_1^{\alpha_1}P_2^{\alpha_2}\left(\dfrac{\alpha_1^{\alpha_1+\alpha_2}+\alpha_2^{\alpha_1+\alpha_2}}{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}}\right)\right]$, because $\alpha_1+\alpha_2=1$,

$=U^0\left[P_1^{\alpha_1}P_2^{\alpha_2}\left(\dfrac{\alpha_1+\alpha_2}{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}}\right)\right]=U^0\left[\dfrac{P_1^{\alpha_1}P_2^{\alpha_2}}{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}}\right]$.

So, $e(P_1,\ P_2,\ U^0)=U^0\cdot\left(\dfrac{P_1}{\alpha_1}\right)^{\alpha_1}\cdot\left(\dfrac{P_2}{\alpha_2}\right)^{\alpha_2}$.

Appendix C. *Demand Functions for One-switch utility functions*

Detailed calculating procedures for Marshallian demand functions, indirect utility function, Hicksian demand functions and expenditure function for four types of one-switch utility functions.

1. the *quadratic* one-switch utility function, $u(x_1,\ x_2)=q(c_1x_1+c_2x_2)^2+d_1x_1+d_2x_2$.

The Marshallian demand functions:

We try to maximize $u(x_1,\ x_2)=q(c_1x_1+c_2x_2)^2+d_1x_1+d_2x_2$,

s.t. $w=P_1x_1+P_2x_2$;

$\mathcal{L}=q(c_1x_1+c_2x_2)^2+d_1x_1+d_2x_2+\lambda(w-P_1x_1-P_2x_2)$

F.O.C.

$\dfrac{d\mathcal{L}}{dx_1}=2qc_1^2x_1+2qc_1c_2x_2+d_1-\lambda P_1=0$,

$\dfrac{d\mathcal{L}}{dx_2}=2qc_1c_2x_1+2qc_2^2x_2+d_2-\lambda P_2=0$.

Then, $\lambda P_1=2qc_1^2x_1+2qc_1c_2x_2+d_1$, and $\lambda P_2=2qc_1c_2x_1+2qc_2^2x_2+d_2$.

So, $\dfrac{P_1}{P_2}=\dfrac{2qc_1^2x_1+2qc_1c_2x_2+d_1}{2qc_1c_2x_1+2qc_2^2x_2+d_2-\lambda P_2}$.

And then, we can have the equation for x_1:

$2qc_1c_2x_1P_1-2qc_1^2x_1P_2=2qc_1c_2x_2P_2-2qc_2^2x_2P_1+d_1P_2-d_2P_1$,

$x_1\cdot(2qc_1c_2P_1-2qc_1^2P_2)=2qc_1c_2x_2P_2-2qc_2^2x_2P_1+d_1P_2-d_2P_1$,

So, $x_1=\dfrac{2qc_1c_2x_2P_2-2qc_2^2x_2P_1+d_1P_2-d_2P_1}{2qc_1c_2P_1-2qc_1^2P_2}$.

In the same manner, the equation for x_2 is:

$2qc_2^2x_2P_1-2qc_1c_2x_2P_2=2qc_1^2x_1P_2-2qc_1c_2x_1P_1+d_1P_2-d_2P_1$,

$x_2\cdot(2qc_2^2P_1-2qc_1c_2P_2)=2qc_1^2x_1P_2-2qc_1c_2x_1P_1+d_1P_2-d_2P_1$,

So, $x_2=\dfrac{2qc_1^2x_1P_2-2qc_1c_2x_1P_1+d_1P_2-d_2P_1}{2qc_2^2P_1-2qc_1c_2P_2}$.

Firstly, we will substitute x_2 in the equation $w=P_1x_1+P_2x_2$.

$$w=P_1x_1+P_2\cdot\frac{2qc_1^2x_1P_2-2qc_1c_2x_1P_1+d_1P_2-d_2P_1}{2qc_2^2P_1-2qc_1c_2P_2}=P_1x_1+\frac{2qc_1^2x_1P_2^2-2qc_1c_2x_1P_1P_2+d_1P_2^2-d_2P_1P_2}{2qc_2^2P_1-2qc_1c_2P_2}$$

$$=P_1x_1+\frac{c_1^2x_1P_2^2-c_1c_2x_1P_1P_2}{c_2^2P_1-c_1c_2P_2}+\frac{d_1P_2^2-d_2P_1P_2}{2qc_2^2P_1-2qc_1c_2P_2}.$$

$$w-\frac{d_1P_2^2-d_2P_1P_2}{2qc_2^2P_1-2qc_1c_2P_2}=P_1x_1+\frac{x_1(c_1^2P_2^2-c_1c_2P_1P_2)}{c_2^2P_1-c_1c_2P_2}=x_1\cdot\left(P_1+\frac{(c_1^2P_2^2-c_1c_2P_1P_2)}{c_2^2P_1-c_1c_2P_2}\right)$$

$$=x_1\cdot\left(\frac{c_1^2P_2^2-c_1c_2P_1P_2+c_2^2P_1^2-c_1c_2P_1P_2}{c_2^2P_1-c_1c_2P_2}\right)=x_1\cdot\frac{(c_1P_2-c_2P_1)^2}{c_2^2P_1-c_1c_2P_2}.$$

So, $x_1^*=\left(w-\dfrac{d_1P_2^2-d_2P_1P_2}{2qc_2^2P_1-2qc_1c_2P_2}\right)\cdot\dfrac{c_2^2P_1-c_1c_2P_2}{(c_1P_2-c_2P_1)^2}=w\cdot\dfrac{c_2^2P_1-c_1c_2P_2}{(c_1P_2-c_2P_1)^2}-\dfrac{d_1P_2^2-d_2P_1P_2}{2q(c_1P_2-c_2P_1)^2}$

$$=\frac{w\cdot(c_2^2P_1-c_1c_2P_2)-\dfrac{d_1P_2^2-d_2P_1P_2}{2q}}{(c_1P_2-c_2P_1)^2}.$$

Secondly, we will substitute x_1 in the equation $w=P_1x_1+P_2x_2$.

$$w=P_1\cdot\frac{2qc_1c_2x_2P_2-2qc_2^2x_2P_1+d_1P_2-d_2P_1}{2qc_1c_2P_1-2qc_1^2P_2}+P_2x_2=\frac{2qc_1c_2x_2P_1P_2-2qc_2^2x_2P_1^2+d_1P_1P_2-d_2P_1^2}{2qc_1c_2P_1-2qc_1^2P_2}+P_2x_2$$

$$=\frac{c_1c_2x_2P_1P_2-c_2^2x_2P_1^2}{c_1c_2P_1-c_1^2P_2}+\frac{d_1P_1P_2-d_2P_1^2}{2qc_1c_2P_1-2qc_1^2P_2}+P_2x_2.$$

$$w-\frac{d_1P_1P_2-d_2P_1^2}{2qc_1c_2P_1-2qc_1^2P_2}=x_2\cdot\left(\frac{c_1c_2P_1P_2-c_2^2P_1^2}{c_1c_2P_1-c_1^2P_2}+P_2\right)=x_2\cdot\left(\frac{c_1c_2P_1P_2-c_2^2P_1^2+c_1c_2P_1P_2-c_1^2P_2^2}{c_1c_2P_1-c_1^2P_2}\right)$$

$$=x_2\cdot\left(\frac{c_2^2P_1^2+c_1^2P_2^2-2c_1c_2P_1P_2}{c_1^2P_2-c_1c_2P_1}\right)=x_2\cdot\frac{(c_1P_2-c_2P_1)^2}{c_1^2P_2-c_1c_2P_1}.$$

So, $x_2^*=\left(w-\dfrac{d_1P_1P_2-d_2P_1^2}{2qc_1c_2P_1-2qc_1^2P_2}\right)\cdot\dfrac{c_1^2P_2-c_1c_2P_1}{(c_1P_2-c_2P_1)^2}=\dfrac{w\cdot(c_1^2P_2-c_1c_2P_1)-\dfrac{d_1P_1P_2-d_2P_1^2}{2q}}{(c_1P_2-c_2P_1)^2}.$

