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Scalar and fermion on-shell amplitudes in generalized Higgs effective field theory

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Beyond-the-standard-model (BSM) particles should be included in effective field theory in order to compute the scattering amplitudes involving these extra particles. We formulate an extension of Higgs effective field theory which contains an arbitrary number of scalar and fermion fields with arbitrary electric and chromoelectric charges. The BSM Higgs sector is described by using the nonlinear sigma model in a manner consistent with the spontaneous electroweak symmetry breaking. The chiral-order counting rule is arranged consistently with the loop expansion. The leading-order Lagrangian is organized in accord with the chiral-order counting rule. We use a geometrical language to describe the particle interactions. The parametrization redundancy in the effective Lagrangian is resolved by describing the on-shell scattering amplitudes only with the covariant quantities in the scalar/fermion field space. We introduce a useful coordinate (normal coordinate), which simplifies the computations of the on-shell amplitudes significantly. We show that the high-energy behaviors of the scattering amplitudes determine the "curvature tensors" in the scalar/fermion field space. The massive spinor–wave function formalism is shown to be useful in the computations of on-shell helicity amplitudes.

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I. INTRODUCTION

Four seemingly independent fundamental energy scales that we know about in elementary particle physics, the Planck scale $\simeq 1.2\times 10^{19}$ GeV (energy scale of gravitational interaction), the cosmological constant $\simeq (2.2\ \text{meV})^4$ (accelerated expansion of the Universe), the weak scale $v\simeq$ 246 GeV (masses of elementary particles), and the QCD scale $\simeq$300 MeV (masses of hadrons).

Among these four known fundamental energy scales, the most well understood one is the QCD scale. High-energy hadronic particle collisions much above the QCD scale can be successfully investigated perturbatively, while the low-energy hadron physics can be described in terms of low-energy effective field theories. The QCD scale is generated dynamically through the dimensional transmutation mechanism in the $SU(3)$ QCD gauge dynamics. Since the scale generation is forbidden at the classical level, the QCD scale is stable against quantum loop corrections. Moreover, the global symmetry structure of QCD allows us to develop systematic expansions in these effective field theories. Specifically, low-energy pion physics can be described in terms of chiral perturbation theory [1–5], in which the pions are treated as pseudo-Nambu-Goldstone bosons associated with the spontaneous breaking of the global chiral symmetry. The low-energy theorems in the pion scattering amplitudes are reproduced in chiral perturbation theory at its leading order. It is also possible to include higher-order corrections in a systematic manner by computing the quantum loop corrections and by introducing higher-order terms in the effective chiral Lagrangian arranged in accord with the chiral-order counting rules. Although the chiral perturbation does not converge above the resonance mass energy scale, the situation can be improved by explicitly introducing resonances such as the

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spin-1 ρ meson in the effective chiral Lagrangian [6–8]. Actually, it is possible to formulate chiral perturbation theory even in the effective chiral Lagrangian including the ρ meson [9–11].

On the other hand, the standard model (SM) of particle physics provides a consistent gauge theory framework to describe the physics associated with the weak scale. A Higgs field and Higgs potential are introduced in the SM to generate the weak scale. It has been shown that the 125 GeV scalar particle discovered at the LHC experiments can be identified successfully as a Higgs particle associated with the SM Higgs field [12]. Unlike the QCD scale generation mechanism, however, in the SM, the Higgs potential responsible for the weak scale is not classically forbidden and the weak scale is subject to huge quantum loop corrections. Fine-tuning of parameters is required in the SM to explain the smallness of the weak scale relative to the Planck scale (naturalness problem). It is a common belief that the certain beyond-the-standard-model (BSM) new physics exists not far above the weak scale, guaranteeing the naturalness of the weak scale. Unfortunately, however, it turned out that the current collider energy is not enough high to reveal the nature of BSM physics. Currently, we have no direct collider physics evidence supporting the existence of BSM physics. Compared with the QCD scale, current understanding of the weak scale physics is highly restricted in this sense, mainly due to the lack of our knowledge about the physics far above the weak scale.

It should be useful if we establish a weak scale analog to chiral perturbation theory. Assuming that BSM is weakly interacting, the standard-model effective field theory (SMEFT) [13], along with recent reviews [14,15] in which the electroweak symmetry $SU(2)\times U(1)$ is realized linearly, can be used for such a purpose. The SMEFT cannot be applied, however, to strongly interacting BSM, in which heavy BSM particles do not decouple from the low-energy physics. For strongly interacting BSM, we can use the Higgs effective field theory (HEFT) [16–31], in which the electroweak symmetry $SU(2)\times U(1)$ is realized nonlinearly.

Weak scale analogs to the resonance chiral perturbation theory have also been studied. Phenomenologies of the weak scale analogs to the spin-1 ρ resonance have been investigated by using the resonance electroweak chiral Lagrangian techniques [32–36]. We have proposed the generalized Higgs effective field theory (GHEFT) framework [37], in which arbitrary number of spin-0 resonances/particles with arbitrary electric charges are introduced in the HEFT Lagrangian. To explain the naturalness of the weak scale, it is tempting to consider BSM scenarios having larger global symmetry and thus with extra pseudo-Nambu-Goldstone particles. Actually, in composite Nambu-Goldstone Higgs models [38], global symmetries larger than the SM gauge group are introduced. There are extra pseudo-Nambu-Goldstone particles in nonminimal composite Higgs models [39–45]. We emphasize that these extra pseudo-Nambu-Goldstone particles can be successfully investigated in the GHEFT framework. We stress here the difference between the weak scale $v\simeq 246$ GeV and the compositeness scale $\Lambda\simeq 4\pi f$ in the composite Higgs scenarios. It is known that f needs to be several times larger than the weak scale in order to keep the consistency with electroweak and flavor precision constraints. In the GHEFT framework, $\Lambda\simeq 4\pi f$ can be regarded as the scale of perturbative unitarity violations, which can be pushed up to high-energy scale independently of v enough to meet these phenomenological constraints. Although the electroweak symmetry is realized nonlinearly in our GHEFT Lagrangian, our theory should be regarded as valid below the compositeness scale $4\pi f$ keeping the perturbative unitarity. This fact motivates us to introduce TeV scale resonances in the GHEFT framework.

There is a difficulty in the studies of effective field theories, i.e., nonuniqueness of its parametrization methods. The Kamefuchi-O'Raifeartaigh-Salam (KOS) theorem [46] tells us that there are equivalent classes of seemingly different effective field theories which describe the same physics. As the KOS theorem suggests, there are many equivalent formulations of effective theories connected with each other through the changes of effective field variables (coordinates). This makes it difficult to compare results computed in an effective field theory with results in seemingly different but equivalent field theories which may be generated more directly from UV physics. The Warsaw basis [47] is often assumed to resolve the nonuniqueness in SMEFT. The Warsaw basis should be understood to be a symptomatic treatment effective only at the lowest order, however. It does not provide a systematic prescription to fix the issue beyond the leading order.

The same problem exists in electroweak resonance chiral perturbation theories. Existing studies of electroweak resonance chiral perturbation theories rely on particular field parametrizations.

In our previous paper on GHEFT [37], we showed that GHEFT (electroweak resonance chiral perturbation theory) can be described by using the covariant tensors of the scalar manifolds, which allows us to parametrize the particle scattering amplitudes and the quantum corrections in a covariant manner under the changes of effective field variables (coordinates) [48,49]. It has been shown that the uses of the normal coordinate simplify the computation of the scattering amplitudes significantly. We have then shown that, once the perturbative unitarity at the tree level is ensured, then one-loop finiteness is automatically guaranteed in the GHEFT framework.

There remains an issue we need to investigate in the electroweak resonance chiral perturbation theory analysis. As far as we know, there have been no studies on the electroweak resonance chiral perturbation theory including fermionic spin-$1/2$ particles strongly coupled with the Higgs sector. It should be emphasized, however, that the SM Higgs particle couples with the top quark (fermion) most strongly. The existence of BSM spin-$1/2$ particles is widely expected in BSM models, thus explaining the naturalness of the weak scale. Actually, in the composite Higgs models, a top-quark partner fermion is usually introduced to explain the mass of the top quark.

In this paper, we generalize the findings we made in our previous paper to include fermionic heavy particles in the GHEFT Lagrangian. The particle scattering amplitudes are expressed using covariant quantities of the bosonic and fermionic field coordinates. The scattering amplitude formulas given in this paper can therefore be easily compared to the formulas computed in other equivalent formulation of fermionic resonance electroweak chiral perturbation theories.

This paper is organized as follows: In Sec. II, we introduce an extended GHEFT Lagrangian including the extra spin-$1/2$ fermionic particles. We then provide a chiral-order counting rule which allows us to perform a systematic expansion in the computation of the scattering amplitudes in a manner similar to the well-known chiral perturbation theory. In Sec. III, the normal coordinate technique is generalized to include fermionic field coordinates. We investigate tree-level spin-0 and spin-$1/2$ particle scattering amplitudes in Sec. IV by applying the normal coordinate technique. It is shown that these scattering amplitudes can be expressed in terms of the covariant quantities of the GHEFT field manifold. We conclude in Sec. V. A quick review on HEFT is given in Appendix A. Notation on the helicity eigenstate wave functions is summarized in Appendix B. Appendix C is for the explicit computations of higher-order coefficients in the normal coordinate expansion, as well as a proof of Bianchi identity.

II. GENERALIZED HIGGS EFFECTIVE FIELD THEORY

We need to incorporate new BSM particles into effective field theories (EFTs) so as to compute production cross sections and decay widths involving these new BSM particles. These new particles are not included in minimal EFTs such as SMEFT [13–15,47] and HEFT [16–31], however. We proposed in our previous paper [37] the GHEFT framework in which an arbitrary number of spin-0 resonances/particles with arbitrary electric charges were introduced. In this section we further generalize our GHEFT framework to incorporate BSM spin-$1/2$ fermions, as well as the 125 GeV Higgs boson, BSM scalar particles, quarks, and leptons.

The electroweak gauge symmetry $G = SU(2)_W \times U(1)_Y$ is spontaneously broken to the electromagnetic $H = U(1)_{\rm em}$ at the electroweak symmetry breaking (EWSB) scale. If the EWSB is triggered by strong new dynamics in BSM, the spontaneously broken symmetry G should be realized nonlinearly at the low-energy scale. Electroweakly charged particles, in such a case, transform nonlinearly under the electroweak gauge symmetry. We use the celebrated Callan-Coleman-Wess-Zumino (CCWZ) formalism [7,50,51] to formulate the low-energy EFT Lagrangian in a manner consistent with the EWSB. We note here that the CCWZ formalism can also be applied even if the electroweak symmetry is broken by perturbative dynamics.

We then provide a chiral-order counting rule in GHEFT which allows us to perform a systematic expansion in the computation of the scattering amplitudes in a manner similar to the well-known chiral perturbation theory [1–5].

A. Leading-order GHEFT Lagrangian

We start the discussion in the gaugeless limit ($g_W = g_Y = 0$) for simplicity. The couplings with the SM gauge fields will be introduced at the end of this subsection. The minimal EFT for strongly interacting EWSB is described by the HEFT Lagrangian [16–31] in the gaugeless limit,

$$\mathcal{L}_{\rm HEFT} = \mathcal{L}_{\rm HEFT,boson} + \mathcal{L}_{\rm HEFT,fermion}, \tag{1}$$

with the bosonic sector Lagrangian $\mathcal{L}_{\rm HEFT,boson}$ being

$$\begin{aligned}\mathcal{L}_{\rm HEFT,boson} = {} & G(h)\mathrm{tr}[\partial^\mu U^\dagger \partial_\mu U] \\ & + \frac{1}{2} G_Z(h)\mathrm{tr}[U^\dagger \partial^\mu U \tau^3]\mathrm{tr}[U^\dagger \partial_\mu U \tau^3] \\ & + \frac{1}{2}(\partial_\mu h)(\partial^\mu h) - V(h).\end{aligned} \tag{2}$$

The fermionic sector Lagrangian $\mathcal{L}_{\rm HEFT,fermion}$ is given in Appendix A. The bosonic sector HEFT Lagrangian (2) should be regarded as the starting point of the GHEFT framework [37]. The 125 GeV Higgs boson field is denoted by h, while U is an exponential function of the Nambu-Goldstone (NG) boson fields,

$$U = \xi_W \xi_Y, \tag{3}$$

where

$$\xi_W(x) = \exp\left(i\sum_{a=1,2}\pi^a(x)\frac{\tau^a}{2}\right), \tag{4}$$

$$\xi_Y(x) = \exp\left(i\pi^3(x)\frac{\tau^3}{2}\right), \tag{5}$$

with τ^a and π^a ($a = 1, 2, 3$) being the Pauli spin matrices and the NG boson fields. $G(h)$, $G_Z(h)$, and $V(h)$ are arbitrary functions of h, which determine the interactions among the 125 GeV Higgs field and the NG boson fields. Custodial symmetry implies $G_Z(h) = 0$. Here we do not impose $G_Z(h) = 0$, however, to keep the generality.

For later convenience, we rewrite the HEFT Lagrangian (2) in terms of the CCWZ formalism, i.e., by using the G/H Lie-algebra-valued Maurer-Cartan (MC) one-forms of the NG boson fields,

$$\alpha^a_{\perp\mu} = \mathrm{tr}\left[\frac{1}{i}\xi_W^\dagger(\partial_\mu \xi_W)\tau^a\right](a=1,2) \tag{6}$$

and

$$\alpha^3_{\perp\mu} = \mathrm{tr}\left[\frac{1}{i}\xi_W^\dagger(\partial_\mu \xi_W)\tau^3\right] + \mathrm{tr}\left[\frac{1}{i}(\partial_\mu \xi_Y)\xi_Y^\dagger\tau^3\right]. \tag{7}$$

The HEFT Lagrangian (2) is expressed as

$$\mathcal{L}_{\mathrm{HEFT,boson}} = \frac{1}{2}G_{ab}(h)\alpha^a_{\perp\mu}\alpha^{b\mu}_{\perp} + \frac{1}{2}(\partial^\mu h)(\partial_\mu h) - V(h), \tag{8}$$

where $G_{11}(h) = G_{22}(h) = G(h)$, $G_{33}(h) = G(h) + G_Z(h)$ and $G_{ab}(h) = 0$ for $a \neq b$. In Eq. (8) and hereafter, summation $\sum_{a=1,2,3}$ is implied whenever an index a is repeated in a product.

The CCWZ formalism allows us to systematically introduce extra BSM scalar particles in the low-energy EFT. Here we introduce extra $(n_R - 1)$ BSM real scalars and n_C BSM complex scalars in addition to the 125 GeV Higgs boson. Therefore, there are $n_s = n_R + 2n_C$ real scalars in total. It is convenient to introduce a real scalar multiplet $\phi^I (I = 1, 2, \ldots, n_s)$ as

$$\phi^I = (\overbrace{\phi^1, \phi^2, \ldots, \phi^{n_N}}^{n_R}, \overbrace{\phi^{n_N+1}\cdots\phi^{n_s}}^{2n_C}), \tag{9}$$

where we identify ϕ^1 as the 125 GeV Higgs boson, $\phi^1 = h$. The $H = U(1)_{\mathrm{em}}$ transformation for the scalar multiplet is defined as

$$\phi^I \xrightarrow{H} [\exp(iQ_\phi\theta_h)]^I{}_J\phi^J, \tag{10}$$

where θ_h is a real constant parameter and the $(n_s \times n_s)$ matrix Q_ϕ is defined as

$$Q_\phi = \begin{array}{cc} \overbrace{\qquad\qquad}^{n_R} & \overbrace{\qquad\qquad\qquad\qquad}^{2n_C} \end{array}$$
$$Q_\phi = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & -q_1\sigma_2 & & \\ & & & & \ddots & \\ & & & & & -q_{n_C}\sigma_2 \end{pmatrix}. \tag{11}$$

Here $\sigma_2 = \tau^2$ and $q_i (i = 1, 2, \ldots n_C)$ denotes the $U(1)_{\mathrm{em}}$ charges of the scalar fields. The $G = SU(2)_W \times U(1)_Y$ transformation of ϕ^I is given by

$$\phi^I \xrightarrow{G} [\rho_\phi]^I{}_J\phi^J, \qquad \rho_\phi = \exp(iQ_\phi\theta_h(\pi, \mathfrak{g}_W, \mathfrak{g}_Y)), \tag{12}$$

where θ_h is a real function of group elements $\mathfrak{g}_W \in SU(2)_W$, $\mathfrak{g}_Y \in U(1)_Y$, and the NG boson fields (π^a). There may be $SU(3)_C$ colored scalar particles such as leptoquark scalars and colored superpartner bosonic particles. The flavor indices I, J are understood to include the color index for these colored bosons. It is straightforward to write the $SU(3)_C$ transformation matrix for ϕ^I.

Since the G transformation matrix ρ_ϕ depends on the NG boson fields, the derivative of the scalar multiplet $\partial_\mu\phi^I$ transforms nonhomogeneously under G,

$$\partial_\mu\phi^I \xrightarrow{G} [\rho_\phi]^I{}_J(\partial_\mu\phi^J) + (\partial_\mu\rho_\phi)^I{}_J\phi^J. \tag{13}$$

Therefore, if ϕ^I contains the charged scalar (namely, $\rho_\phi \neq \mathbf{1}$), the kinetic operator $(\partial_\mu\phi^I)(\partial^\mu\phi^I)$ is not invariant under the G transformation. G-invariant kinetic terms for the charged scalar fields are formulated by introducing the covariant derivative on the G/H coset space. The covariant derivative is defined as

$$\mathcal{D}_\mu\phi^I \coloneqq \partial_\mu\phi^I + i\mathcal{V}^3_\mu[Q_\phi]^I{}_J\phi^J, \tag{14}$$

where

$$\mathcal{V}^3_\mu \coloneqq -\mathrm{tr}\left[\frac{1}{i}(\partial_\mu\xi_Y)\xi_Y^\dagger\tau^3\right] + c\alpha^3_{\perp\mu}, \tag{15}$$

with c being an arbitrary constant. The $\mathcal{V}^3_\mu$ corresponds to the H Lie-algebra-valued MC one-form, which plays the role of the connection field on the G/H coset space. It is straightforward to show that the covariant derivative $\mathcal{D}_\mu\phi^I$ homogeneously transforms under G,

$$\mathcal{D}_\mu\phi^I \xrightarrow{G} [\rho_\phi]^I{}_J(\mathcal{D}_\mu\phi^J). \tag{16}$$

The "covariant" kinetic term $(\mathcal{D}_\mu\phi^I)(\mathcal{D}^\mu\phi^I)$ respects the G invariance.

Using the G-covariant objects $\alpha^a_{\perp\mu}$, ϕ^I, and $\mathcal{D}_\mu\phi^I$, we can systematically write G-invariant Lagrangians. As we will see later, the lowest-order Lagrangian is written as [37]

$$\mathcal{L}_{\mathrm{GHEFT,boson}} = \frac{1}{2}G_{ab}(\phi)\alpha^a_{\perp\mu}\alpha^{b\mu}_{\perp} + G_{aI}(\phi)\alpha^a_{\perp\mu}(\mathcal{D}^\mu\phi^I) + \frac{1}{2}G_{IJ}(\phi)(\mathcal{D}^\mu\phi^I)(\mathcal{D}_\mu\phi^J) - V(\phi). \tag{17}$$

G_{ab}, G_{aI}, G_{IJ}, and V are functions of the scalar fields ϕ^I which homogeneously transform under the G transformation. These functions determine the interactions among the scalar fields. Again, we do not impose the custodial symmetry in Eq. (17) to keep the generality. Once we specify the ultraviolet completion of the EFT, G_{ab}, G_{aI},

G_{IJ}, and V are determined up to the uncertainty associated with the field redefinition.

We next discuss the fermion sector. We need to at least introduce SM quarks and leptons in our EFT framework. Moreover, the existence of BSM spin-1/2 particles is widely expected in BSM models, explaining the naturalness of the weak scale. For this purpose, we incorporate $\hat{n}_M$ Majorana fermions and $\hat{n}_D$ Dirac fermions into the EFT Lagrangian (17). We describe these fermions by using $\hat{n}_f = \hat{n}_M + 2\hat{n}_D$ two-component spinor fields $\psi^{\hat{i}}_\alpha (\hat{i} = \hat{1}, \ldots, \hat{n}_f)$,

$$\psi^{\hat{i}}_\alpha = (\overbrace{\psi^{\hat{1}}_\alpha, \psi^{\hat{2}}_\alpha, \ldots, \psi^{\hat{n}_M}_\alpha}^{\hat{n}_M}, \overbrace{\psi^{\hat{n}_M+1}_\alpha \cdots \psi^{\hat{n}_f}_\alpha}^{2\hat{n}_D}), \tag{18}$$

where α is the spinor index which takes 1 or 2. The Hermitian conjugate of $\psi^{\hat{i}}_\alpha$ is denoted as $\psi^{\dagger\hat{i}^*}_{\dot{\alpha}} \coloneqq (\psi^{\hat{i}}_\alpha)^\dagger$. The $U(1)_{\rm em}$ transformation for the fermion multiplet (18) is defined as

$$\psi^{\hat{i}}_\alpha \xrightarrow{H} [\exp(iQ_\psi \theta_h)]^{\hat{i}}{}_{\hat{j}} \psi^{\hat{j}}_\alpha, \tag{19}$$

where θ_h is a real constant parameter and the $(\hat{n}_f \times \hat{n}_f)$ matrix Q_ψ is defined as

$$Q_\psi = \begin{pmatrix} \overbrace{\begin{matrix} 0 & & \\ & \ddots & \\ & & 0 \end{matrix}}^{\hat{n}_M} & \\ & \overbrace{\begin{matrix} q_{\hat{1}}\sigma_3 & & \\ & \ddots & \\ & & q_{\hat{n}_D}\sigma_3 \end{matrix}}^{2\hat{n}_D} \end{pmatrix}. \tag{20}$$

Here $\sigma_3 = \tau^3$ and $q_{\hat{i}} (\hat{i} = \hat{1}, \hat{2}, \cdots \hat{n}_D)$ denotes the $U(1)_{\rm em}$ charges of the fermion fields. There certainly are $SU(3)_C$ colored spin-1/2 particles. The flavor indices $\hat{i}$, $\hat{j}$ are understood to include the color index for these colored fermions. It is straightforward to write the $SU(3)_C$ transformation matrix for $\psi^{\hat{i}}_\alpha$.

The $G = SU(2)_W \times U(1)_Y$ transformation of $\psi^{\hat{i}}_\alpha$ is given by

$$\psi^{\hat{i}}_\alpha \xrightarrow{G} [\rho_\psi]^{\hat{i}}{}_{\hat{j}} \psi^{\hat{j}}_\alpha, \qquad \rho_\psi = \exp(iQ_\psi \theta_h(\pi, \mathfrak{g}_W, \mathfrak{g}_Y)), \tag{21}$$

where θ_h is a real function of group elements $\mathfrak{g}_W \in SU(2)_W$, $\mathfrak{g}_Y \in U(1)_Y$, and the NG boson fields (π^a). We note that the derivative of the fermion field, $\partial_\mu \psi^{\hat{i}}$, nonhomogeneously transforms under the G transformation as the derivative of the scalar field $\partial_\mu \phi^I$ does. The covariant derivative can be defined as

$$\mathcal{D}_\mu \psi^{\hat{i}}_\alpha \coloneqq \partial_\mu \psi^{\hat{i}}_\alpha + i\mathcal{V}^3_\mu [Q_\psi]^{\hat{i}}{}_{\hat{j}} \psi^{\hat{j}}_\alpha, \tag{22}$$

where the connection field $\mathcal{V}^3_\mu$ is defined in Eq. (15). It is easy to show that the covariant derivative $\mathcal{D}_\mu \psi^{\hat{i}}$ transforms homogeneously under the G transformation,

$$\mathcal{D}_\mu \psi^{\hat{i}}_\alpha \xrightarrow{G} [\rho_\psi]^{\hat{i}}{}_{\hat{j}} (\mathcal{D}_\mu \psi^{\hat{j}}_\alpha). \tag{23}$$

It is now straightforward to construct a G-invariant Lagrangian for the scalar and fermion fields. We can systematically construct G-invariant operators by using the G-covariant objects, $\alpha^a_{\perp\mu}$, ϕ^I, $\psi^{\hat{i}}$, $\mathcal{D}_\mu \phi^I$, and $\mathcal{D}_\mu \psi^{\hat{i}}$. Applying the chiral-order counting rule which we will introduce in Sec. II B, we write down the leading-order Lagrangian of GHEFT as

$$\begin{aligned}
\mathcal{L}_{\rm GHEFT} = {} & \frac{1}{2} G_{ab}(\phi) \alpha^a_{\perp\mu} \alpha^{a\mu}_\perp + G_{aI}(\phi) \alpha^a_{\perp\mu} (\mathcal{D}^\mu \phi^I) + \frac{1}{2} G_{IJ}(\phi) (\mathcal{D}^\mu \phi^I)(\mathcal{D}_\mu \phi^J) - V(\phi) \\
& + \frac{i}{2} G_{\hat{i}\hat{j}^*}(\phi) (\psi^{\dagger\hat{j}^*} \bar{\sigma}^\mu (\mathcal{D}_\mu \psi^{\hat{i}}) - (\mathcal{D}_\mu \psi^{\dagger\hat{j}^*}) \bar{\sigma}^\mu \psi^{\hat{i}}) + V_{\hat{i}\hat{j}^* a}(\phi) \psi^{\dagger\hat{j}^*} \bar{\sigma}^\mu \psi^{\hat{i}} \alpha^a_{\perp\mu} + V_{\hat{i}\hat{j}^* I}(\phi) \psi^{\dagger\hat{j}^*} \bar{\sigma}^\mu \psi^{\hat{i}} (\mathcal{D}_\mu \phi^I) \\
& - \frac{1}{2} M_{\hat{i}\hat{j}}(\phi) \psi^{\hat{i}} \psi^{\hat{j}} - \frac{1}{2} M_{\hat{i}^*\hat{j}^*}(\phi) \psi^{\dagger\hat{i}^*} \psi^{\dagger\hat{j}^*} + \frac{1}{8} S_{\hat{i}\hat{j}\hat{k}\hat{l}}(\phi) (\psi^{\hat{i}} \psi^{\hat{j}})(\psi^{\hat{k}} \psi^{\hat{l}}) + \frac{1}{8} S_{\hat{i}^*\hat{j}^*\hat{k}^*\hat{l}^*}(\phi) (\psi^{\dagger\hat{i}^*} \psi^{\dagger\hat{j}^*})(\psi^{\dagger\hat{k}^*} \psi^{\dagger\hat{l}^*}) \\
& + \frac{1}{4} S_{\hat{i}\hat{j}\hat{k}^*\hat{l}^*}(\phi) (\psi^{\hat{i}} \psi^{\hat{j}})(\psi^{\dagger\hat{k}^*} \psi^{\dagger\hat{l}^*}),
\end{aligned} \tag{24}$$

where we use the spinor-index-free notation for the fermion bilinear operators [52], i.e.,

$$(\psi^{\hat{i}} \psi^{\hat{j}}) \coloneqq \varepsilon^{\alpha\beta} \psi^{\hat{i}}_\beta \psi^{\hat{j}}_\alpha, \tag{25}$$

$$(\psi^{\dagger\hat{i}^*} \psi^{\dagger\hat{j}^*}) \coloneqq \psi^{\dagger\hat{i}^*}_{\dot{\alpha}} \varepsilon^{\dot{\alpha}\dot{\beta}} \psi^{\dagger\hat{j}^*}_{\dot{\beta}}, \tag{26}$$

$$\psi^{\dagger\hat{j}^*} \bar{\sigma}^\mu \psi^{\hat{i}} \coloneqq \psi^{\dagger\hat{j}^*}_{\dot{\alpha}} (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \psi^{\hat{i}}_\alpha, \tag{27}$$

$$\psi^{\hat{i}} \sigma^{\mu\nu} \psi^{\hat{j}} \coloneqq \varepsilon^{\gamma\alpha} \psi^{\hat{i}}_\alpha (\sigma^{\mu\nu})_\gamma{}^\beta \psi^{\hat{j}}_\beta, \tag{28}$$

$$\psi^{\dagger\hat{i}^*} \bar{\sigma}^{\mu\nu} \psi^{\dagger\hat{j}^*} \coloneqq \psi^{\dagger\hat{i}^*}_{\dot{\alpha}} (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \varepsilon^{\dot{\beta}\dot{\gamma}} \psi^{\dagger\hat{j}^*}_{\dot{\gamma}}, \tag{29}$$

with $\varepsilon^{12} = -\varepsilon^{21} = -\varepsilon_{12} = \varepsilon_{21} = 1$. The spinor matrices $\bar\sigma^\mu$, σ^μ, $\sigma^{\mu\nu}$, $\bar\sigma^{\mu\nu}$ are defined as

$$(\bar\sigma^\mu)^{\dot\alpha\alpha} := (\mathbf{1}^{\dot\alpha\alpha}, -(\sigma^a)^{\dot\alpha\alpha}), \tag{30}$$

$$(\sigma^\mu)_{\alpha\dot\alpha} := \varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}(\bar\sigma^\mu)^{\dot\beta\beta}, \tag{31}$$

$$(\sigma^{\mu\nu})_\alpha{}^\beta := \frac{i}{4}[(\sigma^\mu)_{\alpha\dot\gamma}(\bar\sigma^\nu)^{\dot\gamma\beta} - (\sigma^\nu)_{\alpha\dot\gamma}(\bar\sigma^\mu)^{\dot\gamma\beta}], \tag{32}$$

$$(\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} := \frac{i}{4}[(\bar\sigma^\mu)^{\dot\alpha\gamma}(\sigma^\nu)_{\gamma\dot\beta} - (\bar\sigma^\nu)^{\dot\alpha\gamma}(\sigma^\mu)_{\gamma\dot\beta}], \tag{33}$$

where $\mathbf{1}$ and σ^a ($a = 1, 2, 3$) denote a 2×2 unit matrix and the Pauli spin matrices, respectively. Since ϕ^I transforms homogeneously under G transformation, the functions G_{ab}, G_{aI}, G_{IJ}, $G_{\hat\imath\hat\jmath^*}$, $V_{\hat\imath\hat\jmath^* a}$, $V_{\hat\imath\hat\jmath^* I}$, $M_{\hat\imath\hat\jmath}$, $M_{\hat\imath^*\hat\jmath^*}$, $S_{\hat\imath\hat\jmath\hat k\hat l}$, $S_{\hat\imath^*\hat\jmath^*\hat k^*\hat l^*}$, and $S_{\hat\imath\hat\jmath\hat k^*\hat l^*}$ also transform homogeneously under G. They are also assumed to satisfy the index-exchange symmetry,

$$G_{ab}(\phi) = G_{ba}(\phi), \tag{34}$$

$$G_{IJ}(\phi) = G_{JI}(\phi), \tag{35}$$

$$M_{\hat\imath\hat\jmath}(\phi) = M_{\hat\jmath\hat\imath}(\phi), \tag{36}$$

$$M_{\hat\imath^*\hat\jmath^*}(\phi) = M_{\hat\jmath^*\hat\imath^*}(\phi), \tag{37}$$

$$S_{\hat\imath\hat\jmath\hat k\hat l}(\phi) = S_{\hat\jmath\hat\imath\hat k\hat l}(\phi) = S_{\hat\imath\hat\jmath\hat l\hat k}(\phi) = S_{\hat k\hat l\hat\imath\hat\jmath}(\phi), \tag{38}$$

$$S_{\hat\imath\hat\jmath\hat k^*\hat l^*}(\phi) = S_{\hat\jmath\hat\imath\hat k^*\hat l^*}(\phi) = S_{\hat\imath\hat\jmath\hat l^*\hat k^*}(\phi). \tag{39}$$

The Hermiticity of the Lagrangian requires

$$[G_{ab}(\phi)]^* = G_{ab}(\phi), \tag{40}$$

$$[G_{aI}(\phi)]^* = G_{aI}(\phi), \tag{41}$$

$$[G_{IJ}(\phi)]^* = G_{IJ}(\phi), \tag{42}$$

$$[V(\phi)]^* = V(\phi), \tag{43}$$

$$[G_{\hat\imath\hat\jmath^*}(\phi)]^* = G_{\hat\jmath\hat\imath^*}(\phi), \tag{44}$$

$$[V_{\hat\imath\hat\jmath^* a}(\phi)]^* = V_{\hat\jmath\hat\imath^* a}(\phi), \tag{45}$$

$$[V_{\hat\imath\hat\jmath^* I}(\phi)]^* = V_{\hat\jmath\hat\imath^* I}(\phi), \tag{46}$$

$$[M_{\hat\imath\hat\jmath}(\phi)]^* = M_{\hat\jmath^*\hat\imath^*}(\phi), \tag{47}$$

$$[S_{\hat\imath\hat\jmath\hat k\hat l}(\phi)]^* = S_{\hat\imath^*\hat\jmath^*\hat k^*\hat l^*}(\phi), \tag{48}$$

$$[S_{\hat\imath\hat\jmath\hat k^*\hat l^*}(\phi)]^* = S_{\hat k\hat l\hat\imath^*\hat\jmath^*}(\phi). \tag{49}$$

These functions determine the interactions among the scalar bosons and the spin-$1/2$ fermions. The operator $\tilde G_{\hat\imath\hat\jmath^*}(\phi)\mathcal{D}_\mu(\psi^{\dagger\hat\jmath^*}\bar\sigma^\mu\psi^{\hat\imath})$ is absent from the Lagrangian (24) because it can be eliminated by adding the total derivative operator $\partial_\mu(\tilde G_{\hat\imath\hat\jmath^*}(\phi)\psi^{\dagger\hat\jmath^*}\bar\sigma^\mu\psi^{\hat\imath})$ and redefining $V_{\hat\imath\hat\jmath^* a}$ and $V_{\hat\imath\hat\jmath^* I}$ appropriately.

As we will see in Sec. II B, four-fermion operators should be introduced in the leading-order Lagrangian (24), while we do not introduce operators like

$$(\psi^{\hat\imath}\sigma^{\mu\nu}\psi^{\hat\jmath})[\alpha^a_{\perp\mu}, \alpha^b_{\perp\nu}], \tag{50}$$

which seemingly possess lower mass dimensions. The four-fermion operators

$$(\psi^{\dagger\hat\imath^*}\bar\sigma^\mu\psi^{\hat\imath})(\psi^{\dagger\hat\jmath^*}\bar\sigma_\mu\psi^{\hat\jmath}), \qquad (\psi^{\hat\imath}\sigma^{\mu\nu}\psi^{\hat\jmath})(\psi^{\hat k}\sigma_{\mu\nu}\psi^{\hat l}),$$
$$(\psi^{\dagger\hat\imath^*}\bar\sigma^{\mu\nu}\psi^{\dagger\hat\jmath^*})(\psi^{\dagger\hat k^*}\bar\sigma_{\mu\nu}\psi^{\dagger\hat l^*}), \qquad (\psi^{\dagger\hat\imath^*}\bar\sigma^{\mu\nu}\psi^{\dagger\hat\jmath^*})(\psi^{\hat\imath}\sigma_{\mu\nu}\psi^{\hat\jmath})$$

are Fierz rearranged to the standard forms

$$(\psi^{\dagger\hat\imath^*}\psi^{\dagger\hat\jmath^*})(\psi^{\hat\imath}\psi^{\hat\jmath}), \quad (\psi^{\hat\imath}\psi^{\hat\jmath})(\psi^{\hat k}\psi^{\hat l}), \quad (\psi^{\dagger\hat\imath^*}\psi^{\dagger\hat\jmath^*})(\psi^{\dagger\hat k^*}\psi^{\dagger\hat l^*})$$

in the Lagrangian (24).

The HEFT Lagrangian (1) can be reproduced by restricting the particle contents and the structures of the coupling functions. We summarize the relationship between GHEFT (24) and HEFT (1) in Appendix A.

The minimal electroweak gauge interactions are introduced in the EFT Lagrangian by replacing $\partial_\mu\xi_W$ and $\partial_\mu\xi_Y$ with the covariant derivatives,

$$D_\mu\xi_W = \partial_\mu\xi_W - ig_W W^a_\mu\frac{\tau^a}{2}\xi_W, \tag{51}$$

$$D_\mu\xi_Y = \partial_\mu\xi_Y + ig_Y\xi_Y B_\mu\frac{\tau^3}{2}, \tag{52}$$

with W^a_μ ($a = 1, 2, 3$), B_μ, g_W, and g_Y being the $SU(2)_W$ and $U(1)_Y$ gauge fields and gauge coupling strengths. It is also straightforward to introduce minimal QCD interactions with gluons by gauging the bosonic indices I, J and fermionic indices $\hat\imath$, $\hat\jmath$, $\hat\imath^*$, $\hat\jmath^*$ in an appropriate manner. We can also include nonminimal gauge interactions through operators like $g_V(\psi^{\hat\imath}\sigma^{\mu\nu}\psi^{\hat\jmath})V_{\mu\nu}$, with $V_{\mu\nu}$, g_V being the field strength and the coupling strength of the gauge boson. As we will discuss in Sec. II B, however, these operators do not appear at the leading order in the chiral-order counting rule.

B. Chiral-order counting rule

Low-energy effective theories are not renormalizable. They therefore contain infinitely many free parameters. To compute scattering amplitudes while keeping certain predictability in effective theories, we need to introduce an order counting rule which enables us to distinguish phenomenologically relevant parameters from irrelevant ones. If the underlying physics behind the effective theory is a perturbative theory, the operators in the effective theory can simply be organized by their mass dimensions. Higher-dimensional operators decouple from the low-energy physics quickly, and the associated parameters are suppressed by the inverse power of the cutoff scale. SMEFT [13–15,47] is constructed based on this idea.

This idea cannot be applied to nonperturbative physics, however. Chiral perturbation theory describing low-energy pion scattering amplitudes in hadron physics is a well-known example [1–5]. In chiral perturbation theory, the operators in the effective theory are not organized by their mass dimensions. They are instead organized by the number of derivatives. The chiral-order counting rule in chiral perturbation theory is known to be consistent with the expansion in terms of the energy in the scattering amplitudes, with the expansion in terms of loops, and also with the expansion in terms of light quark masses.

How can we organize the chiral-order counting rule in our effective Lagrangian? The rule should be consistent with the expansion in terms of the energy and also with the expansion in terms of loops. To construct such a chiral-order counting rule, we next study divergence structure in the radiative corrections, and we justify the contention that Eq. (24) is regarded as the leading-order Lagrangian in the loop expansion.

We first consider an amputated connected L-loop Feynman diagram $\mathcal{D}$ made only from the interactions in the Lagrangian (24). The diagram possesses I_ϕ scalar internal propagators and I_ψ fermion internal propagators. The vertices in the diagram are labeled by an integer $n = 1, 2, \ldots, N_v$, with N_v being the total number of vertices in the diagram. The superficial degree of divergence for $\mathcal{D}$ is denoted by $d(\mathcal{D})$. It can be calculated from

$$\int (d^4 p)^L p^{(\#\partial)} \left(\frac{1}{p^2}\right)^{I_\phi} \left(\frac{1}{p}\right)^{I_\psi}, \quad (\#\partial) \coloneqq \sum_{n=1}^{N_v} (\#\partial)_n. \tag{53}$$

Here the nth vertex appears from the operator with $(\#\partial)_n$ derivatives and $2 \times (\#\psi\psi)_n$ fermions. We also introduce spurions for later convenience. The nth vertex operator is assigned to have $(\#s)_n$ spurion fields. These numbers for the operators in the Lagrangian (24) are listed in Table I.

We obtain

$$d(\mathcal{D}) = 4L + \sum_{n=1}^{N_v} (\#\partial)_n - 2I_\phi - I_\psi. \tag{54}$$

We next expand the diagram $\mathcal{D}$ in terms of the external momentum p,

TABLE I. The number of derivatives, the number of fermion bilinears, and the number of spurions for operators in the lowest-order Lagrangian.

	G_{ab}	G_{aI}	G_{IJ}	V	$G_{\hat{i}\hat{j}^*}$	$V_{\hat{i}\hat{j}^* a}$	$V_{\hat{i}\hat{j}^* I}$	$M_{\hat{i}\hat{j}}$	$S_{\hat{i}\hat{j}\hat{k}\hat{l}}$
$(\#\partial)_n$	2	2	2	0	1	1	1	0	0
$(\#\psi\psi)_n$	0	0	0	0	1	1	1	1	2
$(\#s)_n$	0	0	0	2	0	0	0	1	0

$$\mathcal{D} = \sum_{(\#p)=0,1,2,\cdots} \mathcal{D}_{(\#p)} p^{(\#p)} = \mathcal{D}_0 + \mathcal{D}_1 p + \mathcal{D}_2 p^2 + \cdots. \tag{55}$$

The superficial degree of divergence for $\mathcal{D}_{(\#p)}$ is thus

$$d(\mathcal{D}_{(\#p)}) = 4L + \sum_{n=1}^{N_v} (\#\partial)_n - (\#p) - 2I_\phi - I_\psi. \tag{56}$$

The number of scalar propagators I_ϕ can be removed from Eq. (56) by using the graph-theoretical Euler formula

$$L + N_v - I_\phi - I_\psi = 1. \tag{57}$$

We find that

$$d(\mathcal{D}_{(\#p)}) = 2L + \sum_{n=1}^{N_v} [(\#\partial)_n - 2] - (\#p) + 2 + I_\psi. \tag{58}$$

We next turn to the renormalization of the effective theory. We assume that the effective theory is nonanomalous. The divergences associated with $d(\mathcal{D}_{(\#p)}) \geq 0$ can thus be subtracted by introducing local operator counterterms $\mathcal{O}$. The number of derivatives, the number of fermions, and the number of spurions in $\mathcal{O}$ are computed as

$$(\#\partial)_{\mathcal{O}} = (\#p), \quad (\#\psi\psi)_{\mathcal{O}} = \sum_{n=1}^{N_v} (\#\psi\psi)_n - I_\psi, \quad (\#s)_{\mathcal{O}} = \sum_{n=1}^{N_v} (\#s)_n. \tag{59}$$

Using the relations above, the inequality $d(\mathcal{D}_{(\#p)}) \geq 0$ can be rewritten as

$$2L + \sum_{n=1}^{N_v} [(\#\partial)_n + (\#\psi\psi)_n + (\#s)_n - 2] \geq (\#\partial)_{\mathcal{O}} + (\#\psi\psi)_{\mathcal{O}} + (\#s)_{\mathcal{O}} - 2. \tag{60}$$

We define the "chiral dimension" of the operator $\mathcal{O}$ as

$$C(\mathcal{O}) \coloneqq (\#\partial)_{\mathcal{O}} + (\#\psi\psi)_{\mathcal{O}} + (\#s)_{\mathcal{O}}. \tag{61}$$

The spurion field dependences $(\#s)_n$ in Table I are determined so as to keep $C(n) = 2$ in the lowest-order Lagrangian. Here we define $C(n)$ as the chiral dimension for the operator from which the nth vertex arises in the Feynman diagram $\mathcal{D}$.

The counterterms $\mathcal{O}$ that we need to introduce to subtract the divergences in the diagram $\mathcal{D}_{(\#p)}$ therefore satisfy an inequality

$$2L + \sum_{n=1}^{N_v} [C(n) - 2] \geq C(\mathcal{O}) - 2. \tag{62}$$

Here the equality corresponds to the logarithmic divergence. Since $C(n) = 2$ for operators in the lowest-order Lagrangian, we obtain

$$2L + 2 \geq C(\mathcal{O}). \tag{63}$$

The divergences in the L-loop diagram made from the lowest-order Lagrangian can thus be subtracted by using a finite number of counterterms having chiral dimensions less than or equal to $2L + 2$.

We need to pay special attention to the four-fermion operators [53–56] in the Lagrangian (24). These four-fermion operators would arise at leading order, for instance, from the exchange of a heavy resonance with a strong coupling. The coefficients of the four-fermion operators *do not decouple* and appear at the leading order in the low-energy effective theory due to the strong interaction with the heavy resonance.[1] Moreover, the SM quarks and leptons may be composite states arising from new strong dynamics. The exchange of common constituents in the composite state naturally produces a large coefficient $(4\pi)^2/\Lambda^2$ for the composite four-fermion operators [58–60]. Even if the SM quarks and leptons are assumed to be elementary fermions, there may also be relatively light partner fermions in the strongly interacting EWSB sector. The four-fermion operators involving these strongly interacting partner fermions appear at the leading order in our effective Lagrangian. These are the reasons why we *did not* assign $(\#s)_n > 0$ for the four-fermion operators in Table I. We also note that the assignments of the spurion field dependences of V and $M_{\hat{i}\hat{j}}$ terms in Table I are determined to balance the chiral dimensions of mass and kinetic terms in scalar and fermion propagators.

There are one-loop divergence which can be subtracted by the counterterm $s\psi^{\hat{i}}\sigma^{\mu\nu}\psi^{\hat{j}}[\alpha^a_{\perp\mu}, \alpha^b_{\perp\nu}]$. Note that, because of the chirality-flip structure of the operator, the divergence appears only with the chirality flipping spurion field s. The chiral dimension of the counterterm is therefore counted as 4. The appearance of the one-loop divergence associated with this operator is consistent with the expectation from the chiral-order counting rule (63). It should also be stressed that, if we had included the operator $s\psi^{\hat{i}}\sigma^{\mu\nu}\psi^{\hat{j}}[\alpha^a_{\perp\mu}, \alpha^b_{\perp\nu}]$ in the lowest-order Lagrangian, we would not be able to perform a systematic expansion in the computation of the amplitudes based on the chiral-order counting rule.

It is now straightforward to construct a systematic expansion of the amplitudes based on the chiral-order counting rule. Note here that the inequality (62) holds even in a general L-loop diagram with $C(n) \geq 2$. It therefore assures us that we can obtain finite amplitude by applying the standard subtraction procedure with these counterterms. The loop expansion should therefore be performed simultaneously with the expansion in terms of the chiral dimension (61).

If we restrict ourselves to the operators with $(\#\psi\psi)_{\mathcal{O}} = 0$, this result is well known in the context of chiral perturbation theory (low-energy effective theory for the QCD pion) [1–5]. Our finding therefore can be regarded as a fermionic generalization of chiral perturbation theory.

Finally, let us comment on the chiral-order counting of the gauge sector. We remark that, in order for the gauge boson kinetic Lagrangian

$$\mathcal{L}_{\text{gauge,kin}} = -\frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \tag{64}$$

to be at the leading order (chiral dimension 2), we need to assign the chiral dimensions of the field strengths as

$$C(W^a_{\mu\nu}) = C(B_{\mu\nu}) = 1. \tag{65}$$

Since $C(\partial) = 1$, Eq. (65) implies

$$C(W^a_\mu) = C(B_\mu) = 0. \tag{66}$$

Furthermore, since the gauge bosons are introduced as Eqs. (51) and (52), the chiral dimension of the gauge coupling parameters should be

$$C(g_W) = C(g_Y) = 1. \tag{67}$$

Computing the one-loop diagrams with an external gauge line, we find that there are divergences in the operator,

$$g_V s(\psi^{\hat{i}}\sigma^{\mu\nu}\psi^{\hat{j}}) V_{\mu\nu}. \tag{68}$$

[1]Four-fermion operators have been ignored in the HEFT approach [16–31], however. This is because that, in the HEFT, quarks and leptons are assumed to couple with the heavy resonance only perturbatively. Therefore, the number of spurions $(\#s)_n$ for the four-fermion operators is assigned to be $(\#s)_n > 0$ in the HEFT approach. They therefore can be treated as the next-to-leading-order Lagrangian in the HEFT [57]. The assumptions made in the HEFT approach need not hold, however, if we do not assume underlying UV physics behind our effective theory.

The chiral dimension of Eq. (68) is

$$C(g_V s(\psi^{\hat{i}}\sigma^{\mu\nu}\psi^{\hat{j}})V_{\mu\nu}) = 4. \tag{69}$$

The appearance of the one-loop divergence associated with this operator is consistent with the expectation from the chiral-order counting rule (63). It should also be stressed that, if we had included the operator $g_V s(\psi^{\hat{i}}\sigma^{\mu\nu}\psi^{\hat{j}})V_{\mu\nu}$ in the lowest-order Lagrangian, we would not be able to perform systematic expansion in the computation of the amplitudes based on the chiral-order counting rule.

C. Geometrical form

The scalar fields in the leading-order GHEFT Lagrangian (24) consist of NG boson fields π^a and the non-NG boson fields ϕ^I. It is convenient to introduce a scalar field multiplet notation ϕ^i without distinguishing the NG bosons from the non-NG bosons,

$$\phi^i = (\pi^a, \phi^I) = (\pi^1, \pi^2, \pi^3, \phi^1, \ldots, \phi^{n_s}). \tag{70}$$

Using the scalar multiplet ϕ^i, the EFT Lagrangian (24) can be expressed in a geometrical form:

$$\begin{aligned}\mathcal{L}_{\text{GHEFT}} = {}& \frac{1}{2}g_{ij}(\phi)(\partial_\mu\phi^i)(\partial^\mu\phi^j) - V(\phi) + \frac{i}{2}g_{\hat{i}\hat{j}^*}(\phi)(\psi^{\dagger\hat{j}^*}\bar{\sigma}^\mu(\partial_\mu\psi^{\hat{i}}) - (\partial_\mu\psi^{\dagger\hat{j}^*})\bar{\sigma}^\mu\psi^{\hat{i}}) + v_{\hat{i}\hat{j}^*i}(\phi)\psi^{\dagger\hat{j}^*}\bar{\sigma}^\mu\psi^{\hat{i}}(\partial_\mu\phi^i) \\ & - \frac{1}{2}M_{\hat{i}\hat{j}}(\phi)\psi^{\hat{i}}\psi^{\hat{j}} - \frac{1}{2}M_{\hat{i}^*\hat{j}^*}(\phi)\psi^{\dagger\hat{i}^*}\psi^{\dagger\hat{j}^*} + \frac{1}{8}S_{\hat{i}\hat{j}\hat{k}\hat{l}}(\phi)(\psi^{\hat{i}}\psi^{\hat{j}})(\psi^{\hat{k}}\psi^{\hat{l}}) + \frac{1}{8}S_{\hat{i}^*\hat{j}^*\hat{k}^*\hat{l}^*}(\phi)(\psi^{\dagger\hat{i}^*}\psi^{\dagger\hat{j}^*})(\psi^{\dagger\hat{k}^*}\psi^{\dagger\hat{l}^*}) \\ & + \frac{1}{4}S_{\hat{i}\hat{j}\hat{k}^*\hat{l}^*}(\phi)(\psi^{\hat{i}}\psi^{\hat{j}})(\psi^{\dagger\hat{k}^*}\psi^{\dagger\hat{l}^*}),\end{aligned} \tag{71}$$

where $g_{\hat{i}\hat{j}^*}$ and $v_{\hat{i}\hat{j}^*i}$ satisfy

$$[g_{\hat{i}\hat{j}^*}(\phi)]^* = g_{\hat{j}\hat{i}^*}(\phi), \tag{72}$$

$$[v_{\hat{i}\hat{j}^*i}(\phi)]^* = v_{\hat{j}\hat{i}^*i}(\phi). \tag{73}$$

The coefficients $g_{\hat{i}\hat{j}^*}$ and $v_{\hat{i}\hat{j}^*i}$ are calculated from $G_{\hat{i}\hat{j}^*}$, $V_{\hat{i}\hat{j}^*a}$, and $V_{\hat{i}\hat{j}^*I}$ as

$$g_{\hat{i}\hat{j}^*} = G_{\hat{i}\hat{j}^*}, \tag{74}$$

$$\begin{aligned}v_{\hat{i}\hat{j}^*1} = {}& V_{\hat{i}\hat{j}^*1} - \frac{1}{2}V_{\hat{i}\hat{j}^*3}\pi^2 \\ & - \frac{1}{6}V_{\hat{i}\hat{j}^*1}\pi^2\pi^2 + \frac{1}{6}V_{\hat{i}\hat{j}^*2}\pi^1\pi^2 + \mathcal{O}((\pi)^3),\end{aligned} \tag{75}$$

$$\begin{aligned}v_{\hat{i}\hat{j}^*2} = {}& V_{\hat{i}\hat{j}^*2} + \frac{1}{2}V_{\hat{i}\hat{j}^*3}\pi^1 \\ & - \frac{1}{6}V_{\hat{i}\hat{j}^*2}\pi^1\pi^1 + \frac{1}{6}V_{\hat{i}\hat{j}^*1}\pi^1\pi^2 + \mathcal{O}((\pi)^3),\end{aligned} \tag{76}$$

$$\begin{aligned}v_{\hat{i}\hat{j}^*3} = {}& V_{\hat{i}\hat{j}^*3} + \frac{1}{2}(G_{\hat{i}'\hat{j}^*}[Q_\psi]^{\hat{i}'}{}_{\hat{i}} + G_{\hat{i}\hat{j}'^*}[Q_\psi]^{\hat{j}'^*}{}_{\hat{j}^*}) \\ & - iV_{\hat{i}\hat{j}^*I}[Q_\phi]^I{}_J\phi^J + \mathcal{O}((\pi)^3),\end{aligned} \tag{77}$$

$$v_{\hat{i}\hat{j}^*I} = V_{\hat{i}\hat{j}^*I}, \tag{78}$$

and the scalar metric tensor g_{ij} are calculated from G_{ab}, G_{aI}, and G_{IJ} [37].

It may also be illuminating to point out a similarity between our GHEFT Lagrangian (71) and the supersymmetric nonlinear sigma model Lagrangian,

$$\begin{aligned}\mathcal{L} = {}& g_{ii^*}(\partial_\mu\phi^i)\eta^{\mu\nu}(\partial_\nu\phi^{\dagger i^*}) - g^{ii^*}P_{,i}P^\dagger_{,i^*} + \frac{i}{2}g_{ii^*}(\psi^{\dagger i^*}\bar{\sigma}^\mu(\partial_\mu\psi^i) - (\partial_\mu\psi^{\dagger i^*})\bar{\sigma}^\mu\psi^i) + \frac{i}{2}(\psi^{\dagger i^*}\bar{\sigma}^\mu\psi^i)(g_{ii^*,j}\partial_\mu\phi^j - g_{ii^*,j^*}\partial_\mu\phi^{\dagger j^*}) \\ & - (P_{,ij} - P_{,i'}g^{i'i^*}g_{ii^*,i})(\psi^i\psi^j) - (P^\dagger_{,i^*j^*} - P^\dagger_{,i'^*}g^{ii'^*}g_{ii^*,j^*})(\psi^{\dagger i^*}\psi^{\dagger j^*}) \\ & + \frac{1}{4}(g_{ii^*,jj^*} - g^{i'i'^*}g_{i'i^*,j^*}g_{ii'^*,j})(\psi^{\dagger i^*}\psi^{\dagger j^*})(\psi^i\psi^j),\end{aligned} \tag{79}$$

where $P(\phi)$ is the superpotential and the Kähler metric $g_{ij^*}(\phi, \phi^\dagger)$ is computed from the Kähler potential $K(\phi, \phi^\dagger)$,

$$g_{ij^*}(\phi, \phi^\dagger) = \frac{\partial^2 K}{\partial\phi^i\partial\phi^{\dagger j^*}}. \tag{80}$$

Here we use a comma-derivative notation

$$g_{ii^*,j} \coloneqq \frac{\partial}{\partial\phi^j}g_{ii^*}, \quad g_{ii^*,j^*} \coloneqq \frac{\partial}{\partial\phi^{\dagger j^*}}g_{ii^*}, \quad \cdots \tag{81}$$

to keep the expression as simple as possible. Not only the scalar and fermion kinetic terms but also the counterparts to

$v_{\hat{i}\hat{j}^* i}(\phi)$ and the four-fermion terms in the GHEFT Lagrangian (71) are expressed in terms of Kähler manifold geometry. Even though the scalar manifold of our GHEFT Lagrangian (71) does not possess Kähler properties, as we will show later, the particle scattering amplitudes in the GHEFT Lagrangian can also be described in terms of the covariant tensors of the manifold.

III. NORMAL COORDINATE

A. Field coordinate transformations

The KOS theorem [46] tells us that seemingly different effective Lagrangians connected through field coordinate transformations can describe the identical scattering amplitudes. An effective Lagrangian therefore cannot be determined uniquely. We summarize here the field coordinate transformation properties in the effective Lagrangian.

We consider a field transformation which keeps the chiral dimension of the fields. Under such a redefinition of the field coordinates $\psi^{\hat{i}}$, $\psi^{\dagger\hat{i}^*}$, ϕ^i,

$$\begin{aligned} \phi^i &\to f^i(\phi), \\ \psi^{\hat{i}} &\to f^{\hat{i}}{}_{\hat{j}}(\phi)\psi^{\hat{j}}, \\ \psi^{\dagger\hat{i}^*} &\to f^{*\hat{i}^*}{}_{\hat{j}^*}(\phi)\psi^{\dagger\hat{j}^*}, \end{aligned} \tag{82}$$

the functions $g_{ij}(\phi)$, $g_{\hat{i}\hat{j}^*}(\phi)$, $v_{\hat{i}\hat{j}^* i}(\phi)$, $M_{\hat{i}\hat{j}}(\phi)$, $M_{\hat{i}^*\hat{j}^*}(\phi)$, $S_{\hat{i}\hat{j}\hat{k}\hat{l}}(\phi)$, $S_{\hat{i}^*\hat{j}^*\hat{k}^*\hat{l}^*}(\phi)$, and $S_{\hat{i}\hat{j}\hat{k}^*\hat{l}^*}(\phi)$ in Eq. (71) transform as

$$g_{ij}(\phi) \to g_{i'j'}(f(\phi))f^{i'}{}_{,i}(\phi)f^{j'}{}_{,j}(\phi), \tag{83}$$

$$g_{\hat{i}\hat{j}^*}(\phi) \to g_{\hat{i}'\hat{j}'^*}(f(\phi))f^{\hat{i}'}{}_{\hat{i}}(\phi)f^{*\hat{j}'^*}{}_{\hat{j}^*}(\phi), \tag{84}$$

$$\begin{aligned} v_{\hat{i}\hat{j}^* i}(\phi) \to{}& v_{\hat{i}'\hat{j}'^* i'}(f(\phi))f^{i'}{}_{,i}(\phi)f^{\hat{i}'}{}_{\hat{i}}(\phi)f^{*\hat{j}'^*}{}_{\hat{j}^*}(\phi) \\ &+\frac{i}{2}g_{\hat{i}'\hat{j}'^*}(f(\phi))(f^{\hat{i}'}{}_{\hat{i},i}(\phi)f^{*\hat{j}'^*}{}_{\hat{j}^*}(\phi) \\ &-f^{\hat{i}'}{}_{\hat{i}}(\phi)f^{*\hat{j}'^*}{}_{\hat{j}^*,i}(\phi)), \end{aligned} \tag{85}$$

$$M_{\hat{i}\hat{j}}(\phi) \to M_{\hat{i}'\hat{j}'}(\phi)f^{\hat{i}'}{}_{\hat{i}}(\phi)f^{\hat{j}'}{}_{\hat{j}}(\phi), \tag{86}$$

$$M_{\hat{i}^*\hat{j}^*}(\phi) \to M_{\hat{i}'^*\hat{j}'^*}(\phi)f^{*\hat{i}'^*}{}_{\hat{i}}(\phi)f^{*\hat{j}'^*}{}_{\hat{j}}(\phi), \tag{87}$$

and

$$S_{\hat{i}\hat{j}\hat{k}\hat{l}}(\phi) \to S_{\hat{i}'\hat{j}'\hat{k}'\hat{l}'}(\phi)f^{\hat{i}'}{}_{\hat{i}}(\phi)f^{\hat{j}'}{}_{\hat{j}}(\phi)f^{\hat{k}'}{}_{\hat{k}}(\phi)f^{\hat{l}'}{}_{\hat{l}}(\phi), \tag{88}$$

$$\begin{aligned} &S_{\hat{i}^*\hat{j}^*\hat{k}^*\hat{l}^*}(\phi) \\ &\quad\to S_{\hat{i}'^*\hat{j}'^*\hat{k}'^*\hat{l}'^*}(\phi)f^{*\hat{i}'^*}{}_{\hat{i}^*}(\phi)f^{*\hat{j}'^*}{}_{\hat{j}^*}(\phi)f^{*\hat{k}'^*}{}_{\hat{k}^*}(\phi)f^{*\hat{l}'^*}{}_{\hat{l}^*}(\phi), \end{aligned} \tag{89}$$

$$\begin{aligned} &S_{\hat{i}\hat{j}\hat{k}^*\hat{l}^*}(\phi) \\ &\quad\to S_{\hat{i}'\hat{j}'\hat{k}'^*\hat{l}'^*}(\phi)f^{\hat{i}'}{}_{\hat{i}}(\phi)f^{\hat{j}'}{}_{\hat{j}}(\phi)f^{*\hat{k}'^*}{}_{\hat{k}'^*}(\phi)f^{*\hat{l}'^*}{}_{\hat{l}'^*}(\phi). \end{aligned} \tag{90}$$

The model parametrization in the GHEFT Lagrangian (71) is, therefore, not unique. Seemingly different Lagrangians can describe the same scattering amplitudes if these Lagrangians are connected with each other through the field redefinitions (82). The GHEFT Lagrangian (71) therefore contains redundancy in its model parametrization, which leads to a lot of inconvenience in its phenomenological analysis. Clearly, we need a method which can uniquely identify the class of GHEFT Lagrangians which describe the same scattering amplitudes.