Substituting demand functions into objective function, gives us **indirect utility function:**

$$v=v(P_1,\ P_2,\ w)=q(c_1x_1^*+c_2x_2^*)^2+d_1x_1^*+d_2x_2^*$$

$$=q\left(c_1\cdot\frac{w\cdot(c_2^2P_1-c_1c_2P_2)-\dfrac{d_1P_2^2-d_2P_1P_2}{2q}}{(c_1P_2-c_2P_1)^2}+c_2\cdot\frac{w\cdot(c_1^2P_2-c_1c_2P_1)-\dfrac{d_1P_1P_2-d_2P_1^2}{2q}}{(c_1P_2-c_2P_1)^2}\right)^2$$

$$+d_1\cdot\frac{w\cdot(c_2^2P_1-c_1c_2P_2)-\dfrac{d_1P_2^2-d_2P_1P_2}{2q}}{(c_1P_2-c_2P_1)^2}+d_2\cdot\frac{w\cdot(c_1^2P_2-c_1c_2P_1)-\dfrac{d_1P_1P_2-d_2P_1^2}{2q}}{(c_1P_2-c_2P_1)^2}.$$

Hicksian demand functions

We try to minimize $w=P_1x_1+P_2x_2$;

s.t. $U^0=q(c_1x_1+c_2x_2)^2+d_1x_1+d_2x_2$.

$\mathcal{L}=P_1x_1+P_2x_2+\lambda[U^0-q(c_1x_2+c_2x_2)^2-d_1x_1-d_2x_2]$.

F.O.C.

$$\frac{d\mathcal{L}}{dx_1}=P_1-\lambda qc_1(2c_1x_1+2c_2x_2)-\lambda d_1=P_1-2\lambda qc_1^2x_1-2\lambda qc_1c_2x_2-\lambda d_1=0,$$

$$\frac{d\mathcal{L}}{dx_2}=P_2-\lambda qc_2(2c_1x_1+2c_2x_2)-\lambda d_2=P_2-2\lambda qc_1c_2x_1-2\lambda qc_2^2x_2-\lambda d_2=0,$$

$$\frac{d\mathcal{L}}{d\lambda}=U^0-q(c_1x_2+c_2x_2)^2-d_1x_1-d_2x_2=0.$$

So, $P_1=2\lambda qc_1^2x_1+2\lambda qc_1c_2x_2+\lambda d_1$;

and $P_2=2\lambda qc_1c_2x_1+2\lambda qc_2^2x_2+\lambda d_2$.

$$\frac{P_1}{P_2}=\frac{2qc_1^2x_1+2qc_1c_2x_2+d_1}{2qc_1c_2x_1+2qc_2^2x_2+d_2},$$

So, $P_1\cdot 2qc_1c_2x_1+P_1\cdot 2qc_2^2x_2+P_1d_2=P_2\cdot 2qc_1^2x_1+P_2\cdot 2qc_1c_2x_2+P_2d_1$,

$2qP_1c_1c_2x_1-2qP_2c_1^2x_1=2qP_2c_1c_2x_2-2qP_1c_2^2x_2+P_2d_1-P_1d_2$,

$$x_1=\frac{2qP_2c_1c_2x_2-2qP_1c_2^2x_2+P_2d_1-P_1d_2}{2qP_1c_1c_2-2qP_2c_1^2}=-\frac{c_2x_2(P_2c_1-P_1c_2)}{c_1(P_1c_2-P_2c_1)}+\frac{P_2d_1-P_1d_2}{2qP_1c_1c_2-2qP_2c_1^2}$$

$$=-\frac{c_2x_2}{c_1}+\frac{P_2d_1-P_1d_2}{2qP_1c_1c_2-2qP_2c_1^2}.$$

In the same manner, $2qP_1c_2^2x_2-2qP_2c_1c_2x_2=2qP_2c_1^2x_1-2qP_1c_1c_2x_1+P_2d_1-P_1d_2$,

$$x_2=\frac{2qP_2c_1^2x_1-2qP_1c_1c_2x_1+P_2d_1-P_1d_2}{2qP_1c_2^2-2qP_2c_1c_2}=-\frac{c_1x_1(P_2c_1-P_1c_2)}{c_2(P_1c_2-P_2c_1)}+\frac{P_2d_1-P_1d_2}{2qP_1c_2^2-2qP_2c_1c_2}$$

$$=-\frac{c_1x_1}{c_2}+\frac{P_2d_1-P_1d_2}{2qP_1c_2^2-2qP_2c_1c_2}.$$

Substitute x_1 in the equation $U^0=q(c_1x_1+c_2x_2)^2+d_1x_1+d_2x_2$.

$$U^0=q\left[c_1\left(-\frac{c_2x_2}{c_1}+\frac{P_2d_1-P_1d_2}{2qP_1c_1c_2-2qP_2c_1^2}\right)+c_2x_2\right]^2+d_1\cdot\left(-\frac{c_2x_2}{c_1}+\frac{P_2d_1-P_1d_2}{2qP_1c_1c_2-2qP_2c_1^2}\right)+d_2x_2,$$

$$=q\left(\frac{P_2d_1-P_1d_2}{2qP_1c_2-2qP_2c_1}\right)^2+d_1\left(\frac{P_2d_1-P_1d_2}{2qP_1c_1c_2-2qP_2c_1^2}\right)-\frac{d_1c_1x_2}{c_1}+d_2x_2,$$

$$=q\left(\frac{P_2d_1-P_1d_2}{2qP_1c_2-2qP_2c_1}\right)^2+d_1\left(\frac{P_2d_1-P_1d_2}{2qP_1c_1c_2-2qP_2c_1^2}\right)+x_2\left(d_2-\frac{d_1c_2}{c_1}\right).$$

$$\text{So, } h_2(P_1,\ \ P_2,\ \ U^0)=x_2=\frac{U^0-q\left(\frac{P_2d_1-P_1d_2}{2qP_1c_2-2qP_2c_1}\right)^2-\frac{P_2d^2-P_1d_1d_2}{2qP_1c_1c_2-2qP_2c_1^2}}{d_2-\frac{d_1c_2}{c_1}}$$

$$=\frac{U^0c_1-qc_1\left(\frac{P_2d_1-P_1d_2}{2qP_1c_2-2qP_2c_1}\right)^2-\frac{P_2d_1^2-P_1d_1d_2}{2qP_1c_2-2qP_2c_1}}{d_2c_1-d_1c_2}.$$

Similarly, substitute x_2 in the equation $U^0=q(c_1x_1+c_2x_2)^2+d_1x_1+d_2x_2$.

$$U^0=q\left[c_1x_1+c_2\left(-\frac{c_1x_1}{c_2}+\frac{P_2d_1-P_1d_2}{2qP_1c_2^2-2qP_2c_1c_2}\right)\right]^2+d_1x_1+d_2\cdot\left(-\frac{c_1x_1}{c_2}+\frac{P_2d_1-P_1d_2}{2qP_1c_2^2-2qP_2c_1c_2}\right),$$

$$=q\left(\frac{P_2d_1-P_1d_2}{2qP_1c_2-2qP_2c_1}\right)^2+d_1x_1+d_2\left(\frac{P_2d_1-P_1d_2}{2qP_1c_2^2-2qP_2c_1c_2}\right)-\frac{d_2c_1x_1}{c_2},$$

$$=q\left(\frac{P_2d_1-P_1d_2}{2qP_1c_2-2qP_2c_1}\right)^2+d_2\left(\frac{P_2d_1-P_1d_2}{2qP_1c_2^2-2qP_2c_1c_2}\right)+x_1\left(d_1-\frac{d_2c_1}{c_2}\right).$$

So, $h_1(P_1,\ P_2,\ U^0)=x_1=\dfrac{U^0-q\left(\dfrac{P_2d_1-P_1d_2}{2qP_1c_2-2qP_2c_1}\right)^2-\dfrac{P_2d_1d_2-P_1d_2^2}{2qP_1c_2^2-2qP_2c_1c_2}}{d_1-\dfrac{d_2c_1}{c_2}}$

$$=\frac{U^0c_2-qc_2\left(\dfrac{P_2d_1-P_1d_2}{2qP_1c_2-2qP_2c_1}\right)^2-\dfrac{P_2d_1d_2-P_1d_2^2}{2qP_1c_2-2qP_2c_1}}{d_1c_2-d_2c_1}.$$