Note that the field redefinitions (82) can be regarded as a general coordinate transformation in the field space manifold. Therefore, the scattering amplitudes are expected to be described in terms of covariantly transforming tensors under these general coordinate transformations. We can consider more general field redefinitions than Eq. (82) such as $\phi \to \phi + \psi\psi$ and $\phi \to \phi + \partial_\mu\phi\partial^\mu\phi$. These field transformations, however, violate the chiral-order counting rule. Amplitudes computed at finite order in the chiral-order counting are affected by these field redefinitions. We therefore restrict ourselves to the field coordinate transformations given in Eq. (82).

In our previous paper [37], we explicitly showed that the scalar scattering amplitudes are described in terms of the Riemann curvature tensor $R_{i_1 i_2 i_3 i_4}(\phi)$ in the scalar field space and the scalar potential $V(\phi)$ and their covariant derivatives. We have also shown that the use of the Riemann normal coordinate (RNC) can reduce the computational tasks significantly. In this paper, we generalize these findings to the fermionic GHEFT Lagrangian (71).

B. Scalar sector

As we showed in Ref. [37], the use of the RNC significantly reduces the computational task of the scalar boson scattering amplitudes. This is because of the fact that, in the RNC, the Taylor expansion of the field metric tensor $g_{ij}(\phi)$ is expressed in terms of covariant quantities, i.e., the Riemann curvature tensor and its covariant derivatives. On the other hand, although the RNC is defined by using the geodesics on the manifold, there is no direct connection between the computation of the scattering amplitudes and the geodesic equations on the scalar manifold. Moreover, the GHEFT Lagrangian that we gave in Eq. (71) contains a complex valued fermion metric $g_{\hat{i}\hat{j}^*}(\phi)$, in addition to the real valued scalar manifold metric $g_{ij}(\phi)$, and the meaning of the geodesic equations in fermionic metric $g_{\hat{i}\hat{j}^*}(\phi)$ is not clear [61]. It should therefore be illuminating, before going to the fermionic sector of the GHEFT Lagrangian, to reconsider the derivation of the normal coordinate in the scalar manifold in a manner not relying on the geodesic equations. Here the normal coordinate is defined as a coordinate in which Taylor expansion coefficients of the metric tensor around the vacuum are all expressed in terms of covariant tensors.

We introduce contravariant and covariant vectors,

$$v^i(\phi), \qquad a_i(\phi), \tag{91}$$

which transform as

$$v^i(\phi) \to v^{i'}(f(\phi)) f^i_{,i'}(\phi), \quad a_i(\phi) \to a_{i'}(f(\phi)) f^{i'}_{,i}(\phi). \tag{92}$$

The covariant derivatives of v^i and a_i are expressed by using the "semicolon" covariant-derivative notation,

$$v^i_{;j} = v^i_{,j} + v^{i'}\Gamma^i_{i'j}, \qquad a_{i;j} = a_{i,j} - a_{i'}\Gamma^{i'}_{ij}, \tag{93}$$

with the bosonic affine connection Γ^i_{jk} being defined as

$$\Gamma^i_{jk} = \frac{1}{2} g^{ii'}(g_{i'j,k} + g_{i'k,j} - g_{jk,i'}), \tag{94}$$

with $g^{ii'}$ being the inverse of the metric tensor,

$$g^{ii'} g_{i'j} = \delta^i_j. \tag{95}$$

We consider

$$\begin{aligned} a_{i;jk} - a_{i;kj} &= a_{i''}(\Gamma^{i''}_{ik,j} - \Gamma^{i''}_{ij,k} + \Gamma^{i''}_{i'j}\Gamma^{i'}_{ik} - \Gamma^{i''}_{i'k}\Gamma^{i'}_{ij}) \\ &= a_{i''} R^{i''}{}_{ijk}, \end{aligned} \tag{96}$$

where we define Riemann curvature tensor as

$$R^{i''}{}_{ijk} = \Gamma^{i''}_{ik,j} - \Gamma^{i''}_{ij,k} + \Gamma^{i''}_{i'j}\Gamma^{i'}_{ik} - \Gamma^{i''}_{i'k}\Gamma^{i'}_{ij}. \tag{97}$$

We also introduce

$$R_{ii'jk} \coloneqq g_{ii''} R^{i''}{}_{i'jk} \tag{98}$$

for later convenience.

We assume that the field values at the vacuum are $\phi^i = 0$. If the eigenvalues of the scalar manifold metric g_{ij} are all positive definite,[2] we are able to find a normal coordinate in which $g_{ij}(\phi)$ is written as

$$\begin{aligned} g_{ij}(\phi) &= \delta_{ij} + \frac{1}{2} G_{ijk_1k_2}\phi^{k_1}\phi^{k_2} + \frac{1}{3!} G_{ijk_1k_2k_3}\phi^{k_1}\phi^{k_2}\phi^{k_3} \\ &\quad + \frac{1}{4!} G_{ijk_1k_2k_3k_4}\phi^{k_1}\phi^{k_2}\phi^{k_3}\phi^{k_4} + \cdots \\ &= \delta_{ij} + \frac{1}{2} G_{ij(k_1k_2)}\phi^{k_1}\phi^{k_2} + \frac{1}{3!} G_{ij(k_1k_2k_3)}\phi^{k_1}\phi^{k_2}\phi^{k_3} \\ &\quad + \frac{1}{4!} G_{ij(k_1k_2k_3k_4)}\phi^{k_1}\phi^{k_2}\phi^{k_3}\phi^{k_4} + \cdots, \end{aligned} \tag{99}$$

with coefficients $G_{ijk_1k_2}, G_{ijk_1k_2k_3}, \cdots$ being expressed in terms of covariant tensors at the vacuum

$$\begin{aligned} R_{ijk_1k_2}\Big|_0 &\coloneqq R_{ijk_1k_2}\Big|_{\phi^i=0}, \\ R_{ijk_1k_2;k_3}\Big|_0 &\coloneqq R_{ijk_1k_2;k_3}\Big|_{\phi^i=0}. \\ &\vdots \end{aligned} \tag{100}$$

Here the indices between the parentheses are understood to be symmetrized, i.e.,

$$G_{ij(k_1k_2)} \coloneqq \frac{1}{2}(G_{ijk_1k_2} + G_{ijk_2k_1}), \tag{101}$$

$$\begin{aligned} G_{ij(k_1k_2k_3)} \coloneqq \frac{1}{3!}(&G_{ijk_1k_2k_3} + G_{ijk_2k_3k_1} + G_{ijk_3k_1k_2} \\ &+ G_{ijk_1k_3k_2} + G_{ijk_2k_1k_3} + G_{ijk_3k_2k_1}). \end{aligned} \tag{102}$$

Since the metric tensor g_{ij} is symmetric under the $i \leftrightarrow j$ exchange, the coefficients $G_{ijk_1k_2}, G_{ijk_1k_2k_3}, \cdots$ need to satisfy

$$G_{ijk_1k_2} = G_{jik_1k_2}, \qquad G_{ijk_1k_2k_3} = G_{jik_1k_2k_3}, \qquad \cdots. \tag{103}$$

The Riemann curvature tensor can be computed as

$$\begin{aligned} R_{i'ijk} &= \frac{1}{2}(G_{i'k(ij)} - G_{ik(i'j)} - G_{i'j(ik)} + G_{ij(i'k)}) + \frac{1}{2}(G_{i'k(ijk_1)} - G_{ik(i'jk_1)} - G_{i'j(ikk_1)} + G_{ij(i'kk_1)})\phi^{k_1} \\ &\quad + \frac{1}{4}(G_{i'k(ijk_1k_2)} - G_{ik(i'jk_1k_2)} - G_{i'j(ikk_1k_2)} + G_{ij(i'kk_1k_2)})\phi^{k_1}\phi^{k_2} \\ &\quad - \frac{1}{4}(G_{i'j'(jk_1)} + G_{jj'(i'k_1)} - G_{i'j(j'k_1)})\delta^{j'j''} \times (G_{j''i(kk_2)} + G_{j''k(ik_2)} - G_{ik(j''k_2)})\phi^{k_1}\phi^{k_2} \\ &\quad + \frac{1}{4}(G_{i'j'(kk_1)} + G_{kj'(i'k_1)} - G_{i'k(j'k_1)})\delta^{j'j''}(G_{j''i(jk_2)} + G_{j''j(ik_2)} - G_{ij(j''k_2)})\phi^{k_1}\phi^{k_2} + \cdots. \end{aligned} \tag{104}$$

[2] This condition guarantees the absence of ghost particles in the GHEFT framework. See [62,63] for discussions of the pseudounitarity in theories with ghost particles.

We therefore obtain an expression for the Riemann curvature tensor at the vacuum,

$$R_{i12j}|_0 = \frac{1}{2}(G_{ij(12)} - G_{1j(i2)} - G_{i2(1j)} + G_{12(ij)}), \tag{105}$$

and thus

$$\begin{aligned} R_{i(12)j}|_0 = \frac{1}{4}(&2G_{ij(12)} - G_{1j(i2)} - G_{1i(j2)} \\ &- G_{i2(1j)} - G_{j2(1i)} + 2G_{12(ij)}). \end{aligned} \tag{106}$$

We here introduced a shorthand abbreviation with which $1, 2, \cdots$ are understood to be $k_1, k_2, \cdots$, respectively. Note that, as we explained earlier, the coefficient $G_{ij(kl\cdots)}$ should be expressed in terms of the Riemann curvature tensor and its covariant derivatives at the vacuum. The form of $G_{ij(12)}$ is uniquely determined as

$$G_{ij(12)} = aR_{i(12)j}|_0, \tag{107}$$

with a being a constant, thanks to the Riemann tensor symmetry

$$\begin{aligned} &R_{1234} + R_{1243} = 0, \\ &R_{1234} + R_{2134} = 0, \\ &R_{1234} - R_{3412} = 0. \end{aligned} \tag{108}$$

Actually, all other index structures, even under the $1 \leftrightarrow 2$ exchange, can be reduced to the form of Eq. (107):

$$R_{ij12} + R_{ij21} = 0, \tag{109}$$

$$R_{i1j2} + R_{i2j1} = -R_{i12j} - R_{i21j} = -2R_{i(12)j}, \tag{110}$$

$$R_{i12j} + R_{i21j} = 2R_{i(12)j}. \tag{111}$$

Plugging Eq. (107) into the rhs of Eq. (106), we obtain

$$R_{i(12)j}|_0 = \frac{3}{2}aR_{i(12)j}|_0, \tag{112}$$

and therefore we find

$$a = \frac{2}{3}. \tag{113}$$

The coefficient $G_{ij(12)}$ is now determined as

$$G_{ij(12)} = \frac{2}{3}R_{i(12)j}|_0. \tag{114}$$

Note that, in our derivation of Eq. (114), we used the $1 \leftrightarrow 2$ symmetrized condition (106) only. We did not use the original condition (105). Since Eq. (105) contains more information than its symmetrized form [Eq. (106)], we should check to see whether or not Eq. (114) satisfies the original condition (105).

Plugging Eq. (114) in the rhs of Eq. (105), we see

$$\begin{aligned} &\frac{1}{2}(G_{ij(12)} - G_{1j(i2)} - G_{i2(1j)} + G_{12(ij)}) \\ &\quad = \frac{1}{3}(2R_{i12j} + R_{i21j} - R_{ij12})|_0 \\ &\quad = R_{i12j}|_0, \end{aligned} \tag{115}$$

and Eq. (105) is actually satisfied with our result [Eq. (114)]. On the last line of Eq. (115), we used the Bianchi identity

$$R_{i123} + R_{i231} + R_{i312} = 0. \tag{116}$$

Since $R_{i12j} \neq R_{i21j}$ in general manifolds, the Bianchi identity plays an essential role for the consistency of the normal coordinates.

The higher-order terms in the Taylor expansion of g_{ij} in the normal coordinate are computed in Appendix C 1. We find that the function $g_{ij}(\phi)$ can be expanded in terms of the covariant tensors as

$$\begin{aligned} g_{ij}(\phi) = \delta_{ij} &+ \frac{1}{2}G_{ij(k_1k_2)}\phi^{k_1}\phi^{k_2} + \frac{1}{3!}G_{ij(k_1k_2k_3)}\phi^{k_1}\phi^{k_2}\phi^{k_3} \\ &+ \frac{1}{4!}G_{ij(k_1k_2k_3k_4)}\phi^{k_1}\phi^{k_2}\phi^{k_3}\phi^{k_4} + \cdots, \end{aligned} \tag{117}$$

with

$$G_{ij(12)} = \frac{2}{3}R_{i(12)j}|_0, \tag{118}$$

$$G_{ij(123)} = \frac{1}{3}[R_{i(12)j;3} + R_{i(23)j;1} + R_{i(31)j;2}]|_0, \tag{119}$$

$$\begin{aligned} G_{ij(1234)} = &\frac{1}{5}[R_{i(12)j;(34)} + R_{i(34)j;(12)} + R_{i(13)j;(24)} + R_{i(24)j;(13)} \\ &+ R_{i(14)j;(23)} + R_{i(23)j;(14)}]|_0 \\ &+ \frac{8}{45}g^{i'j'}[R_{i'(12)i}R_{j'(34)j} + R_{i'(13)i}R_{j'(24)j} \\ &+ R_{i'(14)i}R_{j'(23)j} + R_{i'(34)i}R_{j'(12)j} \\ &+ R_{i'(24)i}R_{j'(13)j} + R_{i'(23)i}R_{j'(14)j}]|_0. \end{aligned} \tag{120}$$

The Taylor expansion of the potential term $V(\phi)$ can also be given in a similar manner. We obtain

$$V_{,12}|_0 = V_{;12}|_0, \tag{121}$$

$$V_{,123}|_0 = V_{;123}|_0, \tag{122}$$

$$V_{,1234}|_0 = V_{;1234}|_0 - \frac{2}{3}[V_{;1i}R^i{}_{(23)4} + V_{;2i}R^i{}_{(13)4} + V_{;3i}R^i{}_{(12)4} + V_{;4i}R^i{}_{(12)3}]|_0, \tag{123}$$

$$\begin{aligned} V_{,12345}|_0 = V_{;12345}|_0 &- \frac{2}{3}\{V_{;34i}R^i{}_{(12)5} + V_{;35i}R^i{}_{(12)4} + V_{;45i}R^i{}_{(12)3} + V_{;24i}R^i{}_{(13)5} + V_{;25i}R^i{}_{(13)4} + V_{;14i}R^i{}_{(23)5} \\ &+ V_{;15i}R^i{}_{(23)4} + V_{;23i}R^i{}_{(14)5} + V_{;13i}R^i{}_{(24)5} + V_{;12i}R^i{}_{(34)5}\}|_0 \\ &+ \frac{1}{6}\{V_{;1i}(R^i{}_{(45)(2;3)} - 5R^i{}_{(23)(4;5)}) + V_{;2i}(R^i{}_{(45)(1;3)} - 5R^i{}_{(13)(4;5)}) + V_{;3i}(R^i{}_{(45)(1;2)} - 5R^i{}_{(12)(4;5)}) \\ &+ V_{;4i}(R^i{}_{(35)(1;2)} - 5R^i{}_{(12)(3;5)}) + V_{;5i}(R^i{}_{(34)(1;2)} - 5R^i{}_{(12)(3;4)})\}|_0, \end{aligned} \tag{124}$$

and therefore

$$V_{,12}|_0 = V_{;(12)}|_0, \tag{125}$$

$$V_{,123}|_0 = V_{;(123)}|_0, \tag{126}$$

$$V_{,1234}|_0 = V_{;(1234)}|_0, \tag{127}$$

$$\begin{aligned} V_{,12345}|_0 &= V_{;(12345)}|_0, \\ &\vdots \end{aligned} \tag{128}$$

in the normal coordinate. The potential term in the Lagrangian (71) can also be expanded in terms of the covariant tensors

$$\begin{aligned} V(\phi) = V\Big|_0 &+ \frac{1}{2}V_{;(k_1k_2)}\Big|_0\phi^{k_1}\phi^{k_2} + \frac{1}{3!}V_{;(k_1k_2k_3)}\Big|_0\phi^{k_1}\phi^{k_2}\phi^{k_3} \\ &+ \frac{1}{4!}V_{;(k_1k_2k_3k_4)}\Big|_0\phi^{k_1}\phi^{k_2}\phi^{k_3}\phi^{k_4} \\ &+ \frac{1}{5!}V_{;(k_1k_2k_3k_4k_5)}\Big|_0\phi^{k_1}\phi^{k_2}\phi^{k_3}\phi^{k_4}\phi^{k_5} + \cdots \end{aligned} \tag{129}$$

in the normal coordinate.

C. Fermion bilinear sector

We next move to the fermion bilinear sector in the GHEFT Lagrangian (71). The fermion bilinear sector depends on the three kinds of coupling functions, $g_{\hat{i}\hat{j}^*}$, $v_{\hat{i}\hat{j}^*i}$, and $M_{\hat{i}\,\hat{j}}$. We define a normal coordinate on the fermion field space so that the coupling functions are expanded in terms of the covariantly transforming tensors.

Before computing the expansion coefficients, we introduce the covariant quantities on the fermion field transformation. We first define the "affine connection" as

$$\Gamma^{\hat{i}}_{j\hat{k}} \coloneqq \frac{1}{2}g^{\hat{i}\hat{l}^*}[g_{\hat{k}\hat{l}^*,j} + g_{j\hat{l}^*,\hat{k}} - g_{j\hat{k},\hat{l}^*}], \tag{130}$$

$$\Gamma^{\hat{i}^*}_{j\hat{k}^*} \coloneqq \frac{1}{2}g^{\hat{l}\hat{i}^*}[g_{\hat{l}\hat{k}^*,j} + g_{j\hat{k},\hat{l}^*} - g_{j\hat{k}^*,\hat{l}}], \tag{131}$$

where $g^{\hat{i}\hat{j}^*}$ is defined as the inverse of $g_{\hat{i}\hat{j}^*}$, i.e.,

$$g^{\hat{i}\hat{k}^*}g_{\hat{j}\hat{k}^*} = \delta^{\hat{i}}_{\hat{j}}, \qquad g^{\hat{k}\hat{i}^*}g_{\hat{k}\hat{j}^*} = \delta^{\hat{i}^*}_{\hat{j}^*}. \tag{132}$$

Got inspiration from the supersymmetric nonlinear sigma model Lagrangian (79), we introduce $g_{i\hat{j}^*,\hat{i}}$ and $g_{\hat{i}\hat{i},\hat{j}^*}$ satisfying

$$v_{\hat{i}\hat{j}^*i}(\phi) = \frac{i}{2}(g_{i\hat{j}^*,\hat{i}}(\phi) - g_{\hat{i}i,\hat{j}^*}(\phi)), \tag{133}$$

which allows us to study supersymmetric theories in the GHEFT framework. There is an ambiguity in the decomposition (133) which will be discussed later. We introduce a function $\chi^{\hat{i}}(\phi)$ and its covariant derivative

$$\chi^{\hat{i}}_{;i} \coloneqq \chi^{\hat{i}}_{,i} + \chi^{\hat{i}'}\Gamma^{\hat{i}}_{i\hat{i}'}. \tag{134}$$

It is easy to show that the derivative (134) covariantly transforms under the field transformation (82). Moreover, the covariant derivative on the fermionic metric satisfies

$$g_{\hat{i}\hat{j}^*;i} \coloneqq g_{\hat{i}\hat{j}^*,i} - g_{\hat{i}'\hat{j}^*}\Gamma^{\hat{i}'}_{i\hat{i}} - g_{\hat{i}\hat{j}'^*}\Gamma^{\hat{j}'^*}_{i\hat{j}^*} = 0. \tag{135}$$

Formulas (130) and (131) are therefore considered to be affine connections.

The covariant derivative of $\chi^{\hat{i}}_{;i}(\phi)$ is also defined as usual:

$$\chi^{\hat{i}}_{;ij} \coloneqq (\chi^{\hat{i}}_{;i})_{,j} + \chi^{\hat{i}'}_{;i}\Gamma^{\hat{i}}_{j\hat{i}'} - \chi^{\hat{i}}_{;i'}\Gamma^{i'}_{ij}. \tag{136}$$

We therefore obtain

$$\begin{aligned} \chi^{\hat{i}}_{;ij} - \chi^{\hat{i}}_{;ji} &= -\chi^{\hat{i}'}[\Gamma^{\hat{i}}_{j\hat{i}',i} - \Gamma^{\hat{i}}_{i\hat{i}',j} + \Gamma^{\hat{i}}_{i\hat{i}''}\Gamma^{\hat{i}''}_{j\hat{i}'} - \Gamma^{\hat{i}}_{j\hat{i}''}\Gamma^{\hat{i}''}_{i\hat{i}'}] \\ &= -\chi^{\hat{i}'}R^{\hat{i}}{}_{\hat{i}'ij}. \end{aligned} \tag{137}$$

Here we define the "Riemann curvature" tensor $R^{\hat{i}}{}_{\hat{j}kl}$ as

$$R^{\hat{i}}{}_{\hat{j}kl} \coloneqq \Gamma^{\hat{i}}_{l\hat{j},k} - \Gamma^{\hat{i}}_{k\hat{j},l} + \Gamma^{\hat{i}}_{k\hat{l}}\Gamma^{\hat{l}}_{l\hat{j}} - \Gamma^{\hat{i}}_{l\hat{l}}\Gamma^{\hat{l}}_{k\hat{j}}. \tag{138}$$

Note that the definition of Riemann curvature tensor leads to

$$R^{\hat{i}}{}_{\hat{j}kl} + R^{\hat{i}}{}_{\hat{j}lk} = 0. \tag{139}$$

It is easy to show that $R^{\hat{i}}{}_{\hat{j}kl}$ transforms covariantly under the coordinate transformation given in Eq. (82). For the latter convenience, we also define

$$R_{\hat{j}^*\hat{i}kl} \coloneqq g_{\hat{l}\hat{j}^*} R^{\hat{l}}{}_{\hat{i}kl}, \tag{140}$$

$$R_{\hat{i}\hat{j}^*kl} \coloneqq -R_{\hat{j}^*\hat{i}kl}. \tag{141}$$

We are now ready to compute the expansion coefficients of the coupling functions in the normal coordinate. The normal coordinate on the fermion field space is defined so that the coupling functions $g_{\hat{i}\hat{j}^*}$ and $v_{\hat{i}\hat{j}^*i}$ are expanded in terms of the covariantly transforming tensors. We first focus on $g_{\hat{i}\hat{j}^*}$. Thanks to the Hermiticity of $g_{\hat{i}\hat{j}^*}$, and since $g_{\hat{i}\hat{j}^*}$ does not depend on the fermion fields, it is always possible to take a fermion coordinate satisfying

$$g_{\hat{i}\hat{j}^*}(\phi) = \delta_{\hat{i}\hat{j}^*}. \tag{142}$$

The expansion of the fermionic metric is therefore trivial.

We next consider the expansion of $v_{\hat{i}\hat{j}^*i}$. Neglecting the anomaly factor only appearing in the loop level, we are allowed to take a coordinate satisfying

$$v_{\hat{i}\hat{j}^*i}(\phi) = A_{\hat{i}\hat{j}^*ij}(\phi)\phi^j, \tag{143}$$

with

$$A_{\hat{i}\hat{j}^*ij}(\phi) = -A_{\hat{i}\hat{j}^*ji}(\phi). \tag{144}$$

We resolve the ambiguity in Eq. (133) as

$$g_{i\hat{j}^*,\hat{i}}(\phi) = -iA_{\hat{i}\hat{j}^*ij}(\phi)\phi^j, \qquad g_{i\hat{i},\hat{j}^*}(\phi) = iA_{\hat{i}\hat{j}^*ij}(\phi)\phi^j. \tag{145}$$

It is now straightforward to obtain

$$\Gamma^{\hat{i}}_{i\hat{j}} = -ig^{\hat{i}\hat{k}^*} A_{\hat{j}\hat{k}^*ij}\phi^j, \qquad \Gamma^{\hat{i}^*}_{i\hat{j}^*} = ig^{\hat{k}\hat{i}^*} A_{\hat{k}\hat{j}^*ij}\phi^j. \tag{146}$$

We next determine the expansion coefficient of $A_{\hat{i}\hat{j}ij}$. Combining Eqs. (138) and (146), we obtain the master formula for the determination of the coefficients:

$$\begin{aligned} R_{\hat{i}\hat{j}^*ij} = {} & i(A_{\hat{i}\hat{j}^*jk}\phi^k)_{,i} - i(A_{\hat{i}\hat{j}^*ik}\phi^k)_{,j} + A_{\hat{i}'\hat{j}^*ik_1} g^{\hat{i}'\hat{j}'^*} A_{\hat{i}\hat{j}'^*jk_2}\phi^{k_1}\phi^{k_2} \\ & - A_{\hat{i}'\hat{j}^*jk_1} g^{\hat{i}'\hat{j}'^*} A_{\hat{i}\hat{j}'^*ik_2}\phi^{k_1}\phi^{k_2}. \end{aligned} \tag{147}$$

Plugging the vacuum condition $\phi^i = 0$ in Eq. (147), we obtain

$$R_{\hat{i}\hat{j}^*ij}|_0 = iA_{\hat{i}\hat{j}^*ji}|_0 - iA_{\hat{i}\hat{j}^*ij}|_0. \tag{148}$$

Since the function $A_{\hat{i}\hat{j}^*ij}(\phi)$ is antisymmetric under the exchange of $i \leftrightarrow j$, Eq. (148) can be expressed as

$$R_{\hat{i}\hat{j}^*ij}|_0 = -2iA_{\hat{i}\hat{j}^*ij}|_0, \tag{149}$$

and we thus obtain

$$A_{\hat{i}\hat{j}^*ij}|_0 = \frac{i}{2} R_{\hat{i}\hat{j}^*ij}|_0. \tag{150}$$

Combining Eq. (150) with Eq. (146), we obtain formulas for the fermionic affine connections and their derivatives at the vacuum,

$$\Gamma^{\hat{i}}_{i\hat{j}}|_0 = 0, \qquad \Gamma^{\hat{i}^*}_{i\hat{j}^*}|_0 = 0, \tag{151}$$

$$\Gamma^{\hat{i}}_{i\hat{j},j}|_0 = \frac{1}{2} g^{\hat{i}\hat{k}^*} R_{\hat{j}\hat{k}^*ij}\bigg|_0, \qquad \Gamma^{\hat{i}^*}_{i\hat{j}^*,j}|_0 = -\frac{1}{2} g^{\hat{k}\hat{i}^*} R_{\hat{k}\hat{j}^*ij}\bigg|_0. \tag{152}$$

The higher-order terms in the Taylor expansion of $v_{\hat{i}\hat{j}^*i}$ in the normal coordinate are computed in Appendix C 2. We find that

$$g_{\hat{i}\hat{j}^*}(\phi) = \delta_{\hat{i}\hat{j}^*} \tag{153}$$

and the function $v_{\hat{i}\hat{j}^*i}(\phi)$ can be expanded in terms of the covariant tensors as

$$\begin{aligned} v_{\hat{i}\hat{j}^*i}(\phi) = {} & A_{\hat{i}\hat{j}^*ik_1}\phi^{k_1} + \frac{1}{2!} A_{\hat{i}\hat{j}^*ik_1k_2}\phi^{k_1}\phi^{k_2} \\ & + \frac{1}{3!} A_{\hat{i}\hat{j}^*ik_1k_2k_3}\phi^{k_1}\phi^{k_2}\phi^{k_3} + \cdots, \end{aligned} \tag{154}$$

with

$$A_{\hat{i}\hat{j}^*i1} = \frac{i}{2} R_{\hat{i}\hat{j}^*i1}|_0, \tag{155}$$

$$A_{\hat{i}\hat{j}^*i12} = \frac{i}{3} R_{\hat{i}\hat{j}^*i(1;2)}|_0, \tag{156}$$

$$\begin{aligned} A_{\hat{i}\hat{j}^*i123} = {} & \frac{i}{4} R_{\hat{i}\hat{j}^*i(1;23)}|_0 + \frac{i}{36} [R_{\hat{i}\hat{j}^*i'1} R^{i'}{}_{(23)i} \\ & + R_{\hat{i}\hat{j}^*i'2} R^{i'}{}_{(31)i} + R_{\hat{i}\hat{j}^*i'3} R^{i'}{}_{(12)i}]|_0, \\ & \vdots \end{aligned} \tag{157}$$

in the normal coordinate.

We next move to $M_{\hat{i}\hat{j}}(\phi)$. Evaluating the affine connection in the normal coordinate, we obtain Eqs. (151) and (152), as well as

$$\Gamma^{\hat{i}}_{1\hat{j},23}|_0 = -\frac{2}{3} R^{\hat{i}}{}_{\hat{j}1(2;3)}\bigg|_0. \tag{158}$$

It is now easy to evaluate

$$M_{\hat{i}\hat{j},1}|_0 = M_{\hat{i}\hat{j};1}|_0, \tag{159}$$

$$M_{\hat{i}\hat{j},12}|_0 = M_{\hat{i}\hat{j};12}|_0 - \frac{1}{2}[M_{\hat{i}'\hat{j}}R^{\hat{i}'}{}_{\hat{i}12} + M_{\hat{i}\hat{j}'}R^{\hat{j}'}{}_{\hat{j}12}]|_0, \tag{160}$$

$$\begin{aligned} M_{\hat{i}\hat{j},123}|_0 = M_{\hat{i}\hat{j};123}|_0 \\ &- \frac{1}{2}[M_{\hat{i}'\hat{j};1}R^{\hat{i}'}{}_{\hat{i}23} + M_{\hat{i}'\hat{j};2}R^{\hat{i}'}{}_{\hat{i}13} + M_{\hat{i}'\hat{j};3}R^{\hat{i}'}{}_{\hat{i}12}]|_0 \\ &- \frac{1}{2}[M_{\hat{i}\hat{j}';1}R^{\hat{j}'}{}_{\hat{j}23} + M_{\hat{i}\hat{j}';2}R^{\hat{j}'}{}_{\hat{j}13} + M_{\hat{i}\hat{j}';3}R^{\hat{j}'}{}_{\hat{j}12}]|_0 \\ &- \frac{2}{3}M_{\hat{i}\hat{j};i}R^{i}{}_{(12)3}|_0 \\ &- \frac{2}{3}[M_{\hat{i}'\hat{j}}R^{\hat{i}'}{}_{\hat{i}1(2;3)} + M_{\hat{i}\hat{j}'}R^{\hat{j}'}{}_{\hat{j}1(2;3)}]|_0, \end{aligned} \tag{161}$$

and therefore

$$M_{\hat{i}\hat{j},1}|_0 = M_{\hat{i}\hat{j};1}|_0, \tag{162}$$

$$M_{\hat{i}\hat{j},12}|_0 = M_{\hat{i}\hat{j};(12)}|_0, \tag{163}$$

$$M_{\hat{i}\hat{j},123}|_0 = M_{\hat{i}\hat{j};(123)}|_0. \tag{164}$$

The fermion mass term in the Lagrangian (71) can also be expanded in terms of the covariant tensors

$$\begin{aligned} M_{\hat{i}\hat{j}}(\phi) = M_{\hat{i}\hat{j}}|_0 + M_{\hat{i}\hat{j};k_1}|_0\phi^{k_1} + \frac{1}{2!}M_{\hat{i}\hat{j};(k_1k_2)}|_0\phi^{k_1}\phi^{k_2} \\ + \frac{1}{3!}M_{\hat{i}\hat{j};(k_1k_2k_3)}|_0\phi^{k_1}\phi^{k_2}\phi^{k_3} + \cdots \end{aligned} \tag{165}$$

in the normal coordinate.

It is worth emphasizing that the introduction of the metriclike objects $g_{i\hat{j}}$ and $g_{i\hat{j}^*}$ in Eq. (133) allows us to express covariant formulas (154) and (165) in compact forms. These metriclike objects, which mix the scalars and fermions, may be understood as "convenient abbreviations" in the present nonsupersymmetric case. They can be regarded as "metrics," however, in a real sense if we embed the theory in supersymmetric models.

D. Holomorphic four-fermion sector

We consider holomorphic four-fermion operators

$$\mathcal{O}^{((\hat{i}_1\hat{i}_2)(\hat{i}_3\hat{i}_4))} \coloneqq (\psi^{\hat{i}_1}_\alpha \varepsilon^{\alpha\beta}\psi^{\hat{i}_2}_\beta)(\psi^{\hat{i}_3}_\gamma \varepsilon^{\gamma\delta}\psi^{\hat{i}_4}_\delta). \tag{166}$$

We put the indices $\hat{i}_1$, $\hat{i}_2$, $\hat{i}_3$, $\hat{i}_4$ in parentheses so as to emphasize the index-exchange symmetry,

$$\mathcal{O}^{((\hat{1}\hat{2})(\hat{3}\hat{4}))} = \mathcal{O}^{((\hat{1}\hat{2})(\hat{4}\hat{3}))} = \mathcal{O}^{((\hat{2}\hat{1})(\hat{3}\hat{4}))} = \mathcal{O}^{((\hat{3}\hat{4})(\hat{1}\hat{2}))}, \tag{167}$$

with $\hat{i}_1, \hat{i}_2, \cdots$ being abbreviated as $\hat{1}, \hat{2}, \cdots$. Furthermore, multiplying the fermion fields $\psi^{\hat{1}}_\alpha\psi^{\hat{2}}_\beta\psi^{\hat{3}}_\gamma\psi^{\hat{4}}_\delta$ to the Schouten identity

$$\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta} + \varepsilon^{\alpha\gamma}\varepsilon^{\delta\beta} + \varepsilon^{\alpha\delta}\varepsilon^{\beta\gamma} \equiv 0, \tag{168}$$

we obtain a Bianchi-like identity

$$\mathcal{O}^{((\hat{1}\hat{2})(\hat{3}\hat{4}))} + \mathcal{O}^{((\hat{1}\hat{3})(\hat{4}\hat{2}))} + \mathcal{O}^{((\hat{1}\hat{4})(\hat{2}\hat{3}))} \equiv 0. \tag{169}$$

Using the Bianchi-like identity (169), we are able to show that

$$\mathcal{O}^{((\hat{1}\hat{1})(\hat{1}\hat{1}))} + \mathcal{O}^{((\hat{1}\hat{1})(\hat{1}\hat{1}))} + \mathcal{O}^{((\hat{1}\hat{1})(\hat{1}\hat{1}))} \equiv 0, \tag{170}$$

and therefore that

$$\mathcal{O}^{((\hat{1}\hat{1})(\hat{1}\hat{1}))} \equiv 0. \tag{171}$$

In a similar manner, we find that

$$\mathcal{O}^{((\hat{1}\hat{2})(\hat{2}\hat{2}))} \equiv 0, \tag{172}$$

$$\mathcal{O}^{((\hat{1}\hat{1})(\hat{2}\hat{2}))} \equiv -2\mathcal{O}^{((\hat{1}\hat{2})(\hat{1}\hat{2}))}. \tag{173}$$

$$\mathcal{O}^{((\hat{1}\hat{1})(\hat{2}\hat{3}))} \equiv -2\mathcal{O}^{((\hat{1}\hat{2})(\hat{1}\hat{3}))}, \tag{174}$$

$$\mathcal{O}^{((\hat{1}\hat{2})(\hat{3}\hat{4}))} \equiv -\mathcal{O}^{((\hat{1}\hat{3})(\hat{2}\hat{4}))} - \mathcal{O}^{((\hat{1}\hat{4})(\hat{2}\hat{3}))}. \tag{175}$$

We next count the independent degrees of freedom (d.o.f.) of the four-fermion operators. The number of independent d.o.f. satisfying the condition (167) is

$$\frac{1}{2}\left[\frac{1}{2}N(N+1)\right]\left[\frac{1}{2}N(N+1)+1\right]. \tag{176}$$

Among them, the four-fermion operators having identical flavor indices automatically vanish, as shown in Eq. (171), which reduces the d.o.f. by N. In a similar manner, the operator identities (172)–(175) reduce the d.o.f. by

$$N(N-1), \qquad \frac{1}{2}N(N-1), \qquad \frac{1}{2}N(N-1)(N-2), \qquad \frac{1}{4!}N(N-1)(N-2)(N-3)$$

accordingly. We therefore find the d.o.f. of the four-fermion operator $\mathcal{O}^{((12)(34))}$ is given by

$$\begin{aligned}&\frac{1}{2}\left[\frac{1}{2}N(N+1)\right]\left[\frac{1}{2}N(N+1)+1\right]-N-N(N-1)\\&\quad-\frac{1}{2}N(N-1)-\frac{1}{2}N(N-1)(N-2)\\&\quad-\frac{1}{4!}N(N-1)(N-2)(N-3)\\&\quad=\frac{1}{12}N^2(N^2-1),\end{aligned} \tag{177}$$

which accords the d.o.f. of the N-dimensional Riemann curvature tensor.

We are now ready to consider the holomorphic four-fermion interactions of the type shown in Eq. (166),

$$\begin{aligned}\mathcal{L}\ni&\frac{1}{8}S_{\hat{i}_1\hat{i}_2\hat{i}_3\hat{i}_4}(\psi_\alpha^{\hat{i}_1}\epsilon^{\alpha\beta}\psi_\beta^{\hat{i}_2})(\psi_\gamma^{\hat{i}_3}\epsilon^{\gamma\delta}\psi_\delta^{\hat{i}_4}).\\&=\frac{1}{8}S_{\hat{i}_1\hat{i}_2\hat{i}_3\hat{i}_4}\mathcal{O}^{((\hat{i}_1\hat{i}_2)(\hat{i}_3\hat{i}_4))}.\end{aligned} \tag{178}$$

Thanks to the index-exchange symmetry (167), we are able to show

$$S_{\hat{i}_1\hat{i}_2\hat{i}_3\hat{i}_4}\mathcal{O}^{((\hat{i}_1\hat{i}_2)(\hat{i}_3\hat{i}_4))}=S_{((\hat{i}_1\hat{i}_2)(\hat{i}_3\hat{i}_4))}\mathcal{O}^{((\hat{i}_1\hat{i}_2)(\hat{i}_3\hat{i}_4))}, \tag{179}$$

with

$$\begin{aligned}S_{((\hat{1}\hat{2})(\hat{3}\hat{4}))}&:=\frac{1}{2}[S_{(\hat{1}\hat{2})(\hat{3}\hat{4})}+S_{(\hat{3}\hat{4})(\hat{1}\hat{2})}],\\S_{(\hat{1}\hat{2})(\hat{3}\hat{4})}&:=\frac{1}{4}[S_{\hat{1}\hat{2}\hat{3}\hat{4}}+S_{\hat{2}\hat{1}\hat{3}\hat{4}}+S_{\hat{1}\hat{2}\hat{4}\hat{3}}+S_{\hat{2}\hat{1}\hat{4}\hat{3}}],\end{aligned}$$

which, of course, satisfies the index-exchange symmetry

$$S_{((\hat{1}\hat{2})(\hat{3}\hat{4}))}=S_{((\hat{2}\hat{1})(\hat{3}\hat{4}))}=S_{((\hat{1}\hat{2})(\hat{4}\hat{3}))}=S_{((\hat{3}\hat{4})(\hat{1}\hat{2}))}. \tag{180}$$

Therefore, the d.o.f. of $S_{((\hat{1}\hat{2})(\hat{3}\hat{4}))}$ is counted as

$$\frac{1}{2}\left[\frac{1}{2}N(N+1)\right]\left[\frac{1}{2}N(N+1)+1\right], \tag{181}$$

which is larger than the d.o.f. of the operator $\mathcal{O}^{((\hat{1}\hat{2})(\hat{3}\hat{4}))}$ counted in Eq. (177). The $S_{((\hat{1}\hat{2})(\hat{3}\hat{4}))}$ parametrization therefore contains redundancy. It is desired to describe the four-fermion interactions in a nonredundant parametrization. For such a purpose, we rewrite the four-fermion interactions as

$$\begin{aligned}&S_{((\hat{i}_1\hat{i}_2)(\hat{i}_3\hat{i}_4))}\mathcal{O}^{((\hat{i}_1\hat{i}_2)(\hat{i}_3\hat{i}_4))}\\&\quad=\frac{2}{3}\left[S_{((\hat{i}_1\hat{i}_2)(\hat{i}_3\hat{i}_4))}-S_{((\hat{i}_1\hat{i}_3)(\hat{i}_2\hat{i}_4))}\right]\mathcal{O}^{((\hat{i}_1\hat{i}_2)(\hat{i}_3\hat{i}_4))},\end{aligned} \tag{182}$$

where we used the Bianchi-like identity (169) and the index-exchange symmetry (180).

We are now ready to introduce a nonredundant parametrization for holomorphic four-fermion interactions,

$$R_{\hat{1}\hat{4}\hat{2}\hat{3}}:=S_{(\hat{1}\hat{2})(\hat{3}\hat{4})}-S_{(\hat{1}\hat{3})(\hat{2}\hat{4})}, \tag{183}$$

which satisfies the index-exchange symmetries

$$R_{\hat{1}\hat{2}\hat{3}\hat{4}}+R_{\hat{1}\hat{2}\hat{4}\hat{3}}=0, \tag{184}$$

$$R_{\hat{1}\hat{2}\hat{3}\hat{4}}+R_{\hat{2}\hat{1}\hat{3}\hat{4}}=0, \tag{185}$$

$$R_{\hat{1}\hat{2}\hat{3}\hat{4}}-R_{\hat{3}\hat{4}\hat{1}\hat{2}}=0, \tag{186}$$

and the Bianchi identity

$$R_{\hat{1}\hat{2}\hat{3}\hat{4}}+R_{\hat{1}\hat{3}\hat{4}\hat{2}}+R_{\hat{1}\hat{4}\hat{2}\hat{3}}=0. \tag{187}$$

The d.o.f. of $R_{\hat{1}\hat{2}\hat{3}\hat{4}}$ is

$$\begin{aligned}&\frac{1}{2}\left[\frac{1}{2}N(N-1)\right]\left[\frac{1}{2}N(N-1)+1\right]\\&\quad-\frac{1}{4!}N(N-1)(N-2)(N-3)\\&\quad=\frac{1}{12}N^2(N^2-1),\end{aligned} \tag{188}$$

which coincides with the d.o.f. of the operators $\mathcal{O}^{((\hat{1}\hat{2})(\hat{3}\hat{4}))}$. The parametrization

$$\mathcal{L}_{\text{four-fermion}}=\frac{1}{12}R_{\hat{i}_1\hat{i}_4\hat{i}_2\hat{i}_3}\mathcal{O}^{((\hat{i}_1\hat{i}_2)(\hat{i}_3\hat{i}_4))} \tag{189}$$

therefore describes the holomorphic four-fermion interactions in a nonredundant manner.

E. Nonholomorphic four-fermion sector

We next consider the nonholomorphic four-fermion operators

$$\mathcal{O}^{(\hat{i}_1\hat{i}_2)(\hat{i}_3^*\hat{i}_4^*)}:=\left(\psi_\alpha^{\hat{i}_1}\epsilon^{\alpha\beta}\psi_\beta^{\hat{i}_2}\right)\left(\psi_{\dot{\alpha}}^{\dagger\hat{i}_3^*}\epsilon^{\dot{\alpha}\dot{\beta}}\psi_{\dot{\beta}}^{\dagger\hat{i}_4^*}\right). \tag{190}$$

Again, we put the indices in parentheses in order to emphasize the index-exchange symmetry

$$\mathcal{O}^{(\hat{1}\hat{2})(\hat{3}^*\hat{4}^*)}=\mathcal{O}^{(\hat{1}\hat{2})(\hat{4}^*\hat{3}^*)}=\mathcal{O}^{(\hat{2}\hat{1})(\hat{3}^*\hat{4}^*)}, \tag{191}$$

with $\hat{i}_1$, $\hat{i}_2$, $\hat{i}_3^*$, $\hat{i}_4^*$ abbreviated as $\hat{1}$, $\hat{2}$, $\hat{3}^*$, $\hat{4}^*$.

The d.o.f. of the four-fermion operators satisfying the conditions (191) are

$$\left[\frac{1}{2}N(N+1)\right]^2. \tag{192}$$

Note that

$$[\mathcal{O}^{(\hat{1}\,\hat{2})(\hat{3}^*\hat{4}^*)}]^\dagger = \mathcal{O}^{(\hat{3}\,\hat{4})(\hat{1}^*\hat{2}^*)}. \tag{193}$$

The d.o.f. as counted in Eq. (192) are therefore regarded as the degrees of freedom counted in *real* parameters. This is in contrast to the d.o.f. of holomorphic four-fermion operators (177) counted in *complex* parameters.

Nonholomorphic four-fermion interactions in the lowest-order GHEFT Lagrangian can be expressed as

$$\begin{aligned}\mathcal{L} &\ni -\frac{1}{4}S_{\hat{i}_1\hat{i}_2\hat{i}_3^*\hat{i}_4^*}(\psi_\alpha^{\hat{i}_1}\varepsilon^{\alpha\beta}\psi_\beta^{\hat{i}_2})(\psi_{\dot{\alpha}}^{\dagger\hat{i}_3^*}\varepsilon^{\dot{\alpha}\dot{\beta}}\psi_{\dot{\beta}}^{\dagger\hat{i}_4^*})\\ &= -\frac{1}{4}S_{\hat{i}_1\hat{i}_2\hat{i}_3^*\hat{i}_4^*}\mathcal{O}^{(\hat{i}_1\hat{i}_2)(\hat{i}_3^*\hat{i}_4^*)}.\end{aligned} \tag{194}$$

Thanks to the index-exchange symmetry (191), we are able to show that

$$S_{\hat{i}_1\hat{i}_2\hat{i}_3^*\hat{i}_4^*}\mathcal{O}^{(\hat{i}_1\hat{i}_2)(\hat{i}_3^*\hat{i}_4^*)} = S_{(\hat{i}_1\hat{i}_2)(\hat{i}_3^*\hat{i}_4^*)}\mathcal{O}^{(\hat{i}_1\hat{i}_2)(\hat{i}_3^*\hat{i}_4^*)}, \tag{195}$$

with

$$S_{(\hat{1}\,\hat{2})(\hat{3}^*\hat{4}^*)} \coloneqq \frac{1}{4}[S_{\hat{1}\,\hat{2}\,\hat{3}^*\hat{4}^*} + S_{\hat{2}\,\hat{1}\,\hat{3}^*\hat{4}^*} + S_{\hat{1}\,\hat{2}\,\hat{4}^*\hat{3}^*} + S_{\hat{2}\,\hat{1}\,\hat{4}^*\hat{3}^*}].$$

It is easy to show that

$$S_{(\hat{1}\,\hat{2})(\hat{3}^*\hat{4}^*)} = S_{(\hat{2}\,\hat{1})(\hat{3}^*\hat{4}^*)} = S_{(\hat{1}\,\hat{2})(\hat{4}^*\hat{3}^*)}. \tag{196}$$

Since the Hermiticity of the Lagrangian requires

$$[S_{(\hat{1}\,\hat{2})(\hat{3}^*\hat{4}^*)}]^* = S_{(\hat{3}\,\hat{4})(\hat{1}^*\hat{2}^*)}, \tag{197}$$

we find the number of d.o.f. of $S_{(\hat{1}\,\hat{2})(\hat{3}^*\hat{4}^*)}$ is $N^2(N+1)^2/4$ real parameters, which agrees with the d.o.f. of the nonholomorphic four-fermion operators (192). Therefore, the nonholomorphic four-fermion interactions can be parametrized by using $S_{(\hat{1}\,\hat{2})(\hat{3}^*\hat{4}^*)}$ in a nonredundant manner.

We finally remark that the nonholomorphic four-fermion operators appear in the supersymmetric nonlinear sigma model as

$$\mathcal{L} \ni \frac{1}{4}R_{\hat{i}_1\hat{i}_3^*\hat{i}_2\hat{i}_4^*}(\psi^{\hat{i}_1}\psi^{\hat{i}_2})(\psi^{\dagger\hat{i}_3^*}\psi^{\dagger\hat{i}_4^*}), \tag{198}$$

where

$$R_{\hat{i}_1\hat{i}_3^*\hat{i}_2\hat{i}_4^*} = g_{\hat{i}_1\hat{i}_3^*,\hat{i}_2\hat{i}_4^*} - g^{\hat{i}'\hat{i}'^*}g_{\hat{i}'\hat{i}_3^*,\hat{i}_4^*}g_{\hat{i}_1\hat{i}'^*,\hat{i}_2}, \tag{199}$$

with $g_{\hat{i}\hat{j}^*}$ being the Kähler metric. We therefore define

$$R_{\hat{1}\hat{3}^*\hat{2}\hat{4}^*} \coloneqq S_{(\hat{1}\,\hat{2})(\hat{3}^*\hat{4}^*)} \tag{200}$$

for the nonholomorphic four-fermion couplings, even in nonsupersymmetric GHEFT.

IV. ON-SHELL AMPLITUDES

The purpose of the GHEFT Lagrangian is to compute the production cross sections and the decay widths involving the new BSM particles. As we showed in the previous section, the nonuniqueness of the parametrization in the effective Lagrangian associated with the KOS theorem [46] can be resolved by using the normal coordinate. The scattering amplitudes can now be computed straightforwardly in the normal coordinate as functions of the covariant tensors. Applying the normal coordinate in the GHEFT, the on-shell amplitudes are expressed by covariant quantities on the coupling functions evaluated at the vacuum.

In this section, we explicitly compute tree-level on-shell helicity amplitudes applying the normal coordinate in the lowest-order GHEFT Lagrangian. In the computation of the on-shell amplitudes, we ignore the gauge boson contributions for simplicity. The computation on the on-shell amplitudes including spin-1 gauge bosons will be published elsewhere. The high-energy behavior of the longitudinally polarized gauge boson scattering amplitudes can be computed even in the gaugeless limit, thanks to the equivalence theorem between the longitudinally polarized gauge boson scattering amplitudes and the corresponding would-be NG boson amplitudes [64–68]. In what follows, we also study the high-energy behaviors of the on-shell amplitudes and discuss their implications.

A. Notation

We express an N-particle invariant amplitude generally as

$$\mathcal{A}_N(12\cdots N). \tag{201}$$

Generalized Mandelstam variables and particle masses are

$$s_{ij} \coloneqq (p_i + p_j)^2, \qquad m_i \coloneqq \sqrt{p_i^2}, \tag{202}$$

where the momentum of the ith particle p_i is understood to be outgoing. For example, the amplitude involving two fermions and one scalar is denoted as

$$\mathcal{A}_3(123) = \mathcal{A}_3(\mathbf{1}^{\lambda_1}, \mathbf{2}^{\lambda_2}, 3), \tag{203}$$

where $\mathbf{1}^{\lambda_1}$, $\mathbf{2}^{\lambda_2}$ denote the momentum, helicity, and flavor quantum numbers for the on-shell fermions, and 3 denotes the momentum and flavor quantum numbers for the on-shell bosons. λ_i labels the helicity of the fermion state.

If the fermion masses can be neglected in the amplitude, we are able to use the celebrated spinor-helicity formalism in the massless limit [69]. The masses of heavy particles including BSM particles cannot be neglected, however, in the GHEFT framework. We therefore employ the Dreiner-Haber-Martin notation [52] for the two-component fermion wave functions in the amplitudes. The fermion wave function carrying three-momentum $\vec{p}$ is expressed by two-component spinors

$$x_\alpha(\vec{p},\lambda), \qquad y_\alpha(\vec{p},\lambda), \tag{204}$$

with $x_\alpha(\vec{p},\lambda)$ and $y_\alpha(\vec{p},\lambda)$ being positive and negative frequency wave functions, respectively. $\lambda = \pm 1$ labels the little-group representation index for the massive spin-1/2 fermion [70]. The explicit forms of the spinor wave functions are summarized in Appendix B.

For later convenience, we introduce square/angle bras and kets denoting massive spinor wave functions,

$$([\mathbf{1}^{\lambda_1})^\alpha \coloneqq y^\alpha(\vec{p}_1,\lambda_1), \tag{205}$$

$$(\mathbf{1}^{\lambda_1}])_\alpha \coloneqq y_\alpha(\vec{p}_1,\lambda_1), \tag{206}$$

$$(\langle\mathbf{1}^{\lambda_1})_{\dot\alpha} \coloneqq x^\dagger_{\dot\alpha}(\vec{p}_1,\lambda_1), \tag{207}$$

$$(\mathbf{1}^{\lambda_1}\rangle)^{\dot\alpha} \coloneqq x^{\dagger\dot\alpha}(\vec{p}_1,\lambda_1), \tag{208}$$

where

$$x^\alpha(\vec{p},\lambda) \coloneqq \varepsilon^{\alpha\beta}x_\beta(\vec{p},\lambda), \tag{209}$$

$$y^\alpha(\vec{p},\lambda) \coloneqq \varepsilon^{\alpha\beta}y_\beta(\vec{p},\lambda), \tag{210}$$

$$x^\dagger_{\dot\alpha}(\vec{p},\lambda) \coloneqq \varepsilon_{\dot\alpha\dot\beta}x^{\dagger\dot\beta}(\vec{p},\lambda), \tag{211}$$

$$y^\dagger_{\dot\alpha}(\vec{p},\lambda) \coloneqq \varepsilon_{\dot\alpha\dot\beta}y^{\dagger\dot\beta}(\vec{p},\lambda). \tag{212}$$

Bracket notations for x_α and $y^\dagger_{\dot\alpha}$ do not need to be introduced, since the amplitudes can be expressed without using x_α and $y^\dagger_{\dot\alpha}$. The inner products among these spinor wave functions are expressed as

$$[\mathbf{1}^{\lambda_1}\mathbf{2}^{\lambda_2}] = y^\alpha(\vec{p}_1,\lambda_1)y_\alpha(\vec{p}_2,\lambda_2), \tag{213}$$

$$\langle\mathbf{1}^{\lambda_1}\mathbf{2}^{\lambda_2}\rangle = x^\dagger_{\dot\alpha}(\vec{p}_1,\lambda_1)x^{\dagger\dot\alpha}(\vec{p}_2,\lambda_2). \tag{214}$$

In the massless limit, these brackets reduce to the massless angle/square brackets,

$$\langle\mathbf{1}^{\lambda_1}\mathbf{2}^{\lambda_2}\rangle \to \begin{cases} \langle 1\,2\rangle & \text{for } \lambda_1=\lambda_2=-1, \\ 0 & \text{otherwise,} \end{cases} \tag{215}$$

$$[\mathbf{1}^{\lambda_1}\mathbf{2}^{\lambda_2}] \to \begin{cases} [1\,2] & \text{for } \lambda_1=\lambda_2=+1, \\ 0 & \text{otherwise.} \end{cases} \tag{216}$$

Here the massless spinor wave functions are denoted by nonbold bras and kets $\langle 1$, $\langle 2$, $[1$, $[2$, $1\rangle$, $2\rangle$, $1]$, $2]$. See Ref. [69] for the details of the massless spinor formalism. Note that the index $\lambda = \pm 1$ in $\langle\mathbf{1}^\lambda$ and $[\mathbf{1}^\lambda$ corresponds to the helicity of the outgoing state.