Expenditure Function

Substituting Hicksian demand functions back into objective function to get expenditure function.

$$e(P_1,\ P_2,\ U^0)=P_1h_1+P_2h_2=P_1\cdot\frac{U^0c_2-qc_2\left(\dfrac{P_2d_1-P_1d_2}{2qP_1c_2-2qP_2c_1}\right)^2-\dfrac{P_2d_1d_2-P_1d_2^2}{2qP_1c_2-2qP_2c_1}}{d_1c_2-d_2c_1}$$

$$+P_2\cdot\frac{U^0c_1-qc_1\left(\dfrac{P_2d_1-P_1d_2}{2qP_1c_2-2qP_2c_1}\right)^2-\dfrac{P_2d_1^2-P_1d_1d_2}{2qP_1c_2-2qP_2c_1}}{d_2c_1-d_1c_2}.$$

2. *linear plus exponential* one-switch utility function

The Marshallian demand functions:

We try to maximize $u(x_1,\ x_2)=c_1x_1+c_2x_2+be^{d_1x_1+d_2x_2}$,

s.t. $w=P_1x_1+P_2x_2$;

$\mathcal{L}=c_1x_1+c_2x_2+be^{d_1x_1+d_2x_2}+\lambda[w-P_1x_1-P_2x_2]$

F.O.C.

$$\frac{d\mathcal{L}}{dx_1}=c_1+bd_1e^{d_1x_1+d_2x_2}-\lambda P_1=0,$$

$$\frac{d\mathcal{L}}{dx_2}=c_2+bd_2e^{d_1x_1+d_2x_2}-\lambda P_2=0.$$

Then, $\lambda P_1=c_1+bd_1e^{d_1x_1+d_2x_2}$, and $\lambda P_2=c_2+bd_2e^{d_1x_1+d_2x_2}$.

So, $\dfrac{P_1}{P_2}=\dfrac{c_1+bd_1e^{d_1x_1+d_2x_2}}{c_2+bd_2e^{d_1x_1+d_2x_2}}$.

And then, we can have equations for x_1 and x_2.

$P_2c_1+P_2bd_1e^{d_1x_1+d_2x_2}=P_1c_2+P_1bd_2e^{d_1x_1+d_2x_2}$,

$e^{d_1x_1+d_2x_2}=\dfrac{P_1c_2-P_2c_1}{P_2bd_1-P_1bd_2}$,

$d_1x_1+d_2x_2=\ln\left(\dfrac{P_1c_2-P_2c_1}{P_2bd_1-P_1bd_2}\right)$,

So, $x_1=\dfrac{\ln\left(\dfrac{P_1c_2-P_2c_1}{P_2bd_1-P_1bd_2}\right)-d_2x_2}{d_1}$, and $x_2=\dfrac{\ln\left(\dfrac{P_1c_2-P_2c_1}{P_2bd_1-P_1bd_2}\right)-d_1x_1}{d_2}$.

Firstly, we will substitute x_1 in the equation $w=P_1x_1+P_2x_2$.

$$w=P_1\cdot\frac{\ln\left(\frac{P_1c_2-P_2c_1}{P_2bd_1-P_1bd_2}\right)-d_2x_2}{d_1}+P_2x_2=\frac{P_1\ln\left(\frac{P_1c_2-P_2c_1}{P_2bd_1-P_1bd_2}\right)}{d_1}-\frac{P_1d_2}{d_1}x_2+P_2x_2,$$

$$w-\frac{P_1\ln\left(\frac{P_1c_2-P_2c_1}{P_2bd_1-P_1bd_2}\right)}{d_1}=x_2\left(P_2-\frac{P_1d_2}{d_1}\right),$$

$$x_2^*=\frac{w-\frac{P_1\ln\left(\frac{P_1c_2-P_2c_1}{P_2bd_1-P_1bd_2}\right)}{d_1}}{P_2-\frac{P_1d_2}{d_1}}=\frac{Ww-P_1\ln\left(\frac{P_1c_2-P_2c_1}{P_2bd_1-P_1bd_2}\right)}{P_2d_1-P_1d_2}.$$

Secondly, we will substitute x_2 in the equation $w=P_1x_1+P_2x_2$.

$$w=P_1x_1+P_2\cdot\frac{\ln\left(\frac{P_1c_2-P_2c_1}{P_2bd_1-P_1bd_2}\right)-d_1x_1}{d_2}=\frac{P_2\ln\left(\frac{P_1c_2-P_2c_1}{P_2bd_1-P_1bd_2}\right)}{d_2}-\frac{P_2d_1x_1}{d_2}+P_1x_1,$$

$$w-\frac{P_2\ln\left(\frac{P_1c_2-P_2c_1}{P_2bd_1-P_1bd_2}\right)}{d_2}=x_1\left(P_1-\frac{P_2d_1}{d_2}\right),$$

$$x_1^*=\frac{w-\frac{P_2\ln\left(\frac{P_1c_2-P_2c_1}{P_2bd_1-P_1bd_2}\right)}{d_2}}{P_1-\frac{P_2d_1}{d_2}}=\frac{wd_2-P_2\ln\left(\frac{P_1c_2-P_2c_1}{P_2bd_1-P_1bd_2}\right)}{P_1d_2-P_2d_1}.$$

Substituting demand functions into objective function, gives us **indirect utility function:**

$$v=v(P_1,\ P_2,\ w)=c_1x_1^*+c_2x_2^*+be^{d_1x_1^*+d_2x_2^*}$$

$$=c_1\cdot\frac{wd_2-P_2\ln\left(\frac{P_1c_2-P_2c_1}{P_2bd_1-P_1bd_2}\right)}{P_1d_2-P_2d_1}+c_2\cdot\frac{wd_1-P_1\ln\left(\frac{P_1c_2-P_2c_1}{P_2bd_1-P_1bd_2}\right)}{P_2d_1-P_1d_2}$$

$$+be^{d_1\cdot\frac{wd_2-P_2\ln\left(\frac{P_1c_2-P_2c_1}{P_2bd_1-P_1bd_2}\right)}{P_1d_2-P_2d_1}+d_2\cdot\frac{wd_1-P_1\ln\left(\frac{P_1c_2-P_2c_1}{P_2bd_1-P_1bd_2}\right)}{P_2d_1-P_1d_2}}.$$

Hicksian demand functions

We try to minimize $w=P_1x_1+P_2x_2$;

s.t. $U^0=c_1x_1+c_2x_2+be^{d_1x_1+d_2x_2}$.

$\mathcal{L}=P_1x_1+P_2x_2+\lambda[U^0-c_1x_1-c_2x_2-be^{d_1x_1+d_2x_2}]$.

F.O.C.

$$\frac{d\mathcal{L}}{dx_1}=P_1-\lambda c_1-\lambda bd_1e^{d_1x_1+d_2x_2}=0,$$

$$\frac{d\mathcal{L}}{dx_2}=P_2-\lambda c_2-\lambda bd_2e^{d_1x_1+d_2x_2}=0,$$

$$\frac{d\mathcal{L}}{d\lambda}=U^0-c_1x_1-c_2x_2-be^{d_1x_1+d_2x_2}=0.$$

So, $P_1 = \lambda c_1 + \lambda b d_1 e^{d_1 x_1 + d_2 x_2}$;

and $P_2 = \lambda c_2 + \lambda b d_2 e^{d_1 x_1 + d_2 x_2}$.

$$\frac{P_1}{P_2} = \frac{\lambda c_1 + \lambda b d_1 e^{d_1 x_1 + d_2 x_2}}{\lambda c_2 + \lambda b d_2 e^{d_1 x_1 + d_2 x_2}},$$

So, $P_1(c_2 + b d_2 e^{d_1 x_1 + d_2 x_2}) = P_2(c_1 + b d_1 e^{d_1 x_1 + d_2 x_2})$,

$$P_1 b d_2 e^{d_1 x_1 + d_2 x_2} - P_2 b d_1 e^{d_1 x_1 + d_2 x_2} = P_2 c_1 - P_1 c_2,$$

$$e^{d_1 x_1 + d_2 x_2} = \frac{P_2 c_1 - P_1 c_2}{P_1 b d_2 - P_2 b d_1},$$

$$d_1 x_1 + d_2 x_2 = \ln\left(\frac{P_2 c_1 - P_1 c_2}{P_1 b d_2 - P_2 b d_1}\right),$$

$$x_1 = \frac{\ln\left(\frac{P_2 c_1 - P_1 c_2}{P_1 b d_2 - P_2 b d_1}\right) - d_2 x_2}{d_1}, \text{ and } x_2 = \frac{\ln\left(\frac{P_2 c_1 - P_1 c_2}{P_1 b d_2 - P_2 b d_1}\right) - d_1 x_1}{d_2}.$$

Substitute x_1 in the equation $U^0 = c_1 x_1 - c_2 x_2 - b e^{d_1 x_1 + d_2 x_2}$.

$$U^0 = c_1 \cdot \frac{\ln\left(\frac{P_2 c_1 - P_1 c_2}{P_1 b d_2 - P_2 b d_1}\right) - d_2 x_2}{d_1} + c_2 x_2 + b e^{d_1 \cdot \frac{\ln\left(\frac{P_2 c_1 - P_1 c_2}{P_1 b d_2 - P_2 b d_1}\right) - d_2 x_2}{d_1} + d_2 x_2}$$

$$= \frac{c_1}{d_1} \cdot \ln\left(\frac{P_2 c_1 - P_1 c_2}{P_1 b d_2 - P_2 b d_1}\right) - \frac{c_1 d_2}{d_1} x_2 + c_2 x_2 + b \cdot e^{\ln\left(\frac{P_2 c_1 - P_1 c_2}{P_1 b d_2 - P_2 b d_1}\right)},$$

$$x_2\left(c_2 - \frac{c_1 d_2}{d_1}\right) = U^0 - \frac{c_1}{d_1} \cdot \ln\left(\frac{P_2 c_1 - P_1 c_2}{P_1 b d_2 - P_2 b d_1}\right) - b \cdot e^{\ln\left(\frac{P_2 c_1 - P_1 c_2}{P_1 b d_2 - P_2 b d_1}\right)},$$

$$h_2(P_1,\ P_2,\ U^0) = x_2 = \frac{U^0 - \frac{c_1}{d_1} \cdot \ln\left(\frac{P_2 c_1 - P_1 c_2}{P_1 b d_2 - P_2 b d_1}\right) - b e^{\ln\left(\frac{P_2 c_1 - P_1 c_2}{P_1 b d_2 - P_2 b d_1}\right)}}{c_2 - \frac{c_1 d_2}{d_1}}.$$

Similarly, substitute x_2 in the equation $U^0 = c_1 x_1 - c_2 x_2 - b e^{d_1 x_1 + d_2 x_2}$.

$$U^0 = c_1 x_1 + c_2 \cdot \frac{\ln\left(\frac{P_2 c_1 - P_1 c_2}{P_1 b d_2 - P_2 b d_1}\right) - d_1 x_1}{d_2} + b e^{d_1 x_1 + d_2 \cdot \frac{\ln\left(\frac{P_2 c_1 - P_1 c_2}{P_1 b d_2 - P_2 b d_1}\right) - d_1 x_1}{d_2}}$$

$$= c_1 x_1 + \frac{c_2}{d_2} \cdot \ln\left(\frac{P_2 c_1 - P_1 c_2}{P_1 b d_2 - P_2 b d_1}\right) - \frac{c_2 d_1}{d_2} x_1 + b \cdot e^{\ln\left(\frac{P_2 c_1 - P_1 c_2}{P_1 b d_2 - P_2 b d_1}\right)}.$$

$$x_1\left(c_1 - \frac{c_2 d_1}{d_2}\right) = U^0 - \frac{c_2}{d_2} \cdot \ln\left(\frac{P_2 c_1 - P_1 c_2}{P_1 b d_2 - P_2 b d_1}\right) - b \cdot e^{\ln\left(\frac{P_2 c_1 - P_1 c_2}{P_1 b d_2 - P_2 b d_1}\right)},$$

$$h_1(P_1,\ P_2,\ U^0) = x_1 = \frac{U^0 - \frac{c_2}{d_2} \cdot \ln\left(\frac{P_2 c_1 - P_1 c_2}{P_1 b d_2 - P_2 b d_1}\right) - b e^{\ln\left(\frac{P_2 c_1 - P_1 c_2}{P_1 b d_2 - P_2 b d_1}\right)}}{c_1 - \frac{c_2 d_1}{d_2}}.$$

Expenditure Function

Substituting Hicksian demand functions back into objective function to get expenditure function.

$$e(P_1,\ P_2,\ U^0)=P_1h_1+P_2h_2=P_1\cdot\frac{U^0-\dfrac{c_2}{d_2}\cdot\ln\left(\dfrac{P_2c_1-P_1c_2}{P_1bd_2-P_2bd_1}\right)-be^{\ln\left(\frac{P_2c_1-P_1c_2}{P_1bd_2-P_2bd_1}\right)}}{c_1-\dfrac{c_2d_1}{d_2}}$$

$$+P_2\cdot\frac{U^0-\dfrac{c_1}{d_1}\cdot\ln\left(\dfrac{P_2c_1-P_1c_2}{P_1bd_2-P_2bd_1}\right)-be^{\ln\left(\frac{P_2c_1-P_1c_2}{P_1bd_2-P_2bd_1}\right)}}{c_2-\dfrac{c_1d_2}{d_1}}.$$

3. *linear times exponential* one-switch utility function

The Marshallian demand functions:

We try to maximize $u(x_1,\ x_2)=(c_1x_1+c_2x_2+b)e^{d_1x_1+d_2x_2}$;

s.t. $w=P_1x_1+P_2x_2$.

$\mathscr{L}=(c_1x_1+c_2x_2+b)e^{d_1x_1+d_2x_2}+\lambda[w-P_1x_1-P_2x_2]$.

$$\frac{d\mathscr{L}}{dx_1}=c_1e^{d_1x_1+d_2x_2}+(c_1x_1+c_2x_2+b)e^{d_1x_1+d_2x_2}\cdot d_1-\lambda P_1=0,$$

$$\frac{d\mathscr{L}}{dx_2}=c_2e^{d_1x_1+d_2x_2}+(c_1x_1+c_2x_2+b)e^{d_1x_1+d_2x_2}\cdot d_2-\lambda P_2=0.$$

Then, $\lambda P_1=c_1e^{d_1x_1+d_2x_2}+(c_1x_1+c_2x_2+b)e^{d_1x_1+d_2x_2}\cdot d_1$, and $\lambda P_2=c_2e^{d_1x_1+d_2x_2}+(c_1x_1+c_2x_2+b)e^{d_1x_1+d_2x_2}\cdot d_2$.

So, $\dfrac{P_1}{P_2}=\dfrac{c_1e^{d_1x_1+d_2x_2}+(c_1x_1+c_2x_2+b)e^{d_1x_1+d_2x_2}\cdot d_1}{c_2e^{d_1x_1+d_2x_2}+(c_1x_1+c_2x_2+b)e^{d_1x_1+d_2x_2}\cdot d_2}$.

And then, we can have the equation for x_1 :

$P_2c_1e^{d_1x_1+d_2x_2}+P_2d_1(c_1x_1+c_2x_2+b)e^{d_1x_1+d_2x_2}=P_1c_2e^{d_1x_1+d_2x_2}+P_1d_2(c_1x_1+c_2x_2+b)e^{d_1x_1+d_2x_2}$,

$P_2c_1+P_2d_1c_1x_1+P_2d_1c_2x_2+P_2d_1b=P_1d_2c_1x_1+P_1d_2c_2x_2+P_1d_2b+P_1c_2$,

$x_1(P_2d_1c_1-P_1d_2c_1)=P_1c_2+P_1d_2c_2x_2+P_1d_2b-P_2c_1-P_2d_1c_2x_2-P_2d_1b$,

So, $x_1=\dfrac{P_1c_2+P_1d_2c_2x_2+P_1d_2b-P_2c_1-P_2d_1c_2x_2-P_2d_1b}{P_2d_1c_1-P_1d_2c_1}$.