Here we briefly summarize the properties of massless spinor wave functions. The massless spinor wave functions satisfy the exchange (anti)symmetries,

$$\begin{aligned} \langle 1\,2\rangle &= -\langle 2\,1\rangle, \\ [1\,2] &= -[2\,1], \\ \langle 1\bar\sigma^\mu 2] &= [2\sigma^\mu 1\rangle, \end{aligned} \tag{217}$$

and the Fierz identity

$$\langle 1\bar\sigma^\mu 2][3\sigma_\mu 4\rangle = 2\langle 1\,4\rangle[3\,2]. \tag{218}$$

Equation (168) leads to the Schouten identities,

$$[1\,2][3\,4] + [1\,3][4\,2] + [1\,4][2\,3] = 0, \tag{219}$$

$$\langle 1\,2\rangle\langle 3\,4\rangle + \langle 1\,3\rangle\langle 4\,2\rangle + \langle 1\,4\rangle\langle 2\,3\rangle = 0. \tag{220}$$

The matrices σ^μ and $\bar\sigma^\mu$ can be decomposed into spinor products as

$$(p])_\alpha(\langle p)_{\dot\alpha} = p_\mu(\sigma^\mu)_{\alpha\dot\alpha}, \quad (p\rangle)^{\dot\alpha}([p)^\alpha = p_\mu(\bar\sigma^\mu)^{\dot\alpha\alpha}. \tag{221}$$

The complex conjugates are given by

$$(\langle 1\,2\rangle)^* = [2\,1], \qquad ([1\,2])^* = \langle 2\,1\rangle. \tag{222}$$

Combining Eqs. (221) and (222), we are able to show that

$$|\langle 1\,2\rangle|^2 = |[1\,2]|^2 = 2p_1\cdot p_2 = (p_1+p_2)^2. \tag{223}$$

Similarly, the massive spinor wave functions satisfy the exchange (anti)symmetries,

$$\begin{aligned} \langle\mathbf{1}^{\lambda_1}\,\mathbf{2}^{\lambda_2}\rangle &= -\langle\mathbf{2}^{\lambda_2}\mathbf{1}^{\lambda_1}\rangle, \\ [\mathbf{1}^{\lambda_1}\mathbf{2}^{\lambda_2}] &= -[\mathbf{2}^{\lambda_2}\mathbf{1}^{\lambda_1}], \\ \langle\mathbf{1}^{\lambda_1}\bar\sigma^\mu\mathbf{2}^{\lambda_2}] &= [\mathbf{2}^{\lambda_2}\sigma^\mu\mathbf{1}^{\lambda_1}\rangle, \end{aligned} \tag{224}$$

and the Fierz identity

$$\langle\mathbf{1}^{\lambda_1}\bar\sigma^\mu\mathbf{2}^{\lambda_2}][\mathbf{3}^{\lambda_3}\sigma_\mu\mathbf{4}^{\lambda_4}\rangle = 2\langle\mathbf{1}^{\lambda_1}\mathbf{4}^{\lambda_4}\rangle[\mathbf{3}^{\lambda_3}\mathbf{2}^{\lambda_2}]. \tag{225}$$

The Schouten identities among the massive spinor wave functions are

$$[\mathbf{1}^{\lambda_1}\mathbf{2}^{\lambda_2}][\mathbf{3}^{\lambda_3}\mathbf{4}^{\lambda_4}] + [\mathbf{1}^{\lambda_1}\mathbf{3}^{\lambda_3}][\mathbf{4}^{\lambda_4}\mathbf{2}^{\lambda_2}] + [\mathbf{1}^{\lambda_1}\mathbf{4}^{\lambda_4}][\mathbf{2}^{\lambda_2}\mathbf{3}^{\lambda_3}] = 0, \tag{226}$$

$$\langle\mathbf{1}^{\lambda_1}\mathbf{2}^{\lambda_2}\rangle\langle\mathbf{3}^{\lambda_3}\mathbf{4}^{\lambda_4}\rangle + \langle\mathbf{1}^{\lambda_1}\mathbf{3}^{\lambda_3}\rangle\langle\mathbf{4}^{\lambda_4}\mathbf{2}^{\lambda_2}\rangle + \langle\mathbf{1}^{\lambda_1}\mathbf{4}^{\lambda_4}\rangle\langle\mathbf{2}^{\lambda_2}\mathbf{3}^{\lambda_3}\rangle = 0. \tag{227}$$

The completeness relations are expressed as

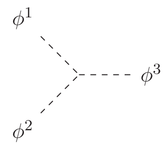

FIG. 1. Feynman diagram for $\mathcal{A}_3(1,2,3)$.

$$\left(\sum_{\lambda=\pm1}\mathbf{1}^{\lambda}]\lambda\langle\mathbf{1}^{-\lambda}\right)_{\alpha\dot{\alpha}} = (p_{1\mu}\sigma^{\mu})_{\alpha\dot{\alpha}}, \tag{228}$$

$$\left(\sum_{\lambda=\pm1}\mathbf{1}^{\lambda}]\lambda[\mathbf{1}^{-\lambda}\right)_{\alpha}^{\ \beta} = m_1\delta_{\alpha}^{\beta}, \tag{229}$$

$$\left(\sum_{\lambda=\pm1}\mathbf{1}^{-\lambda}\rangle\lambda[\mathbf{1}^{\lambda}\right)^{\dot{\alpha}\alpha} = (p_{1\mu}\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha}, \tag{230}$$

$$\left(\sum_{\lambda=\pm1}\mathbf{1}^{-\lambda}\rangle\lambda\langle\mathbf{1}^{\lambda}\right)^{\dot{\alpha}}_{\ \dot{\beta}} = m_1\delta^{\dot{\alpha}}_{\dot{\beta}}. \tag{231}$$

The complex conjugates are given by

$$\begin{aligned}(\langle\mathbf{1}^{\lambda_1}\mathbf{2}^{\lambda_2}\rangle)^* &= [\mathbf{2}^{-\lambda_2}\mathbf{1}^{-\lambda_1}],\\ ([\mathbf{1}^{\lambda_1}\mathbf{2}^{\lambda_2}])^* &= \langle\mathbf{2}^{-\lambda_2}\mathbf{1}^{-\lambda_1}\rangle.\end{aligned} \tag{232}$$

The amplitude $\mathcal{A}_{n_s+n_f}$ depends on the coupling functions at the vacuum, e.g.,

$$\begin{aligned}&V_{;(123)}|_0, \quad V_{;(1234)}|_0, \quad R_{1234}|_0,\\ &R_{1234;5}|_0, \quad M_{\hat{1}\,\hat{2};3}|_0, \quad R_{\hat{1}\,\hat{2}\,34}|_0, \quad \cdots.\end{aligned} \tag{233}$$

Hereafter, we omit the vertical bar symbols so that

$$\begin{aligned}&V_{;(123)}, \quad V_{;(1234)}, \quad R_{1234},\\ &R_{1234;5}, \quad M_{\hat{1}\,\hat{2};3}, \quad R_{\hat{1}\,\hat{2}\,34}, \quad \ldots,\end{aligned} \tag{234}$$

are understood to be evaluated at the vacuum.

B. Three scalars

We start with a three-point scalar amplitude $\mathcal{A}_3(1,2,3)$, which is given by the contact diagram shown in Fig. 1. We have calculated the on-shell three-point scalar amplitude in Ref. [37]. The amplitude is simply given by

$$i\mathcal{A}_3(1,2,3) = -iV_{;(123)}. \tag{235}$$

As we showed in Ref. [37], if we do not use the normal coordinate, we need to perform very involved computations to get the final expression of the amplitude (235).

C. Two fermions and one scalar

We next consider a three-point amplitude with two fermions and one scalar, $\mathcal{A}_3(\mathbf{1}^{\lambda_1},\mathbf{2}^{\lambda_2},3)$. The amplitude is given by the diagram shown in Fig. 2, where the three-point vertex is read from the normal coordinate formula (165). The on-shell amplitude is given as

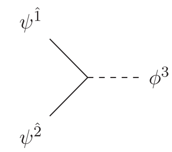

FIG. 2. Feynman diagram for $\mathcal{A}_3(\mathbf{1}^{\lambda_1},\mathbf{2}^{\lambda_2},\mathbf{3})$. We assign the outgoing momenta p_1, p_2, and p_3 to $\psi^{\hat{1}}$, $\psi^{\hat{2}}$, and ϕ^3.

$$i\mathcal{A}_3(\mathbf{1}^{\lambda_1},\mathbf{2}^{\lambda_2},3) = -iM_{\hat{1}\,\hat{2};3}[\mathbf{1}^{\lambda_1}\mathbf{2}^{\lambda_2}] - iM_{\hat{1}^*\hat{2}^*;3}\langle\mathbf{1}^{\lambda_1}\mathbf{2}^{\lambda_2}\rangle. \tag{236}$$

We have confirmed formula (236) without using the normal coordinate technique.

Once we specify the kinematics and helicities of the external states, we can explicitly estimate the spinor inner products in terms of the kinematical variables. See Appendix B for the explicit expressions. For example, we consider the decay process of the scalar 3 into the fermion pair **1** and **2**,

$$\phi^3(-p_3) \to \psi^{\hat{1}}(p_1)\psi^{\hat{2}}(p_2). \tag{237}$$

Initial state momentum is assigned to be $-p_3$ in our notation of the amplitude.

We evaluate the decay amplitudes in the rest frame of ϕ^3. Note that the final state angular momentum should vanish in this process since the interaction vertex does not contain derivatives. The conservation of the total angular momentum implies that the final state spin momentum should also be zero. We therefore expect

$$\mathcal{A}_3(\mathbf{1}^{\pm},\mathbf{2}^{\mp},3) = 0, \tag{238}$$

which actually is confirmed in our explicit computation since $[\mathbf{1}^{\pm}\mathbf{2}^{\mp}] = \langle\mathbf{1}^{\pm}\mathbf{2}^{\mp}\rangle = 0$ in the center-of-mass frame, as we show in Appendix B. On the other hand, $\mathcal{A}_3(\mathbf{1}^{\pm},\mathbf{2}^{\pm},3)$ can be nonzero. The masses of fermions and scalar are denoted as m_1, m_2, and m_3, respectively. For $m_3 \gg m_1, m_2$, we find

$$[\mathbf{1}^{+}\mathbf{2}^{+}] \simeq [1\,2] = -m_3, \qquad [\mathbf{1}^{-}\mathbf{2}^{-}] \simeq 0, \tag{239}$$

$$\langle\mathbf{1}^{-}\mathbf{2}^{-}\rangle \simeq \langle 1\,2\rangle = +m_3, \qquad \langle\mathbf{1}^{+}\mathbf{2}^{+}\rangle \simeq 0, \tag{240}$$

and thus

$$\begin{aligned}\mathcal{A}_3(\mathbf{1}^{+},\mathbf{2}^{+},3) &\simeq m_3M_{\hat{1}\,\hat{2};3},\\ \mathcal{A}_3(\mathbf{1}^{-},\mathbf{2}^{-},3) &\simeq -m_3M_{\hat{1}^*\hat{2}^*;3},\end{aligned} \tag{241}$$

where we ignore the $\mathcal{O}(m_1^2/m_3^2, m_2^2/m_3^2)$ corrections.

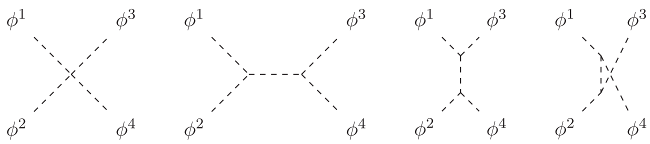

FIG. 3. Feynman diagram for $\mathcal{A}_4(1,2,3,4)$. We assign the outgoing momenta p_1, p_2, p_3, and p_4 to ϕ^1, ϕ^2, ϕ^3, and ϕ^4.

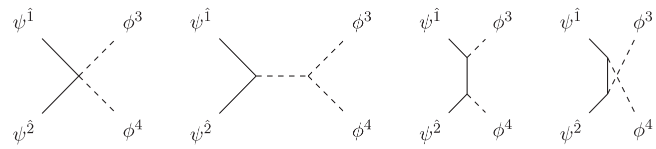

FIG. 4. Feynman diagram for $\mathcal{A}_4(\mathbf{1}^{\lambda_1},\mathbf{2}^{\lambda_2},3,4)$. We assign the outgoing momenta p_1, p_2, p_3, and p_4 to $\psi^{\hat{1}}$, $\psi^{\hat{2}}$, ϕ^3, and ϕ^4.

D. Four scalars

We next consider a four-point scalar amplitude $\mathcal{A}_4(1,2,3,4)$. The amplitude is given by the sum of the contact diagram and the scalar-exchange diagrams, as shown in Fig. 3:

$$\mathcal{A}_4(1,2,3,4)=\mathcal{A}_4^{(c)}(1,2,3,4)+\mathcal{A}_4^{(\phi)}(1,2,3,4). \tag{242}$$

We have estimated the four-point amplitudes in Ref. [37]. The results are

$$\begin{aligned}i\mathcal{A}_4^{(c)}(1,2,3,4)=&-\frac{2i}{3}R_{1(34)2}s_{12}-\frac{2i}{3}R_{1(24)3}s_{13}\\&-\frac{2i}{3}R_{1(23)4}s_{14}-V_{;(1234)}\end{aligned} \tag{243}$$

and

$$\begin{aligned}i\mathcal{A}_4^{(\phi)}(1,2,3,4)=&-\sum_{i,j}V_{;(12i)}[D(s_{12})]^{ij}V_{;(34j)}\\&-\sum_{i,j}V_{;(13i)}[D(s_{13})]^{ij}V_{;(24j)}\\&-\sum_{i,j}V_{;(14i)}[D(s_{14})]^{ij}V_{;(23j)},\end{aligned} \tag{244}$$

where $[D(s)]^{ij}$ denotes the scalar propagator,

$$[D(s)]^{ij}\coloneqq\frac{i}{s-m_i^2}g^{ij}, \tag{245}$$

with m_i being the scalar mass.

The scalar four-point amplitude diverges in the high-energy limit, $s=s_{12}\gg m_1^2$, m_2^2, m_3^2, m_4^2. For example, we consider

$$\phi^1(-p_1)\phi^2(-p_2)\to\phi^3(p_3)\phi^4(p_4). \tag{246}$$

In the high-energy limit, the corresponding scattering amplitude behaves as

$$\mathcal{A}_4(1,2,3,4)\simeq R_{1423}s+\frac{1}{2}R_{1234}s(1+\cos\theta), \tag{247}$$

with θ being the scattering angle in the center of mass. This result implies that, with $R_{1423}\neq 0$ or $R_{1234}\neq 0$, the perturbative unitarity is violated in the high-energy scattering amplitude among the scalar bosons. This observation indicates that the longitudinally polarized gauge boson scattering amplitudes violates the perturbative unitarity if the scalar manifold is curved [37,48,49]. The Lee-Quigg-Thacker sum rules [71,72] for the perturbative unitarity in the $W_LW_L\to W_LW_L$ amplitude can thus be regarded as conditions on the scalar curvature tensor $R_{\pi\pi\pi\pi}=0$.

E. Two fermions and two scalars

We next consider a four-point amplitude with two fermions and two scalars, $\mathcal{A}_4(\mathbf{1}^{\lambda_1},\mathbf{2}^{\lambda_2},3,4)$. As shown in Fig. 4, the amplitude consists of contact, scalar-exchange, and fermion-exchange diagrams,

$$\begin{aligned}\mathcal{A}_4(\mathbf{1}^{\lambda_1},\mathbf{2}^{\lambda_2},3,4)=&\mathcal{A}_4^{(c)}(\mathbf{1}^{\lambda_1},\mathbf{2}^{\lambda_2},3,4)+\mathcal{A}_4^{(\phi)}(\mathbf{1}^{\lambda_1},\mathbf{2}^{\lambda_2},3,4)\\&+\mathcal{A}_4^{(\psi)}(\mathbf{1}^{\lambda_1},\mathbf{2}^{\lambda_2},3,4).\end{aligned} \tag{248}$$

We first focus on the contact diagram $\mathcal{A}_4^{(c)}$, which appears from the vertices, $R_{\hat{i}\hat{j}^*kl}$, $M_{\hat{i}\hat{j};(kl)}$, and $M_{\hat{i}^*\hat{j}^*;(kl)}$ in the normal coordinate. We find that

$$\begin{aligned}&i\mathcal{A}_4^{(c)}(\mathbf{1}^{\lambda_1},\mathbf{2}^{\lambda_2},3,4)\\&=-\frac{i}{2}R_{\hat{2}\hat{1}^*34}(p_3-p_4)_\mu x^\dagger_{\dot\alpha}(\vec{p}_1,\lambda_1)(\bar\sigma^\mu)^{\dot\alpha\beta}y_\beta(\vec{p}_2,\lambda_2)\\&\quad-\frac{i}{2}R_{\hat{2}^*\hat{1}34}(p_3-p_4)_\mu y^\alpha(\vec{p}_1,\lambda_1)(\sigma^\mu)_{\alpha\dot\beta}x^{\dagger\dot\beta}(\vec{p}_2,\lambda_2)\\&\quad-iM_{\hat{1}^*\hat{2}^*;(34)}x^\dagger_{\dot\alpha}(\vec{p}_1,\lambda_1)x^{\dagger\dot\alpha}(\vec{p}_2,\lambda_2)\\&\quad-iM_{\hat{1}\hat{2};(34)}y^\alpha(\vec{p}_1,\lambda_1)y_\alpha(\vec{p}_2,\lambda_2).\end{aligned} \tag{249}$$

Rewriting the amplitude in terms of the angle/square spinors (205)–(208), the spinor wave function structure becomes clearer:

$$\begin{aligned}&i\mathcal{A}_4^{(c)}(\mathbf{1}^{\lambda_1},\mathbf{2}^{\lambda_2},3,4)\\&=-\frac{i}{2}R_{\hat{2}\hat{1}^*34}\Big(\sum_{\lambda_3}\langle\mathbf{1}^{\lambda_1}\mathbf{3}^{-\lambda_3}\rangle\lambda_3[\mathbf{3}^{\lambda_3}\mathbf{2}^{\lambda_2}]-\sum_{\lambda_4}\langle\mathbf{1}^{\lambda_1}\mathbf{4}^{-\lambda_4}\rangle\lambda_4[\mathbf{4}^{\lambda_4}\mathbf{2}^{\lambda_2}]\Big)\\&\quad-\frac{i}{2}R_{\hat{2}^*\hat{1}34}\Big(\sum_{\lambda_3}[\mathbf{1}^{\lambda_1}\mathbf{3}^{\lambda_3}]\lambda_3\langle\mathbf{3}^{-\lambda_3}\mathbf{2}^{\lambda_2}\rangle-\sum_{\lambda_4}[\mathbf{1}^{\lambda_1}\mathbf{4}^{\lambda_4}]\lambda_4\langle\mathbf{4}^{-\lambda_4}\mathbf{2}^{\lambda_2}\rangle\Big)\\&\quad-iM_{\hat{1}^*\hat{2}^*;(34)}\langle\mathbf{1}^{\lambda_1}\mathbf{2}^{\lambda_2}\rangle-iM_{\hat{1}\hat{2};(34)}[\mathbf{1}^{\lambda_1}\mathbf{2}^{\lambda_2}],\end{aligned} \tag{250}$$

where we decomposed $(p_3 - p_4)_\mu \bar{\sigma}^\mu$ and $(p_3 - p_4)_\mu \sigma^\mu$ into products of massive spinor wave functions using Eqs. (228)–(231).

The computation of the scalar-exchange amplitude is easy. It is given by

$$\begin{aligned} i\mathcal{A}_4^{(\phi)}(\mathbf{1}^{\lambda_1}, \mathbf{2}^{\lambda_2}, 3, 4) &= -\sum_{i,j} V_{;(34i)} [D(s_{12})]^{ij} (M_{12;j} [\mathbf{1}^{\lambda_1} \mathbf{2}^{\lambda_2}] \\ &\quad + M_{1^*2^*;j} \langle \mathbf{1}^{\lambda_1} \mathbf{2}^{\lambda_2} \rangle), \end{aligned} \tag{251}$$

where $[D(s)]^{ij}$ is as defined in Eq. (245).

On the other hand, the computation of the fermion-exchange amplitude $\mathcal{A}_4^{(\psi)}$,

$$\begin{aligned} i\mathcal{A}_4^{(\psi)}(\mathbf{1}^{\lambda_1}, \mathbf{2}^{\lambda_2}, 3, 4) = i\mathcal{A}_4^{[]}(\mathbf{1}^{\lambda_1}, \mathbf{2}^{\lambda_2}, 3, 4) \\ + i\mathcal{A}_4^{\langle\rangle}(\mathbf{1}^{\lambda_1}, \mathbf{2}^{\lambda_2}, 3, 4) \\ + i\mathcal{A}_4^{[\rangle}(\mathbf{1}^{\lambda_1}, \mathbf{2}^{\lambda_2}, 3, 4) \\ + i\mathcal{A}_4^{\langle]}(\mathbf{1}^{\lambda_1}, \mathbf{2}^{\lambda_2}, 3, 4), \end{aligned} \tag{252}$$

is a bit involved. Here we organized the amplitude in accord with the spinor structure, i.e.,

$$\begin{aligned} \mathcal{A}_4^{[]}(\mathbf{1}^{\lambda_1}, \mathbf{2}^{\lambda_2}, 3, 4) &= \sum_{\hat{i},\hat{j}} M_{1\hat{i};3} [\mathbf{1}^{\lambda_1} D_{][}^{\hat{i}\hat{j}}(p_{13}) \mathbf{2}^{\lambda_2}] M_{\hat{j}2;4} \\ &\quad + \sum_{\hat{i},\hat{j}} M_{1\hat{i};4} [\mathbf{1}^{\lambda_1} D_{][}^{\hat{i}\hat{j}}(p_{14}) \mathbf{2}^{\lambda_2}] M_{\hat{j}2;3}, \end{aligned} \tag{253}$$

$$\begin{aligned} \mathcal{A}_4^{\langle\rangle}(\mathbf{1}^{\lambda_1}, \mathbf{2}^{\lambda_2}, 3, 4) &= \sum_{\hat{i}^*,\hat{j}^*} M_{1^*\hat{i}^*;3} \langle \mathbf{1}^{\lambda_1} D_{\rangle\langle}^{\hat{i}^*\hat{j}^*}(p_{13}) \mathbf{2}^{\lambda_2} \rangle M_{\hat{j}^*2^*;4} \\ &\quad + \sum_{\hat{i}^*,\hat{j}^*} M_{1^*\hat{i}^*;4} \langle \mathbf{1}^{\lambda_1} D_{\rangle\langle}^{\hat{i}^*\hat{j}^*}(p_{14}) \mathbf{2}^{\lambda_2} \rangle M_{\hat{j}^*2^*;3}, \end{aligned} \tag{254}$$

$$\begin{aligned} \mathcal{A}_4^{[\rangle}(\mathbf{1}^{\lambda_1}, \mathbf{2}^{\lambda_2}, 3, 4) &= \sum_{\hat{i},\hat{j}^*} M_{1\hat{i};3} [\mathbf{1}^{\lambda_1} D_{]\langle}^{\hat{i}\hat{j}^*}(p_{13}) \mathbf{2}^{\lambda_2} \rangle M_{\hat{j}^*2^*;4} \\ &\quad + \sum_{\hat{i},\hat{j}^*} M_{1\hat{i};4} [\mathbf{1}^{\lambda_1} D_{]\langle}^{\hat{i}\hat{j}^*}(p_{14}) \mathbf{2}^{\lambda_2} \rangle M_{\hat{j}^*2^*;3}, \end{aligned} \tag{255}$$

$$\begin{aligned} \mathcal{A}_4^{\langle]}(\mathbf{1}^{\lambda_1}, \mathbf{2}^{\lambda_2}, 3, 4) &= \sum_{\hat{i}^*,\hat{j}} M_{1^*\hat{i}^*;3} \langle \mathbf{1}^{\lambda_1} D_{\rangle[}^{\hat{i}^*\hat{j}}(p_{13}) \mathbf{2}^{\lambda_2}] M_{\hat{j}2;4} \\ &\quad + \sum_{\hat{i}^*,\hat{j}} M_{1^*\hat{i}^*;4} \langle \mathbf{1}^{\lambda_1} D_{\rangle[}^{\hat{i}^*\hat{j}}(p_{14}) \mathbf{2}^{\lambda_2}] M_{\hat{j}2;3}, \end{aligned} \tag{256}$$

with

$$p_{13} \coloneqq p_1 + p_3, \qquad p_{14} \coloneqq p_1 + p_4. \tag{257}$$

In the above expressions, the internal fermion propagators are

$$(D_{][}^{\hat{i}\hat{j}}(p))_\alpha{}^\beta \coloneqq \frac{i m_{\hat{i}}}{p^2 - m_{\hat{i}}^2} \delta^{\hat{i}\hat{j}} \delta_\alpha^\beta, \tag{258}$$

$$(D_{\rangle\langle}^{\hat{i}^*\hat{j}^*}(p))^{\dot{\alpha}}{}_{\dot{\beta}} \coloneqq \frac{i m_{\hat{i}^*}}{p^2 - m_{\hat{i}^*}^2} \delta^{\hat{i}^*\hat{j}^*} \delta_{\dot{\beta}}^{\dot{\alpha}}, \tag{259}$$

$$(D_{]\langle}^{\hat{i}\hat{j}^*}(p))_{\alpha\dot{\beta}} \coloneqq \frac{i}{p^2 - m_{\hat{i}}^2} \delta^{\hat{i}\hat{j}^*} (p])_\alpha (\langle p)_{\dot{\beta}}, \tag{260}$$

$$(D_{\rangle[}^{\hat{i}^*\hat{j}}(p))^{\dot{\alpha}\beta} \coloneqq \frac{i}{p^2 - m_{\hat{i}^*}^2} \delta^{\hat{i}^*\hat{j}} (p\rangle)^{\dot{\alpha}} ([p)^\beta. \tag{261}$$

The amplitudes (253) and (254) are computed by using

$$[\mathbf{1}^{\lambda_1} D_{][}^{\hat{i}\hat{j}}(p_{13}) \mathbf{2}^{\lambda_2}] = (\hat{D}(s_{13}))^{\hat{i}\hat{j}} m_{\hat{i}} [\mathbf{1}^{\lambda_1} \mathbf{2}^{\lambda_2}], \tag{262}$$

$$\langle \mathbf{1}^{\lambda_1} D_{\rangle\langle}^{\hat{i}^*\hat{j}^*}(p_{13}) \mathbf{2}^{\lambda_2} \rangle = (\hat{D}(s_{13}))^{\hat{i}^*\hat{j}^*} m_{\hat{i}^*} \langle \mathbf{1}^{\lambda_1} \mathbf{2}^{\lambda_2} \rangle, \tag{263}$$

with

$$\begin{aligned} (\hat{D}(s))^{\hat{i}\hat{j}} &\coloneqq \frac{i}{s - m_{\hat{i}}^2} \delta^{\hat{i}\hat{j}}, \\ (\hat{D}(s))^{\hat{i}^*\hat{j}^*} &\coloneqq \frac{i}{s - m_{\hat{i}^*}^2} \delta^{\hat{i}^*\hat{j}^*}. \end{aligned} \tag{264}$$

In the computation of the amplitudes (255) and (256), we use identities on the spinor wave functions,

$$\begin{aligned} &[\mathbf{1}^{\lambda_1} D_{]\langle}^{\hat{i}\hat{j}^*}(p_{13}) \mathbf{2}^{\lambda_2} \rangle \\ &\quad = \frac{1}{2} (\hat{D}(s_{13}^2))^{\hat{i}\hat{j}^*} \Big(m_1 \langle \mathbf{1}^{\lambda_1} \mathbf{2}^{\lambda_2} \rangle + m_2 [\mathbf{1}^{\lambda_1} \mathbf{2}^{\lambda_2}] \\ &\qquad + \sum_{\lambda_3 = \pm 1} [\mathbf{1}^{\lambda_1} \mathbf{3}^{\lambda_3}] \lambda_3 \langle \mathbf{3}^{-\lambda_3} \mathbf{2}^{\lambda_2} \rangle \\ &\qquad - \sum_{\lambda_4 = \pm 1} [\mathbf{1}^{\lambda_1} \mathbf{4}^{\lambda_4}] \lambda_4 \langle \mathbf{4}^{-\lambda_4} \mathbf{2}^{\lambda_2} \rangle \Big), \end{aligned} \tag{265}$$

$$\begin{aligned} &\langle \mathbf{1}^{\lambda_1} D_{\rangle[}^{\hat{i}^*\hat{j}}(p_{13}) \mathbf{2}^{\lambda_2}] \\ &\quad = \frac{1}{2} (\hat{D}(s_{13}^2))^{\hat{i}^*\hat{j}} \Big(m_1 [\mathbf{1}^{\lambda_1} \mathbf{2}^{\lambda_2}] + m_2 \langle \mathbf{1}^{\lambda_1} \mathbf{2}^{\lambda_2} \rangle \\ &\qquad + \sum_{\lambda_3 = \pm 1} \langle \mathbf{1}^{\lambda_1} \mathbf{3}^{-\lambda_3} \rangle \lambda_3 [\mathbf{3}^{\lambda_3} \mathbf{2}^{\lambda_2}] \\ &\qquad - \sum_{\lambda_4 = \pm 1} \langle \mathbf{1}^{\lambda_1} \mathbf{4}^{-\lambda_4} \rangle \lambda_4 [\mathbf{4}^{\lambda_4} \mathbf{2}^{\lambda_2}] \Big), \end{aligned} \tag{266}$$

with

$$(\hat{D}(s))^{\hat{i}\hat{j}^*} := \frac{i}{s - m_{\hat{i}}^2} \delta^{\hat{i}\hat{j}^*}, \quad (\hat{D}(s))^{\hat{i}^*\hat{j}} := \frac{i}{s - m_{\hat{i}^*}^2} \delta^{\hat{i}^*\hat{j}}. \tag{267}$$

These identities are derived from

$$p_{13} = \frac{1}{2}(p_1 - p_2 + p_3 - p_4),$$
$$p_{14} = \frac{1}{2}(p_1 - p_2 - p_3 + p_4) \tag{268}$$

and

$$[\mathbf{1}^{\lambda_1}(p_{1\mu}\sigma^\mu) = m_1\langle \mathbf{1}^{\lambda_1}, \qquad (p_{2\mu}\bar{\sigma}^\mu)\mathbf{2}^{\lambda_2}] = -\mathbf{2}^{\lambda_2}\rangle m_2, \tag{269}$$

$$\langle \mathbf{1}^{\lambda_1}(p_{1\mu}\bar{\sigma}^\mu) = m_1[\mathbf{1}^{\lambda_1}, \qquad (p_{2\mu}\sigma^\mu)\mathbf{2}^{\lambda_2}\rangle = -\mathbf{2}^{\lambda_2}]m_2. \tag{270}$$

We are now ready to compute the fermion-exchange amplitude (252). Combining the above formulas, we obtain

$$\begin{aligned} i\mathcal{A}_4^{(\psi)}(\mathbf{1}^{\lambda_1}, \mathbf{2}^{\lambda_2}, 3, 4) = {} & i\mathcal{A}_4^{(\psi,yx)}\left(\sum_{\lambda_3=\pm1}[\mathbf{1}^{\lambda_1}\mathbf{3}^{\lambda_3}]\lambda_3\langle \mathbf{3}^{-\lambda_3}\mathbf{2}^{\lambda_2}\rangle - \sum_{\lambda_4=\pm1}[\mathbf{1}^{\lambda_1}\mathbf{4}^{\lambda_4}]\lambda_4\langle \mathbf{4}^{-\lambda_4}\mathbf{2}^{\lambda_2}\rangle\right) \\ & + i\mathcal{A}_4^{(\psi,xy)}\left(\sum_{\lambda_3=\pm1}\langle \mathbf{1}^{\lambda_1}\mathbf{3}^{-\lambda_3}\rangle\lambda_3[\mathbf{3}^{\lambda_3}\mathbf{2}^{\lambda_2}] - \sum_{\lambda_4=\pm1}\langle \mathbf{1}^{\lambda_1}\mathbf{4}^{-\lambda_4}\rangle\lambda_4[\mathbf{4}^{\lambda_4}\mathbf{2}^{\lambda_2}]\right) \\ & + i\mathcal{A}_4^{(\psi,yy)}[\mathbf{1}^{\lambda_1}\mathbf{2}^{\lambda_2}] + i\mathcal{A}_4^{(\psi,xx)}\langle \mathbf{1}^{\lambda_1}\mathbf{2}^{\lambda_2}\rangle, \end{aligned} \tag{271}$$

with

$$\begin{aligned} i\mathcal{A}_4^{(\psi,yx)} = {} & -\frac{1}{2}\sum_{\hat{i},\hat{j}^*}(M_{\hat{i}\,\hat{1};3}[\hat{D}(s_{13})]^{\hat{i}\hat{j}^*}M_{\hat{j}^*\hat{2}^*;4} \\ & + M_{\hat{i}\,\hat{1};4}[\hat{D}(s_{14})]^{\hat{i}\hat{j}^*}M_{\hat{j}^*\hat{2}^*;3}), \end{aligned} \tag{272}$$

$$\begin{aligned} i\mathcal{A}_4^{(\psi,xy)} = {} & -\frac{1}{2}\sum_{\hat{i},\hat{j}^*}(M_{\hat{j}^*\hat{1}^*;3}[\hat{D}(s_{13})]^{\hat{j}^*\hat{i}}M_{\hat{i}\,\hat{2};4} \\ & + M_{\hat{j}^*\hat{1}^*;4}[\hat{D}(s_{14})]^{\hat{j}^*\hat{i}}M_{\hat{i}\,\hat{2};3}) \end{aligned} \tag{273}$$

and

$$\begin{aligned} i\mathcal{A}_4^{(\psi,yy)} = {} & -\frac{1}{2}m_1\sum_{\hat{i}^*,\hat{j}}(M_{\hat{i}^*\hat{1}^*;3}[\hat{D}(s_{13})]^{\hat{i}^*\hat{j}}M_{\hat{j}\,\hat{2};4} + M_{\hat{i}^*\hat{1}^*;4}[\hat{D}(s_{14})]^{\hat{i}^*\hat{j}}M_{\hat{j}\,\hat{2};3}) \\ & -\frac{1}{2}m_2\sum_{\hat{i},\hat{j}^*}(M_{\hat{i}\,\hat{1};3}[\hat{D}(s_{13})]^{\hat{i}\hat{j}^*}M_{\hat{j}^*\hat{2}^*;4} + M_{\hat{i}\,\hat{1};4}[\hat{D}(s_{14})]^{\hat{i}\hat{j}^*}M_{\hat{j}^*\hat{2}^*;3}) \\ & -\sum_{\hat{i},\hat{j}}m_{\hat{i}}(M_{\hat{i}\,\hat{1};3}[\hat{D}(s_{13})]^{\hat{i}\,\hat{j}}M_{\hat{j}\,\hat{2};4} + M_{\hat{i}\,\hat{1};4}[\hat{D}(s_{14})]^{\hat{i}\,\hat{j}}M_{\hat{j}\,\hat{2};3}), \end{aligned} \tag{274}$$

$$\begin{aligned} i\mathcal{A}_4^{(\psi,xx)} = {} & -\frac{1}{2}m_1\sum_{\hat{i},\hat{j}^*}(M_{\hat{i}\,\hat{1};3}[\hat{D}(s_{13})]^{\hat{i}\hat{j}^*}M_{\hat{j}^*\hat{2}^*;4} + M_{\hat{i}\,\hat{1};4}[\hat{D}(s_{14})]^{\hat{i}\hat{j}^*}M_{\hat{j}^*\hat{2}^*;3}) \\ & -\frac{1}{2}m_2\sum_{\hat{i}^*,\hat{j}}(M_{\hat{i}^*\hat{1}^*;3}[\hat{D}(s_{13})]^{\hat{i}^*\hat{j}}M_{\hat{j}\,\hat{2};4} + M_{\hat{i}^*\hat{1}^*;4}[\hat{D}(s_{14})]^{\hat{i}^*\hat{j}}M_{\hat{j}\,\hat{2};3}) \\ & -\sum_{\hat{i}^*,\hat{j}^*}m_{\hat{i}^*}(M_{\hat{i}^*\hat{1}^*;3}[\hat{D}(s_{13})]^{\hat{i}^*\hat{j}^*}M_{\hat{j}^*\hat{2}^*;4} + M_{\hat{i}^*\hat{1}^*;4}[\hat{D}(s_{14})]^{\hat{i}^*\hat{j}^*}M_{\hat{j}^*\hat{2}^*;3}). \end{aligned} \tag{275}$$

We evaluate the scattering amplitude

$$\phi^3(-p_3)\phi^4(-p_4) \to \psi^{\hat{1}}(p_1)\psi^{\hat{2}}(p_2) \tag{276}$$

in the center-of-mass frame, which implies

$$[\mathbf{1}^+\mathbf{2}^-] = 0, \qquad [\mathbf{1}^-\mathbf{2}^+] = 0, \tag{277}$$

$$\langle \mathbf{1}^+\mathbf{2}^- \rangle = 0, \qquad \langle \mathbf{1}^-\mathbf{2}^+ \rangle = 0. \tag{278}$$

The contact amplitudes (250) dominate in the computation for the high-energy limit $s = s_{12} \gg m_1^2,\ m_2^2,\ m_3^2,\ m_4^2$. We find that

$$[\mathbf{1}^+\mathbf{2}^+] \simeq [1\,2] = -\sqrt{s}, \qquad [\mathbf{1}^-\mathbf{2}^-] \simeq 0, \tag{279}$$

$$\langle \mathbf{1}^-\mathbf{2}^- \rangle \simeq \langle 1\,2 \rangle = +\sqrt{s}, \qquad \langle \mathbf{1}^+\mathbf{2}^+ \rangle \simeq 0, \tag{280}$$

and

$$\begin{aligned}\sum_{\lambda_3} \langle \mathbf{1}^-\mathbf{3}^{-\lambda_3} \rangle \lambda_3 [\mathbf{3}^{\lambda_3}\mathbf{2}^+] &\simeq \langle 1\,3 \rangle [3\,2] \\ &= -\langle 1\,3 \rangle [2\,3] \\ &= \sqrt{s_{13}s_{23}},\end{aligned} \tag{281}$$

$$\begin{aligned}\sum_{\lambda_4} \langle \mathbf{1}^-\mathbf{4}^{-\lambda_4} \rangle \lambda_4 [\mathbf{4}^{\lambda_3}\mathbf{2}^+] &\simeq \langle 1\,4 \rangle [4\,2] \\ &= -\sqrt{s_{14}s_{42}},\end{aligned} \tag{282}$$

$$\begin{aligned}&\sum_{\lambda_3} \langle \mathbf{1}^+\mathbf{3}^{-\lambda_3} \rangle \lambda_3 [\mathbf{3}^{\lambda_3}\mathbf{2}^+] - \sum_{\lambda_4} \langle \mathbf{1}^+\mathbf{4}^{-\lambda_4} \rangle \lambda_4 [\mathbf{4}^{\lambda_4}\mathbf{2}^+] \\ &\quad \simeq -m_1 \frac{s_{13}-s_{14}}{\sqrt{s_{12}}}.\end{aligned} \tag{283}$$

We obtain

$$\begin{aligned}&\mathcal{A}_4(\mathbf{1}^+,\mathbf{2}^+,3,4) \\ &\quad \simeq \sqrt{s}\left(M_{\hat{1}\,\hat{2};(34)} - \frac{1}{2}(m_2 R_{\hat{1}\hat{2}^*34} + m_1 R_{\hat{1}^*\hat{2}34})\cos\theta \right),\end{aligned} \tag{284}$$

$$\mathcal{A}_4(\mathbf{1}^+,\mathbf{2}^-,3,4) \simeq \frac{1}{2} R_{\hat{1}\hat{2}^*34}\, s \sin\theta, \tag{285}$$

with θ being the scattering angle in the center-of-mass frame,

$$\sin\theta \simeq 2\frac{\sqrt{s_{13}s_{14}}}{s_{12}}, \qquad \cos\theta \simeq \frac{s_{13}-s_{14}}{s_{12}}. \tag{286}$$

It is also straightforward to compute the $\mathcal{A}_4(\mathbf{1}^-,\mathbf{2}^-,3,4)$ and $\mathcal{A}_4(\mathbf{1}^-,\mathbf{2}^+,3,4)$ amplitudes. Unless $M_{\hat{1}\,\hat{2};(34)} = M_{\hat{1}^*\hat{2}^*;(34)} = 0$ and $R_{\hat{1}\hat{2}^*34} = R_{\hat{1}^*\hat{2}34} = 0$, these amplitudes eventually violate the perturbative unitarity at the high-energy scale. Considering the equivalence theorem, these results indicate that the scattering amplitudes of the fermion pair scattering to the longitudinally polarized gauge boson violate the perturbative unitarity at a certain high-energy scale unless $M_{\hat{1}\,\hat{2};(34)} = 0$ and $R_{\hat{1}\hat{2}^*34} = 0$. The Appelquist-Chanowitz sum rules [73–78] for the perturbative unitarity in the $W_L W_L \to t\bar{t}$ amplitude can thus be regarded as conditions on $R_{t\bar{t}\pi\pi}$ and $M_{\bar{t}t;(\pi\pi)}$.

F. Four fermions

The four-point fermion amplitude is computed from the Feynman diagram (Fig. 5). We decompose the amplitude into three categories,

$$\begin{aligned}\mathcal{A}_4(\mathbf{1}^{\lambda_1},\mathbf{2}^{\lambda_2},\mathbf{3}^{\lambda_3},\mathbf{4}^{\lambda_4}) &= \mathcal{A}_4^{(c)}(\mathbf{1}^{\lambda_1},\mathbf{2}^{\lambda_2},\mathbf{3}^{\lambda_3},\mathbf{4}^{\lambda_4}) \\ &\quad + \mathcal{A}_4^{(c')}(\mathbf{1}^{\lambda_1},\mathbf{2}^{\lambda_2},\mathbf{3}^{\lambda_3},\mathbf{4}^{\lambda_4}) \\ &\quad + \mathcal{A}_4^{(\phi)}(\mathbf{1}^{\lambda_1},\mathbf{2}^{\lambda_2},\mathbf{3}^{\lambda_3},\mathbf{4}^{\lambda_4}),\end{aligned} \tag{287}$$

where $\mathcal{A}^{(c)}$ and $\mathcal{A}^{(c')}$, respectively, denote the holomorphic and nonholomorphic contact diagrams, and $\mathcal{A}^{(\phi)}$ denotes the scalar-exchange diagram.

Let us first focus on the contact diagram induced from the holomorphic contact interactions ($\psi\psi\psi\psi$ and $\psi^\dagger\psi^\dagger\psi^\dagger\psi^\dagger$). The amplitude is given as

$$\begin{aligned}&i\mathcal{A}_4^{(c)}(\mathbf{1}^{\lambda_1},\mathbf{2}^{\lambda_2},\mathbf{3}^{\lambda_3},\mathbf{4}^{\lambda_4}) \\ &\quad = iS_{\hat{1}\hat{2}\hat{3}\hat{4}}[\mathbf{1}^{\lambda_1}\mathbf{2}^{\lambda_2}][\mathbf{3}^{\lambda_3}\mathbf{4}^{\lambda_4}] + iS_{\hat{1}\hat{3}\hat{4}\hat{2}}[\mathbf{1}^{\lambda_1}\mathbf{3}^{\lambda_3}][\mathbf{4}^{\lambda_4}\mathbf{2}^{\lambda_2}] \\ &\qquad + iS_{\hat{1}\hat{4}\hat{2}\hat{3}}[\mathbf{1}^{\lambda_1}\mathbf{4}^{\lambda_4}][\mathbf{2}^{\lambda_2}\mathbf{3}^{\lambda_3}] + iS_{\hat{1}^*\hat{2}^*\hat{3}^*\hat{4}^*}\langle \mathbf{1}^{\lambda_1}\mathbf{2}^{\lambda_2} \rangle \langle \mathbf{3}^{\lambda_3}\mathbf{4}^{\lambda_4} \rangle \\ &\qquad + iS_{\hat{1}^*\hat{3}^*\hat{4}^*\hat{2}^*}\langle \mathbf{1}^{\lambda_1}\mathbf{3}^{\lambda_3} \rangle \langle \mathbf{4}^{\lambda_4}\mathbf{2}^{\lambda_2} \rangle + iS_{\hat{1}^*\hat{4}^*\hat{2}^*\hat{3}^*}\langle \mathbf{1}^{\lambda_1}\mathbf{4}^{\lambda_4} \rangle \langle \mathbf{2}^{\lambda_2}\mathbf{3}^{\lambda_3} \rangle.\end{aligned} \tag{288}$$

Using the Schouten identities (226) and (227), the amplitude are expressed in terms of nonredundant parameters $R_{\hat{i}\,\hat{j}\,\hat{k}\,\hat{l}}$,

$$\begin{aligned}&i\mathcal{A}_4^{(c)}(\mathbf{1}^{\lambda_1},\mathbf{2}^{\lambda_2},\mathbf{3}^{\lambda_3},\mathbf{4}^{\lambda_4}) \\ &\quad = -\frac{2i}{3}R_{\hat{1}(\hat{3}\hat{4})\hat{2}}[\mathbf{1}^{\lambda_1}\mathbf{2}^{\lambda_2}][\mathbf{3}^{\lambda_3}\mathbf{4}^{\lambda_4}] - \frac{2i}{3}R_{\hat{1}(\hat{2}\hat{4})\hat{3}}[\mathbf{1}^{\lambda_1}\mathbf{3}^{\lambda_3}][\mathbf{4}^{\lambda_4}\mathbf{2}^{\lambda_2}] \\ &\qquad - \frac{2i}{3}R_{\hat{1}(\hat{2}\hat{3})\hat{4}}[\mathbf{1}^{\lambda_1}\mathbf{4}^{\lambda_4}][\mathbf{2}^{\lambda_2}\mathbf{3}^{\lambda_3}] - \frac{2i}{3}R_{\hat{1}^*(\hat{3}^*\hat{4}^*)\hat{2}^*}\langle \mathbf{1}^{\lambda_1}\mathbf{2}^{\lambda_2} \rangle \langle \mathbf{3}^{\lambda_3}\mathbf{4}^{\lambda_4} \rangle \\ &\qquad - \frac{2i}{3}R_{\hat{1}^*(\hat{2}^*\hat{4}^*)\hat{3}^*}\langle \mathbf{1}^{\lambda_1}\mathbf{3}^{\lambda_3} \rangle \langle \mathbf{4}^{\lambda_4}\mathbf{2}^{\lambda_2} \rangle \\ &\qquad - \frac{2i}{3}R_{\hat{1}^*(\hat{2}^*\hat{3}^*)\hat{4}^*}\langle \mathbf{1}^{\lambda_1}\mathbf{4}^{\lambda_4} \rangle \langle \mathbf{2}^{\lambda_2}\mathbf{3}^{\lambda_3} \rangle,\end{aligned} \tag{289}$$

with $R_{\hat{i}\,\hat{j}\,\hat{k}\,\hat{l}}$ being the "curvature" tensors defined in Eq. (183). Similarly, the amplitude from the nonholomorphic contact interaction ($\psi\psi\psi^\dagger\psi^\dagger$) is given by

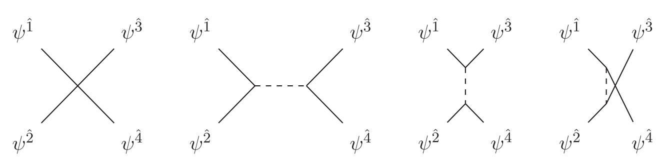

FIG. 5. Feynman diagram for $\mathcal{A}_4(\mathbf{1}^{\lambda_1},\mathbf{2}^{\lambda_2},\mathbf{3}^{\lambda_3},\mathbf{4}^{\lambda_4})$. We assign the outgoing momenta p_1, p_2, p_3, and p_4 to $\psi^{\hat{1}}$, $\psi^{\hat{2}}$, $\psi^{\hat{3}}$, and $\psi^{\hat{4}}$.

$$i\mathcal{A}_4^{(c')}(\mathbf{1}^{\lambda_1},\mathbf{2}^{\lambda_2},\mathbf{3}^{\lambda_3},\mathbf{4}^{\lambda_4}) = iR_{\hat{1}\hat{3}^*\hat{2}\hat{4}^*}[\mathbf{1}^{\lambda_1}\mathbf{2}^{\lambda_2}]\langle\mathbf{3}^{\lambda_3}\mathbf{4}^{\lambda_4}\rangle + iR_{\hat{1}\hat{4}^*\hat{3}\hat{2}^*}[\mathbf{1}^{\lambda_1}\mathbf{3}^{\lambda_3}]\langle\mathbf{4}^{\lambda_4}\mathbf{2}^{\lambda_2}\rangle + iR_{\hat{1}\hat{2}^*\hat{4}\hat{3}^*}[\mathbf{1}^{\lambda_1}\mathbf{4}^{\lambda_4}]\langle\mathbf{2}^{\lambda_2}\mathbf{3}^{\lambda_3}\rangle \\ + iR_{\hat{3}\hat{1}^*\hat{4}\hat{2}^*}\langle\mathbf{1}^{\lambda_1}\mathbf{2}^{\lambda_2}\rangle[\mathbf{3}^{\lambda_3}\mathbf{4}^{\lambda_4}] + iR_{\hat{4}\hat{1}^*\hat{2}\hat{3}^*}\langle\mathbf{1}^{\lambda_1}\mathbf{3}^{\lambda_3}\rangle[\mathbf{4}^{\lambda_4}\mathbf{2}^{\lambda_2}] + iR_{\hat{2}\hat{1}^*\hat{3}\hat{4}^*}\langle\mathbf{1}^{\lambda_1}\mathbf{4}^{\lambda_4}\rangle[\mathbf{2}^{\lambda_2}\mathbf{3}^{\lambda_3}]. \tag{290}$$

Here the curvature tensor $R_{\hat{i}\hat{j}^*\hat{k}\hat{l}^*}$ is as defined in Eq. (200). The scalar-exchange amplitude is computed as

$$\begin{aligned} i\mathcal{A}_4^{(\phi)}(\mathbf{1}^{\lambda_1},\mathbf{2}^{\lambda_2},\mathbf{3}^{\lambda_3},\mathbf{4}^{\lambda_4}) = &-\sum_{i,j}(M_{\hat{1}\hat{2};i}[\mathbf{1}^{\lambda_1}\mathbf{2}^{\lambda_2}] + M_{\hat{1}^*\hat{2}^*;i}\langle\mathbf{1}^{\lambda_1}\mathbf{2}^{\lambda_2}\rangle)[D(s_{12})]^{ij}(M_{\hat{3}\hat{4};i}[\mathbf{3}^{\lambda_3}\mathbf{4}^{\lambda_4}] + M_{\hat{3}^*\hat{4}^*;i}\langle\mathbf{3}^{\lambda_3}\mathbf{4}^{\lambda_4}\rangle) \\ &-\sum_{i,j}(M_{\hat{1}\hat{3};i}[\mathbf{1}^{\lambda_1}\mathbf{3}^{\lambda_3}] + M_{\hat{1}^*\hat{3}^*;i}\langle\mathbf{1}^{\lambda_1}\mathbf{3}^{\lambda_3}\rangle)[D(s_{13})]^{ij}(M_{\hat{2}\hat{4};i}[\mathbf{2}^{\lambda_2}\mathbf{4}^{\lambda_4}] + M_{\hat{2}^*\hat{4}^*;i}\langle\mathbf{2}^{\lambda_2}\mathbf{4}^{\lambda_4}\rangle) \\ &-\sum_{i,j}(M_{\hat{1}\hat{4};i}[\mathbf{1}^{\lambda_1}\mathbf{4}^{\lambda_4}] + M_{\hat{1}^*\hat{4}^*;i}\langle\mathbf{1}^{\lambda_1}\mathbf{4}^{\lambda_4}\rangle)[D(s_{14})]^{ij}(M_{\hat{2}\hat{3};i}[\mathbf{2}^{\lambda_2}\mathbf{3}^{\lambda_3}] + M_{\hat{2}^*\hat{3}^*;i}\langle\mathbf{2}^{\lambda_2}\mathbf{3}^{\lambda_3}\rangle), \end{aligned} \tag{291}$$

where $[D(s)]^{ij}$ denotes the scalar propagator (245).

We are now ready to discuss the high-energy behavior of the four-fermion amplitude,

$$\psi^{\hat{1}}(-p_1)\psi^{\hat{2}}(-p_2) \to \psi^{\hat{3}}(p_3)\psi^{\hat{4}}(p_4), \tag{292}$$

in the center-of-mass frame. Taking the high-energy limit, $s = s_{12} \gg m_1^2,\ m_2^2,\ m_3^2,\ m_4^2$, we find the eight helicity amplitudes

$$\mathcal{A}_4(\mathbf{1}^+,\mathbf{2}^+,\mathbf{3}^+,\mathbf{4}^+), \qquad \mathcal{A}_4(\mathbf{1}^-,\mathbf{2}^-,\mathbf{3}^-,\mathbf{4}^-) \tag{293}$$

and

$$\begin{aligned} &\mathcal{A}_4(\mathbf{1}^+,\mathbf{2}^+,\mathbf{3}^-,\mathbf{4}^-), \qquad \mathcal{A}_4(\mathbf{1}^+,\mathbf{2}^-,\mathbf{3}^+,\mathbf{4}^-), \\ &\mathcal{A}_4(\mathbf{1}^+,\mathbf{2}^-,\mathbf{3}^-,\mathbf{4}^+), \qquad \mathcal{A}_4(\mathbf{1}^-,\mathbf{2}^-,\mathbf{3}^+,\mathbf{4}^+), \\ &\mathcal{A}_4(\mathbf{1}^-,\mathbf{2}^+,\mathbf{3}^-,\mathbf{4}^+), \qquad \mathcal{A}_4(\mathbf{1}^-,\mathbf{2}^+,\mathbf{3}^+,\mathbf{4}^-) \end{aligned} \tag{294}$$

grow as energy squared. We obtain

$$\mathcal{A}_4(\mathbf{1}^+,\mathbf{2}^+,\mathbf{3}^+,\mathbf{4}^+) \simeq R_{\hat{1}\hat{4}\hat{2}\hat{3}}s + \frac{1}{2}R_{\hat{1}\hat{2}\hat{3}\hat{4}}s(1+\cos\theta), \tag{295}$$

$$\mathcal{A}_4(\mathbf{1}^+,\mathbf{2}^+,\mathbf{3}^-,\mathbf{4}^-) \simeq R_{\hat{1}\hat{3}^*\hat{2}\hat{4}^*}s. \tag{296}$$

We also find that 8 of the 16 helicity amplitudes,

$$\begin{aligned} &\mathcal{A}_4(\mathbf{1}^-,\mathbf{2}^+,\mathbf{3}^+,\mathbf{4}^+), \qquad \mathcal{A}_4(\mathbf{1}^+,\mathbf{2}^-,\mathbf{3}^+,\mathbf{4}^+), \\ &\mathcal{A}_4(\mathbf{1}^+,\mathbf{2}^+,\mathbf{3}^-,\mathbf{4}^+), \qquad \mathcal{A}_4(\mathbf{1}^+,\mathbf{2}^+,\mathbf{3}^+,\mathbf{4}^-), \\ &\mathcal{A}_4(\mathbf{1}^+,\mathbf{2}^-,\mathbf{3}^-,\mathbf{4}^-), \qquad \mathcal{A}_4(\mathbf{1}^-,\mathbf{2}^+,\mathbf{3}^-,\mathbf{4}^-), \\ &\mathcal{A}_4(\mathbf{1}^-,\mathbf{2}^-,\mathbf{3}^+,\mathbf{4}^-), \qquad \mathcal{A}_4(\mathbf{1}^-,\mathbf{2}^-,\mathbf{3}^-,\mathbf{4}^+), \end{aligned} \tag{297}$$

behave as $\sqrt{s}$ in the high-energy limit. For example, we obtain

$$\mathcal{A}_4(\mathbf{1}^-,\mathbf{2}^+,\mathbf{3}^+,\mathbf{4}^+) \\ \simeq \frac{\sqrt{s}}{2}\sin\theta(m_1 R_{\hat{1}\hat{2}\hat{3}\hat{4}} - m_3 R_{\hat{2}\hat{1}^*\hat{4}\hat{3}^*} + m_4 R_{\hat{2}\hat{1}^*\hat{3}\hat{4}^*}). \tag{298}$$

Note that the helicity structure determines the difference of the high-energy behaviors.

V. SUMMARY

We have formulated an extension of Higgs effective field theory (GHEFT) which includes an arbitrary number of spin-0 and spin-1/2 particles with arbitrary electric and chromoelectric charges. These particles include the SM quarks and leptons, and also BSM Higgs bosons and fermions. GHEFT can therefore describe the amplitude involving these non-SM particles. This is in contrast to the usual EFT frameworks such as SMEFT and HEFT, which cannot compute the cross sections and decay widths of these new particles, because these new particles are integrated out in these EFTs. The leading-order GHEFT Lagrangian has been expressed in accord with the GHEFT chiral-order counting rule, which clarifies the relationship between the loop expansion and the operator expansion.

The S matrix of a quantum field theory is unchanged by field redefinitions. This fact, known as the KOS theorem, tells us that seemingly different effective Lagrangians connected through field coordinate transformations can describe identical scattering amplitudes. The parametrization of the effective Lagrangian is therefore not unique. In this paper, we proposed using geometric quantities such as the curvature of field space in the GHEFT Lagrangian to resolve the redundancy. We also showed that, by introducing a useful coordinate (normal coordinate) in the field space manifold, the computations of the scattering amplitudes are significantly simplified.

We also estimated tree-level on-shell amplitudes (in Sec. IV). These on-shell amplitudes were expressed in terms of the square and angle bracket notation of the spinor wave function. The high-energy behaviors of the on-shell scattering amplitudes were computed. We found that the four-point scattering amplitudes grow as the scattering energy, and the coefficient of the energy-growing terms relate with covariant tensors such as the curvature tensors on the field space. Perturbative unitarity in the scattering amplitudes requires the flatness in the scalar/fermion field space around the vacuum, i.e.,

$$R_{1234}|_0 = R_{\hat{1}\hat{2}^*34}|_0 = R_{\hat{1}\,\hat{2}\,\hat{3}\,\hat{4}}|_0 = R_{\hat{1}^*\hat{2}^*\hat{3}^*\hat{4}^*}|_0 = R_{\hat{1}\hat{2}^*\hat{3}\hat{4}^*}|_0 = 0.$$

The GHEFT framework should be studied further. To apply the geometrical formulation in phenomenological studies, we need to compute the on-shell amplitudes involving the SM spin-1 particles W and Z in a geometrical language. It should also be emphasized that extra spin-1 particles often appear in models beyond the SM. For example, extra gauge bosons exist in the extensions of the SM gauge group. Spin-1 resonances like techni-ρ may appear in the strong dynamics models of the electroweak symmetry breaking. These spin-1 particles have been studied using electroweak resonance chiral perturbation theories. It will be illuminating to investigate the geometrical formulations for these spin-1 resonances in the GHEFT framework.

Radiative corrections should also be incorporated. To compute the $\gamma\gamma$ decays of the Higgs particles, for an example, we need to investigate radiative corrections in the GHEFT framework. As we showed in this paper, the chiral-order counting rule provides a basis for computing these radiative corrections. A geometrical formulation for the next-to-leading operators will also be useful in such a computation.

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APPENDIX A: HIGGS EFFECTIVE FIELD THEORY

HEFT [16–31] is one of low-energy effective field theories for electroweak symmetry breaking. In the gaugeless limit, the leading-order HEFT Lagrangian is defined as

$$\mathcal{L}_{\rm HEFT} = \mathcal{L}_{\rm HEFT,boson} + \mathcal{L}_{\rm HEFT,fermion}, \tag{A1}$$

where $\mathcal{L}_{\rm HEFT,boson}$ is found in Eq. (2), while $\mathcal{L}_{\rm HEFT,fermion}$ is given as

$$\begin{aligned}\mathcal{L}_{\rm HEFT,fermion} &= q_L^\dagger i\bar{\sigma}^\mu \partial_\mu q_L + q_R^\dagger i\sigma^\mu \partial_\mu q_R \\ &\quad + l_L^\dagger i\bar{\sigma}^\mu \partial_\mu l_L + l_R^\dagger i\sigma^\mu \partial_\mu l_R \\ &\quad + [q_L^\dagger U(\mathcal{F}_{Y_q}(h)\mathbf{1}_2 + \mathcal{F}_{\hat{Y}_q}(h)\tau^3)q_R + \text{H.c.}] \\ &\quad + [l_L^\dagger U(\mathcal{F}_{Y_l}(h)\mathbf{1}_2 + \mathcal{F}_{\hat{Y}_l}(h)\tau^3)l_R + \text{H.c.}],\end{aligned} \tag{A2}$$

with $\mathbf{1}_2$ being a 2×2 unit matrix. h denotes the 125 GeV Higgs boson. $\mathcal{F}$'s are arbitrary functions of h. q_L and l_L denote the $SU(2)_W$ doublet SM quark and lepton fields, respectively. q_R and l_R are vectors defined as

$$q_R = \begin{pmatrix} u_R \\ d_R \end{pmatrix}, \qquad l_R = \begin{pmatrix} 0 \\ e_R \end{pmatrix}, \tag{A3}$$

where u_R, d_R, e_R are the $SU(2)_W$ singlet up quark, down quark, and electron, respectively. Here we consider only one generation for simplicity. It is straightforward to introduce the other generations.

Under the $G = SU(2)_W \times U(1)_Y$ transformation, the fields in Eq. (A1) transforms as

$$U \to \mathfrak{g}_W U \mathfrak{g}_Y^\dagger, \tag{A4}$$

$$h \to h, \tag{A5}$$

$$q_L \to e^{\frac{i}{6}\theta_Y}\mathfrak{g}_W q_L, \qquad q_R \to e^{\frac{i}{6}\theta_Y}\mathfrak{g}_Y q_R, \tag{A6}$$

$$l_L \to e^{-\frac{i}{2}\theta_Y}\mathfrak{g}_W l_L, \qquad l_R \to e^{-\frac{i}{2}\theta_Y}\mathfrak{g}_Y l_R, \tag{A7}$$

where $\mathfrak{g}_W \in SU(2)_W$ and $\mathfrak{g}_Y \in U(1)_Y$. We can easily check to see that the HEFT Lagrangian (A1) respects $G = SU(2)_W \times U(1)_Y$ invariance.