In the same manner, the equation for x_2 is:

$x_2(P_2d_1c_2-P_1d_2c_2)=P_1d_2c_1x_1+P_1d_2b+P_1c_2-P_2c_1-P_2d_1c_1x_1-P_2d_1b$,

So, $x_2=\dfrac{P_1d_2c_1x_1+P_1d_2b+P_1c_2-P_2c_1-P_2d_1c_1x_1-P_2d_1b}{P_2d_1c_2-P_1d_2c_2}$.

Firstly, we will substitute x_2 in the equation $w=P_1x_1+P_2x_2$.

$$w=P_1x_1+P_2\cdot\left[\frac{P_1d_2c_1x_1+P_1d_2b+P_1c_2-P_2c_1-P_2d_1c_1x_1-P_2d_1b}{P_2d_1c_2-P_1d_2c_2}\right]$$

$$=P_1x_1-\frac{P_2c_1}{c_2}x_1+P_2\cdot\left[\frac{P_1c_2-P_2c_1}{(P_2d_1-P_1d_2)c_2}-\frac{b}{c_2}\right],$$

$$\left(P_1-\frac{P_2c_1}{c_2}\right)x_1=w-P_2\cdot\left[\frac{P_1c_2-P_2c_1}{(P_2d_1-P_1d_2)c_2}-\frac{b}{c_2}\right],$$

So, $x_1^*=\dfrac{w-P_2\cdot\left[\dfrac{P_1c_2-P_2c_1}{(P_2d_1-P_1d_2)c_2}-\dfrac{b}{c_2}\right]}{P_1-\dfrac{P_2c_1}{c_2}}=\dfrac{wc_2-P_2\left[\dfrac{P_1c_2-P_2c_1}{P_2d_1-P_1d_2}-b\right]}{P_1c_2-P_2c_1}$.

Secondly, we will substitute x_1 in the equation $w=P_1x_1+P_2x_2$.

$$w=P_1\cdot\left[\frac{P_1c_2+P_1d_2c_2x_2+P_1d_2b-P_2c_1-P_2d_1c_2x_2-P_2d_1b}{P_2d_1c_1-P_1d_2c_1}\right]+P_2x_2$$

$$=P_2x_2-\frac{P_1c_2}{c_1}x_2+P_1\cdot\left[\frac{P_1c_2-P_2c_1}{(P_2d_1-P_1d_2)c_1}-\frac{b}{c_1}\right],$$

$$\left(P_2-\frac{P_1c_2}{c_1}\right)x_2=w-P_1\cdot\left[\frac{P_1c_2-P_2c_1}{(P_2d_1-P_1d_2)c_1}-\frac{b}{c_1}\right],$$

So, $$x_2^*=\frac{w-P_1\cdot\left[\frac{P_1c_2-P_2c_1}{(P_2d_1-P_1d_2)c_1}-\frac{b}{c_1}\right]}{P_2-\frac{P_1c_2}{c_1}}=\frac{wc_1-P_1\left[\frac{P_1c_2-P_2c_1}{P_2d_1-P_1d_2}-b\right]}{P_2c_1-P_1c_2}.$$

Substituting demand functions into objective function, gives us **indirect utility function:**

$$v=v(P_1,\ \ P_2,\ \ w)=(c_1x_1^*+c_2x_2^*+b)e^{d_1x_1^*+d_2x_2^*}$$

$$=\left(c_1\cdot\frac{wc_2-P_2\left[\frac{P_1c_2-P_2c_1}{P_2d_1-P_1d_2}-b\right]}{P_1c_2-P_2c_1}+c_2\cdot\frac{wc_1-P_1\left[\frac{P_1c_2-P_2c_1}{P_2d_1-P_1d_2}-b\right]}{P_2c_1-P_1c_2}\right)\cdot$$

$$e^{d_1\cdot\frac{wc_2-P_2\left[\frac{P_1c_2-P_2c_1}{P_2d_1-P_1d_2}-b\right]}{P_1c_2-P_2c_1}+d_2\cdot\frac{wc_1-P_1\left[\frac{P_1c_2-P_2c_1}{P_2d_1-P_1d_2}-b\right]}{P_2c_1-P_1c_2}}.$$

Hicksian demand functions

We try to minimize $w=P_1x_1+P_2x_2$;

s.t. $U^0=(c_1x_1+c_2x_2+b)e^{d_1x_1+d_2x_2}$.

$$\mathcal{L}=P_1x_1+P_2x_2+\lambda[U^0-(c_1x_1+c_2x_2+b)e^{d_1x_1+d_2x_2}].$$

F.O.C.

$$\frac{d\mathcal{L}}{dx_1}=P_1-\lambda[c_1e^{d_1x_1+d_2x_2}+(c_1x_1+c_2x_2+b)e^{d_1x_1+d_2x_2}d_1]=0,$$

$$\frac{d\mathcal{L}}{dx_2}=P_2-\lambda[c_2e^{d_1x_1+d_2x_2}+(c_1x_1+c_2x_2+b)e^{d_1x_1+d_2x_2}d_2]=0,$$

$$\frac{d\mathcal{L}}{d\lambda}=U^0-(c_1x_1+c_2x_2+b)e^{d_1x_1+d_2x_2}=0.$$

So, $P_1=\lambda[c_1e^{d_1x_1+d_2x_2}+(c_1x_1+c_2x_2+b)e^{d_1x_1+d_2x_2}d_1]$, and $P_2=\lambda[c_2e^{d_1x_1+d_2x_2}+(c_1x_1+c_2x_2+b)e^{d_1x_1+d_2x_2}d_2]$.

$$\frac{P_1}{P_2}=\frac{c_1+d_1(c_1x_1+c_2x_2+b)}{c_2+d_2(c_1x_1+c_2x_2+b)},$$

So, $P_2c_1+P_2d_1c_1x_1+P_2d_1c_2x_2+P_2d_2b=P_1c_2-P_1d_2c_1x_1+P_1d_2c_2x_2+P_1d_2b$,

$$(P_2d_1c_1-P_1d_2c_1)x_1=P_1c_2-P_2c_1+P_1d_2c_2x_2-P_2d_1c_2x_2+P_1d_2b-P_2d_1b,$$

$$x_1=\frac{P_1c_2-P_2c_1+P_1d_2c_2x_2-P_2d_1c_2x_2+P_1d_2b-P_2d_1b}{(P_2d_1-P_1d_2)c_1}=\frac{P_1c_2-P_2c_1}{(P_2d_1-P_1d_2)c_1}-\frac{c_2x_2}{c_1}-\frac{b}{c_1}.$$

Substitute x_1 in the equation $U^0=(c_1x_1+c_2x_2+b)e^{d_1x_1+d_2x_2}$.