Let us introduce $\hat{u}_{L,R}$, $\hat{e}_{L,R}$, and $\hat{\nu}_L$ as

$$\begin{pmatrix} \hat{u}_L \\ \hat{d}_L \end{pmatrix} \coloneqq \exp\left(\frac{i}{6}\pi^3(x)\right)\xi_W^\dagger q_L, \tag{A8}$$

$$\begin{pmatrix} \hat{u}_R \\ \hat{d}_R \end{pmatrix} \coloneqq \exp\left(\frac{i}{6}\pi^3(x)\right)\xi_Y q_R, \tag{A9}$$

$$\begin{pmatrix} \hat{\nu}_L \\ \hat{e}_L \end{pmatrix} \coloneqq \exp\left(-\frac{i}{2}\pi^3(x)\right)\xi_W^\dagger l_L, \tag{A10}$$

$$\begin{pmatrix} 0 \\ \hat{e}_R \end{pmatrix} \coloneqq \exp\left(-\frac{i}{2}\pi^3(x)\right)\xi_Y l_R, \tag{A11}$$

where ξ_W and ξ_Y were introduced in Eqs. (4) and (5). We note that, under the G transformation, the hatted fields transform like the $\psi^{\hat{i}}$ that we introduced:

$$\hat{u}_{L,R} \to \exp(iq_u\theta_Y(\pi, \mathfrak{g}_W, \mathfrak{g}_Y))\hat{u}_{L,R}, \tag{A12}$$

$$\hat{d}_{L,R} \to \exp(iq_d\theta_Y(\pi, \mathfrak{g}_W, \mathfrak{g}_Y))\hat{d}_{L,R}, \tag{A13}$$

$$\hat{\nu}_L \to \hat{\nu}_L, \tag{A14}$$

$$\hat{e}_{L,R} \to \exp(iq_e\theta_Y(\pi, \mathfrak{g}_W, \mathfrak{g}_Y))\hat{e}_{L,R}, \tag{A15}$$

where $(q_u, q_d, q_e) = (2/3, -1/3, -1)$.

It is now easy to see that the GHEFT Lagrangian (24) reproduces the HEFT Lagrangian. The matter particle content of the HEFT corresponds to

$$\phi^i = (\pi^1, \pi^2, \pi^3, h), \tag{A16}$$

$$\psi^{\hat{i}} = (\hat{u}_L, \hat{d}_L, \hat{u}_R^\dagger, \hat{d}_R^\dagger, \hat{\nu}_L, \hat{e}_L, 0, \hat{e}_R^\dagger), \tag{A17}$$

where h is the $U(1)_{\rm em}$ neutral scalar and the $U(1)_{\rm em}$ charges for ψ are assigned as $q_{\hat{i}} = (q_u, q_d, -q_u, -q_d, 0, q_e, 0, -q_e)$. We introduce a zero component (0) in Eq. (A17) for later convenience. We find that, in HEFT, G_{aI}, G_{IJ}, $G_{\hat{i}\hat{j}^*}$, and $M_{\hat{i}\hat{j}}$ are taken as

$$G_{ah} = 0, \tag{A18}$$

$$G_{hh} = 1, \tag{A19}$$

$$G_{\hat{i}\hat{j}^*} = \delta_{\hat{i}\hat{j}^*}, \tag{A20}$$

$$M_{\hat{i}\hat{j}} = \begin{pmatrix} \mathbf{0} & \mathcal{F}_q(h) & \mathbf{0} & \mathbf{0} \\ \mathcal{F}_q(h) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{F}_l(h) \\ \mathbf{0} & \mathbf{0} & \mathcal{F}_l(h) & \mathbf{0} \end{pmatrix}, \tag{A21}$$

where $\mathbf{0}$ denotes 2×2 zero matrix and $\mathcal{F}_{q,l} = \mathcal{F}_{Y_{q,l}}\mathbf{1}_2 + \mathcal{F}_{\hat{Y}_{q,l}}\tau^3$. The other parameters are tuned to be

$$G_{11} = G_{22} = G(h), \qquad G_{33} = G_Z(h), \tag{A22}$$

$$G_{ab} = 0 \quad \text{for } a \neq b, \tag{A23}$$

$$V_{\hat{i}\hat{j}^*a} = -\begin{pmatrix} \tau^a & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \tau^a & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \qquad (a = 1, 2) \tag{A24}$$

$$V_{\hat{i}\hat{j}^*3} = -\begin{pmatrix} \tau^3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \tau^3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} - c\,\mathrm{diag}(q_u, q_d, -q_u, -q_d, 0, q_e, 0, -q_e), \tag{A25}$$

$$V_{\hat{i}\hat{j}^*h} = 0, \tag{A26}$$

where c is the arbitrary parameter which appeared in Eq. (15). Furthermore, the four-fermion operators are assumed to be next-to-leading order in HEFT. Namely, it is assumed that

$$S_{\hat{1}\hat{2}\hat{3}\hat{4}} = 0, \tag{A27}$$

$$S_{\hat{1}^*\hat{2}^*\hat{3}^*\hat{4}^*} = 0, \tag{A28}$$

$$S_{\hat{1}\hat{2}\hat{3}^*\hat{4}^*} = 0 \tag{A29}$$

at the leading order.

APPENDIX B: HELICITY EIGENSTATE

We consider a spin-1/2 field carrying a four-momentum $p^\mu = (E, \vec{p})$, where the direction of $\vec{p}$ is given by $\hat{p} = (\sin\theta_p\cos\phi_p, \sin\theta_p\sin\phi_p, \cos\theta_p)$. The two-component helicity spinor wave functions are given as [52]

$$x_\alpha(\vec{p}, \lambda) = \omega_{-\lambda}(\vec{p})\chi_\lambda(\hat{p}), \tag{B1}$$

$$x^\alpha(\vec{p}, \lambda) = -\lambda\omega_{-\lambda}(\vec{p})\chi_{-\lambda}^\dagger(\hat{p}), \tag{B2}$$

$$y_\alpha(\vec{p}, \lambda) = \lambda\omega_\lambda(\vec{p})\chi_{-\lambda}(\hat{p}), \tag{B3}$$

$$y^\alpha(\vec{p}, \lambda) = \omega_\lambda(\vec{p})\chi_\lambda^\dagger(\hat{p}), \tag{B4}$$

$$x^{\dagger\dot{\alpha}}(\vec{p}, \lambda) = -\lambda\omega_{-\lambda}(\vec{p})\chi_{-\lambda}(\hat{p}), \tag{B5}$$

$$x_{\dot{\alpha}}^\dagger(\vec{p}, \lambda) = \omega_{-\lambda}(\vec{p})\chi_\lambda^\dagger(\hat{p}), \tag{B6}$$

$$y^{\dagger\dot{\alpha}}(\vec{p}, \lambda) = \omega_\lambda(\vec{p})\chi_\lambda(\hat{p}), \tag{B7}$$

$$y_{\dot{\alpha}}^\dagger(\vec{p}, \lambda) = \lambda\omega_\lambda(\vec{p})\chi_{-\lambda}^\dagger(\hat{p}), \tag{B8}$$

where

$$\chi_+(\hat{p}) = \begin{pmatrix} \cos\frac{\theta_p}{2} \\ e^{i\phi_p}\sin\frac{\theta_p}{2} \end{pmatrix}, \tag{B9}$$

$$\chi_-(\hat{p}) = \begin{pmatrix} -e^{-i\phi_p}\sin\frac{\theta_p}{2} \\ \cos\frac{\theta_p}{2} \end{pmatrix}, \tag{B10}$$

and

$$\omega_\pm(\vec{p}) = \sqrt{E \pm |\vec{p}|}, \qquad E = \sqrt{|\vec{p}|^2 + m^2}. \tag{B11}$$

The spinor inner products (213) and (214) are obtained as

$$[\mathbf{1}^+\mathbf{2}^+] = \omega_+(\vec{p}_1)\omega_+(\vec{p}_2) \times \left(e^{-i\phi_1}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - e^{-i\phi_2}\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2}\right), \quad \text{(B12)}$$

$$[\mathbf{1}^-\mathbf{2}^-] = \omega_-(\vec{p}_1)\omega_-(\vec{p}_2) \times \left(e^{i\phi_1}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - e^{i\phi_2}\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2}\right), \quad \text{(B13)}$$

$$[\mathbf{1}^\pm\mathbf{2}^\mp] = \mp\omega_\pm(\vec{p}_1)\omega_\mp(\vec{p}_2) \times \left(\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + e^{\mp i(\phi_1-\phi_2)}\sin\frac{\theta_2}{2}\sin\frac{\theta_1}{2}\right), \quad \text{(B14)}$$

$$\langle\mathbf{1}^+\mathbf{2}^+\rangle = -\omega_-(\vec{p}_1)\omega_-(\vec{p}_2) \times \left(e^{-i\phi_1}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - e^{-i\phi_2}\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2}\right), \quad \text{(B15)}$$

$$\langle\mathbf{1}^-\mathbf{2}^-\rangle = -\omega_+(\vec{p}_1)\omega_+(\vec{p}_2) \times \left(e^{i\phi_1}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - e^{i\phi_2}\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2}\right), \quad \text{(B16)}$$

$$\langle\mathbf{1}^\pm\mathbf{2}^\mp\rangle = \pm\omega_\mp(\vec{p}_1)\omega_\pm(\vec{p}_2) \times \left(\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + e^{\mp i(\phi_1-\phi_2)}\sin\frac{\theta_2}{2}\sin\frac{\theta_1}{2}\right), \quad \text{(B17)}$$

where $\theta_i = \theta_{p_i}$ and $\phi_i = \phi_{p_i}$.

We now confirm that, in the massless limit $m_1 = m_2 = 0$, the spinor inner products become independent of the Lorentz frame:

$$[\mathbf{1}^+\mathbf{2}^+]|_{\text{massless}} = -\sqrt{2p_1 \cdot p_2}, \quad \text{(B18)}$$

$$[\mathbf{1}^-\mathbf{2}^-]|_{\text{massless}} = [\mathbf{1}^\pm\mathbf{2}^\mp]|_{\text{massless}} = 0, \quad \text{(B19)}$$

$$\langle\mathbf{1}^-\mathbf{2}^-\rangle|_{\text{massless}} = +\sqrt{2p_1 \cdot p_2}, \quad \text{(B20)}$$

$$\langle\mathbf{1}^+\mathbf{2}^+\rangle|_{\text{massless}} = \langle\mathbf{1}^\pm\mathbf{2}^\mp\rangle|_{\text{massless}} = 0. \quad \text{(B21)}$$

It should be emphasized that massive spinor products depend on the Lorentz frame. Taking the center of mass (c.m.) frame ($\theta_1 = 0,\ \theta_2 = \pi,\ \phi_1 = \phi_2 = 0$), we obtain

$$[\mathbf{1}^+\mathbf{2}^+]\Big|_{\text{c.m.}} = -\sqrt{(E_1 + |\vec{p}_1|)(E_2 + |\vec{p}_2|)}, \quad \text{(B22)}$$

$$[\mathbf{1}^-\mathbf{2}^-]|_{\text{c.m.}} = \frac{m_1 m_2}{[\mathbf{1}^+\mathbf{2}^+]}\Bigg|_{\text{c.m.}}, \quad \text{(B23)}$$

$$\langle\mathbf{1}^-\mathbf{2}^-\rangle|_{\text{c.m.}} = +\sqrt{(E_1 + |\vec{p}_1|)(E_2 + |\vec{p}_2|)}, \quad \text{(B24)}$$

$$\langle\mathbf{1}^+\mathbf{2}^+\rangle|_{\text{c.m.}} = \frac{m_1 m_2}{\langle\mathbf{1}^-\mathbf{2}^-\rangle}\Bigg|_{\text{c.m.}}, \quad \text{(B25)}$$

and

$$[\mathbf{1}^\pm\mathbf{2}^\mp]|_{\text{c.m.}} = \langle\mathbf{1}^\pm\mathbf{2}^\mp\rangle|_{\text{c.m.}} = 0. \quad \text{(B26)}$$

APPENDIX C: HIGHER-ORDER TERMS IN THE NORMAL COORDINATE

1. Taylor expansion of $g_{ij}(\phi)$

The $G_{ij(123)}$ term in the Taylor expansion (117) can be computed from the first covariant derivative of the Riemann curvature tensor,

$$R_{i'ijk;k_1} = R_{i'ijk,k_1} - R_{k'ijk}\Gamma^{k'}_{i'k_1} - R_{i'k'jk}\Gamma^{k'}_{ik_1} - R_{i'ik'k}\Gamma^{k'}_{jk_1} - R_{i'ijk'}\Gamma^{k'}_{kk_1}. \quad \text{(C1)}$$

Since the affine connection vanishes in the normal coordinate at the vacuum,

$$\Gamma^{i''}_{ik}|_0 = 0, \quad \text{(C2)}$$

we find that

$$R_{i12j;3}|_0 = R_{i12j,3}|_0 = \frac{1}{2}(G_{ij(123)} - G_{1j(i23)} - G_{i2(j13)} + G_{12(ij3)}). \quad \text{(C3)}$$

Symmetrizing under the $1 \leftrightarrow 2$ exchange, we obtain

$$R_{i(12)j;3}|_0 = \frac{1}{4}(2G_{ij(123)} - G_{i1(j23)} - G_{i2(j13)} - G_{1j(i23)} - G_{2j(i13)} + 2G_{12(ij3)}). \quad \text{(C4)}$$

Note that $R_{ij12;3}$ satisfies the second Bianchi identity

$$R_{ij12;3} + R_{ij23;1} + R_{ij31;2} = 0. \quad \text{(C5)}$$

The coefficient $G_{ij(12)3}$ should be expressed in terms of covariant tensors in the normal coordinate. We assume here a form

$$G_{ij(12)3} = aR_{i(12)j;3}|_0 + b[R_{i(13)j;2} + R_{i(23)j;1}]|_0. \quad \text{(C6)}$$

The second Bianchi identity implies

$$R_{i(12)3;j} + R_{3(12)j;i} = 2R_{i(12)j;3} - [R_{i(23)j;1} + R_{i(13)j;2}]. \quad \text{(C7)}$$

Therefore, the fifth-order independent covariant tensors symmetric under the exchanges

$$i \leftrightarrow j, \qquad 1 \leftrightarrow 2 \tag{C8}$$

are exhausted in the assumed form of Eq. (C6). The condition (C4) can be expressed as

$$R_{i(12)j;3}|_0 = \frac{1}{2}[G_{ij(123)} + G_{12(ij3)}] - \frac{1}{4}[G_{i1(j23)} + G_{i2(j13)} + G_{j1(i23)} + G_{j2(i13)}]. \tag{C9}$$

Using

$$G_{ij(123)} = \frac{1}{3}[G_{ij(12)3} + G_{ij(23)1} + G_{ij(31)2}] \tag{C10}$$

and the assumption (C6), the condition (C9) can be rewritten as

$$R_{i(12)j;3} = (a + 2b)R_{i(12)j;3}. \tag{C11}$$

We therefore find that

$$a + 2b = 1, \tag{C12}$$

and thus

$$G_{ij(12)3} = (1 - 2b)R_{i(12)j;3}|_0 + b[R_{i(23)j;1} + R_{i(31)j;2}]|_0 \tag{C13}$$

satisfies Eq. (C4) with an arbitrary b.

We see that the dependence on the parameter b disappears in

$$G_{ij(123)} = \frac{1}{3}[R_{i(12)j;3} + R_{i(23)j;1} + R_{i(31)j;2}]|_0. \tag{C14}$$

The third-order Taylor expansion coefficient in Eq. (99) is therefore uniquely determined in the normal coordinate.

We next check to see whether or not Eq. (C14) satisfies the condition (C3), which is more severe than its symmetrized form [Eq. (C4)]. Plugging the result [Eq. (C14)] into the rhs of Eq. (C3), we see that

$$\begin{aligned}
&\frac{1}{2}(G_{ij(123)} - G_{1j(i23)} - G_{i2(j13)} + G_{12(ij3)}) \\
&\quad = \frac{1}{12}[4R_{i12j;3} + 2R_{i21j;3} - 2R_{ij12;3} \\
&\qquad + [R_{i23j} + 2R_{i32j} + R_{ij23}]_{;1} + [R_{i31j} + 2R_{i13j} + R_{ij31}]_{;2} \\
&\qquad + [R_{j123} + 2R_{j213} + R_{j312}]_{;i} \\
&\qquad + [R_{i213} + 2R_{i123} + R_{i321}]_{;j}]|_0.
\end{aligned} \tag{C15}$$

The Bianchi identity implies

$$R_{i21j} - R_{ij12} = R_{i12j}, \tag{C16}$$

$$R_{i23j} + R_{ij23} = R_{i32j}, \tag{C17}$$

$$R_{i31j} + R_{ij31} = R_{i13j}, \tag{C18}$$

$$R_{j123} + R_{j312} = R_{j213}, \tag{C19}$$

$$R_{i213} + R_{i321} = R_{i123}. \tag{C20}$$

Using these results, the terms on the rhs of Eq. (C15) can be simplified as

$$\begin{aligned}
&\frac{1}{2}(G_{ij(123)} - G_{1j(i23)} - G_{i2(j13)} + G_{12(ij3)}) \\
&\quad = \frac{1}{12}[6R_{i12j;3} + 3(R_{j23i;1} + R_{j213;i}) \\
&\qquad + 3(R_{i13j;2} + R_{i123;j})]\Big|_0.
\end{aligned} \tag{C21}$$

Using the implications of the second Bianchi identity,

$$R_{j23i;1} + R_{j213;i} = R_{j21i;3}, \tag{C22}$$

$$R_{i13j;2} + R_{i123;j} = R_{i12j;3}, \tag{C23}$$

we can simplify Eq. (C21) further as

$$\frac{1}{2}(G_{ij(123)} - G_{1j(i23)} - G_{i2(j13)} + G_{12(ij3)}) = R_{i12j;3}\Big|_0. \tag{C24}$$

We see that the more severe condition (C3) is satisfied by the solution (C14) of the weaker condition (C9).

We next compute the second covariant derivative

$$\begin{aligned}
&R_{i'ijk;k_1k_2}|_0 \\
&\quad = R_{i'ijk,k_1k_2}|_0 - R_{k'ijk}\Gamma^{k'}_{i'k_1,k_2}|_0 - R_{i'k'jk}\Gamma^{k'}_{ik_1,k_2}|_0 \\
&\qquad - R_{i'ik'k}\Gamma^{k'}_{jk_1,k_2}|_0 - R_{i'ijk'}\Gamma^{k'}_{kk_1,k_2}|_0.
\end{aligned} \tag{C25}$$

Using

$$\Gamma^{i''}_{ik,j}|_0 = -\frac{2}{3}g^{i'i''}R_{i'(ik)j}|_0, \tag{C26}$$

we obtain

$$\begin{aligned}
R_{i'ijk;k_1k_2}|_0 = R_{i'ijk,k_1k_2}|_0 &+ \frac{2}{3}R_{k'ijk}g^{k'k''}R_{k''(i'k_1)k_2}\Big|_0 \\
&+ \frac{2}{3}R_{i'k'jk}g^{k'k''}R_{k''(ik_1)k_2}\Big|_0 \\
&+ \frac{2}{3}R_{i'ik'k}g^{k'k''}R_{k''(jk_1)k_2}\Big|_0 \\
&+ \frac{2}{3}R_{i'ijk'}g^{k'k''}R_{k''(kk_1)k_2}\Big|_0.
\end{aligned} \tag{C27}$$

On the other hand, from Eq. (104), we obtain

$$\begin{aligned}
R_{i'ijk,k_1k_2} = & \frac{1}{2}(G_{i'k(ijk_1k_2)} - G_{ik(i'jk_1k_2)} - G_{i'j(ikk_1k_2)} + G_{ij(i'kk_1k_2)}) \\
& - \frac{1}{4}(G_{i'j'(jk_1)} + G_{jj'(i'k_1)} - G_{i'j(j'k_1)})\delta^{j'j''}(G_{j''i(kk_2)} + G_{j''k(ik_2)} - G_{ik(j''k_2)}) \\
& - \frac{1}{4}(G_{i'j'(jk_2)} + G_{jj'(i'k_2)} - G_{i'j(j'k_2)})\delta^{j'j''}(G_{j''i(kk_1)} + G_{j''k(ik_1)} - G_{ik(j''k_1)}) \\
& + \frac{1}{4}(G_{i'j'(kk_1)} + G_{kj'(i'k_1)} - G_{i'k(j'k_1)})\delta^{j'j''}(G_{j''i(jk_2)} + G_{j''j(ik_2)} - G_{ij(j''k_2)}) \\
& + \frac{1}{4}(G_{i'j'(kk_2)} + G_{kj'(i'k_2)} - G_{i'k(j'k_2)})\delta^{j'j''}(G_{j''i(jk_1)} + G_{j''j(ik_1)} - G_{ij(j''k_1)}).
\end{aligned} \tag{C28}$$

We then use Eq. (114)

$$\begin{aligned}
R_{i'ijk,k_1k_2} = & \frac{1}{2}(G_{i'k(ijk_1k_2)} - G_{ik(i'jk_1k_2)} - G_{i'j(ikk_1k_2)} + G_{ij(i'kk_1k_2)}) \\
& - \frac{1}{9}(R_{i'(jk_1)j'} + R_{j(i'k_1)j'} - R_{i'(j'k_1)j})g^{j'j''}(R_{j''(kk_2)i} + R_{j''(ik_2)k} - R_{i(j''k_2)k})\Big|_0 \\
& - \frac{1}{9}(R_{i'(jk_2)j'} + R_{j(i'k_2)j'} - R_{i'(j'k_2)j})g^{j'j''}(R_{j''(kk_1)i} + R_{j''(ik_1)k} - R_{i(j''k_1)k})\Big|_0 \\
& + \frac{1}{9}(R_{i'(kk_1)j'} + R_{k(i'k_1)j'} - R_{i'(j'k_1)k})g^{j'j''}(R_{j''(jk_2)i} + R_{j''(ik_2)j} - R_{i(j''k_2)j})\Big|_0 \\
& + \frac{1}{9}(R_{i'(kk_2)j'} + R_{k(i'k_2)j'} - R_{i'(j'k_2)k})g^{j'j''}(R_{j''(jk_1)i} + R_{j''(ik_1)j} - R_{i(j''k_1)j})\Big|_0.
\end{aligned} \tag{C29}$$

Combining Eqs. (C27) and (C29), we now find that

$$\begin{aligned}
& \frac{1}{2}(G_{i'k(ijk_1k_2)} - G_{ik(i'jk_1k_2)} - G_{i'j(ikk_1k_2)} + G_{ij(i'kk_1k_2)}) \\
& = R_{i'ijk;k_1k_2}|_0 - \frac{2}{3}R_{k'ijk}g^{k'k''}R_{k''(i'k_1)k_2}|_0 - \frac{2}{3}R_{i'k'jk}g^{k'k''}R_{k''(ik_1)k_2}\Big|_0 \\
& - \frac{2}{3}R_{i'ik'k}g^{k'k''}R_{k''(jk_1)k_2}|_0 - \frac{2}{3}R_{i'ijk'}g^{k'k''}R_{k''(kk_1)k_2}\Big|_0 \\
& + \frac{1}{9}(R_{i'(jk_1)j'} + R_{j(i'k_1)j'} - R_{i'(j'k_1)j})g^{j'j''}(R_{j''(kk_2)i} + R_{j''(ik_2)k} - R_{i(j''k_2)k})\Big|_0 \\
& + \frac{1}{9}(R_{i'(jk_2)j'} + R_{j(i'k_2)j'} - R_{i'(j'k_2)j})g^{j'j''}(R_{j''(kk_1)i} + R_{j''(ik_1)k} - R_{i(j''k_1)k})\Big|_0 \\
& - \frac{1}{9}(R_{i'(kk_1)j'} + R_{k(i'k_1)j'} - R_{i'(j'k_1)k})g^{j'j''}(R_{j''(jk_2)i} + R_{j''(ik_2)j} - R_{i(j''k_2)j})\Big|_0 \\
& - \frac{1}{9}(R_{i'(kk_2)j'} + R_{k(i'k_2)j'} - R_{i'(j'k_2)k})g^{j'j''}(R_{j''(jk_1)i} + R_{j''(ik_1)j} - R_{i(j''k_1)j})\Big|_0.
\end{aligned} \tag{C30}$$

When we replace the indices

$$i' \to i, \qquad k \to j, \qquad i \to 1, \qquad j \to 2, \qquad k_1 \to 3, \qquad k_2 \to 4, \tag{C31}$$

the structure of the indices can be made more manifest:

$$\begin{aligned}\frac{1}{2}(G_{ij(1234)}-G_{1j(i234)}-G_{i2(j134)}+G_{12(ij34)}) = R_{i12j;34}|_0 &- \frac{2}{3}g^{i'j'}[R_{i'12j}R_{j'(i3)4}+R_{i'ij2}R_{j'(13)4}+R_{i'ji1}R_{j'(23)4}+R_{i'21i}R_{j'(j3)4}]\Big|_0 \\ &+\frac{1}{9}(R_{i(23)i'}+R_{2(i3)i'}-R_{i(i'3)2})g^{i'j'}(R_{j'(j4)1}+R_{j'(14)j}-R_{1(j'4)j})\Big|_0 \\ &+\frac{1}{9}(R_{i(24)i'}+R_{2(i4)i'}-R_{i(i'4)2})g^{i'j'}(R_{j'(j3)1}+R_{j'(13)j}-R_{1(j'3)j})\Big|_0 \\ &-\frac{1}{9}(R_{i(j3)i'}+R_{j(i3)i'}-R_{i(i'3)j})g^{i'j'}(R_{j'(24)1}+R_{j'(14)2}-R_{1(j'4)2})\Big|_0 \\ &-\frac{1}{9}(R_{i(j4)i'}+R_{j(i4)i'}-R_{i(i'4)j})g^{i'j'}(R_{j'(23)1}+R_{j'(13)2}-R_{1(j'3)2})|_0. \end{aligned} \tag{C32}$$

It is convenient to rewrite formula (C32) as

$$\begin{aligned}\frac{1}{2}(G_{ij(1234)}-G_{1j(i234)}-G_{i2(j134)}+G_{12(ij34)}) = R_{i12j;34}|_0 &- \frac{2}{3}g^{i'j'}[R_{i'12j}R_{j'(i3)4}+R_{i'ij2}R_{j'(13)4}+R_{i'ji1}R_{j'(23)4}+R_{i'21i}R_{j'(j3)4}]\Big|_0 \\ &+\frac{1}{9}g^{i'j'}(R_{i'(23)i}+R_{i'(i3)2}-R_{i'(i2)3})(R_{j'(j4)1}+R_{j'(14)j}-R_{j'(j1)4})\Big|_0 \\ &+\frac{1}{9}g^{i'j'}(R_{i'(24)i}+R_{i'(i4)2}-R_{i'(i2)4})(R_{j'(j3)1}+R_{j'(13)j}-R_{j'(j1)3})\Big|_0 \\ &-\frac{1}{9}g^{i'j'}(R_{i'(j3)i}+R_{i'(i3)j}-R_{i'(ij)3})(R_{j'(24)1}+R_{j'(14)2}-R_{j'(12)4})\Big|_0 \\ &-\frac{1}{9}g^{i'j'}(R_{i'(j4)i}+R_{i'(i4)j}-R_{i'(ij)4})(R_{j'(23)1}+R_{j'(13)2}-R_{j'(12)3})\Big|_0. \end{aligned} \tag{C33}$$

We are now ready to solve Eq. (C33). We first symmetrize the indices (1234) by replacing

$$G_{1j(i234)} \to \frac{1}{4}[G_{1j(i234)}+G_{2j(i134)}+G_{3j(i124)}+G_{4j(i123)}], \tag{C34}$$

$$G_{12(ij34)} \to \frac{1}{6}[G_{12(ij34)}+G_{34(ij12)}+G_{13(ij24)}+G_{24(ij13)}+G_{14(ij23)}+G_{23(ij14)}], \tag{C35}$$

$$R_{i12j;34} \to \frac{1}{6}[R_{i(12)j;(34)}+R_{i(34)j;(12)}+R_{i(13)j;(24)}+R_{i(24)j;(13)}+R_{i(14)j;(23)}+R_{i(23)j;(14)}], \tag{C36}$$

$$R_{i'12j} \to R_{i'(12)j}, \tag{C37}$$

$$R_{j'(i3)4} \to -\frac{1}{2}R_{j'(34)i}, \tag{C38}$$

$$\begin{aligned} g^{i'j'}R_{i'(12)j}R_{j'(34)i} \to \frac{1}{6}g^{i'j'}[&R_{i'(12)j}R_{j'(34)i}+R_{i'(34)j}R_{j'(12)i}+R_{i'(13)j}R_{j'(24)i}+R_{i'(24)j}R_{j'(13)i}+R_{i'(14)j}R_{j'(23)i} \\ &+R_{i'(23)j}R_{j'(14)i}], \end{aligned} \tag{C39}$$

$$R_{j'(13)4} \to 0, \tag{C40}$$

$$R_{i'(i3)2}-R_{i'(i2)3} \to 0. \tag{C41}$$

We then obtain a symmetrized form of Eq. (C33):

$$
\begin{aligned}
&\frac{1}{2}G_{ij(1234)}-\frac{1}{8}[G_{1j(i234)}+G_{2j(i134)}+G_{3j(i124)}+G_{4j(i123)}]-\frac{1}{8}[G_{1i(j234)}+G_{2i(j134)}+G_{3i(j124)}+G_{4i(j123)}]\\
&\quad+\frac{1}{12}[G_{12(ij34)}+G_{34(ij12)}+G_{13(ij24)}+G_{24(ij13)}+G_{14(ij23)}+G_{23(ij14)}]\\
&=\frac{1}{6}[R_{i(12)j;(34)}+R_{i(34)j;(12)}+R_{i(13)j;(24)}+R_{i(24)j;(13)}+R_{i(14)j;(23)}+R_{i(23)j;(14)}]\Big|_0\\
&\quad+\frac{4}{27}g^{i'j'}[R_{i'(12)i}R_{j'(34)j}+R_{i'(13)i}R_{j'(24)j}+R_{i'(14)i}R_{j'(23)j}+R_{i'(34)i}R_{j'(12)j}+R_{i'(24)i}R_{j'(13)j}+R_{i'(23)i}R_{j'(14)j}]\Big|_0. \qquad \text{(C42)}
\end{aligned}
$$

The factor $4/27$ in the expression above is obtained as

$$\left(-\frac{2}{3}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{6}\right)\times 2+\left(\frac{1}{9}\right)\left(\frac{1}{6}\right)\times 2=\frac{4}{27}.$$