$$U^0=\left[c_1\cdot\left(\frac{P_1c_2-P_2c_1}{(P_2d_1-P_1d_2)c_1}-\frac{c_2x_2}{c_1}-\frac{b}{c_1}\right)+c_2x_2+b\right]e^{d_1\cdot\left(\frac{P_1c_2-P_2c_1}{(P_2d_1-P_1d_2)c_1}-\frac{c_2x_2}{c_1}-\frac{b}{c_1}\right)+d_2x_2}$$

$$=\left(\frac{P_1c_1+P_2c_1}{P_2d_1-P_1d_2}\right)e^{d_1\cdot\left(\frac{P_1c_2-P_2c_1}{(P_2d_1-P_1d_2)c_1}-\frac{c_2x_2}{c_1}-\frac{b}{c_1}\right)+d_2x_2}.$$

$$e^{d_1\cdot\left(\frac{P_1c_2-P_2c_1}{(P_2d_1-P_1d_2)c_1}-\frac{c_2x_2}{c_1}-\frac{b}{c_1}\right)+d_2x_2}=U^0\cdot\frac{P_2d_1-P_1d_2}{P_1c_1+P_2c_1},$$

$$d_1\cdot\left(\frac{P_1c_2-P_2c_1}{(P_2d_1-P_1d_2)c_1}-\frac{c_2x_2}{c_1}-\frac{b}{c_1}\right)+d_2x_2=\ln\left(U^0\cdot\frac{P_2d_1-P_1d_2}{P_1c_1+P_2c_1}\right),$$

$$d_2x_2-\frac{d_1c_2}{c_1}x_2=\ln\left(U^0\cdot\frac{P_2d_1-P_1d_2}{P_1c_1+P_2c_1}\right)-\frac{d_1(P_1c_2-P_2c_1)}{(P_2d_1-P_1d_2)c_1}+\frac{d_1b}{c_1},$$

$$h_2(P_1,\ P_2,\ U^0)=x_2=\frac{\ln\left(U^0\cdot\frac{P_2d_1-P_1d_2}{P_1c_1+P_2c_1}\right)-\frac{d_1(P_1c_2-P_2c_1)}{(P_2d_1-P_1d_2)c_1}+\frac{d_1b}{c_1}}{\frac{d_2c_1-d_1c_2}{c_1}}$$

$$=\frac{c_1\cdot\ln\left(U^0\cdot\frac{P_2d_1-P_1d_2}{P_1c_1+P_2c_1}\right)-\frac{d_1(P_1c_2-P_2c_1)}{P_2d_1-P_1d_2}+d_1b}{d_2c_1-d_1c_2}.$$

In the same manner,

$$x_2(P_2d_1c_2-P_1d_2c_2)=P_1c_2-P_2c_1+P_1d_2c_1x_1-P_2d_1c_1x_1+P_1d_2b-P_2d_1b,$$

$$x_2=\frac{P_1c_2-P_2c_1+P_1d_2c_1x_1-P_2d_1c_1x_1+P_1d_2b-P_2d_1b}{c_2(P_2d_1-P_1d_2)}=\frac{P_1c_2-P_2c_1}{c_2(P_2d_1-P_1d_2)}-\frac{c_1x_1}{c_2}-\frac{b}{c_2}.$$

Substitute x_2 in the equation $U^0=(c_1x_1+c_2x_2+b)e^{d_1x_1+d_2x_2}$.

$$U^0=\left[c_1x_1+c_2\cdot\left(\frac{P_1c_2-P_2c_1}{c_2(P_2d_1-P_1d_2)}-\frac{c_1x_1}{c_2}-\frac{b}{c_2}\right)+b\right]e^{d_1x_1+d_2\cdot\left(\frac{P_1c_2-P_2c_1}{c_2(P_2d_1-P_1d_2)}-\frac{c_1x_1}{c_2}-\frac{b}{c_2}\right)}$$

$$=\left(\frac{P_1c_2-P_2c_1}{P_2d_1-P_1d_2}\right)e^{d_1x_1+d_2\cdot\left(\frac{P_1c_2-P_2c_1}{c_2(P_2d_1-P_1d_2)}-\frac{c_1x_1}{c_2}-\frac{b}{c_2}\right)}.$$

$$e^{d_1x_1+d_2\cdot\left(\frac{P_1c_2-P_2c_1}{c_2(P_2d_1-P_1d_2)}-\frac{c_1x_1}{c_2}-\frac{b}{c_2}\right)}=U^0\cdot\frac{P_2d_1-P_1d_2}{P_1c_2-P_2c_1},$$

$$d_1x_1+d_2\cdot\left(\frac{P_1c_2-P_2c_1}{c_2(P_2d_1-P_1d_2)}-\frac{c_1x_1}{c_2}-\frac{b}{c_2}\right)=\ln\left(U^0\cdot\frac{P_2d_1-P_1d_2}{P_1c_2-P_2c_1}\right),$$

$$d_1x_1-d_2\frac{c_1x_1}{c_2}=\ln\left(U^0\cdot\frac{P_2d_1-P_1d_2}{P_1c_2-P_2c_1}\right)-\frac{d_2(P_1c_2-P_2c_1)}{c_2(P_2d_1-P_1d_2)}+\frac{d_2b}{c_2},$$

$$h_1(P_1,\ P_2,\ U^0)=x_1=\frac{\ln\left(U^0\cdot\frac{P_2d_1-P_1d_2}{P_1c_2-P_2c_1}\right)-\frac{d_2(P_1c_2-P_2c_1)}{c_2(P_2d_1-P_1d_2)}+\frac{d_2b}{c_2}}{\frac{d_1c_2-d_2c_1}{c_2}}$$

$$=\frac{c_2\cdot\ln\left(U^0\cdot\frac{P_2d_1-P_1d_2}{P_1c_2-P_2c_1}\right)-\frac{d_2(P_1c_2-P_2c_1)}{P_2d_1-P_1d_2}+d_2b}{d_1c_2-d_2c_1}.$$

Expenditure Function

Substituting Hicksian demand functions back into objective function to get expenditure function.

$$e(P_1,\ P_2,\ U^0)=P_1h_1+P_2h_2=P_1\cdot\frac{c_2\cdot\ln\left(U^0\cdot\dfrac{P_2d_1-P_1d_2}{P_1c_2-P_2c_1}\right)-\dfrac{d_2(P_1c_2-P_2c_1)}{P_2d_1-P_1d_2}+d_2b}{d_1c_2-d_2c_1}$$

$$+P_2\cdot\frac{c_1\cdot\ln\left(U^0\cdot\dfrac{P_2d_1-P_1d_2}{P_1c_1+P_2c_1}\right)-\dfrac{d_1(P_1c_2-P_2c_1)}{P_2d_1-P_1d_2}+d_1b}{d_2c_1-d_1c_2}.$$

4. *Sumex* one-switch utility function

The Marshallian demand functions:

We try to maximize $u(x_1,\ x_2)=ae^{c_1x_1+c_2x_2}+be^{d_1x_1+d_2x_2}$.

s.t. $w=P_1x_1+P_2x_2$.

$\mathcal{L}=ae^{c_1x_1+c_2x_2}+be^{d_1x_1+d_2x_2}+\lambda[w-P_1x_1-P_2x_2]$.

F.O.C.

$$\frac{d\mathcal{L}}{dx_1}=ac_1e^{c_1x_1+c_2x_2}+bd_1e^{d_1x_1+d_2x_2}-\lambda P_1=0,$$

$$\frac{d\mathcal{L}}{dx_2}=ac_2e^{c_1x_1+c_2x_2}+bd_2e^{d_1x_1+d_2x_2}-\lambda P_2=0.$$

So, $\dfrac{P_1}{P_2}=\dfrac{ac_1e^{c_1x_1+c_2x_2}+bd_1e^{d_1x_1+d_2x_2}}{ac_2e^{c_1x_1+c_2x_2}+bd_2e^{d_1x_1+d_2x_2}}$.

$$P_1ac_2e^{c_1x_1+c_2x_2}+P_1bd_2e^{d_1x_1+d_2x_2}=P_2ac_1e^{c_1x_1+c_2x_2}+P_2bd_1e^{d_1x_1+d_2x_2},$$

$$(P_1ac_2-P_2ac_1)e^{c_1x_1+c_2x_2}=(P_2bd_1-P_1bd_2)e^{d_1x_1+d_2x_2},$$

$$\frac{e^{c_1x_1+c_2x_2}}{e^{d_1x_1+d_2x_2}}=\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1},$$

$$e^{c_1x_1+c_2x_2-d_1x_1-d_2x_2}=\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1},$$

$$c_1x_1+c_2x_2-d_1x_1-d_2x_2=\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right),$$

$$x_1(c_1-d_1)=\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)-c_2x_2+d_2x_2=\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)+x_2(d_2-c_2),$$

So, $x_1=\dfrac{\ln\left(\dfrac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)+x_2(d_2-c_2)}{c_1-d_1}$.

In the same manner,

$$x_2(c_2-d_2)=\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)-c_1x_1+d_1x_1,$$

$$x_2=\frac{\ln\left(\dfrac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)+x_1(d_1-c_1)}{c_2-d_2}.$$

Substituting x_1 and x_2 in the equation $w=P_1x_1+P_2x_2$, respectively.

$$w=P_1x_1+P_2\cdot\frac{\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)+x_1(d_1-c_1)}{c_2-d_2}=P_1x_1+\frac{P_2\cdot\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_2-d_2}+\frac{P_2x_1(d_1-c_1)}{c_2-d_2},$$

$$w-\frac{P_2\cdot\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_2-d_2}=x_1\left(P_1+\frac{P_2x_1(d_1-c_1)}{c_2-d_2}\right).$$

So, $x_1^*=\dfrac{w-\dfrac{P_2\cdot\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_2-d_2}}{P_1+\dfrac{P_2x_1(d_1-c_1)}{c_2-d_2}}=\dfrac{w(c_2-d_2)-P_2\cdot\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{P_1(c_2-d_2)+P_2(d_1-c_1)}.$

$$w=P_1\cdot\frac{\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)+x_2(d_2-c_2)}{c_1-d_1}+P_2x_2=\frac{P_1\cdot\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_1-d_1}+\frac{P_1x_2(d_1-c_2)}{c_1-d_1}+P_2x_2,$$

$$w-\frac{P_1\cdot\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_1-d_1}=x_2\cdot\left(\frac{P_1(d_1-c_2)}{c_1-d_1}+P_2\right).$$

So, $x_2^*=\dfrac{w-\dfrac{P_1\cdot\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_1-d_1}}{\left(\dfrac{P_1(d_1-c_2)}{c_1-d_1}+P_2\right)}=\dfrac{w(c_1-d_1)-P_1\cdot\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{P_1(d_2-c_2)+P_2(c_1-d_1)}.$

Substituting demand functions into objective function, gives us **indirect utility function:**

$v=v(P_1,\ P_2,\ w)=ae^{c_1x_1^*+c_2x_2^*}+be^{d_1x_1^*+d_2x_2^*}$

$$=ae^{c_1\cdot\frac{w(c_2-d_2)-P_2\cdot\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{P_1(c_2-d_2)+P_2(d_1-c_1)}+c_2\cdot\frac{w(c_1-d_1)-P_1\cdot\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{P_1(d_2-c_2)+P_2(c_1-d_1)}}$$

$$+be^{d_1\cdot\frac{w(c_2-d_2)-P_2\cdot\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{P_1(c_2-d_2)+P_2(d_1-c_1)}+d_2\cdot\frac{w(c_1-d_1)-P_1\cdot\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{P_1(d_2-c_2)+P_2(c_1-d_1)}}.$$

Hicksian demand functions

We try to minimize $w=P_1x_1+P_2x_2$;

s.t. $U^0=ae^{c_1x_1+c_2x_2}+be^{d_1x_1+d_2x_2}$,

$\mathcal{L}=P_1x_1+P_2x_2+\lambda[U^0-ae^{c_1x_1+c_2x_2}+be^{d_1x_1+d_2x_2}]$.

F.O.C.

$$\frac{d\mathcal{L}}{dx_1}=P_1-\lambda ac_1e^{c_1x_1+c_2x_2}-\lambda bd_1e^{d_1x_1+d_2x_2}=0,$$

$$\frac{d\mathcal{L}}{dx_2}=P_2-\lambda ac_2e^{c_1x_1+c_2x_2}-\lambda bd_2e^{d_1x_1+d_2x_2}=0,$$

$$\frac{d\mathcal{L}}{d\lambda}=U^0-ae^{c_1x_1+c_2x_2}+be^{d_1x_1+d_2x_2}=0.$$

So, $\dfrac{P_1}{P_2}=\dfrac{ac_1e^{c_1x_1+c_2x_2}-bd_1e^{d_1x_1+d_2x_2}}{ac_2e^{c_1x_1+c_2x_2}-bd_2e^{d_1x_1+d_2x_2}}.$

$$P_1ac_2e^{c_1x_1+c_2x_2}+P_1bd_2e^{d_1x_1+d_2x_2}=P_2ac_1e^{c_1x_1+c_2x_2}+P_2bd_1e^{d_1x_1+d_2x_2},$$

$$(P_1ac_2-P_2ac_1)e^{c_1x_1+c_2x_2}=(P_2bd_1-P_1bd_2)e^{d_1x_1+d_2x_2}.$$

$$\frac{e^{c_1x_1+c_2x_2}}{e^{d_1x_1+d_2x_2}}=e^{x_1(c_1-d_1)+x_2(c_2-d_2)}=\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1},$$

$$x_1(c_1-d_1)+x_2(c_2-d_2)=\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right),$$

So, $x_1=\dfrac{\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)-x_2(c_2-d_2)}{c_1-d_1}$, and $x_2=\dfrac{\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)-x_1(c_1-d_1)}{c_2-d_2}$.

Substituting x_1 and x_2 in the equation $U^0=ae^{c_1x_1+c_2x_2}+be^{d_1x_1+d_2x_2}$, respectively.

$$U^0=ae^{c_1x_1+c_2\cdot\frac{\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)-x_1(c_1-d_1)}{c_2-d_2}}+be^{d_1x_1+d_2\cdot\frac{\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)-x_1(c_1-d_1)}{c_2-d_2}}$$

$$=ae^{\frac{c_1x_1(c_2-d_2)+c_2\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)-c_2x_1(c_1-d_1)}{c_2-d_2}}+be^{\frac{d_1x_1(c_2-d_2)+d_2\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)-d_2x_1(c_1-d_1)}{c_2-d_2}}$$

$$=ae^{\frac{x_1(c_2d_1-c_1d_2)+c_2\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_2-d_2}}+be^{\frac{x_1(d_1c_2-d_2c_1)+d_2\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_2-d_2}}$$

$$=e^{\frac{x_1(c_2d_1-c_1d_2)}{c_2-d_2}}\cdot\left(ae^{\frac{c_2\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_2-d_2}}+be^{\frac{d_2\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_2-d_2}}\right),$$

$$e^{\frac{x_1(c_2d_1-c_1d_2)}{c_2-d_2}}=\frac{U^0}{ae^{\frac{c_2\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_2-d_2}}+be^{\frac{d_2\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_2-d_2}}},$$

$$\frac{x_1(c_2d_1-c_1d_2)}{c_2-d_2}=\ln\left(\frac{U^0}{ae^{\frac{c_2\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_2-d_2}}+be^{\frac{d_2\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_2-d_2}}}\right),$$

$$h_1(P_1,\ \ P_2,\ \ U^0)=x_1=\ln\left(\frac{U^0}{ae^{\frac{c_2\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_2-d_2}}+be^{\frac{d_2\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_2-d_2}}}\right)\cdot\frac{c_2-d_2}{x_1(c_2d_1-c_1d_2)}.$$

In the same manner,

$$U^0=ae^{c_1\frac{\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)-x_2(c_2-d_2)}{c_1-d_1}+c_2x_2}+be^{d_1\cdot\frac{\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)-x_2(c_2-d_2)}{c_1-d_1}+d_2x_2}$$

$$=ae^{\frac{c_1\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)-c_1x_2c_2+c_1x_2d_2+c_2x_2(c_1-d_1)}{c_1-d_1}}+be^{\frac{d_1\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)-d_1x_2c_2+d_1x_2d_2+d_2x_2(c_1-d_1)}{c_1-d_1}}$$

$$=ae^{\frac{x_2(c_1d_2-c_2d_1)+c_1\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_1-d_1}}+be^{\frac{x_2(d_2c_1-d_1c_2)+d_1\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_1-d_1}}$$

$$=e^{\frac{x_2(c_1d_2-c_2d_1)}{c_1-d_1}}\cdot\left(ae^{\frac{c_1\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_1-d_1}}+be^{\frac{d_1\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_1-d_1}}\right),$$

$$e^{\frac{x_2(c_1d_2-c_2d_1)}{c_1-d_1}}=\frac{U^0}{ae^{\frac{c_1\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_1-d_1}}+be^{\frac{d_1\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_1-d_1}}},$$

$$\frac{x_2(c_1d_2-c_2d_1)}{c_1-d_1}=\ln\left(\frac{U^0}{ae^{\frac{c_1\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_1-d_1}}+be^{\frac{d_1\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_1-d_1}}}\right),$$