Here the factors $-2/3$ and $1/9$ are from Eq. (C33), the factor $-1/2$ is from the symmetrization (C38), and the factor $1/6$ is from Eq. (C39). We assume here the form

$$
\begin{aligned}
G_{ij(1234)}=&\frac{a}{6}[R_{i(12)j;(34)}+R_{i(34)j;(12)}+R_{i(13)j;(24)}+R_{i(24)j;(13)}+R_{i(14)j;(23)}+R_{i(23)j;(14)}]\Big|_0\\
&+\frac{b}{6}g^{i'j'}[R_{i'(12)i}R_{j'(34)j}+R_{i'(13)i}R_{j'(24)j}+R_{i'(14)i}R_{j'(23)j}+R_{i'(34)i}R_{j'(12)j}+R_{i'(24)i}R_{j'(13)j}+R_{i'(23)i}R_{j'(14)j}]\Big|_0. \qquad \text{(C43)}
\end{aligned}
$$

Using the Bianchi identity and the symmetries of the curvature tensor, it can be shown that

$$G_{1j(i234)}+G_{2j(i134)}+G_{3j(i124)}+G_{4j(i123)}=-G_{ij(1234)}, \qquad \text{(C44)}$$

$$G_{1i(j234)}+G_{2i(j134)}+G_{3i(j124)}+G_{4i(j123)}=-G_{ij(1234)}, \qquad \text{(C45)}$$

$$G_{12(ij34)}+G_{34(ij12)}+G_{13(ij24)}+G_{24(ij13)}+G_{14(ij23)}+G_{23(ij14)}=G_{ij(1234)} \qquad \text{(C46)}$$

under the assumption (C43). Equation (C42) then reads

$$
\begin{aligned}
\left(\frac{1}{2}+\frac{1}{8}+\frac{1}{8}+\frac{1}{12}\right)G_{ij(1234)}=&\frac{1}{6}[R_{i(12)j;(34)}+R_{i(34)j;(12)}+R_{i(13)j;(24)}+R_{i(24)j;(13)}+R_{i(14)j;(23)}+R_{i(23)j;(14)}]\Big|_0\\
&+\frac{4}{27}g^{i'j'}[R_{i'(12)i}R_{j'(34)j}+R_{i'(13)i}R_{j'(24)j}+R_{i'(14)i}R_{j'(23)j}+R_{i'(34)i}R_{j'(12)j}\\
&+R_{i'(24)i}R_{j'(13)j}+R_{i'(23)i}R_{j'(14)j}]\Big|_0 \qquad \text{(C47)}
\end{aligned}
$$

and

$$a=\frac{6}{5}, \qquad b=\frac{16}{15}. \qquad \text{(C48)}$$

We finally obtain

$$
\begin{aligned}
G_{ij(1234)}=&\frac{1}{5}[R_{i(12)j;(34)}+R_{i(34)j;(12)}+R_{i(13)j;(24)}+R_{i(24)j;(13)}+R_{i(14)j;(23)}+R_{i(23)j;(14)}]\Big|_0\\
&+\frac{8}{45}g^{i'j'}[R_{i'(12)i}R_{j'(34)j}+R_{i'(13)i}R_{j'(24)j}+R_{i'(14)i}R_{j'(23)j}+R_{i'(34)i}R_{j'(12)j}+R_{i'(24)i}R_{j'(13)j}+R_{i'(23)i}R_{j'(14)j}]\Big|_0. \qquad \text{(C49)}
\end{aligned}
$$

The result is consistent with the Riemann normal coordinate coefficient

$$G_{ij(k_1k_2)(k_3k_4)} = \frac{6}{5} R_{i(k_1k_2)j;(k_3k_4)}\Big|_0 + \frac{16}{15} R_{i(k_1k_2)l} R^l{}_{(k_3k_4)j}\Big|_0. \tag{C50}$$

2. Taylor expansion of $v_{\hat{i}\hat{j}^* i}(\phi)$

The Taylor expansion coefficient $A_{\hat{i}\hat{j}^* i12}$ in Eq. (154) can be computed from the first covariant derivative of the half-fermionic Riemann tensor:

$$R_{\hat{i}\hat{j}^* ij;k_1} = R_{\hat{i}\hat{j}^* ij,k_1} - R_{\hat{i}'\hat{j}^* ij}\Gamma^{\hat{i}'}_{k_1\hat{i}} - R_{\hat{i}\hat{j}'^* ij}\Gamma^{\hat{j}'^*}_{k_1\hat{j}^*} - R_{\hat{i}\hat{j}^* i'j}\Gamma^{i'}_{k_1 i} - R_{\hat{i}\hat{j}^* ij'}\Gamma^{j'}_{k_1 j}. \tag{C51}$$

Since the affine connections vanish at the vacuum in the normal coordinate, we see that

$$R_{\hat{i}\hat{j}^* ij;k_1}|_0 = R_{\hat{i}\hat{j}^* ij,k_1}|_0. \tag{C52}$$

We then use Eq. (147) to obtain

$$\begin{aligned} R_{\hat{i}\hat{j}^* ij,k_1} = {} & iA_{\hat{i}\hat{j}^* ji,k_1} + iA_{\hat{i}\hat{j}^* jk_1,i} - iA_{\hat{i}\hat{j}^* ij,k_1} - iA_{\hat{i}\hat{j}^* ik_1,j} + i(A_{\hat{i}\hat{j}^* jk_2,ik_1} - A_{\hat{i}\hat{j}^* ik_2,jk_1})\phi^{k_2} \\ & + g^{\hat{i}'\hat{j}'^*}(A_{\hat{i}'\hat{j}^* ik_1}A_{\hat{i}\hat{j}'^* jk_2} + A_{\hat{i}'\hat{j}^* ik_2}A_{\hat{i}\hat{j}'^* jk_1})\phi^{k_2} - g^{\hat{i}'\hat{j}'^*}(A_{\hat{i}'\hat{j}^* jk_1}A_{\hat{i}\hat{j}'^* ik_2} + A_{\hat{i}'\hat{j}^* jk_2}A_{\hat{i}\hat{j}'^* ik_1})\phi^{k_2} \\ & + [g^{\hat{i}'\hat{j}'^*}(A_{\hat{i}'\hat{j}^* ik_3}A_{\hat{i}\hat{j}'^* jk_2} + A_{\hat{i}'\hat{j}^* ik_2}A_{\hat{i}\hat{j}'^* jk_3})]_{,k_1}\phi^{k_2}\phi^{k_3} - [g^{\hat{i}'\hat{j}'^*}(A_{\hat{i}'\hat{j}^* jk_3}A_{\hat{i}\hat{j}'^* ik_2} + A_{\hat{i}'\hat{j}^* jk_2}A_{\hat{i}\hat{j}'^* ik_3})]_{,k_1}\phi^{k_2}\phi^{k_3}. \end{aligned} \tag{C53}$$

Now we are ready to find a relation between $R_{\hat{i}\hat{j}^* ij,k_1}$ and $A_{\hat{i}\hat{j}^* ij,k_1}$ at the vacuum,

$$R_{\hat{i}\hat{j}^* ij,k_1}|_0 = i[A_{\hat{i}\hat{j}^* ji,k_1} + A_{\hat{i}\hat{j}^* jk_1,i} - A_{\hat{i}\hat{j}^* ij,k_1} - A_{\hat{i}\hat{j}^* ik_1,j}]|_0. \tag{C54}$$

Combining the result with Eq. (C52), we obtain

$$R_{\hat{i}\hat{j}^* ij;k_1}|_0 = i[A_{\hat{i}\hat{j}^* jk_1,i} + A_{\hat{i}\hat{j}^* ji,k_1} - A_{\hat{i}\hat{j}^* ij,k_1} + A_{\hat{i}\hat{j}^* k_1 i,j}]|_0. \tag{C55}$$

Since the function $A_{\hat{i}\hat{j}^* ij}(\phi)$ is antisymmetric under the exchange of $i \leftrightarrow j$, Eq. (C55) can be expressed as

$$R_{\hat{i}\hat{j}^* ij;k_1}|_0 = i[A_{\hat{i}\hat{j}^* jk_1,i} + A_{\hat{i}\hat{j}^* k_1 i,j} - 2A_{\hat{i}\hat{j}^* ij,k_1}]|_0. \tag{C56}$$

We consider a Taylor expansion,

$$\begin{aligned} v_{\hat{i}\hat{j}^* i}(\phi) &= A_{\hat{i}\hat{j}^* ij}(\phi)\phi^j \\ &= A_{\hat{i}\hat{j}^* ik_1}|_0\phi^{k_1} + \frac{1}{2}A_{\hat{i}\hat{j}^* i(k_1,k_2)}|_0\phi^{k_1}\phi^{k_2} + \frac{1}{3!}A_{\hat{i}\hat{j}^* i(k_1,k_2k_3)}|_0\phi^{k_1}\phi^{k_2}\phi^{k_3} + \cdots. \end{aligned} \tag{C57}$$

The expansion coefficients $A_{\hat{i}\hat{j}^* ik_1}|_0, A_{\hat{i}\hat{j}^* i(k_1,k_2)}|_0, A_{\hat{i}\hat{j}^* i(k_1,k_2k_3)}|_0, \cdots$ should be written in terms of the covariant tensors in the normal coordinate. Noting the antisymmetry under the $i \leftrightarrow j$ exchange, we assume the form

$$A_{\hat{i}\hat{j}^* ij,k_1}|_0 = aR_{\hat{i}\hat{j}^* ij;1}|_0 + b[R_{\hat{i}\hat{j}^* jk_1;i} + R_{\hat{i}\hat{j}^* k_1 i;j}]. \tag{C58}$$

Plugging the assumed form of Eq. (C58) into the rhs of Eq. (C56), we obtain

$$\text{rhs} = i(a-b)[R_{\hat{i}\hat{j}^* j1;i} + R_{\hat{i}\hat{j}^* 1i;j} - 2R_{\hat{i}\hat{j}^* ij,1}]|_0. \tag{C59}$$

We note that the $R_{\hat{i}\hat{j}^* ij;k}$ satisfies the Bianchi identity

$$R_{\hat{i}\hat{j}^* j1;i} + R_{\hat{i}\hat{j}^* 1i;j} = -R_{\hat{i}\hat{j}^* ij;1}, \tag{C60}$$

as we will discuss in Appendix C 3. Using the Bianchi identity, the expression can be simplified further as

$$\mathrm{rhs} = -3i(a-b)R_{\hat{i}\hat{j}^* ij;1}|_0. \tag{C61}$$

Comparing the result with Eq. (C56), we now obtain

$$a - b = \frac{i}{3}, \tag{C62}$$

and thus

$$A_{\hat{i}\hat{j}^* ij,1}|_0 = \left(\frac{i}{3} + b\right) R_{\hat{i}\hat{j}^* ij;1}\Big|_0 + b[R_{\hat{i}\hat{j}^* j1;i} + R_{\hat{i}\hat{j}^* 1i;j}]|_0. \tag{C63}$$

Note that the b dependence is canceled in the Taylor expansion coefficient $A_{\hat{i}\hat{j}^* i(1,2)}|_0$,

$$A_{\hat{i}\hat{j}^* i(1,2)}\Big|_0 = \left(\frac{i}{3} + b\right) R_{\hat{i}\hat{j}^* i(1;2)}\Big|_0 + b[R_{\hat{i}\hat{j}^*(12);i} - R_{\hat{i}\hat{j}^* i(1;2)}] = \frac{i}{3} R_{\hat{i}\hat{j}^* i(1;2)}|_0. \tag{C64}$$

In the last line of Eq. (C64), we used the Riemann tensor symmetry

$$R_{\hat{i}\hat{j}^* 12} + R_{\hat{i}\hat{j}^* 21} = 0. \tag{C65}$$

We next consider the second covariant derivative of the half-fermionic Riemann curvature tensor,

$$R_{\hat{i}\hat{j}^* ij;k_1k_2} = (R_{\hat{i}\hat{j}^* ij;k_1})_{,k_2} - R_{\hat{i}'\hat{j}^* ij;k_1}\Gamma^{\hat{i}'}_{k_2\hat{i}} - R_{\hat{i}\hat{j}'^* ij;k_1}\Gamma^{\hat{j}'^*}_{k_2\hat{j}^*} - R_{\hat{i}\hat{j}^* i'j;k_1}\Gamma^{i'}_{k_2 i} - R_{\hat{i}\hat{j}^* ij';k_1}\Gamma^{j'}_{k_2 j} - R_{\hat{i}\hat{j}^* i'j;k_1'}\Gamma^{k_1'}_{k_2k_1'}, \tag{C66}$$

which can be computed at the vacuum

$$\begin{aligned} R_{\hat{i}\hat{j}^* ij;k_1k_2}|_0 = {} & R_{\hat{i}\hat{j}^* ij,k_1k_2}|_0 - \frac{1}{2} g^{\hat{i}'\hat{j}'^*} R_{\hat{i}'\hat{j}^* ij} R_{\hat{i}\hat{j}'^* k_1k_2}|_0 + \frac{1}{2} g^{\hat{i}'\hat{j}'^*} R_{\hat{i}\hat{j}'^* ij} R_{\hat{i}'\hat{j}^* k_1k_2}|_0 + \frac{1}{3} g^{i'j'} R_{\hat{i}\hat{j}^* i'j}(R_{j'k_1ik_2} + R_{j'ik_1k_2}) \\ & + \frac{1}{3} g^{i'j'} R_{\hat{i}\hat{j}^* ii'}(R_{j'k_1jk_2} + R_{j'jk_1k_2}). \end{aligned} \tag{C67}$$

Here we used Eqs. (C2), (C26), (151), and (152). We then obtain

$$R_{\hat{i}\hat{j}^* ij;(k_1k_2)}|_0 = R_{\hat{i}\hat{j}^* ij,(k_1k_2)}|_0 - \frac{1}{3} g^{i'j'} R_{\hat{i}\hat{j}^* i'j} R_{j'(k_1k_2)i}|_0 - \frac{1}{3} g^{i'j'} R_{\hat{i}\hat{j}^* ii'} R_{j'(k_1k_2)j}|_0. \tag{C68}$$

We next use Eq. (147) to find

$$\begin{aligned} R_{\hat{i}\hat{j}^* ij,k_1k_2}|_0 = {} & i[A_{\hat{i}\hat{j}^* ji,k_1k_2} + 2A_{\hat{i}\hat{j}^* j(k_1,k_2)i} - A_{\hat{i}\hat{j}^* ij,k_1k_2} - 2A_{\hat{i}\hat{j}^* i(k_1,k_2)j}]|_0 \\ & - \frac{1}{4} g^{\hat{i}'\hat{j}'^*}[R_{\hat{i}'\hat{j}^* ik_1} R_{\hat{i}\hat{j}'^* jk_2} + R_{\hat{i}'\hat{j}^* ik_2} R_{\hat{i}\hat{j}'^* jk_1}]\Big|_0 + \frac{1}{4} g^{\hat{i}'\hat{j}'^*}[R_{\hat{i}'\hat{j}^* jk_1} R_{\hat{i}\hat{j}'^* ik_2} + R_{\hat{i}'\hat{j}^* jk_2} R_{\hat{i}\hat{j}'^* ik_1}]\Big|_0. \end{aligned} \tag{C69}$$

Combining Eqs. (C68) and (C69), we now find a formula which relates the function $A_{\hat{i}\hat{j}^* ij}$ with the Riemann curvature tensor,

$$\begin{aligned} iA|_0 = {} & -R_{\hat{i}\hat{j}^* ij;(k_1k_2)}|_0 - \frac{1}{3} g^{i'j'} R_{\hat{i}\hat{j}^* i'j} R_{j'(k_1k_2)i}\Big|_0 + \frac{1}{3} g^{i'j'} R_{\hat{i}\hat{j}^* i'i} R_{j'(k_1k_2)j}\Big|_0 - \frac{1}{4} g^{\hat{i}'\hat{j}'^*}(R_{\hat{i}'\hat{j}^* ik_1} R_{\hat{i}\hat{j}'^* jk_2} + R_{\hat{i}'\hat{j}^* ik_2} R_{\hat{i}\hat{j}'^* jk_1})\Big|_0 \\ & + \frac{1}{4} g^{\hat{i}'\hat{j}'^*}(R_{\hat{i}'\hat{j}^* jk_1} R_{\hat{i}\hat{j}'^* ik_2} + R_{\hat{i}'\hat{j}^* jk_2} R_{\hat{i}\hat{j}'^* ik_1})\Big|_0. \end{aligned} \tag{C70}$$

Here we introduced a shorthand notation

$$A \coloneqq A_{\hat{i}\hat{j}^* ij,(k_1k_2)} - A_{\hat{i}\hat{j}^* ji,(k_1k_2)} + 2A_{\hat{i}\hat{j}^* i(k_1,k_2)j} - 2A_{\hat{i}\hat{j}^* j(k_1,k_2)i}. \tag{C71}$$

The function $A_{\hat{i}\hat{j}^* ij}(\phi)$ should be expressed in a covariant form in the normal coordinate. We therefore assume that

$$\begin{aligned} A_{\hat{i}\hat{j}^* ij,k_1k_2}|_0 &= aR_{\hat{i}\hat{j}^* ij;(k_1k_2)}|_0 + bg^{i'j'}[R_{\hat{i}\hat{j}^* i'i}R_{j'(k_1k_2)j} - R_{\hat{i}\hat{j}^* i'j}R_{j'(k_1k_2)i}]|_0 + cg^{i'j'}[R_{\hat{i}\hat{j}^* i'k_1}R_{j'k_2ij} + R_{\hat{i}\hat{j}^* i'k_2}R_{j'k_1ij}]|_0 \\ &\quad + dg^{\hat{i}'\hat{j}'^*}(R_{\hat{i}'\hat{j}^* ik_1}R_{\hat{i}\hat{j}'^* jk_2} + R_{\hat{i}'\hat{j}^* ik_2}R_{\hat{i}\hat{j}'^* jk_1})|_0 - dg^{\hat{i}'\hat{j}'^*}(R_{\hat{i}'\hat{j}^* jk_1}R_{\hat{i}\hat{j}'^* ik_2} + R_{\hat{i}'\hat{j}^* jk_2}R_{\hat{i}\hat{j}'^* ik_1})|_0. \end{aligned} \tag{C72}$$

Here a, b, c, d are constants. Plugging the form of assumption (C72) into the definition of the shorthand notation (C71) and using the symmetry structure of $R_{\hat{i}\hat{j}^* ij}$ and

$$R_{\hat{i}\hat{j}^* ij;k_1k_2} - R_{\hat{i}\hat{j}^* ij;k_2k_1} = R_{\hat{i}'\hat{j}^* ij}R^{\hat{i}'}{}_{\hat{i}k_1k_2} + R_{\hat{i}\hat{j}'^* ij}R^{\hat{j}'^*}{}_{\hat{j}^* k_1k_2} + R_{\hat{i}\hat{j}^* i'j}R^{i'}{}_{ik_1k_2} + R_{\hat{i}\hat{j}^* ij'}R^{j'}{}_{jk_1k_2}, \tag{C73}$$

we find that

$$\begin{aligned} A|_0 &= 4aR_{\hat{i}\hat{j}^* ij;(k_1k_2)}|_0 + (-a+b+2c)g^{i'j'}[R_{\hat{i}\hat{j}^* i'i}R_{j'(k_1k_2)j} - R_{\hat{i}\hat{j}^* i'j}R_{j'(k_1k_2)i}]|_0 \\ &\quad + \left(\frac{a}{2} + \frac{3}{2}b + 3c\right)g^{i'j'}[R_{\hat{i}\hat{j}^* i'k_1}R_{j'k_2ij} + R_{\hat{i}\hat{j}^* i'k_2}R_{j'k_1ij}]\Big|_0 \\ &\quad + ag^{\hat{i}'\hat{j}'^*}(R_{\hat{i}'\hat{j}^* ik_1}R_{\hat{i}\hat{j}'^* jk_2} + R_{\hat{i}'\hat{j}^* ik_2}R_{\hat{i}\hat{j}'^* jk_1})|_0 - ag^{\hat{i}'\hat{j}'^*}(R_{\hat{i}'\hat{j}^* jk_1}R_{\hat{i}\hat{j}'^* ik_2} + R_{\hat{i}'\hat{j}^* jk_2}R_{\hat{i}\hat{j}'^* ik_1})|_0. \end{aligned} \tag{C74}$$

Comparing Eqs. (C74) and (C70), we obtain

$$4ia = -1, \tag{C75}$$

$$i(-a+b+2c) = \frac{1}{3}, \tag{C76}$$

$$i\left(\frac{a}{2} + \frac{3}{2}b + 3c\right) = 0, \tag{C77}$$

$$ia = -\frac{1}{4}, \tag{C78}$$

which can be solved as

$$a = \frac{i}{4}, \tag{C79}$$

$$b = -\frac{i}{12} - 2c. \tag{C80}$$

Note that the coefficients c and d are left as undetermined constants. We thus obtain an expression of $A_{\hat{i}\hat{j}^* ij,k_1k_2}$:

$$\begin{aligned} A_{\hat{i}\hat{j}^* ij,k_1k_2}|_0 &= \frac{i}{4}R_{\hat{i}\hat{j}^* ij;(k_1k_2)}|_0 - \left(\frac{i}{12} + 2c\right)g^{i'j'}[R_{\hat{i}\hat{j}^* i'i}R_{j'(k_1k_2)j} - R_{\hat{i}\hat{j}^* i'j}R_{j'(k_1k_2)i}]\Big|_0 \\ &\quad + cg^{i'i''}[R_{\hat{i}\hat{j}^* i'k_1}R_{i''k_2ij} + R_{\hat{i}\hat{j}^* i'k_2}R_{i''k_1ij}]|_0 + dg^{\hat{i}'\hat{j}'^*}(R_{\hat{i}'\hat{j}^* ik_1}R_{\hat{i}\hat{j}'^* jk_2} + R_{\hat{i}'\hat{j}^* ik_2}R_{\hat{i}\hat{j}'^* jk_1})|_0 \\ &\quad - dg^{\hat{i}'\hat{j}'^*}(R_{\hat{i}'\hat{j}^* jk_1}R_{\hat{i}\hat{j}'^* ik_2} + R_{\hat{i}'\hat{j}^* jk_2}R_{\hat{i}\hat{j}'^* ik_1})|_0. \end{aligned} \tag{C81}$$

The dependence on parameters c and d is canceled in

$$\begin{aligned} A_{\hat{i}\hat{j}^* i(1,23)}|_0 &= \frac{i}{4}R_{\hat{i}\hat{j}^* i(1;23)}|_0 - \left(\frac{i}{12}+2c\right)g^{i'j'}\left[R_{\hat{i}\hat{j}^* i' i}R_{j'(123)} - \frac{1}{3}R_{\hat{i}\hat{j}^* i'1}R_{j'(23)i} - \frac{1}{3}R_{\hat{i}\hat{j}^* i'2}R_{j'(31)i} - \frac{1}{3}R_{\hat{i}\hat{j}^* i'3}R_{j'(12)i}\right]\Big|_0 \\ &\quad - \frac{2}{3}cg^{i'j'}[R_{\hat{i}\hat{j}^* i'1}R_{j'(23)i} + R_{\hat{i}\hat{j}^* i'2}R_{j'(31)i} + R_{\hat{i}\hat{j}^* i'3}R_{j'(12)i}]\Big|_0 \\ &= \frac{i}{4}R_{\hat{i}\hat{j}^* i(1;23)}\Big|_0 + \frac{i}{36}g^{i'j'}[R_{\hat{i}\hat{j}^* i'1}R_{j'(23)i} + R_{\hat{i}\hat{j}^* i'2}R_{j'(31)i} + R_{\hat{i}\hat{j}^* i'3}R_{j'(12)i}]\Big|_0. \end{aligned} \tag{C82}$$

The third-order Taylor expansion coefficient in

$$\begin{aligned} v_{\hat{i}\hat{j}^* i}(\phi) &= A_{\hat{i}\hat{j}^* ij}(\phi)\phi^j \\ &= A_{\hat{i}\hat{j}^* ik_1}|_0\phi^{k_1} + \frac{1}{2}A_{\hat{i}\hat{j}^* i(k_1,k_2)}\Big|_0\phi^{k_1}\phi^{k_2} + \frac{1}{3!}A_{\hat{i}\hat{j}^* i(k_1,k_2k_3)}\Big|_0\phi^{k_1}\phi^{k_2}\phi^{k_3} + \cdots \end{aligned} \tag{C83}$$

is therefore uniquely determined in the normal coordinate.

3. A proof on the "half-fermionic Bianchi identity"

The half-fermionic curvature tensor defined in Eq. (138) satisfies a Bianchi type identity

$$R^{\hat{i}}{}_{\hat{j}12;3} + R^{\hat{i}}{}_{\hat{j}23;1} + R^{\hat{i}}{}_{\hat{j}31;2} = 0. \tag{C84}$$

We give a proof of Eq. (C84) in this Appendix.

We first compute $R^{\hat{i}}{}_{\hat{j}12;3}$:

$$\begin{aligned} R^{\hat{i}}{}_{\hat{j}12;3} &= (\Gamma^{\hat{i}}_{2\hat{j},31} - \Gamma^{\hat{i}}_{1\hat{j},23}) + (\Gamma^{\hat{i}}_{1\hat{j}'}\Gamma^{\hat{j}'}_{2\hat{j},3} - \Gamma^{\hat{i}}_{3\hat{j}'}\Gamma^{\hat{j}'}_{1\hat{j},2}) + (\Gamma^{\hat{i}}_{3\hat{j}'}\Gamma^{\hat{j}'}_{2\hat{j},1} - \Gamma^{\hat{i}}_{2\hat{j}'}\Gamma^{\hat{j}'}_{1\hat{j},3}) + (\Gamma^{\hat{i}}_{1\hat{j}',2}\Gamma^{\hat{j}'}_{3\hat{j}} - \Gamma^{\hat{i}}_{2\hat{j}',3}\Gamma^{\hat{j}'}_{1\hat{j}}) + (\Gamma^{\hat{i}}_{1\hat{j}',3}\Gamma^{\hat{j}'}_{2\hat{j}} - \Gamma^{\hat{i}}_{2\hat{j}',1}\Gamma^{\hat{j}'}_{3\hat{j}}) \\ &\quad + (\Gamma^{\hat{i}}_{1\hat{j},i}\Gamma^{i}_{23} - \Gamma^{\hat{i}}_{2\hat{j},i}\Gamma^{i}_{31}) + (\Gamma^{\hat{i}}_{i\hat{j},2}\Gamma^{i}_{31} - \Gamma^{\hat{i}}_{i\hat{j},1}\Gamma^{i}_{23}) + (\Gamma^{\hat{i}}_{3\hat{i}'}\Gamma^{\hat{i}'}_{1\hat{j}'}\Gamma^{\hat{j}'}_{2\hat{j}} - \Gamma^{\hat{i}}_{1\hat{i}'}\Gamma^{\hat{i}'}_{2\hat{j}'}\Gamma^{\hat{j}'}_{3\hat{j}}) + (\Gamma^{\hat{i}}_{2\hat{i}'}\Gamma^{\hat{i}'}_{1\hat{j}'}\Gamma^{\hat{j}'}_{3\hat{j}} - \Gamma^{\hat{i}}_{3\hat{i}'}\Gamma^{\hat{i}'}_{2\hat{j}'}\Gamma^{\hat{j}'}_{1\hat{j}}) \\ &\quad + \Gamma^{\hat{i}}_{i\hat{i}'}(\Gamma^{i}_{23}\Gamma^{\hat{i}'}_{1\hat{j}} - \Gamma^{i}_{31}\Gamma^{\hat{i}'}_{2\hat{j}}) + \Gamma^{\hat{i}'}_{i\hat{j}}(\Gamma^{\hat{i}}_{2\hat{i}'}\Gamma^{i}_{31} - \Gamma^{\hat{i}}_{1\hat{i}'}\Gamma^{i}_{23}). \end{aligned} \tag{C85}$$

The covariant derivatives $R^{\hat{i}}{}_{\hat{j}23;1}$ and $R^{\hat{i}}{}_{\hat{j}31;2}$ are obtained using the replacements $123 \to 231$ and $123 \to 312$ in Eq. (C85).

It is now almost straightforward to see the identity (C84). For example, we see in Eq. (C85) that the second derivative terms of affine connection

$$\Gamma^{\hat{i}}_{2\hat{j},13} - \Gamma^{\hat{i}}_{1\hat{j},23} \tag{C86}$$

are contained in $R^{\hat{i}}{}_{\hat{j}12;3}$. Combined with the contributions from $R^{\hat{i}}{}_{\hat{j}23;1}$ and $R^{\hat{i}}{}_{\hat{j}31;2}$, we see these second derivative terms disappear in $R^{\hat{i}}{}_{\hat{j}12;3} + R^{\hat{i}}{}_{\hat{j}31;2} + R^{\hat{i}}{}_{\hat{j}23;1}$,

$$(\Gamma^{\hat{i}}_{2\hat{j},31} - \Gamma^{\hat{i}}_{1\hat{j},23}) + (\Gamma^{\hat{i}}_{3\hat{j},12} - \Gamma^{\hat{i}}_{2\hat{j},31}) + (\Gamma^{\hat{i}}_{1\hat{j},23} - \Gamma^{\hat{i}}_{3\hat{j},12}) = 0. \tag{C87}$$

Note also that

$$-R_{\hat{i}\hat{j}^*12;3} = R_{\hat{j}^*\hat{i}12;3} = (g_{\hat{k}\hat{j}^*}R^{\hat{k}}{}_{\hat{i}12})_{;3} = g_{\hat{k}\hat{j}^*;3}R^{\hat{k}}{}_{\hat{i}12} + g_{\hat{k}\hat{j}^*}R^{\hat{k}}{}_{\hat{i}12;3} = g_{\hat{k}\hat{j}^*}R^{\hat{k}}{}_{\hat{i}12;3}. \tag{C88}$$

Multiplying $g_{\hat{i}\hat{j}^*}$ by Eq. (C84), we therefore obtain

$$R_{\hat{i}\hat{j}^*12;3} + R_{\hat{i}\hat{j}^*23;1} + R_{\hat{i}\hat{j}^*31;2} = 0. \tag{C89}$$

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