So, $h_2(P_1,\ P_2,\ U^0)=x_2=\ln\left(\frac{U^0}{ae^{\frac{c_1\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_1-d_1}}+be^{\frac{d_1\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_1-d_1}}}\right)\cdot\frac{c_1-d_1}{x_2(c_1d_2-c_2d_1)}.$

Expenditure Function

Substituting Hicksian demand functions back into objective function to get expenditure function.

$$e(P_1,\ P_2,\ U^0)=P_1h_1+P_2h_2=P_1\cdot\ln\left(\frac{U^0}{ae^{\frac{c_2\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_2-d_2}}+be^{\frac{d_2\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_2-d_2}}}\right)\cdot\frac{c_2-d_2}{x_1(c_2d_1-c_1d_2)}$$

$$+P_2\cdot\ln\left(\frac{U^0}{ae^{\frac{c_1\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_1-d_1}}+be^{\frac{d_1\ln\left(\frac{P_2bd_1-P_1bd_2}{P_1ac_2-P_2ac_1}\right)}{c_1-d_1}}}\right)\cdot\frac{c_1-d_1}{x_2(c_1d_2-c_2d_1)}.$$

Appendix D. *Definitions and Models for Buying Price and Selling Price Measures of the Value of Information*

Buying price of information (Bakır and Klutke, 2011, 2013; Bakır, 2017).

Assume a decision maker with a continuous and monotonically increasing utility function $u(w):\mathbb{R}\to\mathbb{R}$ with an initial wealth level w faces a decision on a real valued continuous lottery $\Pi_j:\Omega\to\mathbb{R}$, $j\in\{1,\ \ldots,\ m\}$, where Ω denotes the state space. The lottery Π_j is a random variable mapping the states to monetary outcomes, which may be positive or negative. The decision maker may either accept one of the lotteries or reject the lottery at all. If the decision maker decides to play the lottery, the terminal utility will be $u(w+\Pi)$; if he or she rejects the lottery, the terminal utility is $u(w)$. Before choosing to play the lottery, the decision maker has opportunity to acquire information about whether the disjoint events $\{A_1,\ \ldots,\ A_k\}$ that satisfies $\cup_{i=1}^{k}A_i=\Omega$, will occur or not.

The *buying price* $B(w,\ \mathcal{I},\ u)$ of information bundle $\mathcal{I}$ generated by events $\{A_1,\ \ldots,\ A_k\}$ for a decision maker with utility function u and initial wealth w is the maximum amount of money that decision maker is willing to pay to acquire $\mathcal{I}$. So, the *buying price* $B(w,\ \mathcal{I},\ u)$ of information bundle $\mathcal{I}$ should satisfy the equation:

$$\max_{j\in\{1,\ \ldots,\ m\}}\{u(w),\ E[u(w+\Pi_j)]\}$$
$$=\textstyle\sum_i P(A_i)\cdot\max_{j\in\{1,\ \ldots,\ m\}}\{E[u(x+\Pi_j-B(w,\ \mathcal{I},\ u))|A_i],\ u(w-B(w,\ \mathcal{I},\ u))\}.$$

Selling price of information (Bakır, 2015)

The monetary value of information was assigned by the selling price which measures the increment in wealth level which could make the decision maker be indifferent between acquiring information and not acquiring information while making decisions. With selling price measure of the value of information, the price of information is equal to the minimum monetary compensation which can make the decision maker abandon the opportunity of information acquisition. This implies that with this addition of compensation on the decision maker's wealth, he or she will do not mind obtaining information or not while playing a lottery Π. With the same settings of buying price of information mentioned above, the *selling price* $S(w, \mathcal{I}, u)$ of information bundle $\mathcal{I}$ should satisfy the equation:

$$\max\{u(w+S(w, \mathcal{I}, u)), E[u(w+\Pi+S(w, \mathcal{I}, u))]\} = \sum_i P\{A_i\}\cdot \max\{E[u(w+\Pi)|A_i], u(w)\}.$$

Appendix E. *Calculating Procedures for Functional Equations of One-switch Utility Functions*

All the four one-switch utility functions above can satisfy the functional equations in the Theorem above. Abbas (2018) solely proved the *linear plus exponential* one-switch utility function, $U(w)=aw+be^{-rw}+c$, which can satisfy the functional equation:

$$\begin{aligned} U(w+\delta) &= aw+a\delta+be^{-rw}e^{-r\delta}+c+[be^{-rw}-be^{-rw}] \\ &= [aw+be^{-rw}+c]+be^{-rw}(e^{-rw}-1)+a\delta \\ &= f_1(w)+\phi(\delta)\cdot f_2(w)+k_0(\delta), \end{aligned}$$

where $k_1(\delta)=1$, $\phi(\delta)=b(e^{-r\delta}-1)$, $k_0(\delta)=a\delta$.

In the same manner, the *sumex* one-switch utility function, $U(w)=ae^{-\lambda w}+be^{-rw}+c$, satisfies the functional equation:

$$\begin{aligned} U(w+\delta) &= ae^{-\lambda(w+\delta)}+be^{-r(w+\delta)}+c \\ &= ae^{-\lambda w}e^{-\lambda\delta}+be^{-rw}e^{-r\delta}+c \\ &= e^{-\lambda w}\cdot ae^{-\lambda\delta}+(e^{-\lambda\delta})\cdot(e^{-\lambda\delta})\cdot e^{-rw}\cdot be^{-r\delta}+c \\ &= ae^{-\lambda w}\cdot e^{-\lambda\delta}+e^{-\lambda\delta}\cdot be^{-rw}\cdot e^{(\lambda-r)\delta}+c \\ &= f_1(w)\cdot k_1(\delta)+k_1(\delta)\cdot f_2(w)\cdot\phi(\delta)+k_0(\delta), \end{aligned}$$

where $f_1(w)=ae^{-\lambda w}$, $k_1(\delta)=e^{-\lambda\delta}$, $f_2(w)=be^{-rw}$, $\phi(\delta)=e^{(\lambda-r)\delta}$, and $k_0(\delta)=c$.

In the meantime, the *quadratics* one-switch utility function, $U(w)=aw^2+bw+c$, satisfies the functional equation:

$$\begin{aligned} U(w+\delta) &= a(w+\delta)^2+b(w+\delta)+c \\ &= a(w^2+2w\delta+\delta^2)+bw+b\delta+c \\ &= (aw^2+bw+c)+\delta(2aw+b)+a\delta^2 \\ &= f_1(w)+\phi(\delta)\cdot f_2(w)+k_0(\delta), \end{aligned}$$

where $f_1(w)=(aw^2+bw+c)$, $k_1(\delta)=1$, $\phi(\delta)=\delta$, $f_2(w)=(2aw+b)$, and $k_0(\delta)=a\delta^2$.

Lastly, the *linear times exponential* one-switch utility function, $U(w)=(aw+b)e^{-rw}+c$, satisfies the functional equation:

$$\begin{aligned} U(w+\delta) &= [a(w+\delta)+b]e^{-r(w+\delta)}+c \\ &= aw\cdot e^{-rw}\cdot e^{-r\delta}+a\delta\cdot e^{-rw}\cdot e^{-r\delta}+b\cdot e^{-r\delta}+c \\ &= (aw\cdot e^{-rw})\cdot e^{-r\delta}+e^{-rw}\cdot(a\delta e^{-r\delta}+be^{-r\delta})+c \\ &= (aw\cdot e^{-rw})\cdot e^{-r\delta}+e^{-r\delta}\cdot e^{-rw}\cdot(a\delta+b)+c \end{aligned}$$

$$=f_1(w)\cdot k_1(\delta)+k_1(\delta)\cdot f_2(w)\cdot \phi(\delta)+k_0(\delta),$$

where $f_1(w)=aw\cdot e^{-\gamma w}$, $k_1(\delta)=e^{-\gamma\delta}$, $f_2(w)=e^{-\gamma w}$, $\phi(\delta)=(a\delta+b)$, and $k_0(\delta)=c$.